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Searching for Supersymmetric Cycles: A Quest for Cayley Manifolds in the Calabi–Yau 4-Torus

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by

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Abstract

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Recent results of string theory have shown that while the traditional cycles studied in Calabi-Yau 4-manifolds preserve half the spacetime supersymmetry, the more general class of Cayley cycles are novel in that they preserve only one quarter of it. Moreover, Cayley cycles play a crucial role in understanding mirror symmetry on Calabi-Yau 4-manifolds and Spin_7-manifolds. Nonetheless, only very few nontrivial examples of Cayley cycles are known. In particular, it would be very useful to know interesting examples of Cayley cycles on the complex 4-torus. This thesis will develop key techniques for finding and constructing lattice periodic Cayley manifolds in Euclidean 8-space. These manifolds will project down to the complex 4-torus, yielding nontrivial Cayley cycles.

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0.1 Overview

It appears that there is a theory that generalizes and unifies the various string theories and supergravity. This new theory has been named M-theory. However, many of the specifics of M-theory are not known. The main focus of this thesis is to find examples of nontrivial cycles in the Calabi-Yau complex 4-torus. This will allow the specifics of M-theory to be examined and hopefully better understood.

Conformal field theory, which relies heavily on complex analysis, forms the backbone of string theory. However in M-theory the complex numbers are not sufficient. For this reason we need to employ the more general division algebras of the quaternions and Cayley numbers. Moreover, in order to describe spin structures in higher dimensions, something essential to M-Theory, we will need make use of the Clifford algebras. All these algebras are reviewed in Chapter 1, where we also review a number of deep connections between these structures and the various branches of differential geometry.

Chapter 2 provides some of the connections between physics and geometry, offering motivation for this thesis. Since our focus is to better understand M-theory, we provide a simplified review of string theory and its relationship to M-theory. This will help provide a framework for understanding these problems.

Chapter 3 will focus on Cayley geometry. We first summarise the method of calibrations, paying special attention to the case of the Cayley calibration, and then discuss the specific problems we will address in this thesis. My principle results are roughly divided up into two categories. First in Chapter 4, we discuss our results concerning the angles between various Cayley planes. In particular we derive the Cayley Angle theorem, which is a new result, and important for resolving singularities in intersecting Cayley manifolds.

Very few examples of nontrivial Cayley cycles are known. In fact even locally linear Cayley manifolds, such as polytopes, are unknown and would be interesting. In this thesis we develop a technique for constructing lattice periodic Cayley polytopes, which uses the general theory of finite reflection groups, lattices, and polytopes. We review this and some of the related areas of complex and quaternionic reflections in Chapter 5.

Finally, in Chapter 6 we present our new results on the construction of finite Spin_7 subgroups and lattice periodic Cayley polytopes (called honeycombs). Moreover this technique can be used to find finite quaternionic and special unitary groups in a manner that is easier then previously known techniques. These honeycombs project to non-trivial Cayley cycles in $T^8 \cong T^4_{\mathbb{C}}$, the Calabi-Yau 4-torus.

Chapter 1

Cayley Numbers and Spin Structures

Every mathematician and physicist of the modern day is familiar with the complex numbers, $\mathbb{C}$. They have become an integral part of mathematics and are essential for describing the physical world as we know it. Part of the beauty and power of the complex numbers resides in the natural geometric structure that they possess, namely there is a natural length associated with every complex number which is multiplicative, i.e., $|ab| = |a||b|$ for all $a, b \in \mathbb{C}$. Moreover, multiplication by a unit complex number corresponds to a rotation in the complex plane.

These and other factors caused Hamilton and others in the mid-1800s to explore the possibility of a number system with geometric properties similar to the complex numbers, but for 3-dimensional space. This led to Hamilton's invention of the quaternions in 1843. Attempts to generalize to even higher dimensions resulted in the invention of what we now call the Cayley numbers and the Clifford Algebras. Remarkably these algebraic structures incorporate and help explain the seemingly arbitrary nature of spin in quantum mechanics, and are fundamental for understanding string theory and its generalizations.

1.1 The Division Algebras

A *real algebra* is a vector space over the real numbers equipped with two operations, the commutative addition of the vector space and a (not necessarily associative) multiplication which distributes over the addition,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c), \tag{1.1}$$

for all a, b, c in the algebra. Furthermore, we require that these algebras contain a multiplicative unit, 1.

A real *division algebra* is an algebra additionally equipped with an inner product whose associated norm is multiplicative. Immediate familiar examples are the real and complex numbers. It is possible to show that there is no such three-dimensional algebra, however Hamilton came up with a four-dimensional algebra which he designated the quaternions, denoted $\mathbb{H}$ in homage to Hamilton. The structure of the quaternions mirrors that of the complex numbers. They are the real linear span over four orthogonal vectors $1, \hat{\imath}, \hat{\jmath}, \hat{k}$ where $\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = -1$. We immediately see that there are several copies of $\mathbb{C}$ contained in $\mathbb{H}$. Lastly the multiplication between $\hat{\imath}, \hat{\jmath}, \hat{k}$ mimics the vector cross product, namely $\hat{\imath}\hat{\jmath} = -\hat{\jmath}\hat{\imath} = \hat{k}$ with cyclic permutations. This, together with the distributive law completely specifies the algebra. Notice that the quaternions do not commute while the complex numbers do. Moreover, we can adopt the natural inner product from $\mathbb{R}^4$ for the quaternions. The associated norm is then $|a1 + b\hat{\imath} + c\hat{\jmath} + d\hat{k}|^2 = a^2 + b^2 + c^2 + d^2$. This norm satisfies the multiplicative condition, making $\mathbb{H}$ a division algebra.

Just as a complex number can be broken into its real and imaginary pieces, so too can the quaternions. The real part of a quaternion is the projection onto the 1 axis, while the remaining vector piece is the imaginary portion, denoted $\mathrm{Re}\, h$ and $\mathrm{Im}\, h$, respectively. It is fairly straight forward to verify that for any unit imaginary quaternion u, we have $u^2 = -1$ so that $u^{-1} = -u$. This motivates us to define a quaternion conjugate by $\overline{h} = \mathrm{Re}\, h - \mathrm{Im}\, h$. (In fact this definition makes sense for any algebra with unit 1). Then, since $\mathbb{H}$ is a division algebra, we can determine the inverse of any nonzero element by $h^{-1} = \overline{h}/|h|^2$.

The above process can be repeated to produce an additional division algebra of dimension 8 known as the Cayley numbers, octonions, or octaves, written $\mathbb{O}$. In fact all four of the discussed division algebras can be brought together under what is termed the Cayley-Dickson process. Given an algebra $\mathbb{A}$, we can define a new algebra $\mathbb{B} = \mathbb{A} \oplus \mathbb{A}$ where multiplication in $\mathbb{B}$ is defined by,

$$(a, b)(c, d) = (ac - \overline{d}b, da + b\overline{c}) \tag{1.2}$$

with $a, b, c, d \in \mathbb{A}$. We say $\mathbb{B}$ was obtained from $\mathbb{A}$ via the *Cayley-Dickson Process*. Applying the Cayley-Dickson process to $\mathbb{R}$ yields $\mathbb{C}$, applying it to $\mathbb{C}$ yields $\mathbb{H}$, and applying it to $\mathbb{H}$ yields $\mathbb{O}$. This sequence continues, but the resulting algebras contain zero divisors (i.e., are no longer division algebras). In fact in each application of the Cayley-Dickson process some property is lost. The complex numbers are not ordered, the quaternions are not commutative, and the Cayley numbers are not associative.

This also permits us to view each algebra as a subalgebra of the larger ones, $\mathbb{C} = \mathbb{R} + \mathbb{R}\hat{i}$, $\mathbb{H} = \mathbb{C} + \mathbb{C}\hat{j}$, and $\mathbb{O} = \mathbb{H} + \mathbb{H}\hat{e}$, where the orthogonal basis elements of $\mathbb{O}$ are $1, \hat{i}, \hat{j}, \hat{k}, \hat{e}, \hat{ie}, \hat{je}, \hat{ke}$. In this basis the multiplication table for $\mathbb{O}$ is as in Table 1.1. However, as we know from $\mathbb{H}$, this decomposition is not unique. In fact, in the

	1	$\hat{i}$	$\hat{j}$	$\hat{k}$	$\hat{e}$	$\hat{ie}$	$\hat{je}$	$\hat{ke}$
1	1	$\hat{i}$	$\hat{j}$	$\hat{k}$	$\hat{e}$	$\hat{ie}$	$\hat{je}$	$\hat{ke}$
$\hat{i}$	$\hat{i}$	-1	$\hat{k}$	$-\hat{j}$	$\hat{ie}$	$-\hat{e}$	$-\hat{ke}$	$\hat{je}$
$\hat{j}$	$\hat{j}$	$-\hat{k}$	-1	$\hat{i}$	$\hat{je}$	$\hat{ke}$	$-\hat{e}$	$-\hat{ie}$
$\hat{k}$	$\hat{k}$	$\hat{j}$	$-\hat{i}$	-1	$\hat{ke}$	$-\hat{je}$	$\hat{ie}$	$-\hat{e}$
$\hat{e}$	$\hat{e}$	$-\hat{ie}$	$-\hat{je}$	$-\hat{ke}$	-1	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{ie}$	$\hat{ie}$	$\hat{e}$	$-\hat{ke}$	$\hat{je}$	$-\hat{i}$	-1	$-\hat{k}$	$\hat{j}$
$\hat{je}$	$\hat{je}$	$\hat{ke}$	$\hat{e}$	$-\hat{ie}$	$-\hat{j}$	$\hat{k}$	-1	$-\hat{i}$
$\hat{ke}$	$\hat{ke}$	$-\hat{je}$	$\hat{ie}$	$\hat{e}$	$-\hat{k}$	$-\hat{j}$	$\hat{i}$	-1

Table 1.1: The multiplication table for the Cayley numbers, $\mathbb{O}$.

case of $\mathbb{H}$ if u is any imaginary unit vector then $\mathbb{R} \oplus \mathbb{R}u \cong \mathbb{C}$ as a division algebra.

