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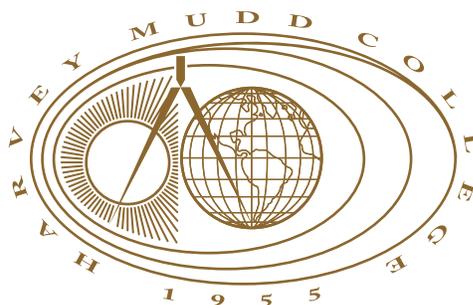
The Hausdorff Dimension of the Julia Set of Polynomials of the Form $zd + c$

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The Hausdorff Dimension of Julia Sets of Polynomials of
the form $z^d + c$

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Abstract

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Complex dynamics is the study of iteration of functions which map the complex plane onto itself. In general, their dynamics are quite complicated and hard to explain but for some simple classes of functions many interesting results can be proved. For example, one often studies the class of rational functions (i.e. quotients of polynomials) or, even more specifically, polynomials.

Each such function f partitions the extended complex plane $\overline{\mathbb{C}}$ into two regions, one where iteration of the function is chaotic and one where it is not. The non-chaotic region, called the *Fatou Set*, is the set of all points z such that, under iteration by f , the point z and all its neighbors do approximately the same thing. The remainder of the complex plane is called the *Julia set* and consists of those points which do not behave like all closely neighboring points.

The Julia set of a polynomial typically has a complicated, self-similar structure. Many questions can be asked about this structure. The one that we seek to investigate is the notion of the dimension of the Julia set. While the dimension of a line segment, disc, or cube is familiar, there are sets for which no integer dimension seems reasonable. The notion of Hausdorff dimension gives a reasonable way of assigning appropriate non-integer dimensions to such sets.

Our goal is to investigate the behavior of the Hausdorff dimension of the Julia

sets of a certain simple class of polynomials, namely $f_{d,c}(z) = z^d + c$. In particular, we seek to determine for what values of c and d the Hausdorff dimension of the Julia set varies continuously with c . Roughly speaking, given a fixed integer $d > 1$ and some complex c , do nearby values of c have Julia sets with Hausdorff dimension relatively close to each other?

We find that for most values of c , the Hausdorff dimension of the Julia set does indeed vary continuously with c . However, we shall also construct an infinite set of discontinuities for each d . Our results are summarized in Theorem 10, Chapter 2.

In Chapter 1 we state and briefly explain the terminology and definitions we use for the remainder of the paper. In Chapter 2 we will state the main theorems we prove later and deduce from them the desired continuity properties. In Chapters 3 we prove the major results of this paper.

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Chapter 1

Definitions and Terms

In this chapter we define many of the terms and concepts used throughout the paper. For a more thorough introduction, see Beardon's book, [2]. For a more detailed discussion of these concepts, see Carleson and Gamelin's book [3] or Milnor's text [7].

1.1 Basic Complex Dynamics

Consider an arbitrary polynomial $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$, acting on the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let f^n denote the n th iterate of f , that is, f composed with itself n times. For each point $z_0 \in \overline{\mathbb{C}}$, we are interested in the behavior of the sequence $z_0, f(z_0), f^2(z_0), \dots, f^n(z_0), \dots$, and in particular what happens as n goes to infinity.

Define the *Fatou set* of f to be the set of $z \in \overline{\mathbb{C}}$ such that there exists some neighborhood of z on which the family of iterates $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. Informally, this is the set of all point which share a common fate with their immediate neighbors. Define the *Julia set* to be the complement of the Fatou set. This is the set of points whose behavior does not resemble that of some of their immediate neighbors. The Julia set and Fatou set thus partition the extended complex plane into the region where the behavior of the function under iteration by f is chaotic and the region where it is not. Note that, by definition, the Fatou set must be open and thus the Julia set is closed.

For example, consider the function $f(z) = z^d$. Under iteration by f , points z_0 in the interior of the unit circle are attracted to the origin, while points z_0 in the exterior are attracted to the point at infinity. Thus both the interior and exterior of the unit circle lie in the Fatou set of $f(z) = z^d$. However, given two points on the unit circle whose angular difference is ε , their iterates both lie on the unit circle but the angular distance between them increases to $d\varepsilon$ under application of f . Thus any two such points are driven apart under iteration by f . Moreover, any neighborhood of a point on the unit circle contains points that converge to 0 and points that escape to infinity, as well as points that remain on the unit circle. Thus the Julia set is the entire unit circle. In general, Julia sets are much more complicated than this, with complex, self-similar structure. Some are even Cantor dusts, perfect sets made up of uncountably many disconnected points. See Appendix C for some pictures of more typical Julia sets.

The Fatou set is conventionally denoted F and the Julia set J ; we will use the notation $F(f)$ and $J(f)$ when we wish to be explicit about what function we are considering. For the family $f_c(z) = z^d + c$ with d a fixed integer greater than 1, we will frequently use the shorthand notation J_c in place of $J(f_c)$.

Note also that the Julia set and Fatou set are both invariant under f , that is, $f(F) = F$ and $f(J) = J$.

The structure of the Julia set is strongly influenced by the behavior of the *critical points* of f . These are the points z for which the pre-images of any given neighborhood of z under f^{-1} are not all distinct. For instance, consider $f(z) = z^2$. For any finite point $\zeta = re^{i\theta} \neq 0$, the neighborhood centered at ζ of radius $\frac{r}{2}$ has two distinct preimages. However, at $\zeta = 0$, all preimages of every neighborhood of 0 must contain 0. Thus for a neighborhood N of 0, if we try to find branches of $f^{-1}(N)$, i.e. connected sets that map bijectively onto N under f (thus allowing a well-defined inverse to be specified), we find that all such branches overlap at 0. Thus 0 is a critical point of $f(z) = z^2$.

In general, a finite point z is a critical point if and only if $f'(z) = 0$. In particular, our class of functions, $f_c(z) = z^d + c$ has a single critical point at $z = 0$. Infinity is also a critical point of any polynomial; however, it can be neglected for most purposes, as its properties are the same for all polynomials. In particular, it is always a critical point and is always fixed under f . We shall see that in general this is sufficient to have it trivially satisfy all requirements of critical points for purposes of our theorems. Thus except when otherwise specified, we will use “critical point” to mean “finite critical point”.

