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Missing Dimensions of Mathematics Instruction

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A few years ago, I began work leading a project, funded by the National Science Foundation, to create a new gateway course to the mathematics major which would be an alternative to Calculus. It is not my purpose here to argue the pros and cons of this idea (interested parties should contact COMAP[1] for sample materials or more extensive descriptions). The original grant had an objective which was consistent with, but not explicitly centered on a humanistic approach. During the work of the grant, I formulated some further ideas for our authors to think about and circulated it as a position paper[2]. In that paper I explicitly addressed questions of a humanistic nature. As the ideas in that paper were not an explicit part of the original proposal and were not mentioned while recruiting authors, I did not feel justified in insisting that those ideas be addressed in writing our curriculum materials. Nonetheless, one member of the Humanistic Mathematics Network, Professor Harald Ness (see the companion article in this issue), feels that our curriculum materials do show a humanistic touch. It is impossible to publish our entire 1-year course here, but readers of this Journal might find my position paper of interest. The present article is a slightly edited version of that original position paper.

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In writing the proposal for this project, my thoughts were mainly on how to express the breadth of mathematics itself. But there are other dimensions along which we can seek breadth in our instruction, dimensions which are little stressed at the present time. These missing dimensions could be decisive in making mathematics attractive to a larger number of students by making mathematics seem less isolated and more tied to thoughts and experiences that our students find familiar and congenial.

These dimensions include:

1. mathematics as an element of culture, evolving as civilization does;
2. mathematics as a social experience as well as a solitary one;
3. the hands-on approach to learning mathematics;
4. the connection of mathematics with major themes in our quest to understand the world about us;
5. the roots of mathematics in student interests and experience; and
6. mathematics as a story.

Other dimensions could perhaps be added. And, of course, the ones presented here are not independent. (For example, the value of applications occurs in a number of these dimensions.)

It is hard enough to teach the basic technicalities of mathematics. Why complicate things with all these other dimensions? Because, by restricting the scope of mathematics instruction to mathematics as an isolated endeavor, we have walled mathematics off from our culture to a degree that its future is problematical. Mathematics instruction mostly ignores these dimensions and is normally presented in a purely rational, emotionally flat and self-contained way, often detached from the interests and experience of ordinary people. The result is an image problem for mathematics and mathematicians. What is worse, many mathematicians acquiesce approvingly in the idiosyncratic image created, as illustrated by the following quote:

> An eloquent mathematician must, from the nature of things, ever remain as rare a phenomenon as a talking fish.... He has to turn his eye ever inwards, to see everything in its driest light, to train and inure himself to a habit

> of internal and impersonal reflection and elaboration of abstract thought, which makes it most difficult for him to touch or enlarge upon any of those themes which appeal to the emotional nature of his fellow men [Sylvester 1991].

I believe that such a "dry" attitude toward mathematics is not inevitable. Imagine what it would have been like to have been a mathematics student of Galileo! Or to have been present at the banquet when Pythagoras (allegedly) sacrificed an ox in honor of the discovery of his famous theorem. Or to have been on deck when the follower of Pythagoras was thrown overboard because of his discovery that the square root of 2 was irrational. Of course, these stories about Pythagoras may be myths, but the people who constructed them surely cared deeply about mathematics because they regarded it as something more than mental gymnastics.

Throughout its history, mathematics was often regarded as deeply meaningful. Consequently, it often aroused passion. We have the example of Gauss being afraid to publish his thoughts on non-Euclidean geometry for fear of ridicule; and Cantor reaping a harvest of ridicule and opposition for his ideas about infinity, perhaps because of their mystical-religious connotations. In our own time, I have heard heated arguments develop over the value (or lack of it) of chaos, fractals, and catastrophe theory. But today's mathematics instruction in our classrooms is mostly about craftsmanship. Our teaching style makes it hard for students to see that we find mathematics deeply meaningful and feel strongly about it.