Similarly there are many subalgebras of $\mathbb{O}$ isomorphic to $\mathbb{H}$. We have the following theorem due to Artin,

Theorem 1.1.1 (Artin) *The subalgebra $\mathbb{A}$ generated by any two elements of $\mathbb{O}$ is isomorphic to a subalgebra of $\mathbb{H}$. In particular, it is associative.*

Moreover, these four algebras exhaust the possible real division algebras:

Theorem 1.1.2 (Hurwitz) *Up to isomorphism, the only normed real division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$.*

For proofs of these theorems I direct the reader to the appendix of [18], which has an extensive discussion of the Cayley numbers. Also, the interested reader might look at [29]. Throughout this text I will frequently switch between the terminology Cayley numbers, octonions, and octaves.

1.2 General Linear Groups

Recall that we can define the algebras of $N \times N$ matrices with real or complex entries, $M_N(\mathbb{R})$ and $M_N(\mathbb{C})$. From these we can define various subgroups, e.g., the subgroups of invertible transformations $GL_N(\mathbb{R})$ and $GL_N(\mathbb{C})$. Using determinants and (hermitian) inner products, with possibly indefinite signature (p, q), we can define the orthogonal and special orthogonal groups, $O(p, q)$ and $SO(p, q)$ and their complex counterparts the unitary and special unitary groups, $U(p, q)$ and $SU(p, q)$. All of this is well known and covered in a variety of standard texts on Lie groups, differential geometry, and special relativity.

It is possible to define these groups because the real and complex numbers are fields. The quaternions, while not a field, share many of the nice properties that fields possess. In fact, if we are careful to mind the order of multiplication, we can construct an algebra out of the $N \times N$ matrices with quaternion values, $M_N(\mathbb{H})$. These acts on the left of vectors in $\mathbb{H}^n$. However in order to consider this a linear space we have to let the scalars act on the right. The group of invertible elements will be denoted $GL_N(\mathbb{H})$.

Just as we can use the symmetric inner product on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ to define a hermitian inner product on $\mathbb{C}^n$, we can similarly define a hermitian symmetric inner product on $\mathbb{H}^n$.

$$\langle x, y\rangle = \overline{x}_1 y_1 + \cdots + \overline{x}_p y_p - \cdots - \overline{x}_{p+q} y_{p+q}. \tag{1.3}$$

The real portion of this inner product is identical to the real inner product on $\mathbb{R}^{4n} \cong \mathbb{H}^n$. The group that preserves this inner product will be denoted $HU(p,q)$ or simply $HU(n)$ when $q = 0$. When $n = 1$ we find that $HU(1) \cong S^3 = \{L_u \mid u \in \mathbb{H} \text{ and } |u| = 1\}$, where L_u denotes the left multiplicative action of $u \in \mathbb{H}$ on $\mathbb{H}^n$. For completeness, let R_u denote the right multiplicative action of $u \in \mathbb{H}$ on $\mathbb{H}^n$, i.e. right multiplication by $\overline{u}$ (conjugation ensures that this is an action: $R_a \circ R_b = R_{ab}$). Similarly we can define the right and left "action" of Cayley numbers on $\mathbb{O}^n$. While defined similarly, the notion of an action in these contexts is malformed since the Cayley numbers are not associative.

Strictly speaking, there is a slightly larger group than $GL_N(\mathbb{H})$ which preserve quaternion lines (i.e., the right $\mathbb{H}$ span of $v \in \mathbb{H}^n$). We can augment this group by right quaternion multiplication by a scalar. In this case the following sequence is exact:

$$1 \longrightarrow \mathbb{R}^* \longrightarrow GL_N(\mathbb{H}) \times \mathbb{H}^* \longrightarrow GL_N(\mathbb{H}) \cdot \mathbb{H}^* \longrightarrow 1, \tag{1.4}$$

where $\mathbb{R}^*$ and $\mathbb{H}^*$ are the subgroups of invertible elements. The new group $GL_N(\mathbb{H}) \cdot \mathbb{H}^*$ is known as the *enhanced* $\mathbb{H}$*-general linear group*. It also has an analogous subgroup $HU(p,q) \cdot HU(1)$ which is defined in the obvious manner.

There is some confusion of notation. Often the groups $HU(n)$ are written $Sp(n)$. However these could equally refer to the real symplectic or skew groups, associated with a skew inner product. To avoid confusion I have decided to use the notation $HU(p,q)$ throughout. This notation is natural as these groups are closely related to the unitary groups. In fact, all these groups can be understood as subgroups of each other: $HU(p,q) \subset U(2p,2q)$, $U(p,q) \subset SO(2p,2q)$.

1.3 The Clifford Algebras, Pin, and Spin

Clifford algebras are another attempt to provide an arbitrary vector space with some sort of algebraic structure. However there is much more to Clifford algebras then their algebraic structure. Ultimately they help provide the mathematical framework that explains the phenomenon of intrinsic spin in physics. Clifford algebras have many deep connections to certain differential forms and the structure of certain types of manifolds. However, they also have a rich structure of their own. Much of this subject matter is far beyond the scope of this thesis. Although we provide an abbreviated summary of the relevant subject matter, the reader will find a more comprehensive treatment in F. Reese Harvey's book [17].

To begin, Clifford algebras are best understood in the context of the exterior algebra and tensors. Given a real vector space V with inner product and associated norm $\langle , \rangle$ and $|| \cdot ||$, respectively, the *tensor algebra*,

$$\otimes V = \sum_{i=0}^{\infty} \otimes^i V,$$

is a graded associative algebra with unit. It also has an associated inner product derived from (and also denoted by) $\langle , \rangle$. If we let $I(V)$ denote the two-sided ideal in $\otimes V$ generated by elements of the form $x \otimes x \in \otimes^2 V$, where $x \in V$, then the *exterior algebra* $\wedge V = \otimes V / I(V)$. In other words, the exterior algebra is the quotient of the tensor algebra by all symmetric or partially symmetric tensors.

Alternately, we can let $I(V)$ denote the two-sided ideal in $\otimes V$ generated by elements of the form $x \otimes x + \langle x, x \rangle$ with $x \in V$. Then the resulting quotient algebra is known as the *Clifford Algebra* of V, denoted $Cl(V)$. It clearly is related to the exterior algebra, yet is a distinct entity. In particular, as vector spaces $Cl(V)$ and $\wedge V$

are isomorphic. These algebras have a number of natural involutions. Consider the isometry on V sending v to $-v$. This extends to an algebra automorphism of $Cl(V)$ which we will denote by $x \mapsto \tilde{x}$. Similarly the anti-automorphism of $\otimes V$ defined by reversing the order of simple products descends to an anti-automorphism of $Cl(V)$. We will denote this by the check involution $x \mapsto \check{x}$. It is straight forward to verify that ˜ and ˇ commute. Thus we can define a third involution by their composition, denoted $x \mapsto \hat{x}$. The automorphism allows us to split $Cl(V)$ into two pieces,

$$Cl^{\text{even}}(V) = \{x \in Cl(V) \mid \tilde{x} = x\}, \qquad Cl^{\text{odd}}(V) = \{x \in Cl(V) \mid \tilde{x} = -x\}. \tag{1.5}$$

Once given these algebras we can consider the multiplicative group, $Cl^*(V)$, much in the same way as we consider $GL(V)$ as a subgroup of linear transformations of V. In fact this group will turn out to be either one or two copies of a general linear group over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. If we consider a non-null vector $u \in V \subset Cl(V)$, (i.e., $|u| \neq 0$) [1] then we have that $u^{-1} = -u/|u|^2$ is the inverse of u. Hence all such u are elements of $Cl^*(V)$.

Definition 1.3.1 *The Pin group is the subgroup of $Cl^*(V)$ generated via multiplication by unit vectors in V. If $V = \mathbb{R}^{p,q}$ then this will be denoted $Pin_{p,q}$.*

Since every vector in $Pin(V)$ is a simple product, each element has a well defined even-odd parity. This motivates the following definition:

Definition 1.3.2 *The group Spin is the even subgroup of Pin. i.e. $Spin_{p,q} = CL^{even} \cap Pin_{p,q}$.*

In what seems like a coincidence, the Spin and Pin groups are closely related to the $SO(V)$ and $O(V)$ groups. If $u \in V$ is a non-null vector (i.e., $|u| \neq 0$), then reflection

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[1] In vector spaces with indefinite inner products it is possible to have vectors with nonzero coordinates, whose norm is zero. For example in $\mathbb{R}^2$ with the indeinite inner product, $|(x, y)|^2 = x^2 - y^2$, any vector of the form $(x, \pm x)$ will have zero norm.

along u is given in terms of Clifford multiplication by $\mathrm{Ref}_u(x) = -uxu^{-1}$ for all $x \in V$. Thus we define the *twisted adjoint representation* $\tilde{Ad}$ of the group $Cl^*(V)$ on $Cl(V)$,

$$\tilde{Ad}_a(x) = \tilde{a}xa^{-1}, \tag{1.6}$$

for all $a \in Cl^*(V)$ and $x \in Cl(V)$. Similarly we can define the adjoint representation by removing the tilde automorphism. On the even algebra the tilde automorphism is the identity so that this distinction is lost. This permits us to project the Spin and Pin groups into the orthogonal groups:

Theorem 1.3.1 *The sequences,*

$$1 \longrightarrow \mathbf{Z}_2 \longrightarrow Pin \xrightarrow{\tilde{Ad}} O(V) \longrightarrow I \tag{1.7}$$

$$1 \longrightarrow \mathbf{Z}_2 \longrightarrow Spin \xrightarrow{\tilde{Ad}} SO(V) \longrightarrow I \tag{1.8}$$

are exact sequences, where $\mathbf{Z}_2 = \{\pm 1\}$.

In other words, Spin and Pin are double covers of $SO(V)$ and $O(V)$ respectively. This has deep consequences which will be made more apparent shortly.

Let us examine the Clifford algebra $Cl(2)$. As a vector space it is the span of these four orthogonal elements, $1, e_1, e_2, e_1e_2$. Notice that the square of any of the last three is minus the identity. In fact we see that this algebra is identical with the algebra of quaternions, $\mathbb{H}$. Moreover making use of the identity $Cl^{\text{even}}(r,s) = Cl(r-1,s)$, we see that $\text{span}\{1, e_1e_2, e_1e_3, e_2e_3\} = Cl^{\text{even}}(3) \cong \mathbb{H}$ as well. It can be shown that $\text{Spin}_3 \cong HU(1)$, is the standard double cover of $SO(3)$. In fact when viewed simply as matrix algebras, the clifford algebras correspond to many of our familiar examples, see Table 1.2.

