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The Kapitza–Dirac Effect: An Approach from QED

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by

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Abstract

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The Kapitza-Dirac effect is similar to the canonical experiment on diffraction of electrons through slits in an opaque screen, except that the diffraction grating has been replaced by a standing wave of light. Remarkably, incident electrons are diffracted by the standing light wave almost as if by a standard diffraction grating. Only recently has this effect been confirmed experimentally in this form [1], although it was originally predicted by Kapitza and Dirac almost 70 years ago. This paper examines the relativistic effects involved in this phenomena using the formalism of quantum field theory.

Table of Contents

List of Figures

Acknowledgments

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Chapter 1

Introduction

In 1933 Paul Dirac and P.L. Kapitza predicted that electrons would be deflected in the presence of a standing wave of light. They theorized that, since light is made up of electromagnetic fields, those fields would necessarily have an effect on the path of an electron. Further, since the fields in a light wave are oscillatory, the deflection resulting from a standing wave of light would be similar to that of a diffraction grating made up of a number of slits in an otherwise opaque sheet. Kaptiza and Dirac predicted that the wave-like diffraction of the electron would create a coherent interference pattern. However, they concluded their publication by writing

”We see, therefore, that the experiment could hardly be carried out with ordinary continuous sources of light, and it seems to us that the only possibility would be to produce the illumination by using an intense spark discharge instead of a mercury arc. In this case, a much larger intensity will be obtained for a short time, but, since the magnitude of the phenomena is proportional to the the square of the intensity, this will increase the number of reflected electrons. The calculation of this case is somewhat difficult, since certain losses will be present due to the broadening of the spectral lines in the spark discharge. Actual experiments will have to be made to find out if it is possible to raise the momentary intensity without undue broadening of the lines, in order to raise the intensity of the reflected beam to an observable value.”

A few important developments have been made since Kapitza and Dirac's prediction. Primary among these is the invention of the laser. The coherent nature of laser light allows the increase in intensity mentioned by Kapitza and Dirac with very little spectral broadening. In 2001, a group led by Herman Batelaan at the University of Nebraska demonstrated for the first time the Kapitza-Dirac effect as it had been predicted nearly 70 years before. Their results (covered in more detail in chapter 2) showed the diffraction pattern of incident electrons to be in good agreement with a numerical solution to the Schrodinger equation for the incident electron in the presence of the oscillatory potential caused by the standing light wave.

Another important development, however, has been in the advances of quantum field theory. At present, our best description of the interaction of electrons with light lies not with the Schrodinger equation but with the formalism of quantum electrodynamics (QED). The Schrodinger equation has proved sufficient for a description of the Batelaan group's experiment, using 10 ns, .2 J laser pulses. Where the Schrodinger equation's description of this phenomena may break down is in the high energy regime, using 'relativistic' power levels in the laser, i.e. power levels at which the electron would be expected to reach a significant fraction of the speed of light in its deflection by the laser. The goal of this paper is to provide a description of the Kapitza-Dirac effect in such relativistic cases using QED, and to predict the power levels at which any relativistic effect would dominate the observed diffraction pattern of the electron.

It is shown, however, that the standard perturbative treatment of QED breaks down at the laser intensities at which the Kapitza Dirac effect has been observed.

Chapter 2 of this paper discusses the experiment and results of the Batelaan group

Chapter 3 will introduce the 'coherent' photon states and demonstrate their use in describing standing waves of light.

Chapter 4 begins the QED description of the Kapitza-Dirac effect.

Chapter 5 will conclude with a summary of our major results.

Appendix A will cover the notation and unit system that will be used for this paper, including a glossary of the commonly used symbols.

Chapter 2

Summary of the Results of the Batelaan group

The experiment conducted by Herman Batelaan's research group at the University of Nebraska was set up as follows, (and as shown in Figure 1): laser light from a Nd:YAG laser is reflected off of a lens to create a standing light wave. The light occurs in 10-ns pulses, each with of 0.2 J of energy and a beam waist of 125 μm. A collimated beam of 380 eV electrons is sent through this grating of light, and a diffraction pattern is observed 24cm downstream of the interaction. The observed maxima of the diffraction pattern occur regularly, with a spacing of 55μm= $2\frac{\lambda_{db}}{\lambda_{opt}}(\times 24cm)$, where λ_{db} is the de Broglie wavelength of the electrons and λ_{opt} is 532 nm, the wavelength of the laser light. Since $\lambda_{db} = \frac{2\pi\hbar}{p}$ and $\lambda_{opt} = \frac{c}{2\pi\omega}$, $2\frac{\lambda_{db}}{\lambda_{opt}} = \frac{2\hbar\omega/c}{p}$ corresponds to two photon recoils separating each peak. The factor of 2 is explained classically by the the Batelaan group as the ratio between the light grating periodicity and the wavelength of the light. Figure 2 shows the data gathered by the Batelaan group, as well as a numerical solution to the Schrodinger equation for their experimental parameters. Both figures (and their captions) are taken from the Nature article authored by the group [1]. The Schrodinger equation proved sufficient to describe the Kapitza-Dirac effect in this experiment, but it may not work as well in all energy regimes for the electrons and intensity ranges for the laser light.

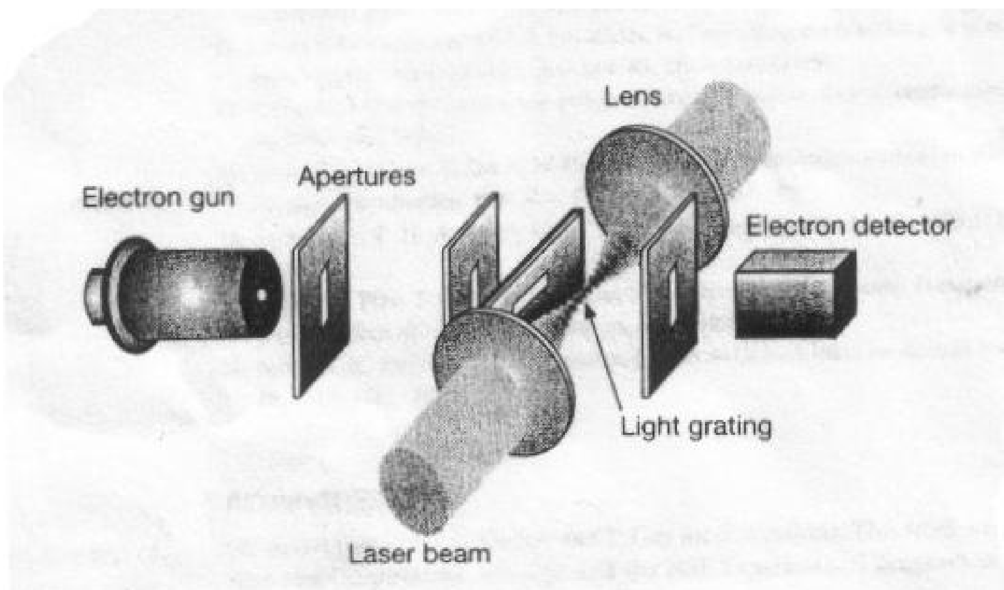

Figure 2.1: Schematic of [Batelaan Group] Apparatus. Electrons are collimated by four molybdenum slits and diffract from a standing wave of light formed by two counter propagating laser beams. The electrons must be described by a quantum mechanical wave while the standing light wave acts as a grating.

