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Superconcentrators

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SUPERCONCENTRATORS*

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Abstract. An n-superconcentrator is an acyclic directed graph with n inputs and n outputs for which, for every $r \leqq n$, every set of r inputs, and every set of r outputs, there exists an r-flow (a set of r vertex-disjoint directed paths) from the given inputs to the given outputs. We show that there exist n-superconcentrators with $39n + O(\log n)$ (in fact, at most $40n$) edges, depth $O(\log n)$, and maximum degree (in-degree plus out-degree) 16.

Key words. superconcentrator, concentrator, directed graph

Superconcentrators were defined by Valiant [1] who showed that there exist n-superconcentrators with at most $238n$ edges. Superconcentrators have proved useful in counterexemplifying conjectures [1] and in demonstrating the optimality of algorithms [2].

Valiant's proof was based on a complicated recursive construction which used a related type of graph, called a "concentrator," as a basic element. Concentrators were defined by Pinsker [3], who showed that there exist (n, m)-concentrators (which we shall not define here), with at most $29n$ edges. Pinsker's proof was based on another rather complicated recursive construction which used a nonconstructive existence theorem concerning bipartite graphs as a basic element. This theorem, though not the recursive construction for concentrators, was also obtained independently by the author [4].

The purpose of this note is to give a sharpened version of the nonconstructive existence theorem and a simple recursive construction, using this theorem as a basic element, for superconcentrators. This yields four benefits. First, the proof that n-superconcentrators with $O(n)$ edges exist is greatly simplified; our construction is simpler than Pinsker's, let alone its composition with Valiant's. Second, our n-superconcentrators have depth $O(\log n)$; Valiant's have depth $O((\log n)^2)$. Third, our superconcentrators have maximum degree (in-degree plus out-degree) 16; Pinsker's concentrators (and thus Valiant's superconcentrators) do not have maximum degree $O(1)$. Finally, our n-superconcentrators have $39n + O(\log n)$ (in fact, at most $40n$) edges.

LEMMA. *For every m, there exists a bipartite graph with $6m$ inputs and $4m$ outputs in which every input has out-degree at most 6, every output has in-degree at most 9, and, for every $k \leqq 3m$ and every set of k inputs, there exists a k-flow (a set of r vertex-disjoint directed paths) from the given inputs to some set of k outputs.*

Proof. Let π be a permutation on $\mathcal{M} = \{0, 1, \cdots, 36m-1\}$. From π we obtain a bipartite graph $G(\pi)$ by taking $\{0, 1, \cdots, 6m-1\}$ as inputs, $\{0, 1, \cdots, 4m-1\}$ as outputs, and, for every x in $\mathcal{M}$, an edge from $(x \bmod 6m)$ to $(\pi(x) \bmod 4m)$. In $G(\pi)$, every input has out-degree at most 6 (since there are only 6 elements of $\mathcal{M}$ in each residue class mod $6m$) and each output has in-degree at most 9 (since there are only 9 elements of $\mathcal{M}$ in each residue class mod $4m$).

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We shall say that a graph $G(\pi)$ is "good" if there do not exist a $k \leqq 3m$, a set A of k inputs, and a set B of k outputs such that every edge directed out of A is directed into B; we shall say that it is "bad" otherwise. If $G(\pi)$ is good, the marriage theorem (see Hall [5]) ensures that for every $k \leqq 3m$ and every set of k inputs, there exists a k-flow from the given inputs to some set of k outputs, so that $G(\pi)$ satisfies the requirements of the lemma. We shall show that there exists such a graph by obtaining an upper bound, less than unity for all m, on the fraction of all permutations π for which $G(\pi)$ is bad.

Any set A of k inputs corresponds to a set $\mathscr{A}$ of $6k$ elements of $\mathscr{M}$, and any set of B of k outputs corresponds to a set $\mathscr{B}$ of $9k$ elements of $\mathscr{M}$. Every edge of $G(\pi)$ directed out of A will be directed into B only if π sends every element of $\mathscr{A}$ into $\mathscr{B}$. Of the $(36m)!$ permutations of $\mathscr{M}$, there are $[9k]_{6k}\,(36m-6k)!$ that satisfy this condition, where $[n]_r = n(n-1)\cdots(n-r+1)$. For a given value of k, there are $\binom{6m}{k}$ possible choices for A and $\binom{4m}{k}$ possible choices for B.