The most obvious effect of critical points is on the connectedness of the Julia set. The Julia set of f is connected if and only if all critical points have bounded forward orbit, that is, if the sequence of points $z_0, f(z_0), f^2(z_0), \dots$ remains bounded. The Julia set is totally disconnected, meaning all components are point components, if and only if all critical points escape to infinity. For functions with only one critical point, such as our polynomials of the form $z^d + c$, these are the only two possibilities. For more complicated functions, other behaviors can occur.

It is useful when considering a class of functions to look at its *parameter space*, in which each point represents a function. For instance, the parameter space of $f_c(z) = z^2 + c$ is isomorphic to the complex plane \mathbb{C} , where the point $c_0 \in \mathbb{C}$ corresponds to the function $f_{c_0} = z^2 + c_0$. Note that the parameter space need not be so simple; for instance, we could have \mathbb{C}^3 as a parameter space in which the triple (c_0, c_1, c_2) corresponds to the function $f(z) = c_2 z^2 + c_1 z + c_0$.

The parameter space allows us to represent various properties of a class of functions in an easy to visualize way. The most common such property is the connectedness of the Julia set. The *connectedness locus* of a family of functions is the set of all points in the parameter space corresponding to functions with connected Julia set. For quadratics of the form $z^2 + c$, the connectedness locus is the famous Mandelbrot set, denoted M . We will be considering functions of the form $z^d + c$ for fixed d . Let us denote the connectedness locus of $z^d + c$ as M_d , which is a subset

of \mathbb{C} . In particular, the Mandelbrot set would be denoted M_2 . Observe that M_d can equivalently be defined as the set of all c for which the orbit of 0 under $z^d + c$ is bounded, as by the earlier remark the Julia set is connected if and only if all critical points remain bounded. See Appendix C for pictures of some typical Julia sets.

Let us now discuss the notion of a *periodic point* of f . A point ζ is a periodic point of order n if $f^n(\zeta) = \zeta$ and whenever $1 \leq k < n$ then $f^k(\zeta) \neq \zeta$. A periodic point of order 1 is called a *fixed point*. The behavior of a periodic point is classified via its action on nearby points. When we describe the behavior of a periodic point of order k , we do so in terms of what happens to nearby points under f^k . Other iterates will have a similar behavior centered at some other point in the orbit of ζ . For instance, 0 is a periodic point of order 2 of the map $f(z) = z^2 - 1$. We shall see that if a point z is sufficiently close to 0, its even iterates $z, f^2(z), f^4(z), \dots$ will be converging to 0, while its odd iterates $f(z), f^3(z), \dots$ will be attracted to $f(0) = -1$ instead.

Let the *multiplier* of a periodic point of order k be defined as $(f^k)'(\zeta)$, the derivative at ζ of the k th iterate of f . If the multiplier has modulus less than 1, the periodic point is called *attracting*, and, in the special case where the multiplier is 0, *superattracting*. In either case, all points in a small neighborhood of the point ζ are attracted to ζ under iteration, that is, for z close enough to ζ , $\lim_{n \rightarrow \infty} |f^n(z) - f^n(\zeta)| = 0$. In other words, the sequence of forward iterates of any nearby point converges to the orbit of ζ . It follows easily that all attracting periodic points lie in the Fatou set, as do all points attracted to a periodic point ζ (points whose forward iterates converge to ζ). An example of an attracting fixed point is the point 0 under the map $f(z) = z^2$; the interior of the unit circle is attracted to 0 under iteration. The above-mentioned map, $f(z) = z^2 - 1$, has an attracting orbit of period 2; points near 0 or -1 are pulled into that cycle. For general polynomials, ∞ is also a superattracting fixed point.

The next type of periodic point ζ is when the multiplier $(f^k)'(\zeta)$ is a root of

unity, that is $(f^k)'(\zeta) = \exp(2\pi im/n)$ for some rational number m/n . In this case, the periodic point is called *parabolic*, and each neighborhood of ζ contains regions of attraction as well as regions where points are repelled from ζ . In finitely many directions, the periodic point is strictly repelling. For z in such a region, $|f^{nk}(z) - \zeta|$ is a strictly increasing function of n . In a like number of directions, the function is strictly attracting. In between these directions, points move along loops which depart from the periodic point tangent to a repelling direction and loop around to reconnect with the point tangent to an attracting direction. Any point in such a loop is thus ultimately attracted to ζ ; the set of points attracted to ζ lying between an attracting direction and an adjacent repelling direction is called a *petal*. Thus any parabolic periodic point lies in the Julia set, but the resulting petals are in the Fatou set.

If the multiplier $(f^k)'(\zeta)$ has modulus 1 but is not a root of unity, the point ζ is *irrationally indifferent*. Points near such a ζ will move along loops around f ; the action of f on points near ζ is similar to an irrational rotation of the unit disk. Thus such ζ lie in the Fatou set.

Finally, we have *repelling* periodic points, which have multiplier $(f^k)'(\zeta)$ greater than 1 in absolute value and drive nearby points away in all directions. Such points lie in the Julia set.

For any periodic point ζ of order k with an attracting region (that is, an attracting or parabolic periodic point) we can define the *basin of attraction*, the set of all points which are eventually attracted to ζ . This region is a collection of disjoint open sets. Some subcollection of these sets will be fixed under f^k ; that is, f^k is an automorphism on those components; such regions are called the *immediate basin* of the point. The immediate basin may be equal to the basin of attraction (as in the case of 0 for $f(z) = z^2$, in which both basins are the open unit disk), but they need not be. Some components of the basin of attraction may be mapped into the immediate basin by some iterate of f , whereafter their image will cycle through the

various components of the immediate basin in a periodic fashion. This behavior is analogous to a point which is strictly preperiodic under f ; its forward iterates become periodic after some point in the sequence.