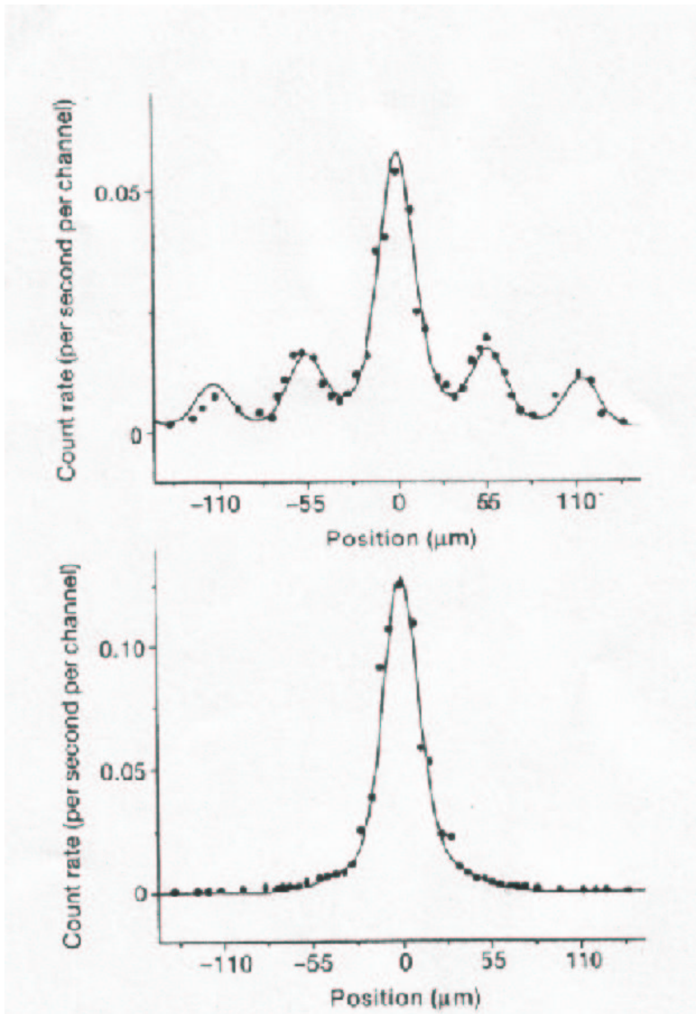

Figure 2.2: Experimental data. The electron detection rate is presented as a function of detector position. [The Batelaan Group's] data (black points) agree reasonably well with a numerical solution of the Schrodinger Equation.... and clearly show diffraction peaks, which is the signature of the Kapitza-Dirac effect. The bottom figure shows the electron beam profile with the laser beams turned off.

Chapter 3

Coherent States and Standing Waves

The first hurdle in finding a quantum field theoretic description of the Kapitza-Dirac effect lies in the determination of the initial state. While the electron can be assumed to begin in a momentum eigenstate while far from the area of the interaction, the photon state is a bit more complicated. A standard photon occupancy state would consist of a definite number of photons with a particular wave vector and indeterminant phase and amplitude. Laser light has within it an indeterminant total number of photons, but the light is known to have definite phase and amplitude. As such, laser light is well described not by an occupancy number, but by a 'coherent' state of light, as defined below.

3.1 Coherent States

A coherent state of light in a particular mode k is defined to be an eigenstate of the photon lowering operator, such that

$$\hat{a_k}|v_k\rangle = v_k|v_k\rangle. \tag{3.1}$$

In terms of the standard photon occupancy (Fock) states for the mode k,

$$|v_k\rangle = e^{-\frac{|v_k|^2}{2}} \sum_{n_k=0}^{\infty} \frac{v_k^{n_k}}{\sqrt{n_k!}} |n_k\rangle \tag{3.2}$$

where $|n_k\rangle$ represents a state with n photons in the mode k

Since the number operator for photons in the mode k is $\widehat{N} = \hat{a}_k^\dagger \hat{a}_k$, the expected number of photons in the state $|v_k\rangle$ is

$$\langle v_k|\widehat{N}|v_k\rangle = \langle v_k|\hat{a}_k^\dagger \hat{a}_k|v_k\rangle = (\langle v_k|\hat{a}_k^\dagger)(\hat{a}_k|v_k\rangle) = v^* v \langle v_k|v_k\rangle = |v_k|^2. \tag{3.3}$$

3.2 The displacement operator

Equation 3.2 can also be written as:

$$|v_k\rangle = e^{-\frac{|v_k|^2}{2}} e^{v_k \hat{a}_k^\dagger} |0\rangle, \tag{3.4}$$

where $|0\rangle$ is the vacuum state.

Since a lowering operator maps the vacuum state to zero, there is no harm in inserting $e^{v_k^* \hat{a}_k}$ next to the vacuum state above. Then

$$\begin{aligned} |v_k\rangle &= e^{-\frac{|v_k|^2}{2}} e^{v_k \hat{a}_k^\dagger} e^{-v_k^* \hat{a}_k} |0\rangle \\ &= e^{v_k \hat{a}_k^\dagger - v_k^* \hat{a}_k} |0\rangle & (3.5) \\ &= \widehat{D}_k(v_k)|0\rangle & (3.6) \end{aligned}$$

where the last line is a definition. The second line above uses the identity that for two operators $\widehat{A}$ and $\widehat{B}$,

$$e^{\widehat{A}+\widehat{B}} = e^{\widehat{A}} e^{\widehat{B}} e^{-\frac{[\widehat{A},\widehat{B}]}{2}} \tag{3.7}$$

provided that $\left[\widehat{A}, [\widehat{A}, \widehat{B}]\right] = \left[\widehat{B}, [\widehat{A}, \widehat{B}]\right] = 0$, which is true for
$\widehat{A} = v_k \hat{a}_k^\dagger$, $\widehat{B} = -v_k^* \hat{a}_k$, $[\widehat{A}, \widehat{B}] = |v|^2$,

