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A Combinatorial Analog of the Poincaré–Birkhoff Fixed Point Theorem

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Abstract

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Results from combinatorial topology have shown that certain combinatorial lemmas are equivalent to certain topologocal fixed point theorems. For example, Sperner's lemma about labelings of triangulated simplices is equivalent to the fixed point theorem of Brouwer. Moreover, since Sperner's lemma has a constructive proof, its equivalence to the Brouwer fixed point theorem provides a constructive method for actually finding the fixed points rather than just stating their existence. The goal of this research project is to develop a combinatorial analogue for the Poincaré-Birkhoff fixed point theorem.

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Chapter 1

Introduction

The study of combinatorial topology examines ways in which methods from combinatorics can be used to obtain topological results. By triangulating a topological space, we can often make that space more tractable for study. For example, by triangulating a space X we can compute the simplicial homology groups of X in a straightforward manner, whereas computing the equivalent singular homology groups for the untriangulated space may be more difficult [9]. In certain situations, we can even prove results from topology using combinatorial theorems. For example, the combinatorial lemma of Sperner can be used to prove the Brouwer fixed point theorem [12], and the combinatorial lemma of Tucker can be used to prove the Borsuk-Ulam theorem [7], [4]. We further discuss the relationship between Sperner's lemma and the Brouwer fixed point theorem below. More recently, methods from combinatorial topology have been used to solve fair division problems. Simmons (see [11]) used Sperner's lemma to solve the classical cake-cutting problem, in which one seeks to divide a cake among n-people such that each person feels that the piece they receive is the most desirable piece.

In this thesis, we begin by discussing the equivalence of Sperner's lemma and the Brouwer fixed point theorem. We then use techniques developed in this discussion to develop a combinatorial analogue of the Poincaré-Birkhoff fixed point theorem.

1.1 Terminology

Before we begin our discussion of the Sperner-Brouwer equivalence, we define some terms that we shall use throughout this paper.

An *(n, d)-polytope* P is the convex hull of n points, $v_1, v_2, ..., v_n$, in $\mathbf{R}^d$. These points are called the *vertices* of P. A *simplex* is a $(d+1, d)$-polytope, the convex hull of $d+1$ affinely independent points in $\mathbf{R}^d$, and may be thought of as a d-dimensional tetrahedron. A *face* of a polytope is the convex hull of any subset of the vertices of that polytope. A face of codimension 1 is called a *facet*.

A *triangulation* T of P is a finite collection of distinct simplices such that: (i) the union of all the simplices in T is P, (ii) the intersection of any two simplices in T is either empty or a face common to both simplices and (iii) every face of a simplex in T is also in T. The vertices $v_1, v_2, ..., v_n$ are called *vertices of P* to distinguish them from the vertices of the triangulation T. A *Sperner labelling* of T is a labelling of the vertices of T that satisfies these conditions: (1) all vertices of P have distinct labels $1, 2, ..., n$ and (2) the label of any vertex of T which lies on a facet of P matches the label of one of the vertices of P that spans that facet. A *full cell* is any d-dimensional simplex in T whose $d+1$ vertices possess distinct labels.

The statement of Sperner's lemma is given below. Proofs of this lemma can be found in [11] and [2].

Theorem 1 (Sperner's Lemma). *Any Sperner labelled triangulation of a simplex contains an odd number of full cells.*

De Loera, Peterson, and Su recently proved a generalization of Sperner's lemma for polytopes [3]. It states:

Theorem 2 (The Polytopal Sperner Lemma). *Any Sperner labelled triangulation of an (n, d)-polytope must contain at least $n - d$ full cells.*

The following figure gives an example of a Sperner labelled simplex and a Sperner labelled polytope.

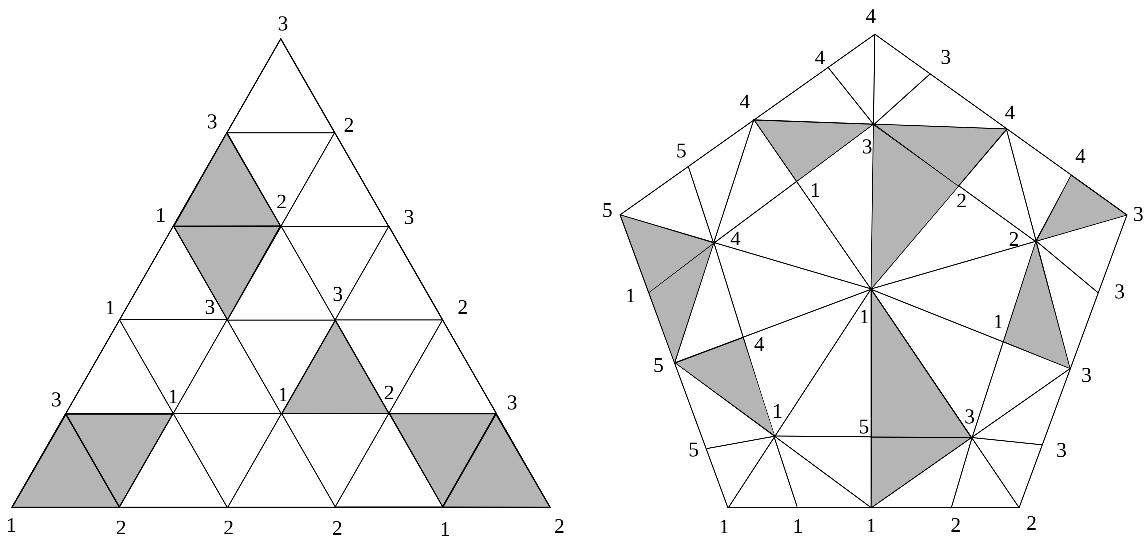

Figure 1.1: A sperner labeled triangle (2-simplex) and pentagon ($(5, 2)$-polytope). Full cells are shaded.

The above theorems are equivalent to the Brouwer fixed point theorem. Similarly, Tucker's lemma, a combinatorial statement about triangulated spheres, is equivalent to the Borsuk-Ulam theorem. Constructive proofs of these combinatorial lemmas have been used to develop algorithms to locate the fixed point guaranteed by the Brouwer theorem or the antipodal points guaranteed by the Borsuk-Ulam theorem [12], [4]. In order to give an idea of how this works, in the next section we prove the Brouwer fixed point theorem via Sperner's lemma and describe how a constructive proof of Sperner's lemma can be used to find the fixed point.