Periodic points interact with critical points as follows: the immediate basin of any attracting periodic point or parabolic periodic point must contain a critical point. This allows us to bound the number of such periodic points, as a polynomial of degree d has at most $d - 1$ critical points, and for the special polynomials we will be considering, which have only one critical point, there is clearly only one parabolic or attracting periodic orbit. If there were more than one the critical point would have to be attracted to more than one periodic orbit, an impossibility. We shall also see that there are important consequences if a critical point is periodic or strictly preperiodic.

A property of critical points related to periodicity is the notion of *recurrence*. A critical point is called *recurrent* if it lies in the closure of the set of its forward iterates. Any periodic critical point is recurrent, as it lies in its forward orbit. Any strictly preperiodic critical point is not, as it doesn't lie in its forward orbit and its forward orbit contains only finitely many points and therefore has no limit points.

1.2 Hyperbolicity

An important class of Julia sets are those which are *hyperbolic*; the dynamics are much more tractable in this case. A Julia set is hyperbolic if f maps it onto itself in a way that is locally expanding in some strong sense. Specifically, a function is hyperbolic on a set X if there are constants $c > 0$ and $\kappa > 1$ such that

$$\|(f^n)'\| \geq c\kappa^n$$

on J_f for all $n \geq 0$, where $\|\cdot\|$ denotes the derivative with respect to the spherical metric. It can be shown ([3]) that the notion of hyperbolicity is equivalent to the condition that every critical point of f is attracted to an attracting cycle. Note

that while this, in general, must hold for all critical points and not just finite ones, infinity is a superattracting fixed point for any polynomial and thus is attracted to an attracting orbit (itself) so can be safely ignored.

For instance $f(z) = z^2 + 1$ has a hyperbolic Julia set, as 0 is attracted to ∞ , an attracting fixed point; in fact any f for which the critical point escapes to infinity has a hyperbolic Julia set.

For Julia sets that are not hyperbolic, it is useful to consider the subsets of them that are hyperbolic. A *hyperbolic subset* of f is a closed subset of $J(f)$ on which the action of f is hyperbolic.

1.3 Dimensions

Because Julia sets have a complicated, self-similar structure, it is useful to look at their *Hausdorff dimension*. The Hausdorff dimension provides a natural way of assigning a noninteger dimension to sets for which no integer dimension seems appropriate. For instance, it is not obvious that the standard center one third Cantor set P has a dimension; nor what that dimension should be. All components of P are single points and P has no length, but clearly it has more substance than a 0 dimensional set such as a few scattered points. For an intuitive idea for what the dimension of such a set should be, we note that if we stretch the universe by the map $z \rightarrow 3z$, a one dimensional set like a line segment gets mapped to 3^1 copies of itself, a two dimensional square to 3^2 copies of itself, and a cube to 3^3 copies of itself. Thus sufficiently simple d -dimensional sets are mapped to 3^d copies of themselves. The Cantor set, though, is mapped to 2 copies of itself; thus the natural dimension to assign is $\log_3 2$. For a precise definition of the Hausdorff dimension of a set, see Appendix A.

We shall denote the Hausdorff dimension of a set U by $\text{Hdim}(U)$.

We can now define the *hyperbolic dimension* of a function. The hyperbolic di-

mension of f is the supremum of the set of Hausdorff dimensions of hyperbolic subsets of f . We shall denote the hyperbolic dimension of f by $\text{hypdim}(f)$. As the Hausdorff dimension of a set is greater than or equal to the Hausdorff dimension of any subset thereof, it follows that $\text{hypdim}(f) \leq \text{Hdim}(J(f))$.

Additionally, the hyperbolic dimension of f has a number of nice properties that the Hausdorff dimension itself does not have. Foremost of these is the fact that the hyperbolic dimension of a family of functions f parameterized analytically by some complex parameter c varies *lower semicontinuously* with respect to c . A function φ is lower semicontinuous at a point a if for every $\varepsilon > 0$ there exists some neighborhood U of a such that for all $x \in U$, $f(x) > f(a) - \varepsilon$. Equivalently a function is lower semicontinuous if the inverse image of every interval of the form (a, ∞) is open. For us, this means that the set of points $c \in \mathbb{C}$ where the hyperbolic dimension of f_c is greater than some N is an open subset of \mathbb{C} .

1.4 *J-stability*

Two complex functions f and g are called *J-equivalent* if there exists some homeomorphism φ conjugating the restriction of f to $J(f)$ to the restriction of g to $J(g)$:

$$\varphi f = g\varphi.$$

A function f is *J-stable* if there exists some neighborhood U of f in the parameter space such that all $g \in U$ are *J-equivalent* to f .

We consider below the set of all functions f that are *J-stable*; following the notation in [5], we will denote this set G , and its complement by K .

The set K is called the locus of *J-instability*. We believe that K is equal to ∂M_d . We know this is true for $d = 2$ (see [11]) but we have been unable to show this equality for general d .

1.5 Other Terminology

For the remainder of the paper let d be a fixed integer greater than 1, and let $f_c(z) = z^d + c$. We also denote the mapping $c \rightarrow \text{Hdim}(J_c)$ by H .

Chapter 2

Results

In this chapter, we state four theorems from Shishikura [11], Urbański [12], and Ruelle [10]. We use these results to show that the mapping $H : c \rightarrow \text{Hdim}(J_c)$ is discontinuous on a subset of K , the locus of J -instability. Additionally, we show continuity at most other points in the parameter plane and briefly discuss the regions of the plane on which the continuity properties of H are still unknown.

2.1 Known Results

Theorems 1 and 2, below, are Theorems 1 and 2 from pages 229 and 230 of [11].

Theorem 1 (Shishikura) *Let $\{f_\lambda | \lambda \in \Lambda\}$ be a complex analytic family of rational maps of degree $d > 1$, where Λ is an open set in \mathbb{C} . Suppose f_{λ_0} ($\lambda_0 \in \Lambda$) is not J -stable in this family. Then*

$$\text{Hdim} \left\{ \lambda \in \Lambda \left| \begin{array}{l} f_\lambda \text{ is not } J\text{-stable and has a hyperbolic} \\ \text{subset containing a forward orbit of a} \\ \text{critical point} \end{array} \right. \right\} \geq \text{hypdim}(f_{\lambda_0}). \quad (2.1)$$

In simple terms, this function bounds the Hausdorff dimension of a subset of the locus of J -instability, and thus the entire locus of J -instability, by the hyperbolic dimension of the functions in that region.