$\widehat{D}_k(v_k)$ is known as the displacement operator. Some important properties of this operator are:

$$\widehat{D}_k^\dagger(v_k) = \widehat{D}_k^{-1}(v_k) = \widehat{D}_k(-v_k) \tag{3.8}$$

and

$$\begin{aligned} \widehat{D}_k^\dagger(v_k)\hat{a}_k\widehat{D}_k(v_k) &= \hat{a}_k - [v_k\hat{a}_k^\dagger - v_k^*\hat{a}_k, \hat{a}_k] = \hat{a}_k + v_k & (3.9) \\ \widehat{D}_k^\dagger(v_k)\hat{a}_k^\dagger\widehat{D}_k(v_k) &= \hat{a}_k + v_k^* & (3.10) \end{aligned}$$

The first of these properties can be proven by noting that $-i(v_k\hat{a}_k^\dagger - v_k^*)\hat{a}_k$ is a manifestly hermitian operator, and that for any hermitian operator $\widehat{O}_h$, $e^{i\widehat{O}_h}$ is unitary.

The conjugation relations can be obtained from the operator identity that whenever $\big[\widehat{A},[\widehat{A},\widehat{B}]\big]=0$,:

$$e^{-\widehat{A}}\widehat{B}e^{\widehat{A}} = \widehat{B} - [\widehat{A},\widehat{B}] \tag{3.11}$$

with $\widehat{A} = v_k\hat{a}_k^\dagger - v_k^*\hat{a}_k$, and $\widehat{B} = \hat{a}_k$ or $\widehat{B} = \hat{a}_k^\dagger$

3.3 Waves

For a coherent state $|v_k\rangle$, the expected value of the electromagnetic vector potential

$$\hat{A}_\mu = \sum_{k',\lambda'} c\sqrt{\frac{2\pi\hbar}{\omega_{k'}V}}\epsilon_\mu(k',\lambda')\big(\hat{a}_{k',\lambda'}e^{i(\vec{k'}\cdot\vec{x}-w_{k'}t)} + \hat{a}^\dagger_{k',\lambda'}e^{-i(\vec{k'}\cdot\vec{x}-w_{k'}t)}\big) \tag{3.12}$$

is given by

$$\langle v_k|\widehat{A}_\mu|v_k\rangle = \langle 0|\widehat{D}^\dagger(v_k)\widehat{A}_\mu\widehat{D}(v_k)|0\rangle.$$

Since $\hat{D}^\dagger(v_k)\hat{a}_{k'}\hat{D}(v_k)|0\rangle = \hat{a}_k + v_k\delta_{k,k'}$, and letting $v = |v|e^{i\phi}$,

$$\begin{aligned}\widehat{D}^\dagger(v_k)\widehat{A}_\mu\widehat{D}(v_k) &= \widehat{A}_\mu + c\sqrt{\frac{2\pi\hbar}{\omega_k V}}\epsilon_\mu(k,\lambda)\big(v_{k,\lambda}e^{i(\vec{k}\cdot\vec{r}-w_kt)} + v^*_{k,\lambda}e^{-i(\vec{k}\cdot\vec{x}-w_kt)}\big)\\ &= \widehat{A}_\mu + 2c\sqrt{\frac{2\pi\hbar}{\omega_k V}}|v_k|\epsilon_\mu(k,\lambda)\cos(\vec{k}\cdot\vec{x}-w_kt+\phi). \end{aligned} \tag{3.13}$$

Hence, since $\langle 0|\hat{A}_\mu|0\rangle = 0$,

$$\langle v_k|\widehat{A}_\mu|v_k\rangle = 2c\sqrt{\frac{2\pi\hbar}{\omega_k V}}|v_k|\epsilon_\mu(k,\lambda)\cos(\vec{k}\cdot\vec{x}-w_kt+\phi) \tag{3.14}$$

That is, the expected value of the vector potential for a coherent state $|v_k\rangle$ is a plane wave of definite phase ϕ and amplitude proportional to $|v|$ traveling in the k direction. This is a good description of a laser beam, which has definite phase and amplitude.

In the case of the Kapitza-Dirac effect, the standing wave of light is created by two counter propagating laser beams. Following the above logic, the combined state of these beams is given by $|v_k, v_{-k}\rangle$. The expectation value of $\widehat{A}_\mu$ is given by

$$\langle v_k, v_{-k}|\widehat{A}_\mu|v_k, v_{-k}\rangle = \langle 0|\widehat{D}^\dagger(v_k)\widehat{D}^\dagger(v_{-k})\widehat{A}_\mu\widehat{D}(v_{-k})\widehat{D}(v_k)|0\rangle.$$

As before, since $\hat{D}^\dagger(v_k)\hat{a}_{k'}\hat{D}(v_k)|0\rangle = \hat{a}_k + v_k\delta_{k,k'}$, and letting $v = |v|e^{i\phi}$,

$$\widehat{D}^\dagger(v_k)\widehat{D}^\dagger(v_{-k})\widehat{A}_\mu\widehat{D}(v_{-k})\widehat{D}(v_k) = \widehat{A}_\mu + c\sqrt{\frac{2\pi\hbar}{\omega_k V}}\epsilon_\mu(k,\lambda^+)\big(v_{k,\lambda^+}e^{i(\vec{k}\cdot\vec{x}-w_k t)} + v^*_{k,\lambda^+}e^{-i(\vec{k}\cdot\vec{x}-w_k t)}\big) + c\sqrt{\frac{2\pi\hbar}{\omega_{-k}V}}\epsilon_\mu(-k,\lambda^-)\big(v_{-k,\lambda^-}e^{i(-\vec{k}\cdot\vec{x}-w_{-k}t)} + v^*_{-k,\lambda^-}e^{-i(-\vec{k}\cdot\vec{x}-w_{-k}t)}\big).$$