The vector space that the Clifford algebras act upon when viewed as matrix algebras is known as the space of *Pinors*. Similarly, the space that the even Clifford algebras act upon is known as the space of *Spinors*. Additionally we have the identity $Cl^{\text{even}}(r,s) \cong Cl^{\text{even}}(s,r)$. Given an orientation for $V = \mathbb{R}^{p,q}$ we can define

p	0	1	2	3	4	5	6	7
$q=0$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$M_8(\mathbb{R})$	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$
$q=1$	$\mathbb{R}\oplus\mathbb{R}$	$M_2(\mathbb{R})$	$M_2(\mathbb{C})$	$M_2(\mathbb{H})$	$M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$	$M_4(\mathbb{H})$	$M_8(\mathbb{C})$	$M_{16}(\mathbb{R})$
$q=2$	$M_2(\mathbb{R})$	$M_2(\mathbb{R}) \oplus M_2(\mathbb{R})$	$M_4(\mathbb{R})$	$M_4(\mathbb{C})$	$M_4(\mathbb{H})$	$M_4(\mathbb{H}) \oplus M_4(\mathbb{H})$	$M_8(\mathbb{H})$	$M_{16}(\mathbb{C})$

Table 1.2: The Matrix algebras isomorphic to the Clifford algebras for $V = \mathbb{R}^{p,q}$. Note that $Cl(r+8,q) = Cl(r,q)\otimes M_{16}(\mathbb{R})$.

$\lambda \in Cl(V)$ to be the unit volume form. Then $a \in Cl(V)$ is said to be *self dual* if $\lambda a = a$ and *anti-self dual* if $\lambda a = -a$. In the cases where $Cl(V)$ is split into two simple matrix algebras $M_N(\mathbb{F}) \oplus M_N(\mathbb{F})$, then these two algebras can be realized as the self dual and anti-self dual portions of the Clifford algebra. The induced representations of Spin or Pin will be referred to as the *positive* and *negative* representations of these groups, acting on the positive or negative spinors or pinors, $S_\pm, P_\pm$. In low dimensions the Spin and Pin groups are often isomorphic with the orthogonal group of their space of spinors or pinors, as was for the Spin_3 case above. However, in higher dimensions these groups can be highly nontrivial subgroups of the larger groups, e.g., Spin_7 below.

1.4 The Connection to the Division Algebra $\mathbb{O}$

We saw in the last section that the quaternions re-emerge in the study of the Clifford algebras. One might hope that a similar re-emergence might happen for the octonions as well. However one will quickly be disappointed. All the Clifford algebras, since they can be realized as matrix algebras, are associative. Nonetheless, the octonions do provide a useful model for the Clifford algebras $Cl(8)$ and $Cl(7)$.

In eight dimensions the Clifford algebra has three inequivalent eight-dimensional representations. More precisely the following sequences are exact and correspond to inequivalent representations,

$$1 \longrightarrow \mathbf{Z}_2 = \{1, \lambda\} \longrightarrow \mathrm{Spin}_8 \xrightarrow{\rho_+} SO(8) \longrightarrow 1 \tag{1.9}$$

$$1 \longrightarrow \mathbf{Z}_2 = \{1, -\lambda\} \longrightarrow \mathrm{Spin}_8 \xrightarrow{\rho_-} SO(8) \longrightarrow 1 \tag{1.10}$$

$$1 \longrightarrow \mathbf{Z}_2 = \{1, -1\} \longrightarrow \mathrm{Spin}_8 \xrightarrow{\tilde{Ad}=\chi} SO(8) \longrightarrow 1. \tag{1.11}$$

They are the positive, negative and vector representation. There are outer automorphisms of Spin_8 that exchange these representations known as the *triality* automorphisms. All three of the vector spaces $V = \mathbb{R}^8, S_+, S_-$ can be identified with the octonions $\mathbb{O}$. For example, $V \subset Cl(8)$ can be identified as,

$$V = \left\{ \begin{pmatrix} 0 & -R_u \\ -R_{\overline{u}} & 0 \end{pmatrix} \mid u \in \mathbb{O} \right\}. \tag{1.12}$$

In this context the unit volume element for $Cl(8)$ and the even Clifford algebra are given by

$$\lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad Cl^{\mathrm{even}}(8) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathrm{End}_{\mathbb{R}}(\mathbb{O}) \right\}. \tag{1.13}$$

Octonion multiplication helps unify and relate the three eight-dimensional representations of Spin_8 as follows:

Theorem 1.4.1 (The Triality Theorem) *Let (g_+, g_-, g_0) be a triplet of orthogonal linear maps on $\mathbb{O}$. Then $(g_+, g_-) \in Spin_8$ with g_0 the vector representation of (g_+, g_-) if and only if $g_+(x \cdot y) = g_-(x) \cdot g_0(y)$ for all $x, y \in \mathbb{O}$.*

For a proof of this theorem and a more comprehensive discussion see [17].

The octonion model for Spin_8 helps us construct such a model for Spin_7. Recall that the subgroup of Spin_n that fixes a vector in the vector representation is isomorphic to Spin_{n-1}. Thus we have the following representations (with the important standard isomorphisms):

$$\begin{aligned}
\mathrm{Spin}_7 &= \left\{g \in \mathrm{Spin}_8 \mid g_0(1) = 1\right\} && (1.14)\\
\mathrm{Spin}_6 &= \left\{g \in \mathrm{Spin}_8 \mid g_0(1) = 1 \text{ and } g_0(\hat{i}) = \hat{i}\right\} \cong SU(4) && (1.15)\\
\mathrm{Spin}_5 &= \left\{g \in \mathrm{Spin}_8 \mid g_0(1) = 1, g_0(\hat{i}) = \hat{i}, \text{ and } g_0(\hat{j}) = \hat{j}\right\} \cong HU(2) && (1.16)\\
\mathrm{Spin}_4 &= \left\{g \in \mathrm{Spin}_8 \mid g_0(1) = 1, g_0(\hat{i}) = \hat{i}, g_0(\hat{j}) = \hat{j} \text{ and } g_0(\hat{k}) = \hat{k}\right\} && (1.17)\\
&\cong HU(1) \times HU(1) && (1.18)
\end{aligned}$$

Because of the Triality Theorem, we can get another useful characterization of Spin_7. Namely setting $y = 1 \in \mathbb{O}$ yields $g_+(x) = g_-(x)$ for all $x \in \mathbb{O}$ and $(g_+, g_-) \in \mathrm{Spin}_7$. Finally, returning to the octonion model we find that,

$$\mathrm{Spin}_7 \text{ is generated by } \begin{pmatrix} R_u & 0 \\ 0 & R_u \end{pmatrix} \text{ with } u \in \mathrm{Im}\,\mathbb{O}. \tag{1.19}$$

There are other associations which are worth mentioning. First of all the octonions are closely related to the exceptional Lie group G_2. In fact this is the group of algebra automorphism of the octonions, ($g \in G_2$ if and only if $g(uv) = g(u)g(v)$). In particular this means that any element of G_2 must fix $1 \in \mathbb{O}$, so that $G_2 \subset SO(\mathrm{Im}\,\mathbb{O}) = SO(7)$. In fact G_2 is precisely the subgroup of Spin_7 that fixes $1 \in \mathbb{O}$. Recalling Theorem 1.4.1, we see that G_2 is the subgroup of Spin_7 that is identical to its vector representation, $g_+ = g_- = g_0$.

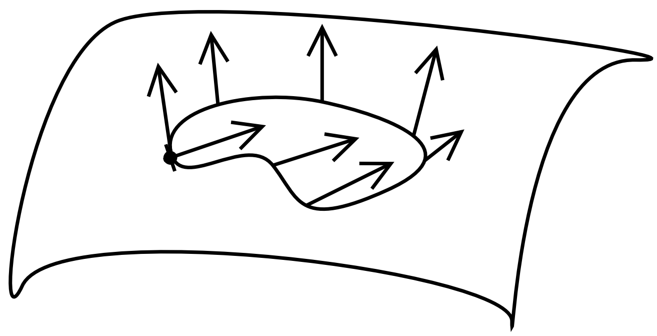

Figure 1.1: Parallel Transport on a Manifold

1.5 Holonomy and Spin Structures

Let (M, g) be a Riemannian Manifold, with connection ∇. This connection is not necessarily compatible with the metric. Let $\gamma : [0, 1] \to M$ be smooth, such that $\gamma(0) = p$. Then γ is a path starting at p. Recall that, using the metric and the connection, there is a well defined map, known as parallel transport, from the tangent space at p to the tangent space at $\gamma(t)$ for all $t \in [0, 1]$, $P_{\gamma,t} : T_pM \to T_{\gamma(t)}M$. Let γ be a smooth path, such that $\gamma(0) = \gamma(1) = p \in M$. Then γ is a loop based at p. When restricted to loops the associated parallel transport maps are invertible linear maps and form a subgroup $Hol_p(\nabla) \subset GL(T_pM)$ under loop composition $P_\gamma \circ P_\beta = P_{\gamma \circ \beta}$. Moreover it can be shown for connected manifolds M that this group is independent of base point, see [21].

Recall that there is a unique connection ∇_g compatible with the metric g, known as the Levi-Civita connection. When the connection is this unique connection, then parallel transport preserves the length (i.e., the metric) on the tangent plane. In this case the associated group $Hol_p(\nabla_g)$ is a subgroup of $O(T_pM)$, and is known as the *holonomy* group. While certain types of manifolds have holonomy that is the full orthogonal group, most have holonomy that is a proper subgroup of some kind.

