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Nicholas Pippenger
Harvey Mudd College

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GENERALIZED CONNECTORS*

NICHOLAS PIPPENGER†

Abstract. An n-connector is an acyclic directed graph having n inputs and n outputs and satisfying the following condition: given any one-to-one correspondence between inputs and distinct outputs, there exists a set of vertex-disjoint paths that join each input to the corresponding output. It is known that the minimum possible number of edges in an n-connector lies between lower and upper bounds that are asymptotic to $3n \log_3 n$ and $6n \log_3 n$ respectively. A generalized n-connector satisfies the following stronger condition: given any one-to-many correspondence between inputs and disjoint sets of outputs, there exists a set of vertex-disjoint trees that join each input to the corresponding set of outputs. It is shown that the minimum number of edges in a generalized n-connector is asymptotic to the minimum number in an n-connector.

Imagine an information transmission network intended to mediate between n sources of information and n users of this information. At any time, any of the users may wish to be connected with any of the sources; a user can be connected with only one source at a time, but many users may wish to be connected with the same source. This paper deals with an idealized version of the problem of designing a network capable of providing any such pattern of simultaneous connections.

An (n, m)-*graph* is an acyclic directed graph with a set of n distinguished vertices called *inputs* and a disjoint set of m distinguished vertices called *outputs*. An n-graph is an (n, n)-graph.

An n-*connector* is an n-graph satisfying the following condition: given any one-to-one correspondence between inputs and distinct outputs, there exists a set of vertex-disjoint paths that join each input to the corresponding output. (A *path* joining an input to an output is a directed path whose origin is the input and whose destination is the output.) Let $c(n)$ denote the minimum possible number of edges in an n-connector; it is known that

$$3n \log_3 n \leqq c(n) \leqq 6n \log_3 n + O(n)$$

(see Pippenger and Valiant [4, Remark 2.2.6]).

A *generalized* n-*connector* is an n-graph satisfying the following stronger condition: given any one-to-many correspondence between inputs and disjoint sets of outputs, there exists a set of vertex-disjoint trees that join each input to the corresponding set of outputs. (A *tree* joining an input to a set of outputs is a directed tree whose root is the input and whose leaves are the outputs.) Let $d(n)$ denote the minimum possible number of edges in a generalized n-connector; that

$$d(n) \leqq 10n \log_2 n + O(n)$$

for n a power of 2 is implicit in the work of Ofman [1]. Thompson [5] has recently shown that

$$d(n) \leqq 12n \log_3 n + O(n)$$

for n a power of 3.

The object of this note is to show that

$$d(n) = c(n) + O(n),$$

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† Mathematical Sciences Department, IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598.

and thus that

$$d(n) \sim c(n).$$

It is clear that

$$d(n) \geqq c(n);$$

thus it will suffice to show that

$$(1) \qquad d(n) \leqq c(n) + O(n).$$

This will be done by means of a new type of graph which will be called a generalizer. An *n-generalizer* is an n-graph that satisfies the following condition: given any correspondence between inputs and nonnegative integers that sum to at most n, there exists a set of vertex-disjoint trees that join each input to the corresponding number of distinct outputs. Let $g(n)$ denote the minimum possible number of edges in an n-generalizer; it will be shown below that

$$(2) \qquad g(n) \leqq 120n + O((\log n)^2),$$

so that in particular

$$g(n) = O(n).$$

A generalized n-connector can be obtained from an n-generalizer and an n-connector by identifying the outputs of the generalizer with the inputs of the connector, as shown in Fig. 1. It is obvious that this yields a generalized n-connector: the generalizer provides the appropriate number of copies of each input, and the connector joins these copies to the appropriate outputs. Thus

$$\begin{aligned} d(n) &\leqq c(n) + g(n) \\ &\leqq c(n) + O(n), \end{aligned}$$

which completes the proof of (1).

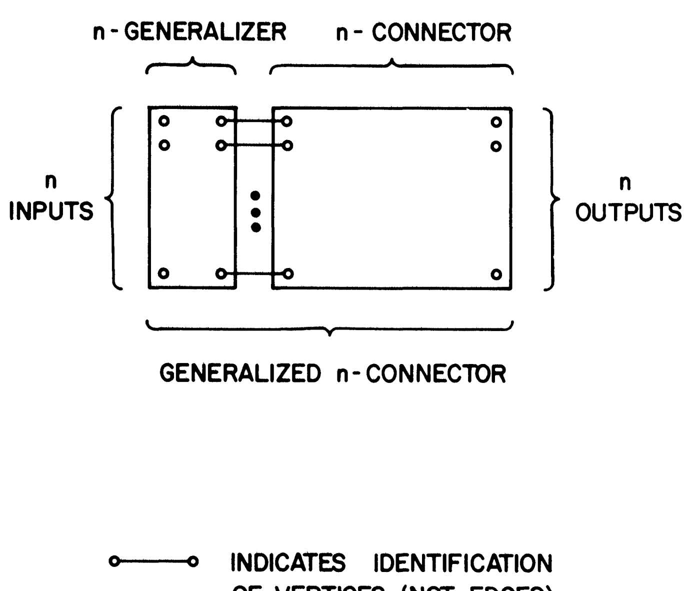

FIG. 1.