Thus an upper bound on the fraction of all permutations π for which $G(\pi)$ is bad is

$$I_m = \sum_{1 \leqq k \leqq 3m} \binom{6m}{k}\binom{4m}{k}\frac{[9k]_{6k}(36m-6k)!}{(36m)!} = \sum_{1 \leqq k \leqq 3m} \frac{\binom{6m}{k}\binom{4m}{k}\binom{9k}{6k}}{\binom{36m}{6k}}$$

We shall show that I_m is less than unity.

1. We first observe that

$$\binom{36m}{6k} \geqq \binom{6m}{k}\binom{4m}{k}\binom{26m}{4k},$$

for the number of ways of choosing $6k$ out of $36m$ objects is not less than the number of ways of choosing k out of the first $6m$, k out of the next $4m$, and $4k$ out of the last $26m$. Thus I_m is at most

$$J_m = \sum_{1 \leqq k \leqq 3m} \frac{\binom{9k}{6k}}{\binom{26m}{4k}}.$$

2. To find the largest term in J_m, we set

$$L_k = \frac{\binom{9k}{6k}}{\binom{26m}{4k}}$$

and observe that the ratio of successive terms can be written as

$$\frac{L_{k+1}}{L_k}=\frac{(9k+9)\cdots(9k+7)(9k+6)\cdots(9k+1)(4k+4)(4k+3)\cdots(4k+1)}{(6k+6)\cdots(6k+1)\qquad(3k+3)\cdots(3k+1)(26m-4k)\cdots(26m-4k-3)}.$$

Each vertically aligned factor or pair of factors is an increasing function of k. Thus L_{k+1}/L_k is increasing, $L_{k-1}L_{k+1}/L_k^2$ is greater than unity, and the largest term of J_m must be either the first (L_1) or the last (L_{3m}).

3. If the largest term is the first, then J_m is at most

$$3mL_1=3m\frac{\binom{9}{6}}{\binom{26m}{4}}=\frac{3024}{13(26m-1)(26m-2)(26m-3)},$$

which is less than unity for all $m\geqq 1$.

4. If the largest term is the last, then J_m is at most

$$3mL_{3m}=3m\frac{\binom{27m}{18m}}{\binom{26m}{12m}}=3m\frac{(27m)!\,(12m)!\,(14m)!}{(18m)!\ (9m)!\ (26m)!}.$$

We shall use Stirling's formula in the form

$$(2\pi n)^{1/2}\,e^{-n}n^n\leqq n!\leqq e^{1/12n}(2\pi n)^{1/2}\,e^{-n}n^n$$

(see Robbins [6]), together with

$$e^x\leqq\frac{1}{1-x}\qquad (x\leqq 1)$$

which implies

$$n!\leqq\left(\frac{12n}{12n-1}\right)(2\pi n)^{1/2}\,e^{-n}n^n.$$

These inequalities give

$$3mL_{3m}\leqq 3m\left(\frac{324m}{324m-1}\right)\left(\frac{144m}{144m-1}\right)\left(\frac{168m}{168m-1}\right)\cdot\left(\frac{27\ 12\ 14}{18\ \ 9\ 26}\right)^{1/2}\left(\frac{27^{27}\ 12^{12}\ 14^{14}}{18^{18}\ \ 9^9\ 26^{26}}\right)^m,$$

which is less than unity for all $m\geqq 3$. (The bound for $m=3$ is easily evaluated with a table of logarithms and a calculator. Futhermore, the bound is a decreasing function of m, since if m is increased by 1, the first factor increases by a factor of at most $4/3$, the next three factors decrease, and the last factor decreases by a factor exceeding 2.)