The holonomy group of a manifold is an extremely useful concept. It allows us to gain a great amount of knowledge about the manifold. One reason it is so powerful is that it captures information about both the topology and curvature of the manifold. Berger has classified the possible holonomy groups for compact connected Riemannian manifolds. Of particular interest is when the manifold is not a symmetric space. In this case the holonomy group falls into one of seven types:

Theorem 1.5.1 (Berger) *Suppose (M, g) is a Riemannian manifold of dimension n that is irreducible*[2] *and not a symmetric space. The connected component of the holonomy group is isomorphic to one of the following:*

1. $Hol(\nabla_g) = SO(n)$,

2. $n = 2m$, $m \geq 2$, *and* $Hol(\nabla_g) = U(m) \subset SO(2m)$,

3. $n = 2m$, $m \geq 2$, *and* $Hol(\nabla_g) = SU(m) \subset SO(2m)$,

4. $n = 4m$, $m \geq 2$, *and* $Hol(\nabla_g) = HU(m) \subset SO(4m)$,

5. $n = 4m$, $m \geq 2$, *and* $Hol(\nabla_g) = HU(m) \cdot HU(1) \subset SO(4m)$,

6. $n = 7$, *and* $Hol(\nabla_g) = G_2 \subset SO(7)$, *or*

7. $n = 8$, *and* $Hol(\nabla_g) = Spin_7 \subset SO(8)$.

For a proof of the above theorem see [21].

[2]It can be shown that when the holonomy group is reducible, e.g. $SO(m) \times SO(n-m)$, then the manifold is isomorphic to a product of manifolds. Each of the resulting manifolds can be seen to have irreducible holonomy, exactly matching the irreducible parts of the original holonomy. Hence we only wish to consider manifolds that cannot be decomposed as a simple product.

There is a substantial amount that can be determined about the manifold from its holonomy group. For example there is a 1-1 correspondence between constant[3] tensors on the manifold and invariants of the associated holonomy group:

Theorem 1.5.2 *Let S be a (p,q)-tensor on the Riemannian manifold (M,g). S is a constant tensor, so that $\nabla S = 0$, if and only if the restriction of S to the tangent plane at $x \in M$, S_x, is left invariant under $Hol_x(\nabla)$*

For proof, see [21]. This naturally induces certain constant differential forms on the given manifold, depending on its holonomy group.

There is a family of manifolds known as *spin manifolds* which are closely related to the holonomy groups. If we are given an oriented Riemannian manifold (M,g), it naturally induces a unique principal $SO(n)$-bundle over M, TM. Sometimes we can construct a Spin_n-bundle over this same manifold M, that is locally based on the projection $\pi : \text{Spin}_n \to SO(n)$. With this spin structure, we can then define a vector bundle over M associated with the spin structure. This vector bundle will have fibers that are the vector space of spinors associated with Spin_n (alternatively we could use the vector space of complexified spinors $S \otimes \mathbb{C}$). This is known as the (complex) Spin Bundle. Spin structures do not exist on all manifolds, nor are they always unique. There exist characterizations that allow one to determine which manifolds admit spin structure, but they are fairly technical and off topic.

The Levi-Civita connection induces a natural connection on the Spin Bundle, and so we can discuss the notion of constant spinors. As we have seen Spinors are closely related to tensors and so it is not surprising that we are able to find a similar 1-1 correspondence between certain holonomy groups and the number of constant complexified spinors. In particular there are only covariantly constant spinors in the holonomy cases listed in Table 1.5, and only in the numbers listed.

[3] Constant tensors (and as we shall see shortly, spinors) are also known as *parallel* tensors (and spinors). The defining condition is that $\nabla S = 0$, where S is the tensor (or spinor).

Holonomy	Parallel Forms	Num. Par. Spinors
$SO(n)$	On $\mathbb{R},\ v_M = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$	n/a
$U(n)$	On $\mathbb{C}^n,\ \omega = \frac{i}{2}(dz_1 \wedge d\overline{z}_1 + \cdots + dz_n \wedge d\overline{z}_n)$	n/a
$SU(n)$	On $\mathbb{C}^n,\ \omega_i$ and $\Omega = dz_1 \wedge \cdots \wedge dz_n$	$N_+ = 2,$ $N_- = 0,\ n = 2m$ $N_+ = 1,$ $N_- = 1,\ n = 2m+1$
$HU(n) \cdot HU(1)$	On $\mathbb{H}^n,\ \Theta = \omega_I^2 + \omega_J^2 + \omega_K^2$	n/a
$HU(n)$	On $\mathbb{H}^n$, $\omega_I = \frac{\hat{i}}{2}(dz_1 \wedge d\overline{z}_1 + \cdots + dz_n \wedge d\overline{z}_n)$ $\sigma_I = \omega_J - \hat{i}\omega_K$	$N_+ = n+1,\ N_- = 0$
Spin_7	On $\mathbb{R}^8,\ \Phi = 1^* \wedge \phi + \star\phi$	$N_+ = 1,\ N_- = 0$
G_2	On $\mathbb{R}^7,\ \phi$ and $\star\phi$	$N = 1$

In even dimensions the complexified spinors spilt into odd and even parts $S_\pm$. $N_\pm$ is the number of even/odd constant spinors for a particular choice of Spin structure. Here switching the choice of orientation switches N_+ and N_-.

Table 1.3: Constant Forms and Spinors for particular Holonomy

Even more can be said based on the Holonomy group. On certain manifolds, the Ricci curvature is identically zero. In this case the manifold is called *Ricci flat*. It turns out the Ricci flat manifolds are precisely the ones with holonomy $SU(n), HU(n), G_2$ and Spin_7. These are remarkably the same holonomies that admit constant spinors! See [21] and Table 1.5.

1.6 More Connections to the Division Algebras

As we saw in the last section, much of the structure of a Riemannian manifold is determined by its holonomy group. In general we found that manifolds with smaller holonomy admitted more constant differential forms and often more constant spinors. These different manifolds can also be understood in the context of the division algebras.

Recall that the division algebras $\mathbb{R}$ and $\mathbb{C}$ can naturally be used as the scalars for the vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$. Similarly we were able to define a scalar action of $\mathbb{H}$ on $\mathbb{H}^n$. However, since the quaternions are non commutative this action is only a right action. In all of these generalized vector spaces we can define a notion of isomorphism, $g : V \to V$, by

$$\langle g(u), g(v) \rangle = \langle u, v \rangle, \tag{1.20}$$

$$g(ux) = g(u)x. \tag{1.21}$$

where $\langle , \rangle$ is the real norm, $u, v \in \mathbb{A}^n$, and $x \in \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. The octonions, since they are not associative, do not have an associated action on $\mathbb{O}$, however we can still make the above definition sensible when $\mathbb{A} = \mathbb{O}$, if we restrict ourselves to the case $n = 1$.

Since all of these division algebras admit a vector-space-like structure, it is natural to ask whether this can be generalized to the case of manifolds, i.e., can we have manifolds such that their tangent planes admit structures analogous to the division algebra and that vary differentiably. In this context it is better to expand the above definition to allow the isomorphism to vary:

Definition 1.6.1 *Suppose the V is a normed linear $\mathbb{A}$-space of rank n. An $\mathbb{R}$-linear isometry g of V is called a twisted isomorphism if there exists $\theta \in SO(\mathbb{A})$ such that*

$$g(vx) = g(v)\theta(x), \tag{1.22}$$

for all $v \in V$ and $x \in \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

Hence we can determine the group of twisted isomorphisms for each algebra, labeled $G_{\mathbb{A}}$. It can be shown that $G_{\mathbb{R}} = O(n)$, $G_{\mathbb{C}} = U(n)$, $G_{\mathbb{H}} = HU(n) \cdot HU(1)$, and $G_{\mathbb{O}} = \mathrm{Spin}_7$. See [22] for details. This permits us to immediately make the generalization we wished:

Definition 1.6.2 *A Riemannian manifold (M, g) is called a Riemannian $\mathbb{A}$-manifold if the holonomy group is a subgroup of $G_{\mathbb{A}}(n) \subset O(m)$ with $m = dimM = ndim\mathbb{A}$.*

We immediately see that all these cases occur.

There is a very natural actions of $G_{\mathbb{A}}(n)$ on the algebra $\mathbb{A}$, given by:

$$\lambda_{\mathbb{R}}(g)(x) = x\ det_{\mathbb{R}}(g), \tag{1.23}$$

$$\lambda_{\mathbb{C}}(g)(x) = x\ det_{\mathbb{C}}(g), \tag{1.24}$$

$$\lambda_{\mathbb{H}}(g)(x) = xq \text{ where } (\alpha, q) \in HU(n) \cdot HU(1), \tag{1.25}$$

$$\lambda_{\mathbb{O}}(g)(x) = g(x)\ g \in \mathrm{Spin}_7 \subset SO(8) \text{ as above}, \tag{1.26}$$

where $g \in G_{\mathbb{A}}(n)$ and $x \in \mathbb{A}$. If this map fixes $1 \in \mathbb{A}$ then g is known as *special*. Thus we get the following associations between related holonomy groups, different kinds of differential Riemannian geometry, and different normed division algebras, outlined in Table 1.4. It is remarkable that all the most important geometries arise through this method.

Div. Algebra	Riemannian $\mathbb{A}$-Manifold	Special Riemannian $\mathbb{A}$-Manifold
$\mathbb{R}$	$O(n)$ Riemannian manifold	$SO(n)$ Oriented Riemannian manifold
$\mathbb{C}$	$U(n)$ Kähler manifold	$SU(n)$ Calabi-Yau manifold
$\mathbb{H}$	$HU(n) \cdot HU(1)$ Quaternionic-Kähler manifolds	$HU(n)$ Hyperkähler manifolds
$\mathbb{O}$	Spin_7 Spin_7-manifold	G_2 G_2-manifold

Table 1.4: The Relationship between Holonomy Groups, Geometries, and Division Algebras

Chapter 2

Geometry and Physics

Ever since the formulation of general relativity, geometry and physics have been inseparable. This is even more true today. The basic structure of string theory and M-theory are essentially geometric in nature. Moreover, the physical nature of string theory has helped provide insight into unexpected phenomena in mathematics. Most notably, due to the physical insights of string theory the discovery of mirror symmetry was possible.

Furthermore string theory and M-theory provide a wealth of interesting unsolved problems, fueling the on going developments in modern geometry. For this reason, we will review some of the ways that geometry has emerged as essential to physics and discuss some of the connections of string theory to the geometries discussed in Chapter 1.

2.1 A Brief Introduction to the Geometry of Physics

Prior to the 1900s, Newtonian mechanics dominated physics. Here time and space are completely separated. The physical space that we live in is taken to be $\mathbb{R}^3$, and objects move through this space in accordance with our everyday intuition. However, beginning with Einstein's, Lorentz's, and Minkowski work on special relativity we have learned that space and time are really inseparable. From the geometric point of view, special relativity can be understood in terms of an indefinite metric on $\mathbb{R}^4$, with signature $(+1, +1, +1, -1)$, $ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$. In this way time and space can be united through a judicious choice of metric on a larger manifold. This is a basic philosophical approach to modern physics: different aspects of physics can be united, with better understanding of each, by looking at a larger space with a special structure.