Note that $\omega_k = \omega_{-k}$ Then, assuming that the counter-propagating laser modes have the same magnitude (i.e. $|v_k| = |v_{-k}|$), with phases $\phi^+$and ϕ^-, (and that both modes are polarized in the same direction, so that $\epsilon_\mu(k,\lambda^+) = \epsilon_\mu(-k,\lambda^-) = \epsilon_\mu(k)$),

$$\begin{aligned}\hat{D}^\dagger(v_k)\hat{D}^\dagger(v_{-k})\hat{A}_\mu\hat{D}(v_{-k})\hat{D}(v_k) &= \hat{A}_\mu + c\sqrt{\frac{2\pi\hbar}{\omega_k V}}|v|\epsilon_\mu(k)\big(e^{i\phi^+}e^{i(\vec{k}\cdot\vec{x}-w_k t)} + e^{-i\phi^+}e^{-i(\vec{k}\cdot\vec{x}-w_k t)} + e^{i\phi^-}e^{i(-\vec{k}\cdot\vec{x}-w_k t)} + e^{-i\phi^-}e^{-i(-\vec{k}\cdot\vec{x}-w_k t)}\big)\\ &= \hat{A}_\mu + 4c\sqrt{\frac{2\pi\hbar}{\omega_k V}}|v|\epsilon_\mu(k)\cos{(\vec{k}\cdot\vec{x}+\phi^+-\phi^-)}\cos{(w_k t - \frac{\phi^-+\phi^+}{2})}.\end{aligned}$$

Again, since $\langle 0|\hat{A}_\mu|0\rangle = 0$ the expected value of the vector potential is a standing wave in space:

$$\langle v_k, v_{-k}|\hat{A}_\mu|v_k, v_{-k}\rangle = 4c\sqrt{\frac{2\pi\hbar}{\omega_k V}}|v|\epsilon_\mu(k)\cos{(\vec{k}\cdot\vec{x}+\phi^+-\phi^-)}\cos{(w_k t - \frac{\phi^-+\phi^+}{2})}$$

Chapter 4

The Kaptiza-Dirac Effect.

We are now ready to attempt a quantum field theoretic description of the Kapitz-Dirac effect.

Let $\hat{H}(t) = -ie\int \widehat{\bar{\psi}}(\vec{x},t)\gamma^{\mu}\widehat{\psi}(\vec{x},t)\widehat{A}_{\mu}\mathrm{d}^3x$ be the standard quantum electrodynamic interaction hamiltonian (see appendix for definition of $\widehat{\psi}$). The interaction time development operator $\hat{U}(t,t_0)$ is then given by the integral equation:

$$\widehat{U}(t,t_0) = \widehat{I} - i\int_{t_0}^{t}\mathrm{d}t'\widehat{U}(t,t')\widehat{H}(t'). \tag{4.1}$$

For a general operator $\widehat{O}$, let $\widehat{\widetilde{O}} = \widehat{D}^{\dagger}(v_k)\widehat{D}^{\dagger}(v_{-k})\widehat{O}\widehat{D}(v_{-k})\widehat{D}(v_k)$, and for a ket $|i\rangle$, let $|\tilde{i}\rangle = \widehat{D}^{\dagger}(v_{-k})\widehat{D}^{\dagger}(v_k)|i\rangle$ so that for arbitrary initial and final states $|i\rangle$ and $\langle f|$

$$\langle f|\widehat{U}(t,t_0)|i\rangle = \langle\tilde{f}|\widehat{\widetilde{U}}(t,t_0)|\tilde{i}\rangle. \tag{4.2}$$

Then

$$\widehat{\widetilde{U}}(t,t_0) = \widehat{I} - i\int_{t_0}^{t}\mathrm{d}t'\widehat{\widetilde{U}}(t,t')\widehat{\widetilde{H}}(t') \tag{4.3}$$

Note that in the case of the Kapitza-Dirac effect, $|i\rangle = |v_k, v_{-k}, e^-_{p_i}\rangle$, so $|\tilde{i}\rangle = |e^-_{p_i}\rangle$. This simplifies somewhat the calculation involved in determining the time development of this state, as shall be seen.

Now since $\widehat{D}$ acts only on the space of photons, the operators $\widehat{\psi}$ and $\widehat{\bar{\psi}}$ act only on the space of electrons,and so $\widehat{\widetilde{H}} = e\int \widehat{\bar{\psi}}(\vec{x},t)\gamma^{\mu}\widehat{\psi}(\vec{x},t)\widehat{\widetilde{A}}_{\mu}\mathrm{d}^3x$. As we saw in chapter 5, however,

$$\widehat{\widetilde{A}}_{\mu} = \widehat{A}_{\mu} + 4c\sqrt{\frac{2\pi\hbar}{\omega_k V}}|v|\epsilon_{\mu}(k)\cos(\vec{k}\cdot\vec{x}+\phi^{+}-\phi^{-})\cos(w_k t - \frac{\phi^{-}+\phi^{+}}{2}) \tag{4.4}$$

so

$$\widehat{\widehat{H}} = \widehat{H} - ie \int 4c\sqrt{\frac{2\pi\hbar}{\omega_k V}}|v|\epsilon_\mu(k)\cos(\vec{k}\cdot\vec{x}+\phi^+-\phi^-)\cos(w_k t - \frac{\phi^- + \phi^+}{2})\widehat{\bar{\psi}}(\vec{x},t)\gamma^\mu\widehat{\psi}(\vec{x},t)\mathrm{d}^3x \tag{4.5}$$

Define the second term of the above expression, describing the interaction of an electron with a classical standing wave of light, as $\widehat{F}$, so that

$$\widehat{\widehat{H}}(t) = \widehat{H}(t) + \widehat{F}(t) \tag{4.6}$$

Then

$$\begin{aligned} \widehat{\widehat{U}}(t,t_0) &= \widehat{I} - i\int_{t_0}^{t} dt'\widehat{\widehat{U}}(t,t')(\widehat{H}(t')+\widehat{F}(t')) \\ &= \widehat{I} - i\int_{t_0}^{t} dt'\widehat{\widehat{U}}(t,t')\widehat{H}(t') - i\int_{t_0}^{t} dt'\widehat{\widehat{U}}(t,t')\widehat{F}(t') \end{aligned} \tag{4.7}$$

A standard approach to perturbative quantum electrodynamics at this point would be to use this equation to substitute iteratively for $\widehat{\widehat{U}}$. Unfortunately, in the case of the Kapitza-Dirac Effect, the perturbative approach to quantum field theory is largely ineffective. The perturbative approach relies upon the condition that the interaction hamiltonian is small compared to unity. Since the hamiltonian for the interaction is given by $-ie\int\widehat{\bar{\psi}}(\vec{x},t)\gamma^\mu\widehat{\psi}(\vec{x},t)\widehat{A}_\mu\mathrm{d}^3x$, this requirement is usually met because of the factor of $e=\sqrt{\alpha}\approx\sqrt{\frac{1}{137}}$ that causes each term in the expansion to be geometrically smaller than the last. However, in the Kapitza-Dirac effect, $\widehat{F}$ is proportional to $|v|$, which is the squareroot of the number of photons expected in each mode of the standing wave. Hence, if this expected number is on the order of about 137 or greater, the perturbative approach can no longer be applied, as the series expansion may not even converge. Since a laser has, in general, an expected $|v|^2=\frac{E}{\hbar\omega}$ photons, where E is the energy of the laser pulse, this approach is rarely useful. For example, in the Batelaan group's experiment,$E = 0.2J$, $\lambda=\frac{2\pi c}{\omega}=532nm$, so $|v|^2\approx 5.7\times 10^{17}$, far outside of a perturbative regime. There is, however, a form of $\widehat{\widehat{U}}(t,t_0)$ that will allow some qualitative arguments to be made in comparing the QED description of the effect to the classical description.