5. In the remaining cases, $m=1$ and $m=2$, I_m can be evaluated with a table of binomial coefficients (for example, Miller [7]), and is less than unity. □

COROLLARY. *For every m, there exists a bipartite graph with* $4m$ *inputs and* $6m$ *outputs in which every input has out-degree at most* 9, *every output has in-degree at most* 6, *and, for every* $k \leqq 3m$ *and every set of k outputs, there exists a k-flow to the given outputs from some set of k inputs.*

Proof. Exchange the roles of inputs and outputs and reverse the directions of edges and flows in the lemma. □

Let $s(n)$ denote the minimum possible number of edges in an n-superconcentrator. Let

$$\theta(n) = 4\left\lceil \frac{n}{6} \right\rceil,$$

where $\lceil \cdot \rceil$ denotes "the smallest integer not less than".

THEOREM. *For any n,* $s(n) \leqq 13n + s(\theta(n))$.

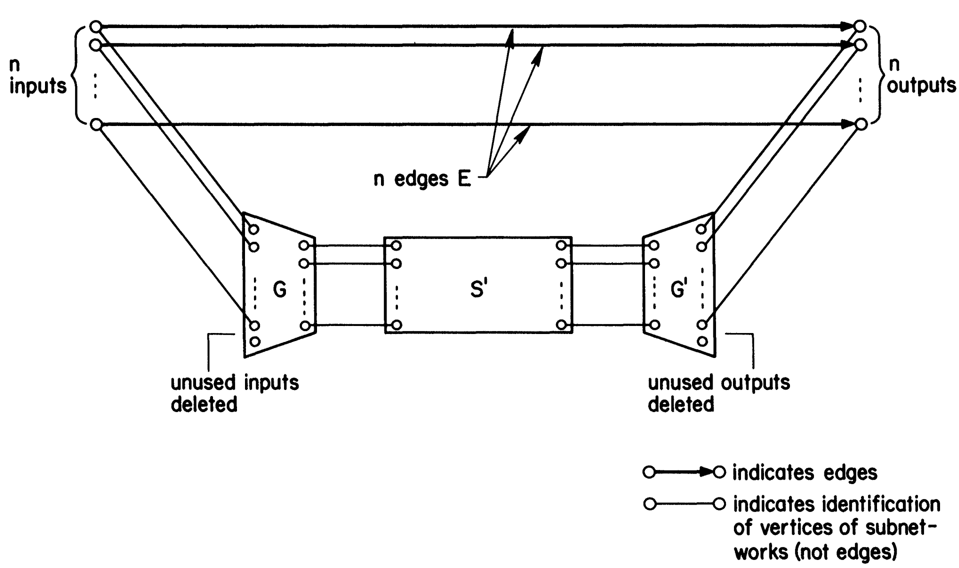
Proof. Let

$$m = \left\lceil \frac{n}{6} \right\rceil.$$

Let G and G' be bipartite graphs satisfying the lemma and corollary, respectively, and let S' be a $4m$-superconcentrator with $s(4m)$ edges. The graph S is obtained by deleting $6m - n$ inputs (and the edges directed out of them) from G, identifying the outputs of G with the inputs of S', identifying the outputs of S' with the inputs of G', deleting $6m - n$ outputs (and the edges directed into them) from G', and adding a set E of n edges from the surviving inputs of G to the surviving outputs of G'. This is illustrated in the figure below.

The graph S clearly has $13n + s(\theta(n))$ edges. All that remains is to verify that S is an n-superconcentrator.

For some $r \leqq n$, let X be a set of r inputs and let Y be a set of r outputs. Let X be partitioned into two parts: X_0, the vertices of X that correspond through E to



vertices in Y, and X_1, the vertices of X that correspond through E to vertices not in Y. Similarly, let Y be partitioned into Y_0 (corresponding to vertices in X) and Y_1 (corresponding to vertices not in X). There is an l-flow from X_0 through E to Y_0, where l is the common cardinality of X_0 and Y_0. The set X_1 corresponds through E to a set of vertices disjoint from and equinumerous with Y_1. Thus X_1 and Y_1 have a common cardinality $k \leqq n/2 \leqq 3m$. By the lemma, there is a k-flow from X_1 to some set X' of k outputs of G, and by the corollary, there is a k-flow from some set Y' of k inputs of G' to Y_1. Finally, by inductive hypothesis, there is a k-flow from X' through S' to Y'. These four flows together constitute an r-flow from X to Y. □

From this theorem it is clear that $s(n) \leqq 39n + O(\log n)$, and that this can be accomplished by graphs with depth $O(\log n)$ and maximum degree 16. Since it is often helpful to have an explicit bound, we shall show that $s(n) \leqq 40n$.