Let me elaborate on some of these thoughts about the missing dimensions of instruction in mathematics. I leave as an unsolved problem how items on this list can influence the design of a broad one-year introduction to college mathematics.

1. Mathematics as an Element of Culture

Mathematics has played a central role in our culture, but we mostly ignore this in our teaching. It is also true (but less well understood) that culture influences how mathematics evolves. A course that reveals this interplay can help dispel the notion that mathematics is a static body of revealed knowledge. Consider, as an example, the fact that the relatively modern study of connectivity (in topology and especially in graph theory) parallels the fact that there is, in the modern world, an unprecedented degree to which human beings and their institutions and towns are connected. This changes human consciousness. A traveler returning from China today does not make news, but when Marco Polo did so, this caused a sensation that reverberates in the history books. The telephone company advertising jingle, "We're all connected" could be a subtitle for large chunks of modern mathematics.

An important aspect of this is that, as culture evolves, so does mathematics. In a time when much of the world's geography has been explored, and space exploration is restricted to astronauts, mathematics offers fertile ground for exploring the unknown. A broad approach to mathematics instruction should include much modern mathematics, including some which displays the symbiosis between mathematics and culture.

2. Social Meaning of Mathematics

One of the many revolutions that challenge us is the development of computer software capable of doing manipulations that were once the hallmark of a capable and promising student. It has been remarked that for a modest sum, one can buy a pocket computer which could get a high grade on a traditional Calculus final. Our initial response is to change what we teach—more understanding and less rote. This is an excellent response but it is not

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a complete solution. Artificial Intelligence programs that can solve word problems have already been devised, and computers are being taught to prove theorems as well. The program SRE+VE, developed by L. Hines and W. Bledsoe, can prove that the sum of two continuous functions is continuous [*Notices* 1991]. Perhaps in the long run computers will be able to get high grades on anything that most of our students can be taught to

do. If we depend upon mechanical competence to distinguish us from machines, we will, at best, be forever one step ahead of the Artificial Intelligence researchers. We need instead to insist that an essential component of mathematics is that it is done by human beings, as a social activity, to which meaning is attached and about which people feel deeply. Although we need to take advantage of computers, we cannot define mathematics to be just that which cannot be done by machines.

One way in which one might emphasize the social meaning of mathematics, is to have mathematics be learned as a social activity instead of a solitary one, perhaps using small-group learning methods, which are currently under investigation. Is it possible to structure a text so that it can equally well be used for traditional instruction and for small-group instruction as well?

3. A Hands on Approach

Mathematics is already a hands on subject since we are always manipulating symbols to do calculations or construct proofs. But we can supplement this with "real" objects for experiment: puzzles, soap films, physical devices giving rise to chaos, computer software, etc. Thurston, Doyle and Conway [1991] have devised a "Geometry and the Imagination" course at Princeton exploiting the possibilities inherent in paper, tape and scissors. In England, Celia Hoyles has been the mathematical host of a prime-time TV program on mathematical puzzles [Howson and Kahane 1990].

In our pencil and paper work with symbols, we need to make sure that students understand the symbols they are manipulating. We are often preoccupied with "covering" the material. We should be mindful of the double meaning of "cover" and concentrate instead on revealing the mathematics. (I am indebted to Bill Thurston, Peter Doyle and John Conway for this neat linguistic distinction.)

4. Stress Major Themes and Imagination as well as Craftsmanship

Einstein has remarked [1950] that the hallmark of science and mathematics, curiosity to understand the world around us, is inherent in children and is sometimes extinguished as they grow up. But we can try to reawaken this childlike curiosity. One way to be successful at this is to link our instruction to major themes of perennial interest, such as the following three.

A. One such major theme is our desire to predict phenomena in the world around us. Today we can predict the arrivals of comets and solar eclipses with precision far beyond that needed for most practical purposes, but we can't tell whether it will rain tomorrow. This extra-ordinary unevenness of scientific development could be a useful "hook" to enlist student interest in dynamical systems. Devaney has written a recent book [1990] that presents some of these ideas using no more than high school mathematics. The idea of sensitive dependence on initial conditions can be communicated without lots of mathematical machinery. Examples where the initial conditions are not very crucial can be shown for comparison. This is an obvious example where computational assistance, especially computer graphics, can be valuable.