4.1 Qualitative arguments about the Kapitza Dirac effect

Define a new unitary operator $\widehat{M}(t, t_0)$ by the differential equation:

$$i\frac{\partial}{\partial t}\widehat{M}(t, t_0) = \widehat{F}(t)\widehat{M}(t, t_0). \tag{4.8}$$

$\widehat{M}(t, t_0)$ describes the time evolution of a quantized Dirac particle in the presence of a classical standing wave potential. Let $\widehat{U}_M(t, t_0) = \widehat{M}(t, t_0)^\dagger \widehat{\widehat{U}}(t, t_0)$ Then $\widehat{\widehat{U}}(t, t_0) = \widehat{M}(t, t_0)\widehat{U}_M(t, t_0)$ and

$$\begin{aligned} i\frac{\partial}{\partial t}(\widehat{U}_M(t, t_0)) &= i\frac{\partial}{\partial t}(\widehat{M}(t, t_0)^\dagger \widehat{\widehat{U}}(t, t_0)) \\ &= i\frac{\partial}{\partial t}(\widehat{M}(t, t_0)^\dagger) * \widehat{\widehat{U}}(t, t_0) + \widehat{M}(t, t_0)^\dagger * i\frac{\partial}{\partial t}(\widehat{\widehat{U}}(t, t_0)) \\ &= -\widehat{M}(t, t_0)^\dagger \widehat{F}(t) * \widehat{\widehat{U}}(t, t_0) + \widehat{M}(t, t_0)^\dagger * (H(t) + F(t))\widehat{\widehat{U}}(t, t_0) \\ &= \widehat{M}(t, t_0)^\dagger * H(t)\widehat{\widehat{U}}(t, t_0) \end{aligned} \tag{4.9}$$

Where the third line makes use of the adjoint of the definition of $\widehat{M}$:

$$-i\frac{\partial}{\partial t}\widehat{M}(t, t_0)^\dagger = \widehat{M}(t, t_0)^\dagger \widehat{F}(t).$$

Defining $\widehat{H}_M(t, t_0) = \widehat{M}(t, t_0)^\dagger \widehat{H}(t)\widehat{M}(t, t_0)$, it is evident that

$$i\frac{\partial}{\partial t}\widehat{U}_M(t, t_0) = \widehat{H}_M(t, t_0)\widehat{U}_M(t, t_0). \tag{4.10}$$

The purpose of all this symbol pushing now becomes more clear: Since $\widehat{M}$ is unitary, $\widehat{\psi}\widehat{M}$ is normalized in the same fashion as $\widehat{\psi}$. Hence, $\widehat{H}_M = \widehat{M}^\dagger(t, t_0)(-ie\int \widehat{\bar{\psi}}(\vec{x}, t)\gamma^\mu \widehat{\psi}(\vec{x}, t)\widehat{A}_\mu \mathrm{d}^3x)\widehat{M}(t, t_0)$ is of the same order as $\widehat{H}$ (i.e. $\sqrt{\alpha}$) and can be treated perturbatively. This allows a meaningful series expansion to be written for the time development of the system:

$$
\begin{aligned}
\widehat{\widehat{U}}(t,t_0) &= \widehat{M}(t,t_0)\widehat{U}_M(t,t_0) \\
&= \widehat{M}(t,t_0)\big[\widehat{I} - i\int_{t_0}^{t}\mathrm{d}t'\widehat{H}_M(t') + \\
&\quad (-i)^2\int_{t_0}^{t}\mathrm{d}t_1\int_{t_1}^{t}\mathrm{d}t'\widehat{H}_M(t')\widehat{H}_M(t_1) + \\
&\quad (-i)^3\int_{t_0}^{t}\mathrm{d}t_2\int_{t_2}^{t}\mathrm{d}t_1\int_{t_1}^{t}\mathrm{d}t'\widehat{H}_M(t')\widehat{H}_M(t_1)\widehat{H}_M(t_2) + \ldots\big] \\
&= \widehat{M}(t,t_0) - i\int_{t_0}^{t}\mathrm{d}t'\widehat{M}(t,t')\widehat{H}(t')\widehat{M}(t',t_0) + \\
&\quad (-i)^2\int_{t_0}^{t}\mathrm{d}t_1\int_{t_1}^{t}\mathrm{d}t'\widehat{M}(t,t')\widehat{H}(t')\widehat{M}(t',t_1)\widehat{H}(t_1)\widehat{M}(t_1,t_0) + \\
&\quad (-i)^3\int_{t_0}^{t}\mathrm{d}t_2\int_{t_2}^{t}\mathrm{d}t_1\int_{t_1}^{t}\mathrm{d}t'\widehat{M}(t,t')\widehat{H}(t')\widehat{M}(t',t_1)\widehat{H}(t_1)\widehat{M}(t_1,t_2)\widehat{H}(t_2)\widehat{M}(t_2,t_0) + \ldots
\end{aligned}
$$

If the (transformed) initial state is a single electron, this form has the interesting physical interpretation that the electron is propagating in the presence of a classical field, and at various times (t',t_1, etc.), interacts with the quantized electromagnetic field by emitting or absorbing a photon (or a virtual photon). Each term in the expansion represents a possible number of interactions with the quantized field, and the integrals are taken over all possible times for these interactions.