Theorem 2 (Shishikura) *Suppose that a rational map f_0 of degree $d > 1$ has a parabolic fixed point ζ with multiplier $\exp(2\pi ip/q)$ with $p, q \in \mathbb{Z}$ and $(p, q) = 1$, and that the immediate parabolic basin of ζ contains only one critical point of f_0 . Then for any $\varepsilon > 0$*

and $\beta > 0$, there exists a neighborhood \mathcal{N} of f_0 in the space of rational maps of degree d , a neighborhood V of ζ in $\overline{\mathbb{C}}$, and positive integers N_1 and N_2 such that if $f \in \mathcal{N}$, and if f has a fixed point in V with multiplier $\exp(2\pi i\alpha)$ where

$$q\alpha = p \pm \frac{1}{a_1 \pm \frac{1}{a_2 + \beta}}$$

with integers $a_1 \geq N_1$, $a_2 \geq N_2$, and $\beta \in \mathbb{C}$, $0 \leq \operatorname{Re}(\beta) < 1$, $|\operatorname{Im}(\beta)| \leq b$, then $\operatorname{hypdim}(f) > 2 - \varepsilon$.

We will show that for the functions which we are interested in, the condition that every parabolic basin contains a fixed point will always hold so long as there is a parabolic fixed point. Theorem 2 then constructs a set of functions arbitrarily close to the given function with hyperbolic dimension as large as we want. We can therefore use this theorem to construct a sequence of functions with parameters in K , the locus of J -stability, that converges to any given function P_c that has a parabolic periodic point in such a way that the hyperbolic dimensions of these functions converge to 2.

Theorem 3 is Theorem 7.15 from [12], restricted to polynomials.

Theorem 3 (Urbański) *If $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is a polynomial map with no nonperiodic recurrent critical points, then $\operatorname{Hdim}(J(f)) < 2$.*

Theorem 4 is Corollary 6 from [10].

Theorem 4 (Ruelle) *If the Julia set J of a rational function f is hyperbolic, then the Hausdorff dimension of J depends real-analytically on f .*

It follows that H , the Hausdorff dimension of J_c , is a *continuous* function of the parameter $c \in \mathbb{C}$.

2.2 New Theorems

As an immediate consequence of Theorem 1, we can prove this close analog of Corollary 3(i) on page 231 of Shishikura's paper, [11]:

Theorem 5 *If U is an open set containing $c \in K$, then*

$$\begin{aligned} \text{Hdim}(K \cap U) &\geq \text{Hdim}\{c \in K \cap U \mid 0 \text{ is not recurrent under } P_c\} \\ &\geq \text{hypdim}(P_c). \end{aligned}$$

Proof: The first inequality follows from the fact that if $V \subset W$, then $\text{Hdim}(V) \leq \text{Hdim}(W)$.

The second inequality follows from Theorem 1 with $\lambda = c$ and $\Lambda = U$. It is clear from the definition that a hyperbolic subset cannot contain a critical point. Therefore, if the forward orbit of the critical point lies within a hyperbolic subset, the critical point cannot lie in its forward orbit. Further, as hyperbolic subsets are closed, the critical point cannot lie in the closure of its forward orbit either. Thus the set of points constructed in Theorem 1 is a subset of the set of J -unstable critical points with the critical point non-recurrent. Then the Hausdorff dimension of the set of functions with 0 non-recurrent is at least the Hausdorff dimension of the set of functions with a hyperbolic set containing a critical point, which, by Theorem 1, is greater than or equal to the hyperbolic dimension of P_c , providing the desired inequality.

The first inequality in Theorem 5, along with Theorem 3, yields the following.

Theorem 6 *There exists a nonempty subset S of K such that for all $c \in S$,*

$$\text{Hdim}(J(P_c)) < 2.$$

Proof: Let S be the set of points c where 0 is non-recurrent under P_c . By Theorem 5, S has Hausdorff dimension 2, and is *a fortiori* nonempty (and in fact, uncountably

infinite). Since 0 is the only critical point of P_c , all critical points of $P_c(z) = z^d + c$ are non-recurrent, so there are no periodic critical points. Theorem 3 now implies that for all $c \in S$, the Hausdorff dimension of the Julia set $J(P_c)$ is strictly less than 2.

In Chapter 3 we show that Theorem 2 implies the following analog of Corollary 3(ii) in [11].

Theorem 7 *If P_c has a parabolic periodic point, then there exists a sequence c_n in K such that $c_n \rightarrow c$ and $\text{hypdim}(P_{c_n}) \rightarrow 2$.*

In Chapter 2 we also prove Theorem 11, that parabolic periodic points are dense in the locus of J -instability K .

We can now prove the following interesting side result:

Theorem 8 *For any open set U which intersects K , $\text{Hdim}(K \cap U) = 2$. Further, $\text{Hdim}(K) = 2$.*

Proof: Since parameters c with a parabolic periodic point are dense in K , there exists such a point in $K \cap U$; call it c_0 . By Theorem 7, there exists some sequence c_n in K converging to c such that $\text{hypdim}(P_{c_n}) \rightarrow 2$. Since U is open, there exists some neighborhood of c_0 that lies completely in U ; but all neighborhoods of c_0 contain infinitely many of the c_n . By Theorem 5, $\text{Hdim}(K \cap U)$ must be at least the supremum of the hyperbolic dimension $\text{hypdim}(P_{c_n})$ over all $P_{c_n} \in U$. Since the c_n have hyperbolic dimension tending to 2, the supremum is at least 2; thus $\text{Hdim}(K \cap U) \geq 2$. Since all these sets are subsets of the plane, the Hausdorff dimension must be less than or equal to 2, the dimension of the plane, giving us the desired result that $\text{Hdim}(K \cap U) = 2$. Setting $U = \mathbb{C}$ yields the second result.