New understanding can be gained form such an approach. In special relativity an inertial reference frame corresponds to three vectors in this space that behave as Euclidean space does along with an orthogonal "time-like" vector. Then the transformations that change inertial reference frames are precisely those that preserve this unusual indefinite metric (known as *Lorentz transformations*). They include rotations among the three Euclidean "space-like" vectors as well as *Lorentz boosts* which accelerate the reference frame. However, due to the indefinite metric vectors of the form $(|v|/c, \vec{v})$ where $\vec{v} \in \mathbb{R}^3$ remain invariant (mod rotations). Since the norm of such a four vector is zero with this metric these are known as null vectors. They correspond to the trajectory of an object moving at velocity c, the speed of light. In other words, the speed of light is the same in all inertial reference frames.

The move to general relativity is made by simply replacing the flat spacetime of $\mathbb{R}^4$ by a four manifold equipped with an indefinite metric of signature $(+1, +1, +1, -1)$ and making the assumption that the curvature of the manifold is proportional to the amount of "stuff" (mass and energy) in the manifold at the point in question. In particular, we can show that Einstein's field equations must be satisfied:

$$R_{ij} - \frac{R}{2}g_{ij} + \Lambda g_{ij} = 8\pi \frac{G}{c^4} T_{ij}, \tag{2.1}$$

where R_{ij} is the Ricci curvature, R is the scalar curvature, g_{ij} is the metric, G is the gravitational constant, T_{ij} is the stress-energy-momentum tensor (a measurement of the density of energy and matter), and Λ is the cosmological constant[1].

Unlike Newtonian mechanics or special relativity, where objects move along simple straight lines, in general relativity space is curved according to equation

[1]Experimental evidence has shown that a non-zero cosmological constant is needed to explain some phenomenon, although it is very close to zero

(2.1), and objects move along geodesics. Hence, in the vicinity of matter an objects inertial trajectory will be bent according to the curvature of spacetime. This would look like the object was being pulled toward the matter. Hence general relativity explains gravity in terms of geometry.

Similarly, certain aspects of quantum mechanics (or quantum field theory, QFT) can be explained in terms of geometrical structures. Developed by countless physicists starting around 1905, the basic premises of QFT are that all the particles we know of are governed by a variety of locally defined fields. In order to accurately describe the world, these fields not only have to take on tensor values, but also spinor values. Hence physics must take place on a manifold which admits a Spin bundle, a very geometric constraint. Moreover these field take on additional internal structure, which accounts for the various forces seen in nature[2].

These fields can then be used to determine the probability of seeing a particular sort of event, although precisely determined predictions are impossible. For example we may have a particle which is described by the spinor-valued function ψ tensored with an additional $U(1)$ structure. In the physics notation this would be denoted: $|\psi \otimes e^{i\theta} >$. The the probability of finding the particle in a region $\Omega \subset \mathbb{R}^3$ at time t is given by,

$$\text{Probability in } \Omega = \langle \psi \otimes e^{i\theta} | \psi \otimes e^{i\theta} \rangle = \int_\Omega |\psi(x,t)|^2 dx. \tag{2.2}$$

Notice that the phase of the $U(1)$ structure doesn't matter in this situation. In the full context of QFT this $U(1)$ structure would correspond to a force analogous to electricity and magnetism. Such an internal structure is known in the physics literature as a *gauge symmetry*. Understanding the structure of all such gauge symmetries is a very important unsolved problem. Professor Tian has shown the importance of Cayley manifolds in this context, [27].

Three of the four fundamental forces (electromagnetism, strong and weak nu-

[2]Excluding gravity. QFT cannot be consistently used to explain gravity

clear forces) can be incorporated into QFT in this way. However, gravity cannot. It can be shown that any theory that uses point like interactions leads to unresolvable difficulties when gravity is incorporated. Moreover the geometric characterization of general relativity is incompatible with the quantum nature of particles in QFT.

Furthermore, QFT offers no explanation for the existence of these various internal structures, nor for the large number of elementary particles that we observe in nature. One important attempt to solve this is to assume that the tensor and spinor particles undergo a sort of symmetry among each other. If this symmetry exists globally, then it corresponds to the existence of a constant spinor, Q, on the spacetime manifold. Such a spinor relates particles in the following way: If ϕ is a scalar particle field, then the quantity $\psi = Q\phi$ is a new spinor field that is determined by the constant spinor Q and the field ϕ. This symmetry is known as *supersymmetry*[3].

As we have seen, the existence of constant spinors is intimately related to the geometry of the manifold, and particularly to the holonomy group of the manifold. This inevitably incorporates the curvature of the manifold, and hence gravity, which we noted was incompatible with QFT. However, supersymmetric theories have proved to be very important, and it is believed that supersymmetry is an actual approximate symmetry.

Attempts to explain the existence of the internal symmetries of QFT have resulted in studying higher dimensional spaces. In these spaces the extra dimensions, which we don't observe, are curled into some sort of manifold, hence the whole space locally looks like $\mathbb{R}^{3,1} \times M$. This is known as *compactification*. The various internal structures of quantum mechanics are then explained by various geometric structure of the manifold M. For example, if the holonomy of M is one of the Ricci flat holonomies, then M admits a constant spinor. Hence the theory

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[3]One consequence of super symmetry is that the paired particles, e.g., ϕ and $\psi = Q\phi$, have identical masses. This is note observed in nature. However supersymmetry solves many additional problems with quantum mechanics, and so is still a desirable symmetry. Hence it is believed that some sort of approximate supersymmetry still exists in nature.

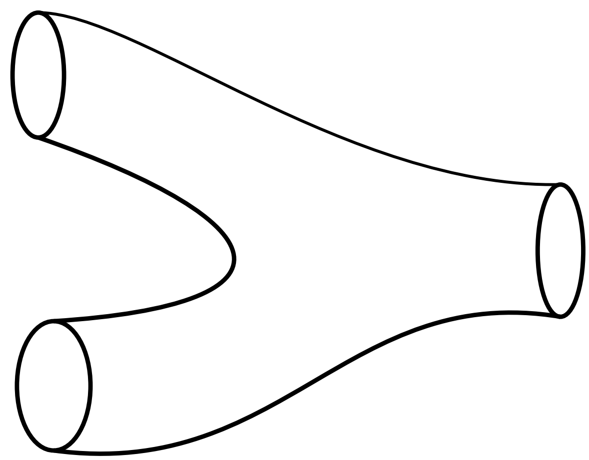


Figure 2.1: The non-local character of interacting strings

naturally incorporates supersymmetry.

2.2 String Theory, M-Theory, and F-Theory

String Theory has become popular partly because it solves many of these difficulties between quantum mechanics and gravity in a very natural way. Moreover it is essentially the only theory that does so (of course it brings in its own problems). The basic idea is to replace the point particles of quantum field theory with very small one-dimensional strings. There is only one type of these strings. The large number of particles that we observe are really different vibrational modes of the string. One of the most surprising and inspirational results is that in all consistent string theories one gets a particular mode that behaves exactly as a gravitational particle should behave. Moreover since the particles are not point particles, the theory is manifestly non-local and avoids the problems encountered with other quantum theories of gravity.

For technical reasons, string theories require that there be ten dimensions, nine space-like and one time-like. Since we do not observe ten dimensions in the real

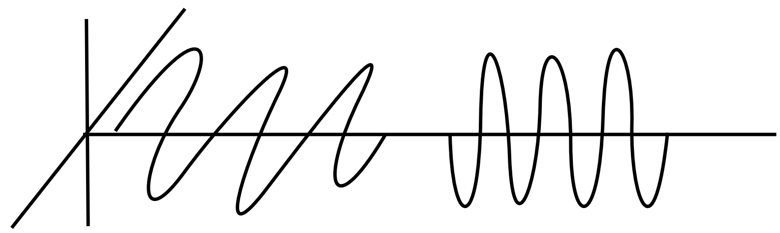


Figure 2.2: Vibrations of a string can be decomposed in Euclidean space

world, only four, string theories require the remaining space to be compactified on a six dimensional manifold. This idea has many additional benefits. For example, in Euclidean space the vibrations of a string can be decomposed into independent orthogonal directions, e.g. in the x-, y- and z-directions. However in a curved manifold this is not always possible. In a curved manifold, vibrations in one direction can potentially effect the vibrations in other directions. Since these vibrations are supposed to correspond to particles, curved space allows for these particles to interact with one another, an essential ingredient for any theory expected to describe the real world.

In physics the use of symmetry is extremely important. By assuming a slightly simplified model with additional symmetries, many ground breaking insights can be made. For example the most important solution to the field equations of general relativity was made by Schwarzchild. He assumed spherical symmetry and that space was a vacuum ($T_{ij} = 0$). This led to the condition that the Ricci curvature must vanish as well. He was then able to solve this case. This is the basic model that is used in almost all black hole calculations in general relativity. Similarly, we must assume certain symmetries in string theory (and M-theory). These symmetric models will give us insight into the more general case. In particular the compactified manifolds are assumed to be Ricci flat and hence admit a constant spinor, i.e., supersymmetry.

The basic model for string theory involves compactifying the additional six dimensions down to a manifold with holonomy $SU(3)$. Hence this manifold will be a Calabi-Yau manifold. However this does not uniquely define the string theory. There are actually five distinct string theories that fit this model.

However it has been shown that these five ten-dimensional string theories undergo a number of dualities which relate them to one another. One such symmetry is mirror symmetry, which relates type II A string theory to type II B sting theory. Moreover, Witten has demonstrated that it appears the they are all special limits of one eleven-dimensional theory now called M-Theory. Moreover it has been shown that eleven-dimensional supergravity appears to be another limit of this theory. For example, if we consider a a two dimensional torus embedded in an eleven dimensional space, we can imagine two natural limits. If we shrink the size of the torus to zero, then it becomes a point object; this limit is eleven dimensional supergravity. If instead, we let the torus be wrapped along one of the compactified dimensions and we let this dimension shrink to zero the we get a string in a ten dimensional space. It can be shown that this corresponds to type II A string theory. Hence it seems that the string theories are really limits of a larger theory.