1.2 Sperner Implies Brouwer

The importance of the Sperner lemma is seen through its equivalence to the Brouwer fixed point theorem. We are mainly interested in one direction of this equivalence, namely the proof that Sperner's lemma implies the Brouwer fixed point theorem. The Brouwer fixed point theorem states:

Theorem 3 (The Brouwer Fixed Point Theorem). *Let* $\mathbf{B}^n$ *denote the n-dimensional ball. Then every continuous map* $f : \mathbf{B}^n \to \mathbf{B}^n$ *has a fixed point.*

To prove this theorem via Sperner's lemma, we shall triangulate a regular n-simplex S, which is homeomorphic to the ball $\mathbf{B}^n$, use the map f to provide a Sperner labelling for that triangulation, and use the existence of a full cell to show the existence of a fixed point of f.

Proof. Begin by giving the vertices of S distinct labels. To determine the labels of the remaining vertices, we define an auxiliary function r on the vertices in the following way: for each vertex v, we draw the ray from $f(v)$ through v. We define the point at which this ray intersects the boundary to be $r(v)$. Now, we label the vertex v with the same label as the vertex of S to which $r(v)$ lies the closest. If $r(v)$ lies the same distance from more than one vertex of S, then we give v the label of one of these vertices. This procedure is shown in Figure 1.2.

This labeling by the retract r is a Sperner labeling since all of the vertices of S have distinct labels and for any vertex v on the boundary of S, $r(v) = v$, so v will be given the same label as one of the vertices of S that span the face of the S that contains v.

Now, suppose that the map f has no fixed points. Then the retract r is defined on each point of S. Since the map f is continuous, the map r must also be continuous. In fact, since S is compact, r must be uniformly continuous. Now, let $\epsilon > 0$ be

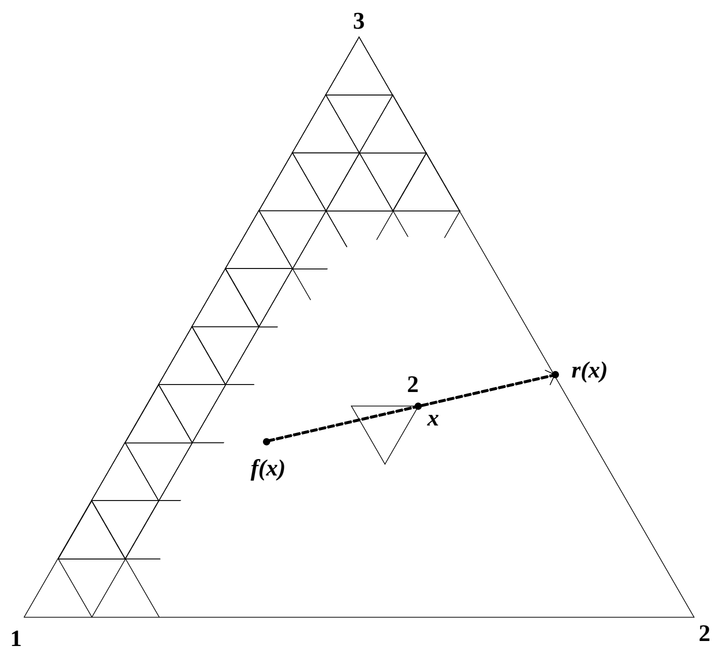

Figure 1.2: The vertex x is labelled via $r(x)$.

less than one half the side length of S, and choose $\delta > 0$ according to the definition of uniform continuity of r.

To arrive at a contradiction, we give S a triangulation T such that the mesh size of T is smaller than δ. Label the vertices of T according to r. This is a Sperner labeling, and thus it contains a full cell. Since the vertices of a full cell all have distinct labels, r maps the vertices of this cell to the boundary of S in such a way that the images of no two vertices are closest to the same vertex of S. Thus, there must be a pair of vertices of the full cell whose images are at least half of the side length of S apart. That is, there are vertices less than δ apart whose images under r are more than ϵ apart. Thus, r cannot be uniformly continuous, which is a contradiction. Therefore, the map f must have a fixed point.

$\square$

That the existence of a full cell implies the existence of a fixed point makes intuitive sense. For in order for a cell of the triangulation to have all distinct labels, r spreads the vertices of this cell out in different directions which, in turn, means that f maps the vertices of this cell in different directions. So, since f is continuous, there must be point near the full cell that is not moved at all by f, or a fixed point. Thus, fixed points can be found near full cells.

The usefulness of the Sperner-Brouwer equivalence comes from the fact that Sperner's lemma can be proved constructively. That is, the proof of Sperner's lemma provides a method of actually locating the full cells. We will outline the procedure for a 2-simplex or a triangle. It is easily generalized to higher dimensions through induction, and a detailed proof can be found in [11].

Given a Sperner labelled 2-simplex S, we show the existence of a full cell via a *path following* argument. Suppose the vertices of the triangulation T of S are given labels 1, 2, or 3. We may view cells of this triangulation as rooms, and edges labeled with a 1 and a 2, which we will call $(1,2)$-edges, as doors to these rooms. Now, a given cell has 0, 1, or 2 doors since it is possible for a cell to contain as many $(1,2)$-edges. Notice that the only cells that contain exactly one door are full cells. Now, we will construct a path by moving from room to room via the $(1,2)$-doors without backtracking. We start the paths at any door on the boundary of S. These doors will lie on the edge of S spanned by the labels 1 and 2. Now, we walk through a boundary door into a room. This room is either has another door or it does not. If it does not, then it is a full cell. If it does, then we walk through that door into the next room, and so on. Paths starting from a boundary door can either terminate at a full cell, or a one door room, or can exit S through another boundary door. Now, on the side of S spanned by 1 and 2, as we move from the vertex labelled 1 to the vertex labeled 2, we must make an odd number of transitions between the labels 1 and 2. Thus, there are an odd number of $(1,2)$-edges along this side. Thus, there is an odd number of boundary doors in S. Since paths that both enter and exit through a boundary door pair off an even number of the boundary doors, there must be at least one path starting from a boundary door that leads to a full cell. Hence, by following paths from the boundary doors, we are able to locate at least one full cell. This procedure is illustrated in Figure 1.3. A path following procedure can also be used to locate the full cells given by the polytopal Sperner lemma. See [3].