In Chapter 3 we also prove the following:

Theorem 9 *The set of parameters c with $\text{Hdim}(J_c) = 2$ is dense in the locus of J -instability K .*

We can now prove our desired continuity results, parts (i) through (iii) of Theorem 10. Recall that H is the map from c to $\text{Hdim}(J_c)$, the Hausdorff dimension of the Julia set of $P_c(z) = z^d + c$.

Theorem 10 (i) *The function H is discontinuous on a nonempty subset of K , specifically the set of $c \in K$ with Hausdorff dimension less than 2.*

(ii) *H is continuous at all $c \in K$ with hyperbolic dimension 2.*

(iii) *H is continuous at every hyperbolic point c .*

Proof: By Theorem 6, the set of points $c \in K$ such that the Hausdorff dimension of the Julia set, $\text{Hdim}(J_c)$, is *less than 2* is nonempty. By Theorem 9, the set of points with Hausdorff dimension equal to 2 is dense in K , implying that h is discontinuous at every point in K with Hausdorff dimension less than 2. This proves part (i).

Part (ii) follows from the result on page 229 of [11] that the mapping $f \rightarrow \text{hypdim}(f)$ is lower semicontinuous; that is, for any function f and $\varepsilon > 0$ there exists some neighborhood on which all points have hyperbolic dimension greater than $\text{hypdim}(f) - \varepsilon$. Since the Hausdorff dimension of J_c is at least the hyperbolic dimension of P_c , we get that for any function f and $\varepsilon > 0$ there is a neighborhood U of f on which for all $g \in U$, $\text{Hdim}(J(g)) > \text{hypdim}(f) - \varepsilon$. Then since the Hausdorff dimension cannot exceed 2, if $\text{hypdim}(f) = 2$, we have that $\text{Hdim}(J_g)$ is within ε of 2 everywhere in U . So f satisfies the epsilon-delta definition of continuity, so the Hausdorff dimension is continuous in this case. This proves (ii).

Part (iii) is immediate from Theorem 4.

We have thus constructed an infinite set of discontinuities of the map H , and shown that H is continuous for much of the remainder of the parameter plane. It is unknown whether there are any points that do not fall in one of the three categories detailed by Theorem 10. We now discuss possible indeterminate regions.

We believe it is the case that $K = \partial M_d$. This is implied to be true in [11], but we have been unable to locate a proof. It is, however, definitely true when $d = 2$. If this is true, then continuity is characterized almost completely by the above. It is conjectured that for $d = 2$, the hyperbolic points in \mathbb{C} make up the complement of ∂M_2 [7].

It is also reasonable to conjecture this for higher dimensions; if true it would mean that all places where continuity is unknown lie in ∂M_d , which we believe coincides with K .

It is unknown whether it is possible to have a point with Hausdorff dimension 2 but hyperbolic dimension less than 2; however, at any such point we cannot at present characterize the continuity of H .

Chapter 3

New Results

We now prove the three results mentioned in Chapter 2. First, we show that parameters with a parabolic periodic point are dense in K . We then prove Theorem 7, that we can construct a sequence converging to any parabolic periodic point such that the hyperbolic dimension converges to 2. Finally, we prove Theorem 9, that functions whose Julia set has dimension 2 are dense in the locus of J -instability.

3.1 Density of Parabolic Periodic Points

We prove the following:

Theorem 11 *Parameters c for which $f_c(z) = z^d + c$ has a parabolic periodic point are a dense subset of K .*

Proof: By page 77 of [5], K is equivalent to the closure of the set of functions for which there are some point ζ and some integer $m \geq 1$ such that $\left. \frac{d(f^m)}{dz} \right|_{\zeta} = 1$ and $f^m(\zeta) = \zeta$. A parabolic periodic point of order n with multiplier $\exp(2\pi i \frac{p}{q})$ satisfies this when $m = nq$. Thus the set of parabolic periodic points is a subset of K . To show that the set of such points is dense, we note that if ζ is a fixed point of f^m , then ζ is a periodic point of order k of f , for some k that divides m . Therefore we can write $m = kp$ for some integer p . Then the multiplier of the periodic point is a p th root of unity, so all such periodic points are parabolic. The result is then immediate from the fact that sets are dense in their closures.

3.2 Proof of Theorem 7

We now prove Theorem 7, which states that if P_c has a parabolic periodic point, we can find a sequence $c_n \in K$ converging to c with hyperbolic dimension $\text{hypdim}(P_{c_n})$ converging to 2.

Let $f_c(z) = z^d + c$ have a parabolic periodic point of order n , that is, a point z_1 such that

$$f^n(z_1) = z_1 \text{ and } \frac{d}{dz}(f^n(z_1)) = e^{2\pi i \frac{p}{q}}.$$

Let $z_k = f(z_{k-1})$ for $k \geq 2$. Note that as z_1 is periodic of order n , we have $z_k = z_{k-n}$. Thus, there are only n unique orbit points z_k , which we will express as z_1, z_2, \dots, z_n . Let us denote by P_k the immediate parabolic basin of z_k under f^n . Then f maps P_k injectively (one-to-one) onto P_{k+1} .

The immediate parabolic basin of the parabolic cycle z_1, z_2, \dots, z_n is simply the union of the P_k for $k = 1, 2, \dots, n$. Thus 0 , the unique critical point of f , must lie in some P_k . Without loss of generality, let us say $0 \in P_n$.

Now, any critical point of $f^n(z)$ must satisfy

$$\begin{aligned} 0 &= (f^n)'(z) \\ &= (f(f^{n-1}))'(z) \\ &= f'(f^{n-1}z) \cdot (f^{n-1})'(z). \end{aligned}$$

Expanding this recursively, we find that the critical points of f^n are the roots of

$$f'(f^{n-1}(z))f'(f^{n-2}(z)) \cdots f'(f(z))f'(z) = 0.$$

Since $f'(z) = 0$ if and only if $z = 0$, this means the critical points of f^n must solve $f^k(z) = 0$ for some $k \in \{0, 1, \dots, n-1\}$. Thus any critical point is of the form $f^{-k}(0)$.