For small values of n we shall use a "rearrangeable connection network" or "permutation network." Such a network contains an n-flow following any prescribed mapping from its inputs to its outputs, and is, *a fortiori*, an n-superconcentrator. A well-known recursive construction for these networks gives

$$s(n) \leqq 3n(2\lceil \log_3 n \rceil - 1)$$

(see Beneš [8, Thm. 3.1]; in the outer stages use 3-by-3 switches, with at most one smaller switch when n is not a multiple of 3; in the inner stages use this construction recursively). This gives $s(n) \leqq 39n$ for $n \leqq N = 3^7 = 2187$.

For large values of n we shall apply the theorem recursively. Define

$$\theta^0(n) = n,$$
$$\theta^{t+1}(n) = \theta(\theta^t(n)).$$

Then applying the theorem $t+1$ times gives

$$s(n) \leqq 13(\theta^0(n) + \theta^1(n) + \cdots + \theta^t(n)) + s(\theta^{t+1}(n)).$$

Let us choose t such that

$$\theta^t(n) > N \geqq \theta^{t+1}(n).$$

Then by the result of the preceding paragraph

$$s(n) \leqq 13(\theta^0(n) + \theta^1(n) + \cdots + \theta^t(n)) + s(\theta^{t+1}(n)).$$

We note that

$$\begin{aligned} \theta(n) &= 4\left\lceil \frac{n}{6} \right\rceil \\ &\leqq 4\left(\frac{n}{6} + \frac{5}{6}\right) \\ &= \frac{2}{3}n + \frac{10}{3}, \end{aligned}$$

and $\theta(n)$ is even. Furthermore, if n is even,

$$\theta(n)=4\left\lceil\frac{n}{6}\right\rceil$$
$$\leqq 4\left(\frac{n}{6}+\frac{2}{3}\right)$$
$$=\frac{2}{3}n+\frac{8}{3},$$

and again $\theta(n)$ is even. Thus, by induction on t,

$$\theta^t(n)\leqq\left(\frac{2}{3}\right)^t n+8.$$

Applying this to the result of the preceding paragraph gives

$$s(n)\leqq 39n+104(t+3).$$

Next we note that if $n\geqq N=3^7=2187$,

$$\theta(n)=4\left\lceil\frac{n}{6}\right\rceil$$
$$\leqq 4\left(\frac{n}{6}+\frac{5}{6}\right)$$
$$=\left(\frac{2}{3}+\frac{10}{3n}\right)n$$
$$\leqq\frac{4384}{6561}n.$$

Thus, by induction on t, if $\theta^0(n), \theta^1(n), \cdots, \theta^{t-1}(n)\geqq N$,

$$\theta^t(n)\leqq\left(\frac{4384}{6561}\right)^t n.$$

From the condition defining t it follows that

$$t\leqq\frac{\log\frac{n}{2187}}{\log\frac{6561}{4384}}.$$

Now

$$\log\tfrac{6561}{4384}\geqq\tfrac{1}{3}\log 3,$$

and therefore

$$t\leqq 3\log_3 n-21.$$

Furthermore, if $n\geqq N$,

$$\frac{3\log_3 n}{n}\leqq\frac{3\log_3 N}{N}=\frac{7}{729},$$

and therefore

$$t \leqq \frac{7}{729} n - 21,$$

or

$$104(t+3) \leqq \frac{728}{729} n - 1872.$$

Combining this with the result of the preceding paragraph gives

$$s(n) \leqq 40n.$$

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