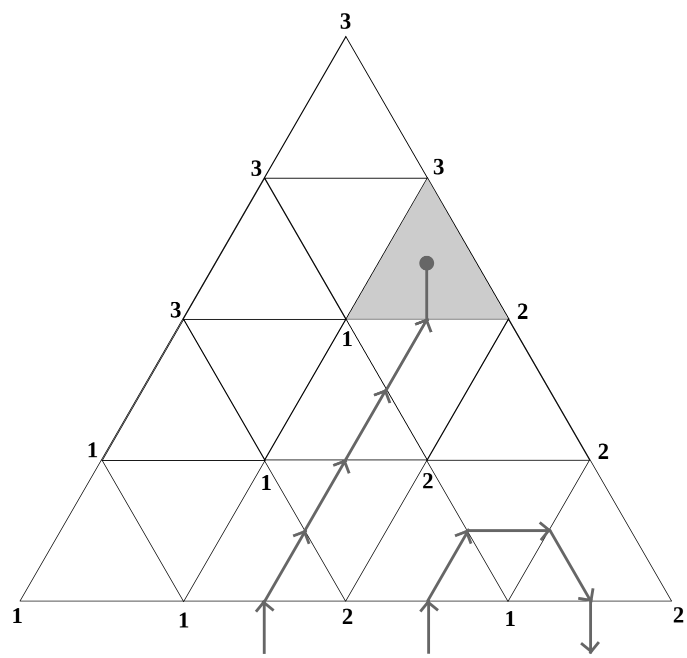

Figure 1.3: Path following in a Sperner labelled 2-simplex.

Given a continuous map $f : \mathbf{B}^n \to \mathbf{B}^n$, we can find a homeomorphism $h : \mathbf{B}^n \to S$ where S is a regular n-simplex. We can then triangulate S via the retract r and use the path following technique to find a full cell. If we triangulate S with a sequence of finer and finer triangulations, some subsequence of full cells converges to a single point. We may then map this point back to $\mathbf{B}^n$ using h^{-1} in order to locate the fixed point of f.

1.3 The Poincaré-Birkhoff Fixed Point Theorem

We now provide some background information on the Poincaré-Birkhoff fixed point theorem before we embark on developing a combinatorial analogue for it.

Theorem 4 (The Poincaré-Birkhoff Fixed Point Theorem). *Any continuous self map $f : A \to A$ of the annulus $A = S^1 \times [0,1]$ that is area preserving and satisfies a boundary twist condition, which states that f advances points on the outer edge of A positively and points on the inner edge negatively, must have at least two fixed points.*

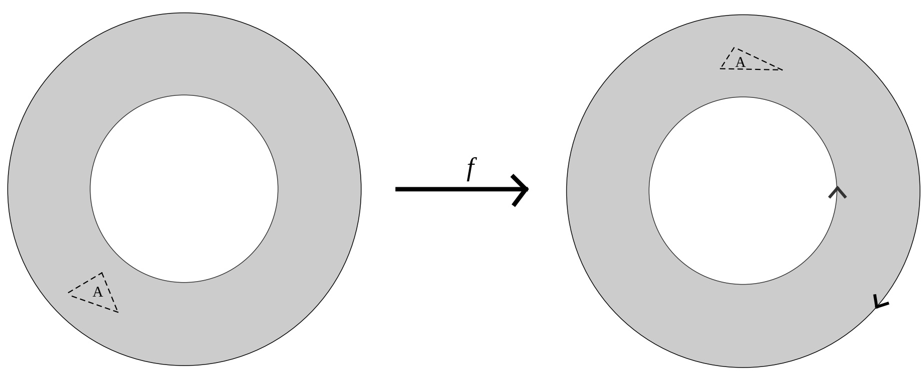

Figure 1.4: The Poincaré-Birkhoff fixed point theorem.

The original proof of this theorem can be found in [1]. There is a generalization of this theorem due to Franks [5] that states that a homeomorphism f from the open annulus $A = S^1 \times (0,1)$ to itself such that every point in A is non-wandering under f and there is a lift of f to the universal covering space, $\mathbf{R} \times (0,1)$ that possesses both a positively and negatively returning disk, then f has at least one fixed point. A corollary that follows from this theorem states that self map of the closed annulus that satisfies the same conditions will contain two fixed points.

Chapter 2

A Combinatorial Analogue to the Poincaré-Birkhoff Fixed Point Theorem

In this chapter we seek to develop a combinatorial analogue to the Poincaré-Birkhoff fixed point theorem. To do so we will employ some of the techniques from the Sperner-Brouwer equivalence developed in the previous chapter. We shall state and prove a combinatorial lemma, called the NWSE lemma, and then show that it implies the Poincaré-Birkhoff fixed point theorem for a certain class of maps.

2.1 The NWSE Lemma

To prove the Brouwer fixed point theorem via Sperner's lemma we first triangulated the ball $\mathbf{B}^n$ by mapping it to an n-simplex. We then used the map f, which we wanted to show had a fixed point, to generate a labeling of the triangulation by using the retract r. The existence of a full cell of the triangulation contradicts the uniform continuity of r, proving the existence of a fixed point of f. In the same way, to develop a combinatorial analogue to the Poincaré-Birkhoff fixed point theorem we wish to triangulate the annulus A, label the vertices of the triangulation according to the map f, which we want to prove has two fixed points, in such a way that the resuting labeling will have two full cells. Then we will use the existence of these full cells to contradict the uniform continuity of an auxiliary function associated to f, thus proving the existence of two fixed points of the map f.

In order to triangulate the annulus A we may think of A as a rectangle with two opposite edges, say the right and left edges, identified. This space is a cylinder which is homeomorphic to an annulus. Next, we triangulate the rectangle by dividing it into smaller rectangles via grid lines, then dividing each square into two triangles across the diagonal connecting the bottom left and top right corners. Call this triangulation T. Next, we construct a labeling scheme for the vertices of T that is determined by the map f. That is, we construct a set of rules for the labeling the rectangle such that if we were to suppose that these labels were determined by a map f then the labels would emulate the area preserving and boundary twist conditions imposed on f. This labeling scheme should provide two full cells since the Poincaré-Birkhoff theorem guarantees two fixed points.