Recall that 0 lies in P_n . Since f maps P_{n-1} injectively onto P_n , there is precisely one value of $f^{-1}(0)$ in P_{n-1} . Proceeding inductively we find that P_{n-k} contains precisely one value of $f^{-k}(0)$. Further, as P_k is mapped injectively onto P_{k+1} by f , no $f^{-k}(0)$ can lie in P_{n-j} for $j \neq k$. Thus each P_k contains precisely one critical point of f^n .

Recall that for a general polynomial f , every parabolic orbit must contain a critical point. Since there is only one critical point of f , f has at most one parabolic cycle. Thus z_1, z_2, \dots, z_n is the only parabolic cycle of f . Therefore, the P_k are the only immediate parabolic basins of f^n , so every parabolic basin of f contains exactly one critical point. This means we can apply Theorem 2 to f_c^n if f_c has a parabolic periodic point of order n , as f_c^n is clearly a rational map of order greater than 1. Let us denote the parabolic periodic point by ζ .

Proceeding along these lines, let us take $\varepsilon = 1/m$ for some positive integer m , and let b be arbitrary; say $b = 1$. Then by Theorem 2, there exists a neighborhood \mathcal{N} of f_c^n in the space of functions of degree nd , a neighborhood V of ζ in $\overline{\mathbb{C}}$, and positive integers N_1 and N_2 such that if $a_1 > N_1$, $a_2 > N_2$, $0 \leq \operatorname{Re}(\beta) < 1$, $|\operatorname{Im}(\beta)| \leq b$, and $g \in \mathcal{N}$ has parabolic periodic point with multiplier α satisfying

$$q\alpha = p \pm \frac{1}{a_1 \pm \frac{1}{a_2 + \beta}},$$

then $\operatorname{hypdim}(g) > 2 - \varepsilon$. In particular, if we take $\beta = 0$ and $g = f_{c'}^n$ for c' close to c and with multiplier in the correct form, we find that for each positive integer m this allows us to construct a point in the parameter space which is “near” f_c and has hyperbolic dimension of at least $2 - \frac{1}{m}$. If we additionally require that $a_1 > m$, we thus generate a sequence of points c_m converging to c with hyperbolic dimension converging to 2. Further, since we took $\beta = 0$ in all cases, α is rational. Therefore, all c_m are parabolic periodic points in their own right. Then, by Theorem 11, all parabolic periodic points lie in K , so we have constructed the desired sequence in K converging to c , with hyperbolic dimension converging to 2.

3.3 Proof of Theorem 9

We now prove that the set of points with Hausdorff dimension $\text{Hdim}(J_c) = 2$ is dense in K . Let

$$R_n = \left\{ c \in K \mid \text{hypdim}(P_c) > 2 - \frac{1}{n} \right\}$$

for any positive integer n . Then $R = \bigcup_{n=0}^{\infty} R_n$ is the set of points c with hyperbolic dimension 2. Since the Hausdorff dimension of the Julia set is at least the hyperbolic dimension of the function and cannot exceed 2, R is a subset of the set of points with Hausdorff dimension 2. Thus it suffices to show that R is dense in K .

To do so, we note that by semicontinuity of the map $f \rightarrow \text{hypdim}(f)$, all the R_n are open. Further, we show that they are dense in K . To prove this, we show that any open set intersecting K contains a point in some R_n . Note that if U intersects K , then there is some parabolic periodic point c_0 in $K \cap U$. By Theorem 7, there exists a sequence c_n converging to c_0 with hyperbolic dimension tending to 2. Then since K is open, the tail of this sequence lies in K ; hence we can find a point with hyperbolic dimension arbitrarily close to 2. It follows that K intersects every R_n , as desired. Thus R is the union of a countable number of dense open sets, which by Baire's Theorem is dense. Thus the points c such that $\text{Hdim}(J_c) = 2$ are dense in K , as desired.

Chapter 4

Conclusion

We now revisit our original question: for polynomials of the form $z^d + c$, at what values of c is the mapping $H : c \rightarrow \text{Hdim}(J_c)$ discontinuous? This question has been largely resolved in the case $d = 2$, thanks to the work of Shishikura. We have generalized this result to show that there is a sizable subset of the locus of J -instability K on which H is discontinuous. Additionally, we have shown that H is continuous wherever the action of f on its Julia set is hyperbolic, as well as any point in the locus of J -instability with hyperbolic dimension 2. This, however, leaves a number of questions unresolved, which we now briefly discuss.

First and foremost, there is the question of whether the locus of J -instability is equal to the boundary of the connectedness locus. We believe this to be the case, and it is strongly implied by a number of papers; however we have been unable to find or construct a proof of this fact. Both [5] and [6] have material relating to this question, but neither has a complete proof of it. If true, this would create a much cleaner statement of our results, as the connectedness locus is easier to deal with than the locus of J -instability.

Second, it is unknown whether H is continuous at any point c which is J -stable but not hyperbolic. It is conjectured for $d = 2$ that there are no such points, though proving this is so is a major unsolved problem in complex dynamics. Thus discussing continuity in this case seems futile at this point.

Finally, we can say nothing about any points c in the locus of J -instability K with hyperbolic dimension less than 2 but Hausdorff dimension 2. It may well require new tools to show any continuity results in this case. However, it is unknown

whether this case is even possible, so an interesting first step might be to try to find such points or prove their non-existence.