However it should be stressed that the specifics of this theory are poorly defined, and much concerning it is still conjectural. In this theory G_2-manifolds play a central role, analogous to the role played by Calabi-Yau manifolds. Furthermore some of the limiting processes in M-Theory are more natural than others, leaving some aspects of these string theories unexplained. A clear global picture has yet to emerge. Vafa, [28], has been able to explain some of the more unnatural dualities in terms of an analogous twelve-dimensional theory known as 'F-Theory'. In this context the critical dimension of the compactified manifold is 8 real dimensions. Here Spin_7-manifolds play a crucial role analogous to G_2- and Calabi-Yau manifolds. While Calabi-Yau manifolds are relatively well understood, very little is known about general G_2- and Spin_7-manifolds. To this end the focus of this thesis

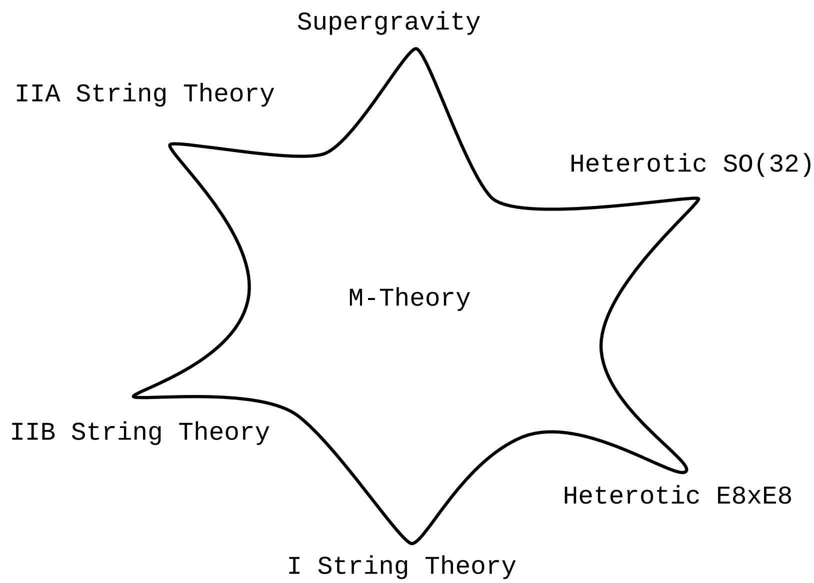


Figure 2.3: String Theories as the Limits of M-Theory

has been to provide a rich set of tools for studying these objects, particularly in the Spin_7 case.

Finally, string theory and M-theory are not just theories of strings. Rather, higher dimensional objects naturally emerge. These are known as p-branes. One particularly important example is known as a D-brane. This arises from boundary conditions imposed by the compactification and through the string dualities. In what is known about M-theory, we have found that these membranes actually play a role as fundamental as strings. In particular, minimal surfaces play an extremely important role in understanding the specifics of the theory and choice of compactified manifold. We now understand black holes as a particular type of minimal surface that exhibits a self-singularity. For this reason it is important to study the structure of minimal surfaces and their intersections.

There are many powerful tools that are being developed to accomplish this. Many algebraic geometric techniques have been developed for counting the intersections of minimal surfaces (e.g. Seiberg-Witten theory). Also, there are variational methods that examine how minimal manifolds develop singularities. One particularly elegant method for investigating minimal surfaces in general is known as calibrated geometry. This will be the topic of the next chapter.

Chapter 3

The Cayley Calibration

The field of calibrated geometry began with the work of Wirtinger [30] in the 1930s, de Rham [12] in the 1950s, and Federer [14] in the 1960s, who used Kähler forms and their powers to prove that compact *complex submanifolds* of Kähler manifolds are volume-minimizing in their homology classes. In the early 1970s, Berger [2] extended this approach to quaternionic forms. In the early 1980s, Harvey and Lawson wrote a monumental work [18] on this subject. They exhibited and studied several beautiful geometries of minimal subvarieties other than complex submanifolds which include associative geometry, coassociative geometry and Cayley geometry.

As we will see, calibrations and their calibrated submanifolds are intimately connected to the geometry of the manifold. Much of the structure of the manifold can be understood in terms of the structure of the calibrated submanifolds. It is in this context that mirror symmetry has its most profound influence on mathematics.

3.1 Method of Calibrations

Let M be a Riemannian manifold, a *calibration* on M is a closed p-form ϕ such that

$$\phi(\hat{e}_1, \ldots, \hat{e}_p) \leq 1 \tag{3.1}$$

on all orthonormal p-tuples of tangent vectors at all points of M, i.e., on all tangent p-planes $e_1 \wedge e_2 \wedge \cdots \wedge e_p$ with $|e_1 \wedge e_2 \wedge \cdots \wedge e_p| = 1$. A tangent plane is *calibrated* if ϕ achieves the maximal value 1 on it. A p-dimensional submanifold of M is called calibrated if all of its oriented tangent planes are calibrated. The crucial result, known as the fundamental theorem of calibrated geometry, states that any calibrated closed oriented p-dimensional cycle $N \subset M$ is of absolutely minimal volume in its homology class.

This is an immediate consequence of Stokes's theorem. Let $N \subset M$ be a closed oriented p-dimensional cycle, calibrated by the p-calibration ϕ. Let $N' \subset M$ be any other p-dimensional, oriented cycle in N's homology class. Thus there exists a $(p+1)$-dimensional manifold $B \subset M$ such that the boundary of B is N' with the proper orientation and N with the opposite orientation. From the definition of a calibration we have the following result,

$$Vol(N') \geq \int_{N'} \phi, \tag{3.2}$$

with equality if and only if N' is calibrated by ϕ. In particular, $Vol(N) = \int_N \phi$. Thus,

$$Vol(N') - Vol(N) \geq \int_{N'} \phi - \int_N \phi = \int_{\partial B} \phi = \int_B d\phi = 0, \tag{3.3}$$

where the last two equalities are due to Stokes's theorem and the fact that the calibration ϕ is closed. Thus the volume of N is less than or equal to the volume of any other cycle in its homology class.

Recall that the holonomy of a manifold determines the number and type of constant differential forms on that manifold, see Table 1.3. Furthermore, these holonomy groups are closely related to the particular geometry of that manifold. It happens each of these constant forms is a calibration that distinguishes certain submanifolds. For example, if the holonomy is $U(n)$ then the manifold admits a Kähler form, ω. The n'th powers of this form calibrate complex n-dimensional submanifolds, showing that indeed complex submanifolds are volume minimizing. Calabi-Yau manifolds have the additional constant form Ω, which defines a one parameter family of *special Lagrangian* calibrations, Re $(e^{i\theta}\Omega)$. The manifolds calibrated by these new forms are consequently called special Lagrangian manifolds. In the exceptional cases, Spin_7 and G_2, the calibrating form Φ is known as the Cayley calibration, while the special forms ϕ and $\star\phi$ are known as the associative and coassociative calibrations, respectively. These will be discussed in more detail in the next section.

Mirror symmetry is perhaps one of the largest triumphs of string theory. It is related to the one of the string theory dualities, and was predicted based on unrigorous physical arguments. To the surprise of the mathematical community, mirror symmetry has to a large extent been realized, although many conjectures still remain. Heuristically, mirror symmetry is a map between two Calabi-Yau manifolds which preserves much of the structure, e.g., their cohomology rings are dual in a particular sense. Moreover these Calabi-Yau manifolds can be radically different from one another; one smooth, for example, while the other contains several singularities. In the context of string theory this is a map from type II A string theory to type II B string theory.

Calibrated geometry received renewed attention in 1996 when the role of the special Lagrangian geometry in mirror symmetry was discovered by Strominger, Yau, and Zaslow (see [26]). The reader might also consult [4], [20], [17], and [3]. Here Calabi-Yau 3-manifolds are realized as special Lagrangian torus fibrations over some base manifold.

$$\begin{array}{ccl} \text{S.L. } T^3 & \hookrightarrow & M^6_{CY} \\ & & \downarrow \\ & & B^3 \quad (\text{e.g., } S^3) \end{array}$$

In this context it is possible to define a dual torus fibration over the same base manifold, and to construct a map between the two Calabi-Yau 3-manifolds. This is precisely the mirror map. The mirror map exchanges complex and special Lagrangian manifolds, sending the complex submanifolds of the first Calabi-Yau to the special Lagrangian of the second, and the complex submanifolds of the second to the special Lagrangian of the first[1]. Moreover the cup products of the Calabi-Yau manifolds' cohomologies are preserved under this mirror map. Hence the cohomologies of these manifolds share related structures.

There has been new interest recently in the geometry of Cayley cycles. Following [26], the roles of exceptional geometries in mirror symmetry were first investigated in [1], From the physics point of view, the authors showed that the Cayley cycles in Spin_7 holonomy eight-manifolds and the associative and coassociative cycles in G_2 holonomy seven-manifolds preserve half of the space-time supersymmetry. They discovered that while the complex and special Lagrangian cycles in Calabi-Yau 4-folds preserve half of the space-time supersymmetry, the Cayley submanifolds are novel as they preserve only one quarter of it. They also conjectured as to what kind of roles Cayley cycles will play in mirror symmetry for Calabi-Yau 4-folds (in contrast to the roles of complex and special Lagrangian cycles in the mirror symmetry of Calabi-Yau 3-folds) and proposed the problem of finding explicit examples of Cayley cycles to demonstrate the above conjectured phenomenon.

In particular, while in Calabi-Yau 3-folds the complex and special Lagrangian cycles fill out all the $H^{p,p}$ cohomology classes, it is not known if this holds for Calabi-Yau 4-folds. It is quite possible that complex and Special Lagrangian cycles fall short of filling out all of the $H^{2,2}$ cohomology classes. One hope is that Cayley cycles could fill out the remaining classes, (complex and Special Lagrangian cycles actually being subcases of Cayley cycles). See [1] for the details of their conjecture.