Before we develop a labeling scheme for T, we state a few definitions. A *chain* in the triangulation T is a sequence of distinct vertices $v_1, v_2, ..., v_n$ and edges $e_1, e_2, ..., e_{n-1}$ where each vertex v_i is connected to the next, v_{i+1}, by the edge e_i. A *closed chain* is a chain in T whose beginning and end points are the same. A *cycle* is a closed chain in T whose beginning and end point are the same vertex on the identified edge of A. That is, a cycle is simply a loop of vertices and edges of T that wraps around the cylinder represented by our rectangle.

We now state a labeling condition for the vertices of T. Suppose a rectangle A is triangulated with triangulation T and that each vertex of T is given one of the four labels N, W, S, or E, subject to the following conditions: (1) The labels on the left edge of A are identical to the labels on the right edge of A, (2) The vertices on the top edge of A are all labeled W (resp. E) and the vertices on the bottom edge of A are all labeled E (resp. W), and (3) there are no cycles in T all of whose vertices are labeled S and there are no cycles in T all of whose vertices are labeled N. Such a labeling will be called an *NWSE labeling*.

Theorem 5 (The NWSE Lemma). *Any triangulated rectangle whose vertices are given an NWSE labeling will either contain two full cells or will contain a cycle whose vertices will be labeled with only E's and W's.*

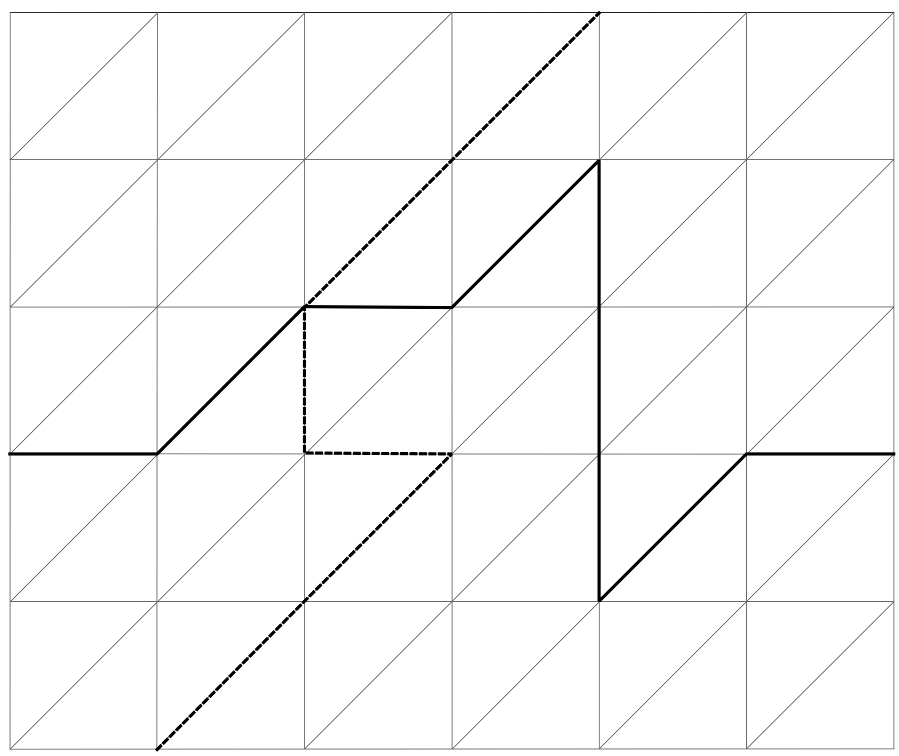

Figure 2.1: Examples of a chain and a cycle.

Proof. To prove this lemma, we shall make use of a *piecewise linear* or PL map from A to another rectangle R whose corners are labeled N, E, S, and W read clockwise. The PL map maps each vertex of T to the vertex of R that has the same label, and extends linearly across each simplex of T. As such, the PL map is continuous.

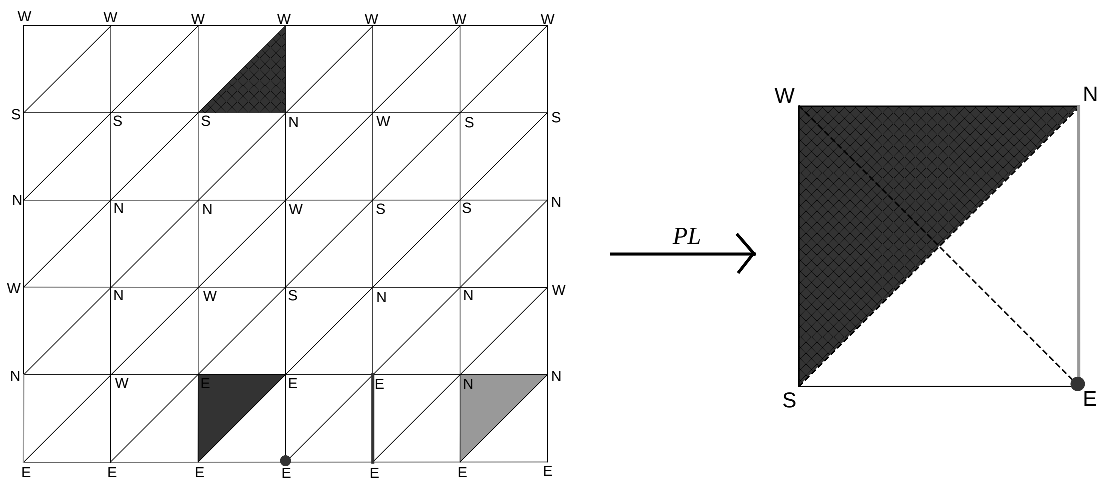

Figure 2.2: An illustration of the PL map.

We now state and prove a few useful lemmas.