It remains to prove (2). To do this, two more types of graphs, called concentrators and superconcentrators, will be needed.

An n-superconcentrator is an n-graph that satisfies the following condition: given any set of inputs and any equinumerous set of outputs, there exists a set of vertex-disjoint paths that join the given inputs in a one-to-one fashion to the given outputs. Let $s(n)$ denote the minimum possible number of edges in an n-superconcentrator; that

$$s(n) \leqq 234n$$

was shown by Valiant [6], who first defined superconcentrators. Pippenger [3] subsequently showed that

$$s(n) \leqq 39n + O(\log n).$$

An (n, m)-*concentrator* is an (n, m)-graph that satisfies the following condition: given any set of m or fewer inputs, there exists a set of vertex-disjoint paths that join the given inputs in a one-to-one fashion to distinct outputs. Let $r(n, m)$ denote the minimum possible number of edges in an (n, m)-concentrator; that

$$r(n, m) \leqq 29n$$

was shown by Pinsker [2], who first defined concentrators. It will now be shown that

$$r(n, \lfloor n/2 \rfloor) \leqq 20n + O(\log n), \tag{3}$$

where $\lfloor \cdots \rfloor$ denotes "the greatest integer less than or equal to ...".

A $(n, \lfloor n/2 \rfloor)$-concentrator can be obtained by combining $\lfloor n/2 \rfloor$ edges with an $\lceil n/2 \rceil$-superconcentrator (where $\lceil \cdots \rceil$ denotes "the least integer greater than or equal to ..."), as shown in Fig. 2. It is obvious that this yields an $(n, \lfloor n/2 \rfloor)$-

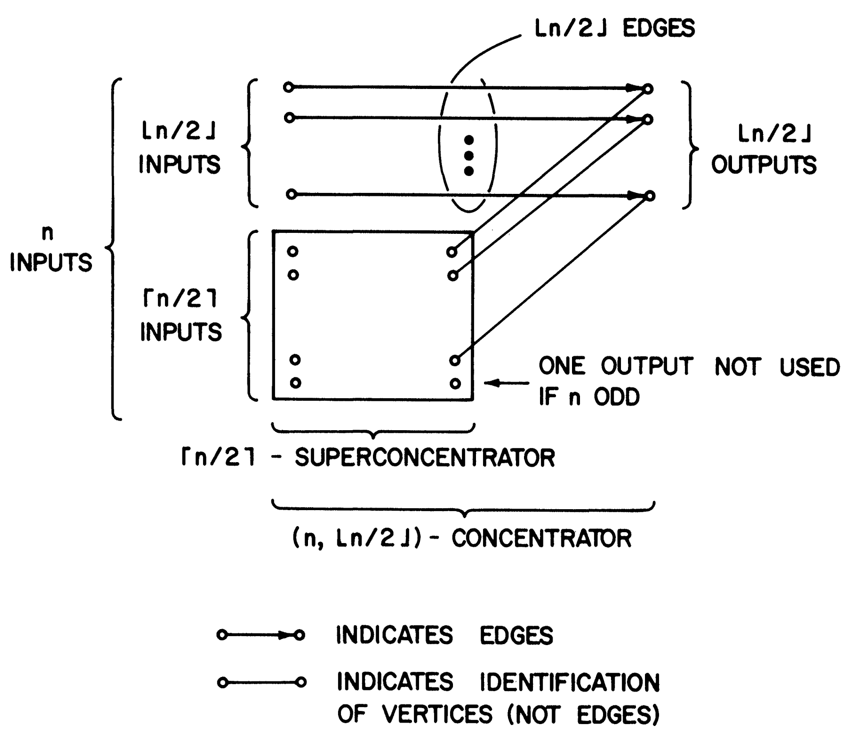

FIG. 2.

concentrator: those of the given inputs that lie among the upper $\lfloor n/2 \rfloor$ inputs can be joined to distinct outputs through the edges; those that lie among the lower $\lceil n/2 \rceil$ can be joined to other distinct outputs through the superconcentrator. Thus

$$\begin{aligned} r(n, \lfloor n/2 \rfloor) &\leqq \lfloor n/2 \rfloor + s(\lceil n/2 \rceil) \\ &\leqq \lfloor n/2 \rfloor + 39\lceil n/2 \rceil + O(\log \lceil n/2 \rceil) \\ &\leqq 20n + O(\log n), \end{aligned}$$

which completes the proof of (3).