In the above context this would be realized as associative or coassociative fibrations over some base manifold in the G_2 case and Cayley cycle fibrations in the Spin_7 case. The nature of these calibrated submanifolds determines the overall structure of the fibration. For example, in the Calabi-Yau manifolds if the special

[1]This is not true as stated as special Lagrangian manifolds are 3-dimensional and complex submanifolds are even dimensional. However the special Lagrangian submanifolds will be sent to a stable complex bundle (i.e. a complex submanifold plus some additional one dimensional piece) which defines the complex submanifold.

$$
\begin{array}{llcc}
G_2 : & \text{Associative } C^3 & \hookrightarrow & M^7_{G_2} \\
& & & \downarrow \\
& & & B^4 \\
& \text{Co-associative } C^4 & \hookrightarrow & M^7_{G_2} \\
& & & \downarrow \\
& & & B^3 \\
\text{Spin}_7 : & \text{Cayley } C^4 & \hookrightarrow & M^8_{\text{Spin}_7} \\
& & & \downarrow \\
& & & B^4
\end{array}
$$

Figure 3.1: The Conjectured Fibration Structure of G_2 and Spin_7 manifolds.

Lagrangian fibers do not intersect each other, then the fibration will be a simple product manifold. Thus understanding the structure of these possible intersections is crucial for understanding the structure of the manifolds. While there has been substantial work on this in the Calabi-Yau case, essentially none has been done in the G_2 and Spin_7 cases. The key to understanding this problem lies in understanding the linear case. Hence the next chapter provides a detailed classification of the allowed angles between Cayley 4-planes.

Cayley submanifolds and Cayley cycles have another important application in the further development of gauge theory as discussed by Gang Tian in his paper [27]. Identifying Cayley cycles in the complex 4-dimensional torus (a Calabi-Yau 4-fold) is a problem suggested to my advisor, Professor W. Gu, by Professor G. Tian. There are methods for construction Cayley cycles on general Spin_7-manifolds using the solutions on the complex four-torus, further motivating this case.

Recall a flat torus $T^8 \cong T^4_{\mathbb{C}}$ can be identified as $T^8 \cong \mathbb{R}^8/\Lambda$, where Λ is a lattice in $\mathbb{R}^8$. Any Cayley cycle in $\mathbb{R}^8$ lifts to a Λ-periodic Cayley cycle in $\mathbb{R}^8$. Unfortunately, few non trivial Cayley submanifolds or Cayley cycles are known even in $\mathbb{R}^8$. Even nontrivial cases where the Cayley manifold is locally linear (e.g. a space filling polytopes) are extremely interesting. In Chapters 5 and 6 we will provide the background and develop a technique for constructing lattice periodic Cayley manifolds which are locally linear.

3.2 Cayley Calibration on $\mathbb{R}^8$

The Cayley Calibration is closely related to associative and coassociative calibrations, which are the special forms associated with the Cayley geometry. The following discussion on these calibrations mainly follows from [18] and [22].

In the *associative geometry*, we use the associative calibration (i.e. the 3-form) $\phi(x, y, z) =< x, yz >$ on $x, y, z \in \mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$. It is called associative because the local system of differential equations for this geometry is essentially deduced from the vanishing of the *associator* $[x, y, z] \equiv (xy)z - x(yz)$. This is a natural generalization to the nonassociative Cayley numbers of the familiar commutator $[x, y] = xy - yx$. It measures the associativity of of x, y, z.

In the *coassociative geometry*, we use the coassociative form ψ on $\mathrm{Im}\,\mathbb{O}$, denoted $\star\phi$, where $\star : \wedge^3\mathbb{R}^7 \to \wedge^4\mathbb{R}^7$ is the usual hodge star map of Euclidean spaces, mapping simple forms to simple forms. It is called coassociative because it is the dual geometry of the associative geometry.

In *Cayley geometry*, we use the Cayley calibration $\Phi \in \wedge^4\mathbb{O}^*$ on $\mathbb{O} \cong \mathbb{R}^8$. It is given $1^* \wedge \phi + \psi$. Cayley geometry is the most complex and fascinating geometry discussed in [18], as it contains all the other geometries as subcases. In particular if you fix a complex structure $R_u \in S^7 = \{R_u \mid u \in \mathrm{Im}\,\mathbb{O},\ |u| = 1\}$ where R_u is right-Cayley multiplication, then $\Phi = \mathrm{Re}\,(\Omega) - \frac{1}{2}\omega^2$.

In local coordinates, Φ can be written as

$$\Phi(x, y, z, w) = \langle x, y \times z \times w \rangle \tag{3.4}$$

where $x, y, z, w \in \mathbb{O}$. Here we make use of the triple cross product of Cayley numbers (see [18], def. B.3). One can verify that Φ is alternating, closed, and has a maximum value of one on normed four-planes. Moreover, we can equivalently make use of the Cayley quadruple cross product to obtain,

$$\Phi(x, y, z, w) = \mathrm{Re}\,(x \times y \times z \times w). \tag{3.5}$$

Theorem 3.2.1 *$\Phi(\zeta) \leq 1$ for all $\zeta \in G(4,8) \subset \wedge^4\mathbf{O}$, with equality if and only if ζ is a Cayley 4–plane (i.e. ζ or $-\zeta$ is a complex 2–plane with respect to one of the complex structures determined by a two-plane contained in ζ.)*

A 4-plane is called a Cayley plane if Φ achieves the value 1 on it. We call a 4-manifold a Cayley manifold if all its tangent planes are Cayley planes. We use $G(\Phi)$ to denote the set of Cayley planes.

Recall that $Spin_7$ is the subgroup of $SO(8)$ generated by $S^6 \equiv \{R_u : u \in \mathrm{Im}\,\mathbb{O} \text{ and } |u| = 1\}$, where R_u is the right Cayley multiplication. There are several alternate definitions of Spin_7 which are particularly useful,

$$Spin_7 = \{g \in SO(8) : g^*\Phi = \Phi\}. \tag{3.6}$$

$$\mathrm{Spin}_7 = \left\{ g \in SO_8 \middle| g(uv) = g(u)\overline{\chi}_g(v) \right\} \qquad \forall u, v \in \mathbb{O} \tag{3.7}$$

where $\overline{\chi}_g : \mathrm{Spin}_7 \to SO(\mathrm{Im}\,\mathbb{O}) \cong SO_7$ is defined by $\overline{\chi}_g(v) = g(g^{-1}(1) \cdot v)$ for all $v \in \mathbb{O}$, which is the standard double cover of SO_7 by Spin_7. Notice in particular that Spin_7 is precisely the subgroup of $SO(8)$ that fixes the Cayley calibration.

Theorem 3.2.2 *The action of $Spin_7$ on $G(\Phi)$ is transitive with isotropy subgroup $K = SU(2) \times SU(2) \times SU(2)/\mathbf{Z}_2 = HU(1) \times HU(1) \times HU(1)/\mathbf{Z}_2$. Thus $G(\Phi) \cong Spin_7/K$.*

Remark 3.2.1 *The geometry of Cayley submanifolds includes several other geometries:*

1. *A submanifold M which lies in $\mathrm{Im}\,\mathbb{O} \subset \mathbb{O}$ is Cayley if and only if M is coassociative.*

2. *A submanifold M of $\mathbb{O}$ of the form $\mathbb{R} \times N$, where N is a submanifold of $\mathrm{Im}\,\mathbb{O}$, is Cayley if and only if N is associative.*

3. *Fix a unit imaginary quaternion $u \in S^6 \subset \mathrm{Im}\,\mathbb{O}$. Consider the complex structure $J_u = R_u$ and let $\mathbb{O} \cong \mathbb{C}^4$. Each complex surface in $\mathbb{O}$, with the reverse orientation, is a Cayley submanifold.*

4. *In addition to choosing one of the distinguished complex structures J_u (as in (3)) choose a quaternion subalgebra $\tilde{\mathbb{H}}$ of $\mathbb{O}$ orthogonal to u and identify $\mathbb{R}^4 \subset \mathbb{C}^4$ with $\tilde{\mathbb{H}} \subset \mathbb{O}$. Each special Lagrangian submanifold of $\mathbb{C}^4 \cong \mathbb{O}$ is a Cayley submanifold.*

There are few known Cayley submanifolds which are not holomorphic or special Lagrangian.

3.3 Partial Differential Equations of Cayley Manifolds

One special type of Cayley submanifold we can look for is the graph of a function $f : \Omega \subset \mathbb{H} \to \mathbb{H}$. That is, manifolds parametrized as $(x, f(x)) \in \mathbb{H} \oplus \mathbb{H} = \mathbb{O}$. The local system of partial differential equations for this case is deduced in [18].

We denote a point in $\mathbb{H}$ by $x = x_1 + x_2\hat{i} + x_3\hat{j} + x_4\hat{k}$.

Definition 3.3.1 *The* Dirac *operator D is defined on f as*

$$Df = \frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\hat{i} - \frac{\partial f}{\partial x_3}\hat{j} - \frac{\partial f}{\partial x_4}\hat{k}. \tag{3.8}$$

The first order Monge-Ampere operator *on f is defined as*

$$\sigma f = \left(\frac{\partial f}{\partial x_2} \times \frac{\partial f}{\partial x_3} \times \frac{\partial f}{\partial x_4}\right) + \left(\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_3} \times \frac{\partial f}{\partial x_4}\right)\hat{i} - \left(\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_2} \times \frac{\partial f}{\partial x_4}\right)\hat{j} + \left(\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_2} \times \frac{\partial f}{\partial x_3}\right)\hat{k} \tag{3.9}$$

and a third operator is defined by

$$\delta f = \mathrm{Im}\left[\left(\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_2} - \frac{\partial f}{\partial x_3} \times \frac{\partial f}{\partial x_4}\right)\hat{i}\right] +$$

$$\mathrm{Im}\left[\left(\frac{\partial f}{\partial x_1}\times\frac{\partial f}{\partial x_3}+\frac{\partial f}{\partial x_2}\times\frac{\partial f}{\partial x_4}\right)\hat{j}+\left(\frac{\partial f}{\partial x_1}\times\frac{\partial f}{\partial x_4}-\frac{\partial f}{\partial x_2}\times\frac{\partial f}{\partial x_3}\right)\hat{k}\right] \qquad (3.10)$$

Theorem 3.3.1 *Suppose $f : \Omega \subset \mathbb{H} \to \mathbb{H}$ is C^1. The graph of f is a Cayley manifold if and only if f satisfies the differential equations*

$$Df = \sigma f \qquad (3.11)$$

$$\delta f = 0 \qquad (3.12)$$

Note that the resulting PDEs are only first order, however, the σf term is highly nonlinear. No one knows how to solve this system in general.