Lemma 1. *If the triangulation T of A contains no cycle labeled entirely with N's, then there is a chain in T connecting the bottom edge and top edge of A that has no vertex labeled with an N. Likewise, if there are no cycles in T labeled entirely with S's, then there is a chain from the top to the bottom of A that has no vertices labeled with an S.*

Proof. Given a vertex x of T, let the *neighborhood* of x, denoted n_x be the interior of the polytope spanned by the vertices that are adjacent to x. Then n_x is an open neighborhood of x. Let U be the union of the neighborhoods of all vertices of T not labeled N, and let B be the component of U that contains the bottom edge of

A. Such a component will exist since the bottom edge contains no vertices labeled with an N. Suppose that there is no chain in T connecting the top and bottom edges that contains no vertices labeled with N's. Then B does not intersect the top edge of A. Hence, the boundary of B must be a set of disjoint chains. The vertices of these chains must all be labeled with N's, otherwise they would be contained in B. Since the top and bottom edges of A are not connected by a chain with no vertices labeled N, some N-chain in the boundary of B must separate the top edge from the bottom edge. This N-chain must be a cycle, thus contradicting the assumption that there were no N-cycles in T. Thus, there must be a chain in T connecting the top and bottom edges of A that contains no N-vertices.

A similar argument shows that there exists a chain from the top edge to the bottom edge of A that contains no vertices labeled S. □

By lemma 1 we may find an chain from the bottom edge to the top edge of A that contains no vertices labeled N, and one that contains no vertices labeled S. Call these chains c_N and c_S respectively. Now, since c_N leads from the top edge of A, which without loss of generality is labeled entirely with W's, to the bottom edge, which is labeled entirely with E's, and since the PL map is continuous, $PL(c_N)$ will be a continuous path connecting the E corner and W corner of R.

Since c_N contains no vertices labeled N, $PL(c_N)$ will either cover the E-W diagonal of R or it will cover the W-S and S-E edges of R, or both. Likewise, $PL(c_S)$ will either cover the E-W diagonal of R or will cover the E-N and N-W edges of R. Construct a closed chain c by connecting the ends of c_N and c_S along the top and bottom edges of A. Then $PL(c)$ will do one of the following: (a) $PL(c)$ will cover the boundary of R with degree of at least one, (b) $PL(c)$ will cover the boundary of the WSE or the WNE simplex of R with degree of at least one, or (c) $PL(c)$ will only cover the E-W edge of R.

Lemma 2. *Let $f : A \to R$ be a continuous map. If the image of A under f covers the* *boundary of R with degree of at least one, then f is surjective.*

For a proof of this lemma, see [3]

Now, if (a) is true, then PL is surjective by Lemma 2, and thus at least two of the full simplices in R (NWS, NES, ESW, and ENW) will have preimages in A. That is, the triangulation T will contain at least two full cells.

If (b) is true then PL maps onto of the full simplices of R, again by Lemma 2, and thus that simplex in R has a preimage in A. So, there is at least one full cell in T. Once this full cell is located we use a path following argument to find another full cell. In the proof of Sperner's lemma we viewed 1-2 edges as doors through which we could walk into the neighbooring cell. For this proof, we view E-W edges as doors. Any cell in T will either have 0, 1, or 2 E-W doors and cells with exactly one E-W edge are full cells. We now construct a path from cell to cell by walking through E-W doors without backtracking. See Figure 2.3. If (b) is true, the full cell we are given by the PL map provides a starting point for the path, since it has only one E-W door. The resulting path cannot form a cycle, that is it cannot terminate in the same cell in which it started, for that cell only has one E-W door. Thus, the path must terminate in another full cell. Hence, we have found two full cells.

Now we consider the case where (c) is true. If PL covers the E-W edge of R, then that edge will have a preimage in T. Thus, we can find an E-W edge in T. We may use this edge, or rather the cells that share this edge, as starting points for a path. This path either has two end points, which are full cells, or it forms a loop of simplices around the cylinder represented by A whose edges will form an E-W cycle.

Thus, T will either contain two full cells or it will contain an E-W cycle. □

Notice that a path following argument also works for N-S edges. Hence, in the case that (a) is true, once one full cell is found, we can find another by following N-S or E-W edges. So, there will always be an even number of full cells (possibly

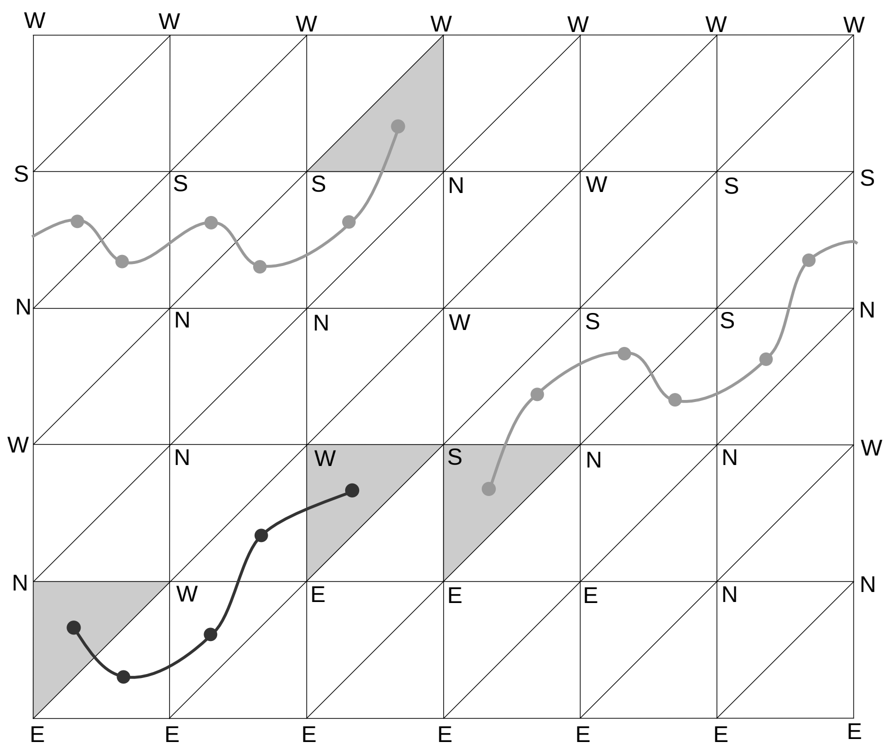

Figure 2.3: Path following.

zero).