Appendix A

The Hausdorff Dimension

We now present the details of the Hausdorff dimension. To do so, we first define the *t-dimensional Hausdorff measure* of a set. For a set E , the t dimensional Hausdorff measure $m_t(E)$ is given by

$$m_t(E) = \lim_{\delta \rightarrow 0} \left(\inf \left\{ \sum_j |A_j|^t : |A_j| < \delta, E \in \bigcup_j A_j \right\} \right).$$

In simpler terms, for any $\delta > 0$ we can cover the set with neighborhoods A_j of diameter less than δ . For any such cover, we sum the t^{th} power of the diameters of the neighborhoods of the cover. We then take the infimum over all such covers, and take the limit of this value as δ goes to 0. What we find, though, is that for most t , this limit is either 0 or infinity. If the number of neighborhoods required to cover the set grows faster than δ^{-t} , the limit is infinite; if it grows more slowly, the limit will be 0. Thus we can define the *Hausdorff dimension* to be the supremum of all t such that $m_t(E) = 0$. The Hausdorff dimension is then the precise number d such that if you shrink δ by a factor of k , the number of neighborhoods required to cover E is asymptotically equal to k^d . In the case of a purely fractal set, i.e., one which can be scaled to create a number of identical copies of itself, this coincides with the intuitive dimension we would assign such a set. However, this dimension is clearly much more general, and can be applied to any set, regardless of self-similarity properties.

Appendix B

Examples

In this paper, we showed that there is an infinite class of points in the locus of J -instability with Hausdorff dimension strictly less than 2. We now explicitly construct such a point. The specific requirement was that 0 be non-recurrent. The easiest way to guarantee this is to look at points where 0 is strictly preperiodic. The simplest form of preperiodic is prefixed. Thus, we look for points where the forward orbit of 0 is eventually fixed. Now, the forward iterates of 0 are c , $c^d + c$, $(c^d + c)^d + c$, and so on. We first check for c -values where $c^d + c = c$; however this clearly yields $c = 0$, and thus 0 is fixed rather than pre-fixed. Thus, we consider the case where $(c^d + c)^d + c = c^d + c$. We then find that

$$\begin{aligned}(c^d + c)^d + c &= c^d + c \quad \text{so,} \\ c^d(c^{d-1} + 1)^d &= c^d \quad \text{so,} \\ (c^{d-1} + 1)^d &= 1\end{aligned}$$

as we are only interested in the case where $c \neq 0$. Thus $c^{d-1} + 1$ is a d th root of unity. Further, it must be a root of unity other than 1, as 1 yields the solution $c = 0$ again. Thus

$$c^{d-1} + 1 = e^{\frac{2\pi ik}{d}}$$

for $k = 1, 2, \dots, d - 1$. Thus the functions where $c^d + c$ is fixed and nonzero are precisely those with

$$c = \left(e^{\frac{2\pi ik}{d}} - 1 \right)^{\frac{1}{d-1}}.$$

For $d = 2$, this is the point $c = -2$, the left endpoint of the Mandelbrot set. In general, these points seem to be located at the endpoints of antennas of the connectedness locus, though we have been unable to prove this.

Appendix C

Images

In this chapter we present a number of images generated by Brian Roney. These show the structure and shape of the connectedness locus of z^d+c for small values of c . Figure 1 contains pictures of M_d for $d = 2, 3, 4, 6$. Figure 2 is a series of successive zooms around a bulb of the connectedness locus for $d = 3$. Figures 3 and 4 are pictures of some typical Julia sets. In these pictures, the green represents the filled Julia set, that is, the Julia set together with all components of the Fatou set that lie within it. The Julia set itself is the boundary of the green region.

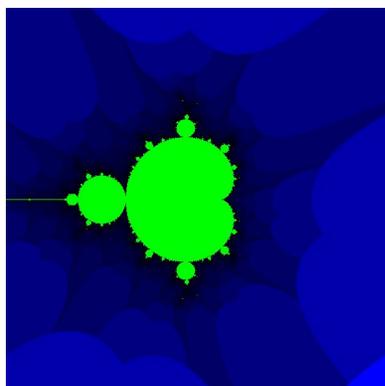
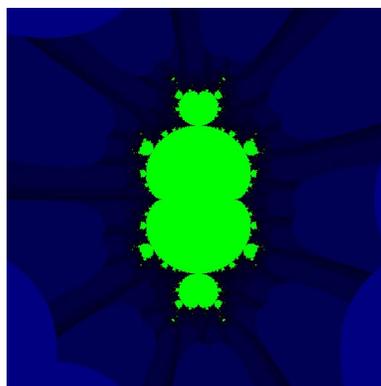
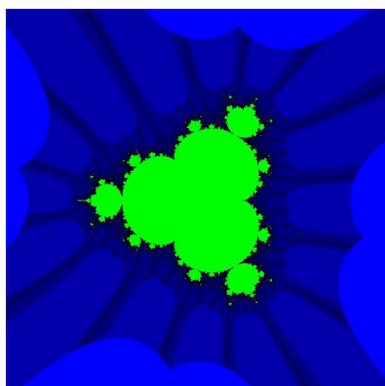
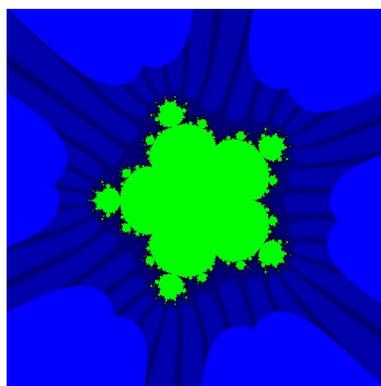
(a) $z^2 + c$ ($d = 2$)(b) $z^3 + c$ ($d = 3$)(c) $z^4 + c$ ($d = 4$)(d) $z^6 + c$ ($d = 6$)

Figure C.1: Images of the connectedness locus for $d = 2, 3, 4, 6$. Thanks to Brian Roney for the images.