If the initial and final photon states are the same, and $|i\rangle = |v_k, v_{-k}, e^-_{p_i}\rangle$, then $|\tilde{i}\rangle = |e^-_{p_i}\rangle$, and $\langle\tilde{f}| = \langle e^-_{p_f}|$, so to second order in $\sqrt{\alpha}$

$$
\begin{aligned}
\langle\tilde{f}|\widehat{\widehat{U}}(t,t_0)|\tilde{i}\rangle &= \langle e^-_{p_f}|\widehat{\widehat{U}}(t,t_0)|e^-_{p_i}\rangle \qquad (4.11)\\
&\approx \langle e^-_{p_f}|\widehat{M}(t,t_0)|e^-_{p_i}\rangle \\
&\quad -i\int_{t_0}^{t}\mathrm{d}t'\langle e^-_{p_f}|\widehat{M}(t,t')\widehat{H}(t')\widehat{M}(t',t_0)|e^-_{p_i}\rangle \\
&\quad +(-i)^2\int_{t_0}^{t}\mathrm{d}t_1\int_{t_1}^{t}\mathrm{d}t'\langle e^-_{p_f}|\widehat{M}(t,t')\widehat{H}(t')\widehat{M}(t',t_1)\widehat{H}(t_1)\widehat{M}(t_1,t_0)|e^-_{p_i}\rangle.
\end{aligned}
$$

Since

$$
\widehat{H} = -ie\int \mathrm{d}^3x\widehat{\bar{\psi}}(\vec{x},t)\gamma^\mu\widehat{\psi}(\vec{x},t)\sum_{k',\lambda'} c\sqrt{\frac{2\pi\hbar}{\omega_{k'}V}}\epsilon_\mu(k',\lambda')\big(\hat{a}_{k',\lambda'}e^{i(\vec{k'}\cdot\vec{x}-w_{k'}t)}+\hat{a}^\dagger_{k',\lambda'}e^{-i(\vec{k'}\cdot\vec{x}-w_{k'}t)}\big)
$$

is linear in the photon creation and annihilation operators, and $\widehat{M}$ has no effect on the photon state, the first order term above is identically zero. Hence, the first correction to $\langle e^-_{p_f}|\widehat{M}(t,t_0)|e^-_{p_i}\rangle$ comes in at a factor of $(\sqrt{\alpha})^2 \approx \frac{1}{137}$ smaller than the zeroth order term.

The largest contribution to the electron's behavior from interactions with the quantized vector potential $\widehat{A}_\mu$ occurs at first order in $\sqrt{\alpha}$ with the electron emitting a single photon into the mode k_f. This process is commonly known as bremsstrahlung. In this case, $\langle \tilde{f}| = \langle e^-_{p_f}|\hat{a}_{k_f}$, so

$$\begin{aligned}
\langle \tilde{f}|\widehat{\widetilde{U}}(t,t_0)|\tilde{i}\rangle &= \langle e^-_{p_f}|\hat{a}_{k_f}\widehat{\widetilde{U}}(t,t_0)|e^-_{p_i}\rangle \approx -i\int_{t_0}^{t}\mathrm{d}t'\langle e^-_{p_f}|\hat{a}_{k_f}\widehat{M}(t,t')\widehat{H}(t')\widehat{M}(t',t_0)|e^-_{p_i}\rangle \\
&= -i\int_{t_0}^{t}\mathrm{d}t'\langle e^-_{p_f}|\hat{a}_{k_f}\widehat{M}(t,t')\big(-ie\int \mathrm{d}^3x'\widehat{\bar{\psi}}(\vec{x},t)\gamma^\mu\widehat{\psi}(\vec{x},t) \\
&\quad \times\sum_{k'} c\sqrt{\frac{2\pi\hbar}{\omega_{k'}V}}\epsilon_\mu(k')\hat{a}^\dagger_{k'}e^{-i(\vec{k}'\cdot\vec{x}'-w_{k'}t')}\big)\widehat{M}(t',t_0)|e^-_{p_i}\rangle \\
&= -ec\sqrt{\frac{2\pi\hbar}{\omega_{k_f}V}}\epsilon_\mu(k_f)\int_{t_0}^{t}\mathrm{d}t'\int \mathrm{d}^3x' e^{-i(\vec{k}_f\cdot\vec{x}'-w_{k_f}t')} \\
&\quad \times\langle e^-_{p_f}|\widehat{M}(t,t')\widehat{\bar{\psi}}(\vec{x}',t')\gamma^\mu\widehat{\psi}(\vec{x}',t')\widehat{M}(t',t_0)|e^-_{p_i}\rangle,
\end{aligned} \tag{4.12}$$

where the last step follows from the fact that $\widehat{M}$ does not act on the photon state.

The probability amplitudes of each of these processes can then be calculated dependent upon knowledge of M, the time development operator for the classical standing wave interaction with the quantized Dirac field.

Unfortunately, while this term can be understood qualitatively, it is difficult, given the strength of the electromagnetic field, to find a solution to the differential equation $i\frac{\partial}{\partial t}\widehat{M}(t,t_0) = \widehat{F}(t)\widehat{M}(t,t_0)$. However, the qualitative results above do show that to a good approximation, the vector potential can be treated classically for the Kapitza-Dirac Effect.

4.2 The Perturbative regime

While it is not a commonly realized physical situation, it is interesting to investigate the behavior of an electron in the presence of a standing wave of light with a low expected number of photons. In this case, the equation

$$\widehat{\widetilde{U}}(t,t_0) = \widehat{I} - i\int_{t_0}^{t} dt'\widehat{\widetilde{U}}(t,t')(\widehat{H}(t') + \widehat{F}(t')) \tag{4.13}$$

becomes, in a series expansion:

$$\begin{aligned}\widehat{\widetilde{U}}(t,t_0) = \widehat{I} - i\int_{t_0}^{t} \mathrm{d}t'(\widehat{H}(t') + \widehat{F}(t')) + (-i)^2\int_{t_0}^{t}\mathrm{d}t_1\int_{t_1}^{t}\mathrm{d}t'(\widehat{H}(t') + \widehat{F}(t'))(\widehat{H}(t_1) + \widehat{F}(t_1)) + \\ (-i)^3\int_{t_0}^{t}\mathrm{d}t_2\int_{t_2}^{t}\mathrm{d}t_1\int_{t_1}^{t}\mathrm{d}t'(\widehat{H}(t') + \widehat{F}(t'))(\widehat{H}(t_1) + \widehat{F}(t_1))(\widehat{H}(t_2) + \widehat{F}(t_2)) + \ldots.\end{aligned}$$

If the number of photons expected in each mode of the standing wave is small compared to $\frac{1}{\alpha} \approx 137$, then each term in this series is smaller than the last. If the transformed initial and final states are, respectively, $|e^-_{p_i}\rangle$ and $\langle e^-_{p_f}|$, then, since H must either create or destroy a photon and F acts only on the electron state, to second order in $\sqrt{\alpha}$ the series becomes:

$$\begin{aligned}\widehat{\widetilde{U}}(t,t_0) \;&\approx\; \langle e^-_{p_f}|e^-_{p_i}\rangle - i\int_{t_0}^{t}\mathrm{d}t'\langle e^-_{p_f}|\widehat{F}(t')|e^-_{p_i}\rangle \\ &+(-i)^2\int_{t_0}^{t}\mathrm{d}t_1\int_{t_1}^{t}\mathrm{d}t'(\langle e^-_{p_f}|\widehat{H}(t')\widehat{H}(t_1)|e^-_{p_i}\rangle + \langle e^-_{p_f}|\widehat{F}(t')\widehat{F}(t_1)|e^-_{p_i}\rangle).\end{aligned}$$