It still remains to prove (2). This will be done by means of a recursive construction: an n-generalizer can be obtained by combining an $(n, \lfloor n/2 \rfloor)$-concentrator, an $\lfloor n/2 \rfloor$-generalizer, $2\lfloor n/2 \rfloor$ edges, and an n-superconcentrator, as shown in Fig. 3. This can be seen to yield an n-generalizer as follows. If an input is to be joined to x

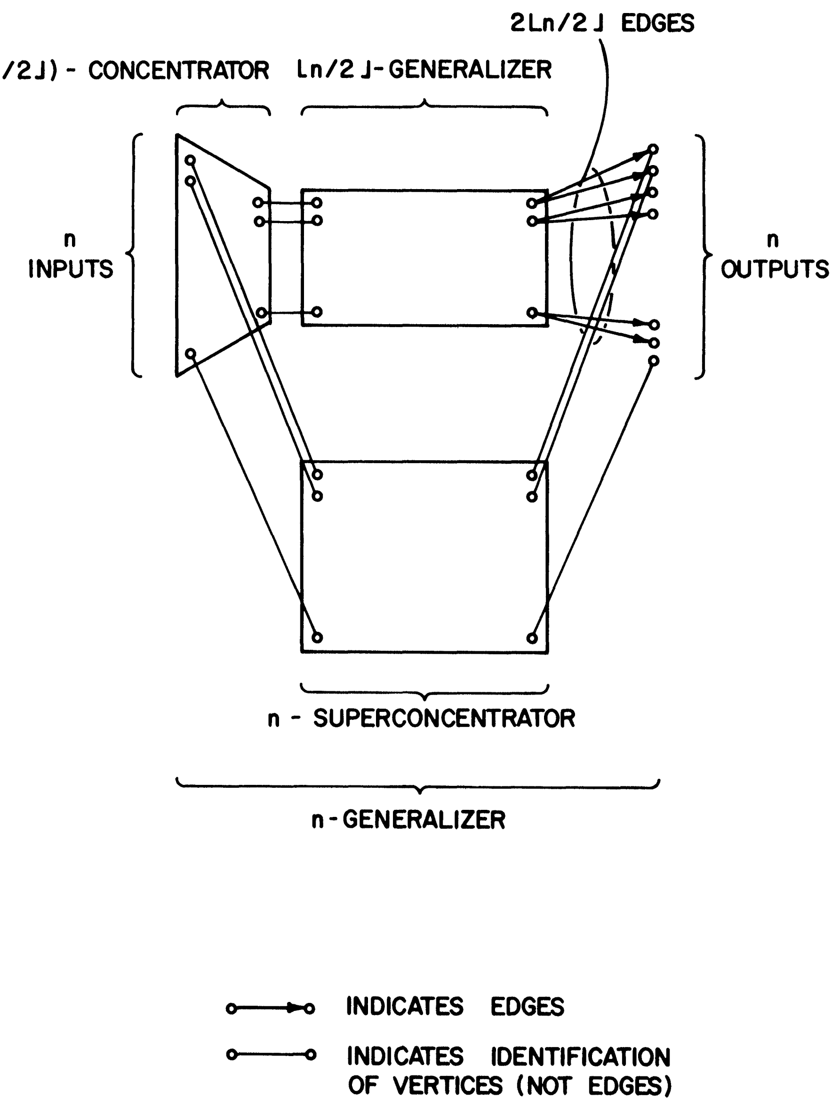

FIG. 3.

distinct outputs, one can write $x = 2y + z$, where y is a nonnegative integer and z is either 0 or 1. Since the x's sum to at most n, there can be at most $\lfloor n/2 \rfloor$ inputs for which y is greater than 0. Each of these inputs can therefore be joined to a distinct output of the concentrator, thence to y distinct outputs of the $\lfloor n/2 \rfloor$-generalizer, and finally to $2y$ distinct outputs of the n-generalizer. All that remains is to join the inputs for which z is 1 to other distinct outputs; this can be done through the superconcentrator. Thus

$$\begin{aligned} g(n) &\leqq g(\lfloor n/2 \rfloor) + r(n, \lfloor n/2 \rfloor) + 2\lfloor n/2 \rfloor + s(n) \\ &\leqq g(\lfloor n/2 \rfloor) + 20n + O(\log n) + 2\lfloor n/2 \rfloor + 39n + O(\log n) \\ &\leqq g(\lfloor n/2 \rfloor) + 60n + O(\log n) \\ &\leqq 120n + O((\log n)^2), \end{aligned}$$

which completes the proof of (2).

The result of this note is satisfying from a theoretical point of view: information-theoretic considerations suggest that since

$$\log n^n = \log n! + O(n)$$

one should have

$$d(n) = c(n) + O(n),$$

as has indeed been shown to be the case. The proof technique used in this note, however, does not endow the result with any practical significance: $120n$ exceeds $6n \log_3 n$ until n exceeds $3^{20} = 3{,}486{,}784{,}401$.

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