B. Another major theme in mathematics that often evokes wonder in ordinary people is the concept of infinity. Infinity is inherently mysterious (and psychologically difficult) and has philosophical and religious resonances for many students. None of this is exploited in Calculus where the treatment is quick and matter-of-fact. If we give students a variety of experiences with infinity we may be able to make it more understandable while still exploiting its potential for intriguing students. We could dwell on geometric series, contrasting it with the harmonic series. We could describe the philosophical problems Zeno raised. We could show some fractal pictures. Although our goal is to build understanding and not create confusion, we might even display some of Cantor's paradoxical discoveries.

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C. A third major theme concerns one of the great intellectual watersheds of the twentieth century: the

discovery that the physical world doesn't conform too well to common sense. When dealing with the very small or the very large (quantum theory or relativity), only mathematics is capable of dealing with phenomena that appear odd to common sense. A small corner of this story can be approached through consideration of non-Euclidean geometry. Prenowitz and Jordan [1989] begin their geometry book with a short, informal, high school level introduction to non-Euclidean geometry.

5. Root Our Instruction in Student Interests and Experience

In beginning courses, we need to stress what students actually care about. The usual arrangement between professors and students is for students to learn to care about what the professor says they should care about. When market demand propels students into courses, they eagerly absorb professorial values. But market demand for mathematics is invisible to many of today's students. Consequently, we need to show them how mathematics is relevant to things they care about.

Partly this means applications, since seeing the usefulness of mathematics gives a hint of market demand (even though students would prefer to see "mathematician wanted" ads in the newspapers). The desirability of an applied flavor for some of our courses has been widely accepted by our community and needs little further elaboration here. However, I would like to urge that the applications come first as motivations for beginning students. Calculus has always been an applications-oriented course, but some calculus books have begun with a review of the formal properties of the real numbers. (A recent book begins with a discussion of an automobile odometer and the relation of its readings and the number of miles driven, and this is a great improvement.)

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Probability is a wonderful subject through which we can make contact with the vital interests of ordinary people. Human beings have very poor intuition about matters of chance. There are many reasons for this, not the least of which are emotional biases that prevent us from feeling in our hearts what calculation tells us to be true. Do we not all know someone who is afraid of flying but has no fear of the much greater danger of driving the same mileage in an automobile? Students instinctively realize that they live in a world of probabilistic hazards and opportunities and that proper management of these risks can bring great dividends. Students who do not enter careers in mathematics also realize that if improving their risk management requires remembering theorems and algorithms throughout their lives, they will have to do without the benefit. Might it be possible to use the theorems and algorithms to sharpen intuition; something which might remain even if the technical basis gets rusty?

6. Mathematics as a Story

Can we make a mathematics book which is a "page-turner"— one where students are wondering what happens next? The characteristic of a page-turner is that it is a good story. Many aspects of mathematics can be presented as good stories.

Mathematicians have many of the same passions as poets and politicians and so mathematics partakes of the timeless human dramas of intellectual curiosity and pursuit of beauty, of ambition, rivalry and egotism, even the drive for power and wealth. There are historical aspects to the story too: traditions we followed, and later cast off in the realization that they were blinders. Here and now there are axes being ground and philosophical apple carts being upset. New ideas come along and the old guard beats them back. But some ideas survive anyhow and change things. These are the stories that need to be told.

Here are three great stories from our history, all of which are still evolving. Can we find a way to tell some of these stories at the freshman level?