2.2 The NWSE Lemma Implies Poincaré-Birkhoff for the Class of Regular Maps

We now show that the NWSE lemma implies the Poincaré-Birkhoff fixed point theorem for a certain class of maps. To do this we will use an approach similar to the one used to show that Sperner's lemma implies the Brouwer fixed point theorem. That is, given a map f that satisfies the conditions of the Poincaré-Birkhoff fixed point theorem, we suppose that f has fewer than two fixed points, use f to generate a labeling of an annulus A, show that the labeling is an NWSE labeling, and use information gained from the NWSE lemma to arrive at a contradiction.

In order to define precisely the class of maps for which the NWSE lemma applies, we first describe the procedure by which we generate a labeling of the vertices of a triangulation according to a map f.

Let A be an annulus and let $f : A \to A$ be a map satisfying the conditions of the Poincaré-Birkhoff fixed point theorem. Define an auxiliary map $g : A \to [0, 2\pi)$ in the following way: take a point x of A, and examine the ray starting at x that passes through $f(x)$. Let $g(x)$ be the angle that this ray makes with the horizontal.

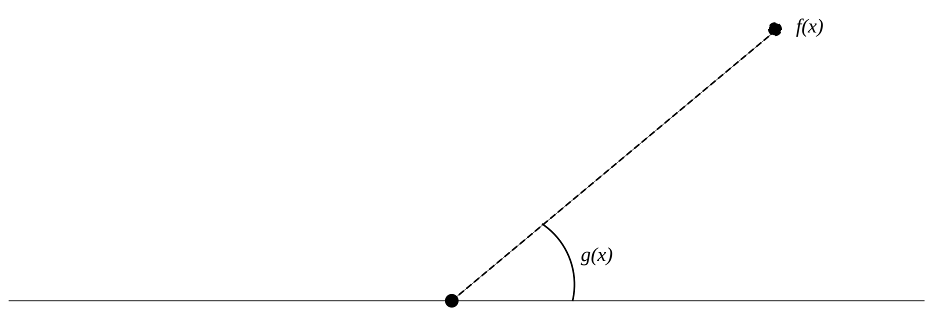

Figure 2.4: An illustration of the auxiliary map g.

Since the right and left edges of A are identified, there are many possible rays starting at x that pass throught $f(x)$, and so we need to be precise about which one we choose. To do this, we examine the covering space of A, $\mathbf{R} \times I$, where I is the interval $[0,1]$. Choose a lift f_l of f into this covering space such that f_l advances points on the top and bottom edges of A in opposite directions. In the covering space, there is only one ray from x to $f_l(x)$, and so we may use this lift to determine $g(x)$.

We label a vertex v of a triangulation T N(orth), W(est), S(outh), or E(ast) according to which direction f moves v the most. That is, we divide the plane into quadrants, one for each direction, and label v by the quadrant in which $f(v)$ lies. This corresponds to giving v a label of N if $g(v) \in [\pi/4, 3\pi/4)$, or a label of W if $g(v) \in [3\pi/4, 5\pi/4)$ and so on. Figure 2.5 illustrates this procedure.

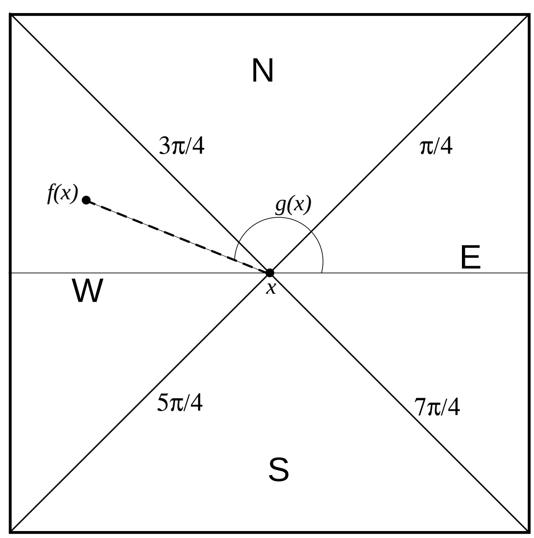

Figure 2.5: The vertex x is labeled with a W because $f(x)$ lies in the W quadrant, or $g(x) \in [3\pi/4, 5\pi/4)$.

Now we define the class of maps for which the NWSE lemma implies the

Poincaré-Birkhoff fixed point theorem. We shall call a cycle c in a triangulation T *regular* if there is no vertical line that passes through c at more than one point. A cycle is *irregular* if it is not regular. A map f is *regular* if there exists a $\delta > 0$ such that if the mesh size of a triangulation T is smaller than δ, then the auxiliary map g associated to f induces a labeling of the vertices of T that contains no irregular N or S-cycles. We shall prove the following theorem using the NWSE lemma.

Theorem 6. *If $f : A \to A$ is a regular map satisfying the conditions of the Poincaré-Birkhoff fixed point theorem then f has at least two fixed points.*

Proof. Suppose the map f has at most one fixed point, p. Then the map g is defined and continuous at every point of A except possibly p. In fact, since A is a compact set, g is uniformly continuous on A except possibly in a neighborhood around p. This is to say, if f has no fixed points, then g is uniformly continuous on A, and if f has one fixed point p and U is an open neighborhood of p, then $g \mid_{A-U}$ is uniformly continuous, since $A - U$ is a closed subset of a compact set, and thus is itself compact.

Let $0 < \epsilon < \pi/4$. Since g (resp. $g \mid_{A-U}$) is uniformly continuous, $\exists \delta > 0$ such that if $\| x_1 - x_2 \| < \delta$ then $\| g(x_1) - g(x_2) \| < \epsilon$ where $x_1, x_2 \in A$ (resp. $A - U$). Triangulate A with a triangulation T that has a mesh size smaller than δ. We now will use the auxiliary map g to label the vertices of T.

We now show that this labeling of T is an NWSE labeling. First, notice that the labeling by g will certainly satisfy conditions (1) and (2) since the vertices on the left and right edges of A are identified and since the boundary twist condition of the Poincaré-Birkhoff fixed point theorem will require all of the vertices on the top edge of A to be labeled E (or W) and the vertices on the bottom edge to be labeled W (or E).