The scattering matrix $S_{fi} = \langle f|\widehat{U}(\infty,-\infty)|i\rangle = \langle\tilde{f}|\widehat{\widetilde{U}}(\infty,-\infty)|\tilde{i}\rangle$. To second order, this is:

$$\begin{aligned}S_{fi} \;&\approx\; \langle e^-_{p_f}|e^-_{p_i}\rangle - i\int_{-\infty}^{\infty}\mathrm{d}t'\langle e^-_{p_f}|\widehat{F}(t')|e^-_{p_i}\rangle \\ &+(-i)^2\int_{-\infty}^{\infty}\mathrm{d}t_1\int_{t_1}^{\infty}\mathrm{d}t'(\langle e^-_{p_f}|\widehat{H}(t')\widehat{H}(t_1)|e^-_{p_i}\rangle + \langle e^-_{p_f}|\widehat{F}(t')\widehat{F}(t_1)|e^-_{p_i}\rangle).\end{aligned}$$

The first order correction term in this expansion is:

$$-i\int_{-\infty}^{\infty} \mathrm{d}t' \langle e^-_{p_f}|\widehat{F}(t')|e^-_{p_i}\rangle = -i\int \mathrm{d}^4x' \langle e^-_{p_f}|\widehat{\bar{\psi}}(\vec{x}', t')\gamma^\mu \widehat{\psi}(\vec{x}', t')|e^-_{p_i}\rangle \times$$
$$(-ie4c\sqrt{\frac{2\pi\hbar}{\omega_k V}}|v|\epsilon_\mu(k)\cos(\vec{k}\cdot\vec{x} + \phi^+ - \phi^-)\cos(w_k t - \frac{\phi^- + \phi^+}{2})).$$

Now

$$\langle e^-_{p_f}|\widehat{\bar{\psi}}(\vec{x}', t')\gamma^\mu \widehat{\psi}(\vec{x}', t')|e^-_{p_i}\rangle = \frac{mc^2}{V}\bar{u}^{(s_f)}(p_f)e^{-i(\frac{p_f\cdot\vec{x}-E_f t}{\hbar})}\gamma^\mu u^{(s_i)}(p_i)e^{-i(\frac{p_i\cdot\vec{x}-E_i t}{\hbar})},$$

so, using the identity $\int \mathrm{d}x' e^{kx} = 2\pi\delta(k)$,

$$-i\int_{-\infty}^{\infty} \mathrm{d}t' \langle e^-_{p_f}|\widehat{F}(t')|e^-_{p_i}\rangle = -e\bar{u}^{(s_f)}(p_f)\gamma^\mu\epsilon_\mu(k,\lambda)u^{(s_i)}(p_i)\frac{m\sqrt{2pi}|v|(2\pi)^4}{\sqrt{E_f E_i \omega_k V^3}} \times$$
$$\big[\delta(\vec{k} - \vec{p}_f + \vec{p}_i)[\delta(\frac{E_f - E_i}{\hbar} - \omega_k)e^{i\phi^+} + \delta(\frac{E_f - E_i}{\hbar} + \omega_k)e^{-i\phi^-}]$$
$$+\delta(-\vec{k} - \vec{p}_f + \vec{p}_i)[\delta(\frac{E_f - E_i}{\hbar} - \omega_k)e^{i\phi^-} + \delta(\frac{E_f - E_i}{\hbar} + \omega_k)e^{-i\phi^+}]\big].$$

Qualitatively, this term shows the electron momentum changing by one photon recoil in each direction. The energy delta functions show the electron gaining or losing the energy of a single photon of frequency ω as it emits or absorbs a photon from the modes k or $-k$ of the standing wave.

Similarly, the second order term includes $(-i)^2\int_{-\infty}^{\infty}\mathrm{d}t_1\int_{t_1}^{\infty}\mathrm{d}t'\langle e^-_{p_f}|\widehat{H}(t')\widehat{H}(t_1)|e^-_{p_i}\rangle$, a mass correction term that will result in no momentum change, and $(-i)^2\int_{-\infty}^{\infty}\mathrm{d}t_1\int_{t_1}^{\infty}\mathrm{d}t'\langle e^-_{p_f}|\widehat{F}(t')\widehat{F}(t_1)|e^-_{p_i}\rangle$, a term that will result in momentum delta functions corresponding to two photon recoils (for a net momentum change of 0 or $2k$).

We would expect, then, that the electron diffraction pattern upon passing through a low intensity standing wave would have peaks corresponding to integer numbers of photon recoils. Comparing this prediction with the experimental data of the Batelaan group, the finite width of the observed peaks can be attributed to the non-idealities of the system not taken into account by this simple model. For instance, the electron should be described by a gaussian wave packet rather than a single momentum eigenstate. This accounts for the smooth appearance of the diffraction pattern. One notes, however, the clear discrepancy between the low intensity theory and the high intensity experiment that while the peaks are predicted to occur at integer numbers of photon recoils, they actually occur at only even numbers of recoils. This is a problem that bears further investigation. The resolution of this issue, if it is not a fundamental change in the behavior in low and high intensity versions of the experiment, may lie in a more thorough treatment of the polarization of the beam and the phases ϕ^+ and ϕ^- of the counter-propagating waves.

Chapter 5

Conclusion

While we have met with some success in describing the initial state of the system for the light in the Kapitza-Dirac effect as a direct product of counter-propagating coherent states, the direct approach to the understanding of the Kapitza-Dirac effect via a perturbative expansion has been stymied by the formally divergent series that results from the effective hamiltonian $\widehat{H} + \widehat{F}$ being large compared to unity. However, indirect approaches to the problem, such as those explored in chapter 4, look to be promising. The full quantum electrodynamic time development matrix
$U_{fi}(t, t_0) = \langle f|\widehat{U}(t, t_0)|i\rangle$ in the case of the Kapitza Dirac effect appears to be equivalent to first order in $\sqrt{\alpha}$ to the time development matrix for an electron interacting with a classical standing wave vector potential. This qualitative agreement is bolstered by the apparent success of the classical Schrodinger equation in describing the data seen by the Batelaan group.