A. Mathematics and the Social Sciences

The great triumph of Newtonian physics in the seventeenth and eighteenth centuries led intellectuals to wonder whether the domain of human affairs could not be made predictable and understandable through mathematics. Condorcet,

one of the towering figures of the French Enlightenment wrote:

> This advance of the physical sciences ... could not be observed without enlightened men seeking to follow it up in the other sciences; at each step it held out to them the model to be followed

In a eulogy for a friend and mentor, Turgot, he also writes of views he absorbed from this friend:

> A great man whose teaching and example, and above all whose friendship I shall always mourn, was convinced that the truths of the moral and political sciences are susceptible of the same certainty as those forming the system of the physical sciences, even those branches of the these sciences which, like astronomy, seem to approach mathematical certainty .... It was for him that I undertook this work subjecting to calculation questions relating to the common utility. [Baker 1975, p. 197]

Condorcet believed that probability theory would be a mechanism for rational design of laws and the consequent perfection of society. His optimism has not borne fruit. But the story is not over. There continues to be slow but steady application of mathematics to social science. Will mathematics ever be as effective in this area as in the physical sciences? And if not, can we give a mathematical account of where the limitation lies (an "impossibility theorem")? This is one of the major unfinished stories of mathematics.

B. Geometry and the Nature of Space

Modern mathematicians have an understanding of the word "geometry" that would be surprising, perhaps even appalling to Euclid. Gauss, one of the greatest of all mathematicians, was afraid to publish his thoughts on non-Euclidean geometry because they seemed so radical. Similarly, the idea of n-dimensional space is a modern revolution. We have mentioned that non-Euclidean geometry can be approached with a high school mathematics background. Likewise the powerful ideas of dimensionality have been made "popular" by many talented expositors such as Abbot, Banchoff and Rucker. How about revealing some of this to our students?

C. Functions

The concept of function that we teach today is a relative newcomer to mathematics. For the longest time, function meant polynomial with relatively low degree or one of the trigonometric functions. Eventually exponential and logarithmic functions arrived. Still later, we have functions obtained from clever limiting processes or defined as solutions to differential equations. Today we often contemplate, especially in discrete mathematics and computer science, functions that do not deal with numbers (e.g., boolean functions). This increasing generality is part of mathematics' increasing power. Could we tell some of this story?

This increasing generality in the nature of functions we can deal with parallels the increasing generality of geometric structures we deal with. Euclid's geometry is all about rectilinear figures and a few (very few) curvilinear ones—chiefly the conic sections. Oddly, most of the shapes we see about us in the world of nature are neither rectilinear nor conic. But today, we can deal with fractal representations of clouds, mountains, human faces. How is it possible that the geometry of Euclid, which describes so little of our natural world, was so successful? There is a story here. And will our new-found ability to deal with clouds, mountains and faces, be a great turning point in the story or just a little twist in a minor subplot? The promise and uncertainty in this development has many mathematicians on the edge of their seats with anticipation. Shouldn't students know about this current and hot story?

Can we write mathematics in the style described here and make it "work"? The view of mathematical writing described here is only partly new. Great expositors have been stressing these missing dimensions for years, especially in books for the layman. Of course, these books for the layman leave out the precision, rigor, and skill building that we have stressed in our textbooks. I think our students are not so different in motivation from laymen and so it would be useful to combine the two writing styles into one grand synthesis. High-quality books for the layman have been influential in attracting many of today's mathematicians to the field, and, for me, that is good evidence that including the missing dimensions does work.

[1] Available now for field-testing from COMAP, 57 Bedford, Lexington MA 02173, or r.slade @comap.com. To be published by Springer-Verlag.

[2] Originally published in Math 101-102: A New Start For college Mathematics, ed. by Walter Meyer, COMAP, 1992.

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To Stephen W. Hawking

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(originally in Bulgarian)

The space-time collapse,
in the space-time fist,
rises in waves and spins.
Where is Now? When is Here?

Rock at the edge
of human want, chance, and mystery,
what are our faults today?
Where is the eternal Fire?

The Question looks for an Answer.
The Answer asks a new Plan.
Universe of dreams, great
is your temple of stars.

Space-time of the spirit
of desire and eclipse,
what are your strivings?
When will come the Spring?