Our intuition tells us that this labeling by f should also satisfy condition (3), or rather that a labeling not satisfying condition (3) would not be generated by a map that satisfies the area preserving condition of the Poincaré-Birkhoff fixed point theorem. Examine the simple example illustrated in Figure 2.6.

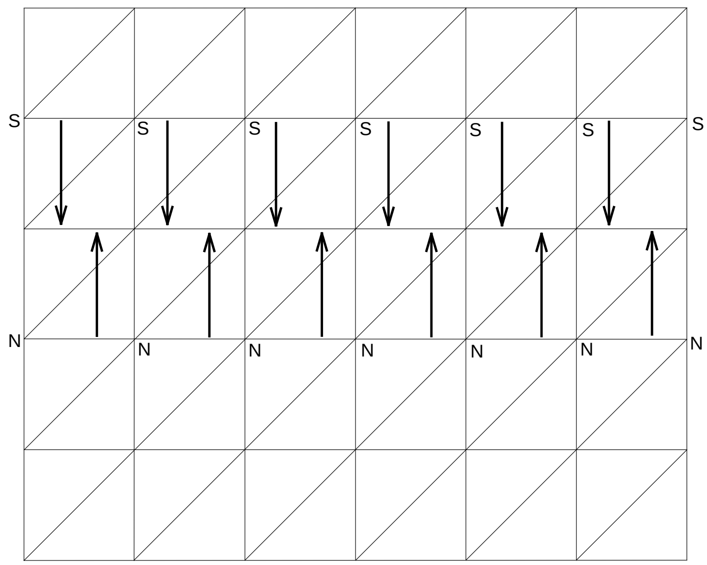

Figure 2.6: A labeling with an N-cycle and an S-cycle is not induced by an area preserving map.

Points between the N-cycle and the S-cycle are being pushed together, and thus a map that induces this labeling cannot be area preserving. Similar situations can arise when there is any arbitrary N or S cycle present in the labeling. We will now prove that our intuition is correct.

Suppose that the labeling of T generated by the auxiliary map g contained an N-cycle or an S-cycle. Without loss of generality, suppose it is an N-cycle c_n. Because f is a regular map, c_n is a regular cycle. If f has no fixed points or if the neighborhood U of the potential fixed point p does not intersect c_n, then, since the mesh size of T is smaller than δ, any point x on c_n must lie within δ of a vertex v of c_n. The map g is uniformly continuous on c_n, thus $g(x)$ must lie within $[\pi/4, 3\pi/4)$ since v is labeled N. Hence for each $x \in c_n$, $f(x)$ lies higher in the vertical direction. Note that f cannot map a point of A that lies above c_n to a point that lies below c_n. This is because a point y above c_n is connected to the top edge of A by a path that does not cross c_n. Since f is continuous, $f(y)$ must be connected to the top edge of A by a path that does not cross $f(c_n)$. Therefore, the area between c_n and the top edge of A cannot be preserved by f, which is a contradiction.

Now, if the neighborhood U of p does intersect c_n, then we cannot say that g is uniformly continuous on c_n and so the above argument does not apply. So, we choose a smaller neighborhood U_1 of p, find a new (smaller) δ, say δ_1. Construct a subtriangulation of T that preserves the grid structure of the original triangulation and has mesh size smaller than the δ_1. The cycle c_n will still be present in this subtriangulation, except now it will have more vertices and U_1 may not intersect c_n. If U_1 does not intersect c_n, then the above argument applies and we arrive at a contradiction. If U_1 does intersect c_n, the we repeat this procedure by choosing a smaller neighborhood U_2 of p, finding an new bound on the mesh size δ_2 and retriangulating, and so on. If there is an k such that U_k does not intersect c_n, then we have a contradiction as argued above. If not, then p lies on c_n and every other point of c_n is mapped vertically higher by f, and so we again have that f is not area preserving.

Thus, we have shown that the labeling of T generated by g cannot contain any N-cycles or S-cycles, and so we conclude that this labeling is indeed an NWSE labeling.

Note that the above arguments may not hold if f is not a regular map. Specifically an area preserving map f may induce a labeling of T that contains an irregular N-cycle. See Figure 2.7.

The dark line is an N-cycle and the light line is its image under f. Notice that the area above the cycle and the area above its image are potentially the same. This type of behavior, however, cannot occur if f is a regular map.

The NWSE lemma then guarantees either two full cells or an E-W cycle. We wish to show that the existence of these objects violates the uniform continuity of

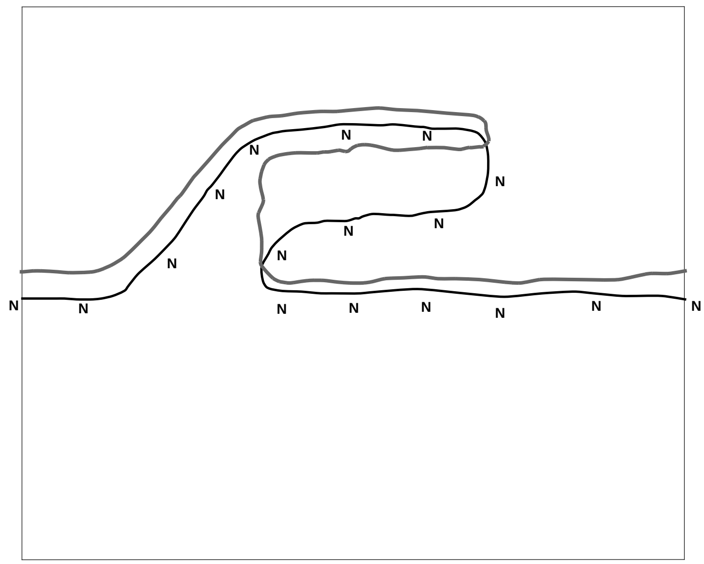

Figure 2.7: The area above the irregular N-cycle and its image are the same.

g. To do so, we will use the following lemma.