An issue which requires resolution is the behavior of the system at low laser intensities, which has peaks predicted at all integer numbers of photon recoils. This is qualitatively different than the peaks observed only at even integer numbers of photon recoils by the Batelaan group at high laser intensities. Either there exists some transition between these two behaviors as the laser intensity increases, or the low energy prediction must be reexamined in order to account for this missing factor of 2. The next step toward a quantum electrodynamic description of the Kapitza Dirac effect would appear to be in solving the differential equation for the classical potential time development operator $M(t,t_0)$, i.e.

$$i\frac{\partial}{\partial t}\widehat{M}(t,t_0) = \widehat{F}(t)\widehat{M}(t,t_0). \tag{5.1}$$

Given a solution to this equation, the scattering amplitudes involved in the Kapitza-Dirac effect can be approximated to second order in $\sqrt{\alpha}$ using the straightforward (if symbolically complicated) prescription of eqs. 4.11 and 4.12.

Appendix A

Notation

This paper will use the standard Dirac, (or Bra(c)ket) notation to represent the states of the particles discussed. Linear operators will be represented with hats (e.g. $\widehat{O}$), and their adjoints by daggers (e.g. $\widehat{O}^\dagger$). Equations and normalization factors are given in gaussian units. The notation of quantum electrodynamics can quickly become overwhelming. For convenience, there follows here a list of definitions of common symbols used in this paper.

$\hat{a}_k$ is the lowering operator for the mode k. For a Fock occupancy state $|n\rangle_k$ of n photons in mode k, $\hat{a}_k|n\rangle_k = \sqrt{n}|n-1\rangle_k$. $\hat{a}_k^\dagger$ is the raising operator for the mode k. For a Fock occupancy state $|n\rangle_k$ of n photons in mode k, $\hat{a}_k^\dagger|n\rangle_k = \sqrt{n+1}|n+1\rangle_k$.

The commutation relations for $\hat{a}_k^\dagger$ and $\hat{a}_k$ are:

$$\begin{aligned} [\hat{a}_k, \hat{a}_{k'}^\dagger] &= \delta_{k,k'} \\ [\hat{a}_k, \hat{a}_{k'}] &= 0 \\ [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] &= 0 \end{aligned}$$

$\widehat{A}_\mu$ is the quantized vector potential operator:
$\widehat{A}_\mu = \sum_{k',\lambda'} c\sqrt{\frac{2\pi\hbar}{\omega_{k'}V}}\epsilon_\mu(k',\lambda')\left(\hat{a}_{k',\lambda'}e^{i(\vec{k'}\cdot\vec{x}-w_{k'}t)} + \hat{a}_{k',\lambda'}^\dagger e^{-i(\vec{k'}\cdot\vec{x}-w_{k'}t)}\right)$

$\epsilon_\mu(k,\lambda)$ is the polarization vector associated with a particular mode k,λ of the quantized electromagnetic field.

$\widehat{F}$ is used in this paper to denote the electron space operator resultant from using the classical standing wave potential for $\widehat{A}_\mu$ in the formula $-ie\int\widehat{\overline{\psi}}(\vec{x},t)\gamma^\mu\widehat{\psi}(\vec{x},t)\widehat{A}_\mu \mathrm{d}^3x$ for the interaction hamiltonian. i.e.,
$\widehat{F}=-ie\int 4c\sqrt{\frac{2\pi\hbar}{\omega_k V}}|v|\epsilon_\mu(k)\cos{(\vec{k}\cdot\vec{x}+\phi^+-\phi^-)}\cos{(w_k t-\frac{\phi^-+\phi^+}{2})}\widehat{\overline{\psi}}(\vec{x},t)\gamma^\mu\widehat{\psi}(\vec{x},t)\mathrm{d}^3x$

$\widehat{\psi}$ is the quantized Dirac wave operator:
$\sum_{\vec{p'},s'}\sqrt{\frac{mc^2}{EV}}\left[\hat{b}^{(s')}_{\vec{p'}}u^{(s')}(\vec{p'})e^{i(\frac{\vec{p}\cdot\vec{x}}{\hbar}-\frac{E't}{\hbar})}+\hat{d}^{(s')\dagger}_{\vec{p'}}v^{(s')}(\vec{p'})e^{-i(\frac{\vec{p}\cdot\vec{x}}{\hbar}-\frac{E't}{\hbar})}\right]$
where $\hat{b}$ is the electron annihilation operator, $\hat{d}^\dagger$ is the positron creation operator, and the sum is over all possible momenta p and spins s for the electrons and positrons. u and v are bispinors satisfying

$$\begin{aligned}(i\gamma_\mu p^\mu+mc)u^{(s)}(\vec{p}) &= 0\\ (-i\gamma_\mu p^\mu+mc)v^{(s)}(\vec{p}) &= 0\end{aligned}\tag{A.1}$$

where γ_μ with $\mu=1,2,3,4$ are 4×4 matrices satisfying the anticommutation relations

$$\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}\tag{A.2}$$

$\hat{b}$ and $\hat{d}$ satisfy the anticommutation relations:

$$\begin{aligned}\{\hat{b}^{(s)}_{\vec{p}},\hat{b}^{(s')\dagger}_{\vec{p'}}\} &= \delta_{p,p'}\delta_{s,s'}\\ \{\hat{d}^{(s)}_{\vec{p}},\hat{d}^{(s')\dagger}_{\vec{p'}}\} &= \delta_{p,p'}\delta_{s,s'}\\ \{\hat{d}^{(s)}_{\vec{p}},\hat{d}^{(s')}_{\vec{p'}}\} &= \{\hat{d}^{(s)\dagger}_{\vec{p}},\hat{d}^{(s')\dagger}_{\vec{p'}}\}\\ &= \{\hat{b}^{(s)}_{\vec{p}},\hat{b}^{(s')}_{\vec{p'}}\}=\{\hat{b}^{(s)\dagger}_{\vec{p}},\hat{b}^{(s')\dagger}_{\vec{p'}}\}\\ &= \{\hat{d}^{(s)}_{\vec{p}},\hat{b}^{(s')\dagger}_{\vec{p'}}\}=\{\hat{d}^{(s)\dagger}_{\vec{p}},\hat{b}^{(s')}_{\vec{p'}}\}\\ &= \{\hat{d}^{(s)\dagger}_{\vec{p}},\hat{b}^{(s')\dagger}_{\vec{p'}}\}=\{\hat{d}^{(s)}_{\vec{p}},\hat{b}^{(s')}_{\vec{p'}}\}=0\end{aligned}\tag{A.3}$$

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