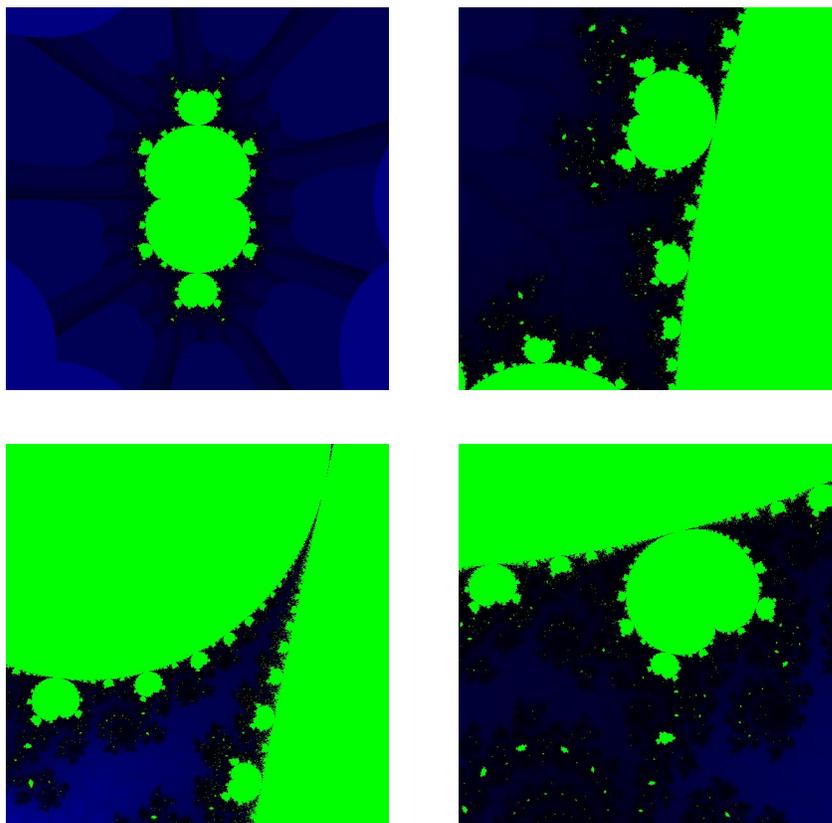


Figure C.2: Successive zooms on the $\frac{5}{14}$ th bulb of the connectedness locus of $z^3 + c$ ($d = 3$). Thanks to Brian Roney for the images.

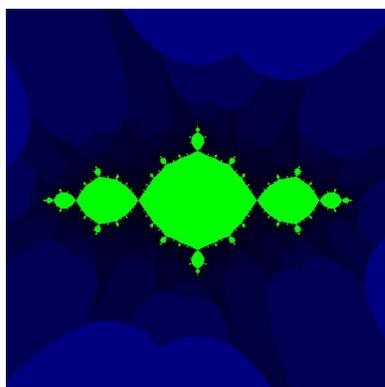
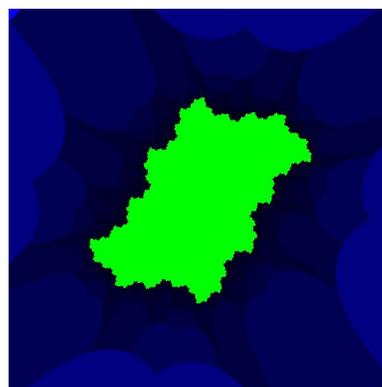
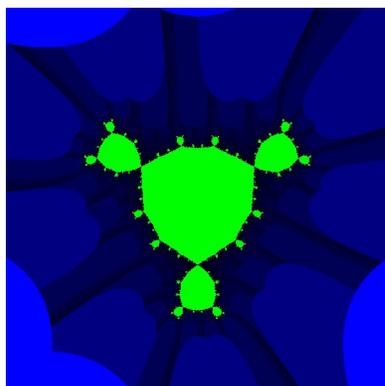
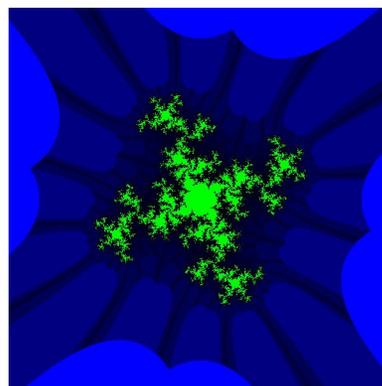
(a) $z^2 - 1$ (b) $z^2 + .5i$ (c) $z^3 + i$ (d) $z^4 - .83i$

Figure C.3: Some typical filled Julia sets. Thanks to Brian Roney for the images.

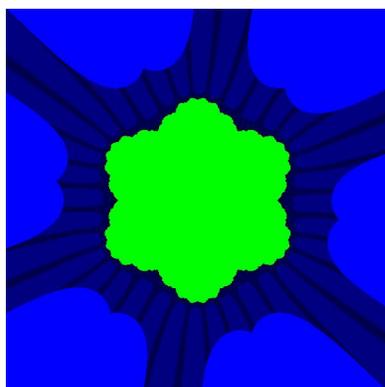
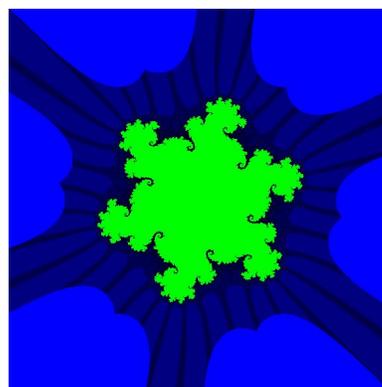
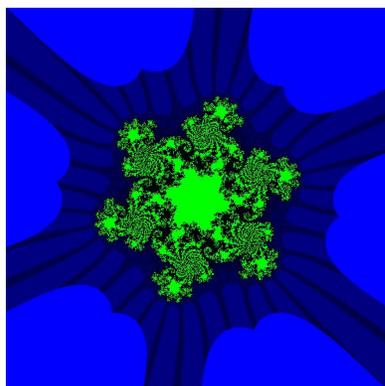
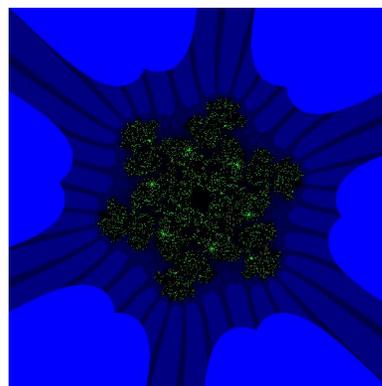
(a) $z^6 + .5i$ (b) $z^6 + .77i$ (c) $z^6 + .773i$ (d) $z^6 + .78i$

Figure C.4: Some typical filled Julia sets. Thanks to Brian Roney for the images.

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