Lemma 3. *If the labeling of the triangulation T induced by the map g contains an E-W or N-S edge, then g is not uniformly continuous.*

Proof. Any two points sharing an edge in T cannot be more than δ far apart, however a point labeled E and a point labeled W must have images under g that differ by at least $\pi/2$. Hence, an E-W edge provides two points that violate the uniform continuity of g. The same holds for N-S edges. $\square$

Suppose the NWSE lemma gives us an E-W cycle. This cycle will contain more than one E-W edge. Now, one of the E-W edges may intersect the neighborhood U of p, where g is not uniformly continuous, and so one E-W edge may not violate the uniform continuity of $g \mid_{U-A}$. However, δ was chosen so that the E-W edges cannot both intersect U. Hence, the existence of two E-W edges necessarily violates the uniform continuity of g or $g \mid_{U-A}$.

Now suppose that the NWSE lemma gives us two full cells. If these full cells do not share an edge, then they will each contain either an E-W or an N-S edge, which violates the uniform continuity of g as argued above.

Finally, suppose that the two full cells given by the NWSE lemma share an edge e_1, say an E-W edge. If T contains no other E-W or N-S edges, then e_1 must be an edge in a cycle c_1 that is labeled with all N's or all S's, say S's without loss of generality, except for the vertices of e_1. In this case, we construct a subtriangulation of T with smaller mesh size. This subtriangulation either contains another N-S or E-W edge, which leads to a contradiction of the uniform continuity of g, or it contains only one E-W edge, e_2 that is part of a cycle c_2 whose other vertices are labeled with S's. If the latter is true, we repeat the process of subtriangulation.

Either this process terminates after a given number of steps, having produces more than one E-W or N-S edge, or we construct a sequence of edges e_k and a sequence of cycles c_k. It follows from the uniform continuity of g that the subsequence of regular cycles of c_k is equicontinuous and thus converges to some cycle c. The edges e_k in this subsequence converge to a point p which must be fixed. Hence, the label p is given by g is arbitrary, and we may choose to label p with an S. This way, the cycle c is a regular S-cycle, which contradicts our assumption that f was area preserving.

Thus, the supposition that f has at most one fixed point leads to a violation of either the uniform continuity of g or the regularity of f. Therefore, we have shown that f has at least two fixed points.

$\square$

Chapter 3

Conclusion, Future Work

In our proof of the Poincaré-Birkhoff fixed point theorem via the NWSE lemma, we did not employ the full strength of the NWSE lemma. Though the NWSE lemma possibly gives two full cells, we were only interested in the existence of two E-W or N-S edges. So, if we only wanted the existence of two E-W or N-S edges, why did we provide the stronger version of the NWSE lemma? To answer this question, we again look to the Sperner-Brouwer equivalence. When showing that Sperner's lemma implied the Brouwer fixed point theorem , we noticed that full cells of the triangulation closely corresponded to fixed points of the map. The same can be said for the NWSE-Poincaré-Birkhoff relationship. Fixed points of the map f will be located near full cells of the triangulation T of A. The case where there is only an E-W cycle rather than two full cells is likely to correspond to the case where the map f produces fixed circle in the annulus. The reason we did not use the full strength of the NWSE lemma in our proof is that our proof is not constructive. A problem for future research is to find a constructive proof of the Poincaré-Birkhoff fixed point theorem via the NWSE lemma. Such a proof may use the full strength of the NWSE lemma. Moreover, it may provide a method of locating the fixed points of f as is the case with constructive proof of the Brouwer fixed point theorem via Sperner's lemma.

In this paper, we have only shown that the NWSE lemma implies the Poincaré-Birkhoff fixed point theorem for the class of regular maps. Another problem for the future is to modify the NWSE lemma so that it applies to non-regular maps.

In this paper, we discussed only one direction of the equivalence between the

NWSE lemma and the Poincaré-Birkhoff fixed point theorem. An open problem is to develop a proof that the Poincaré-Birkhoff fixed point theorem implies the NWSE lemma, thus showing that the two theorems are in fact *equivalent* in the sense that each implies the other. To do this, one could take a similar approach to proving the Poincaré-Birkhoff fixed point theorem via NWSE, but in reverse. Here we would start with an NWSE labeled triangulation, use it to construct a map f satisfying the conditions of the Poincaré-Birkhoff fixed point theorem, then use the existence of two fixed points to prove the existence of two full cells or use the existence of a fixed circle to prove the existence of an E-W cycle. It is unclear how this map f is to be constructed, however. So, while this direction of the equivalence is less useful and seemingly more difficult, it would be nice to have for theoritical completeness.

Yet another interesting problem for future research would be to find a combinatorial analogue for the Lefschetz fixed point theorem. The Lefschetz theorem is very general in the sense that it provides conditions under which a continuous self map of *any* finite complex will have a fixed point. It states that if f is a continuous self map of a complex $|K|$ and the Lefschetz number of f, denoted by $\Lambda(f)$, is not zero, then f has a fixed point. To get a handle on this problem, I propose to begin by examining self maps of surfaces which are easy to represent as triangulated objects. The n-holed torus, for example, can be represented as a $4n$-gon, with certain pairs of edges identified. In this way, we can triangulate the torus by triangulating its respective polygon. We may then examine labelings of these triangulations by maps, and determine what types of maps induce labelings that contain full cells.

Examining the relationship between combinatorics and the Lefschetz number yields some more profound questions as well. $\Lambda(f)$ is determined by the homology groups $H_p(K)$ and the induced map f_*, and thus $\Lambda(f)$ depends only on the homotopy class of f and the underlying space $|K|$. Hence, knowledge of the map f and the Lefschetz number yields a good deal of information about the space $|K|$. For example, if the map f is simply the identity map, it can be shown that $\Lambda(f)$ is equal to the Euler characteristic of the complex K, $\chi(K)$. Now, $\chi(K)$ can be determined in a combinatorial fashion by examining a triangulation of the space $|K|$. It can be shown that $\chi(K) = V - E + F$, where V, E, and F are the number of vertices, edges, and faces of the triangulation, respectively. I would like to investigate ways in which $\Lambda(f)$ could be determined in a combinatorial fashion in the more general situation where the map f is a simplicial map, or an arbitrary continuous map.

Finding combinatorial analogues of fixed point theorems is not only useful, in that it provides a way to actually locate fixed points rather than just stating their existence, but it also provides a means of investigating a potentially very interesting and profound connection between combinatorics and topology.

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