3.4 Cayley Manifolds with Symmetry

In some of the author's previous work, he a Professor W. Gu were able to provide several non-trivial solutions to the Cayley PDEs. The principle method was to look for solutions with a novel symmetry. By using such symmetry groups the system of PDEs governing the Cayley graphs were usually simplified to an ODE, in much the same way a surface of revolution is determined by its values on a 1-dimensional curve. We summarize the main ideas below.

We define a *symmetric submanifold* of $\mathbb{R}^8$ is a manifold $M \subset \mathbb{O}$, equipped with a subgroup of the special orthogonal group, $G \subset SO(8)$, such that M is invariant under the action of this group. In other words, given $A \in G$ and $x \in M$ we have $A(x) \in M$. In principle we could choose an alternate group from which to draw our symmetries, such as the full orthogonal group $O(8)$ or the affine reflection group $Aff(\mathbb{R}^8)$, however for the purposes of this paper, $SO(8)$ is sufficient.

The symmetry of such a submanifold allows us to specify the submanifold at fewer points in the following sense: Let M be a symmetric submanifold of $\mathbb{R}^8$ with symmetry group G. Suppose that the map $f : U \subset \mathbb{R}^n \to M \subset \mathbb{R}^8$, with U open in $\mathbb{R}^8$, is a local parametrization of M at a point $p \in f(U) \subset M \subset \mathbb{R}^8$.

Since M is invariant under the action of the group G, for all $A \in G$, $A(f(U)) \subset M$. Moreover, the map $A \circ f : U \to M$ is a local parametrization of M at the point $A(p)$. Since we are assuming that $G \subset SO(8)$, if M is oriented, this new coordinate parametrization will preserve this orientation. Consequently, we only need to specify the submanifold M on a subset such that the orbit of this subset, under G, results in the whole manifold. This leads to the familiar result that a surface of revolution in $\mathbb{R}^3$ is fully determined by its values along a planar curve.

We are particularly interested in the case when the manifold M is Cayley. The the author and Professor W. Gu were able to show in their paper, [15], that if $A \in G$ is an element of the symmetry group of the 4-submanifold M, and $\zeta_p \in G(4,8)$ is the tangent 4-plane at $p \in M$, then the tangent plane at $A(p) \in M$ is determined by $\zeta_{A(p)} = A(\zeta_p)$. Hence we have $\Phi(\zeta_{A(p)}) = \Phi(A(\zeta_p)) = A^*\Phi(\zeta_p)$. This implies that if the symmetry group is a subset of Spin_7 and $N \subset M$ is such that its orbit is all of M, then if ζ_x is Cayley for all $x \in N$, M is a Cayley submanifold.

As noted above, one particularly interesting example is when the submanifold M is locally the graph over $\mathbb{H} \subset \mathbb{O}$ of a function $f : \mathbb{H} \to \mathbb{H}$. In this case, we have the additional restriction that the symmetry group leaves the two orthogonal subspaces $\mathbb{H}$ and $\mathbb{H}e$ invariant, and is hence a subgroup of the isotropy subgroup K. This result is the basis of [15], which examines some important three dimensional subgroups of K. Several new Cayley manifolds were discovered, which were previously unknown.

During the summer of 2002, Professor W. Gu and the author were able to expand the work of [15], to include a complete classification of the three dimensional subgroups of K. Then each case was carefully examined. The simplifications in the first paper, [15], were removed.

This approach has been quite fruitful and and has provided a series of novel Cayley manifolds with symmetry. Physics is intimately connected to symmetries. In the future it would be interesting to explore what the physical significance of these symmetries is in the relevant M-theory.

3.5 Importance of the Choice of Complex Structure

One particularly interesting result I have been able to demonstrate concerns the importance of the choice of the complex structure when considering whether a plane (or submanifold) is complex or special Lagrangian:

Theorem 3.5.1 *Let $\zeta \in G(4,8)$ be a Cayley 4-plane. Then there exists complex structures $J_1, J_2 \in \{J_u = R_u \mid u \in \mathrm{Im}\,\mathbb{O}\} \cong S^6$, such that under J_1 ζ is special Lagrangian and under J_2, ζ is complex with the opposite orientation.*

Here R_u represents right multiplication by $u \in \mathbb{O}$.

Proof: We first quote a lemma from Harvey and Lawson [18]:

Lemma 3.5.1 *Suppose $\zeta \in G(4,8) \subset \wedge^4\mathbb{O}$. Then ζ is Cayley if and only if $-\zeta$ is a complex 2-plane with respect to one (or all) of the complex structures $J_{y\times x} = R_{y\times x}$ where $x \wedge y$ is a real 2-plane in ζ.*

All that remains to show is that the same plane is special Lagrangian under the appropriate choice of complex structure. Notice that the Cayley plane $\zeta_0 = 1 \wedge i \wedge j \wedge k = \mathbb{R}^4 \subset \mathbb{C}^4 \cong \mathbb{O}$ is special Lagrangian with respect to the complex structure $J_e = R_e$. In other words for all $x \in \zeta_0$, $J_e(x) \perp \zeta_0$. Now since Spin_7 is transitive, any other Cayley manifold can be written $\zeta = A\zeta_0$ with $A \in \mathrm{Spin}_7$.

We can now consider the complex structure $\tilde{J} = A \circ J_e \circ A^{-1}$. Under this complex structure we see that for all $Ax \in \zeta$, $\tilde{J}(Ax) \perp A\zeta_0 = \zeta$. Thus we see that under this choice of complex structure ζ is Lagrangian. Since ζ is also Cayley, we have that ζ is special Lagrangian.

Notice that any Calabi-Yau manifold comes equipped with a natural complex structure, and it is this complex structure which is relevant for the conjectures of

[1]. However this leads to several new directions. It immediately begs the question of whether every Cayley manifold can be locally written as a complex, or special Lagrangian with a locally chosen complex structure (of course it would still be possible that no such global complex structure could be chosen). In other words, I can attempt to prove or disprove the existence of a Cayley manifold that cannot locally be special Lagrangian or complex with respect to any of the six-sphere of important complex structures. Alternatively, this leads us to try to reformulate the Cayley equations in terms of some sort of local complex structure or weak form of it.

3.6 Alternate Equations

Along the lines discussed above, I was able to obtain the following result,

Theorem 3.6.1 *Let the manifold $M \subset \mathbb{O} \cong \mathbb{R}^8$ be the graph of a function $f : \mathbb{H} \to \mathbb{H}$. If f satisfies the following three conditions,*

1. *Span*$\{\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\} \perp$ *Span*$\{\frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}\}$
2. $Df = 0$
3. $\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial x_3} \times \frac{\partial f}{\partial x_4} = 0$,

then M is a Cayley graph.

The first of the above conditions is very strong and is related to the Cauchy-Riemann equations of complex analysis and geometry.

Proof: Let $M = \{(x, f(x)) \in \mathbb{O} \,|x \in \Omega \subset \mathbb{H}\}$. Let $u_1 = 1 + \frac{\partial f}{\partial x_1}e$, $u_2 = i + \frac{\partial f}{\partial x_2}e$, etc. be the tangent vectors to M at p. Now suppose that M satisfies the above three conditions. Then, Span$\{u_1, u_2\} \perp$ Span$\{u_3, u_4\}$ and

$$\begin{aligned} u_1 \times u_2 + u_3 \times u_4 &= -(\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial x_3} \times \frac{\partial f}{\partial x_4}) - ((Df)i)e && (3.13) \\ &= 0 && (3.14) \end{aligned}$$

by the assumed conditions. Now, without loss of generality, we can choose an orthonormal basis for the tangent plane of M at p, v_1, v_2, v_3, v_4 such that Span$\{u_1, u_2\} =$ Span$\{v_1, v_2\}$ and similarly for v_3, v_4. Furthermore, we have that $\overline{v_2}v_1 = v_1 \times v_2 = -v_3 \times v_4 = \overline{v_3}v_4$. Hence, $v_1 = v_2(\overline{v_3}v_4)$ and $\Phi(v_1 \wedge v_2 \wedge v_3 \wedge v_4) = 1$, and the tangent plane is Cayley.

Chapter 4

The Cayley Angle Theorem

In three dimensions, the notion of an angle begins with two intersecting lines. However this is easily generalized to angles between intersecting planes or between an intersecting line and plane. The angle between a line and a plane, for example, is the smallest angle between the line, ℓ_1, and every line contained within the plane, P. Since the angle lies between zero and $\pi/2$ by convention, the minimum angle, α, maximizes $\cos\alpha$.

In higher dimensions the process becomes slightly more complicated. For example in six dimensions, we can consider two 3-planes, ξ and η, intersecting at the origin. It is possible for these 3-planes to coincide, intersect on a 2-space, intersect on a line, or merely intersect at the origin. Given this extra freedom, we can define three angles between these planes. First choose unit vectors $u_1 \in \text{Span}\{\xi\}$ and $v_1 \in \text{Span}\{\eta\}$ such that $< u_1, v_1 >= \cos\alpha_1$ is maximized. This is well de-

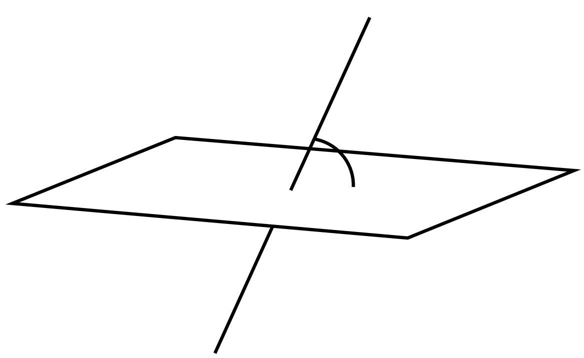


Figure 4.1: Line and Plane Intersect at an Angle

fined because the set of unit vectors in the span of a 3-plane, ie S^2, is compact. Thus α_1 defines the first angle. Note also that if $\alpha \neq 0$ this defines a unique vector $v_1' \perp \text{Span}\{\xi\}$ such that $v_1 = u_1 \cos\alpha_1 + v_1' \sin\alpha_1$.

Next define unit vectors $u_2 \in \text{Span}\{\xi\}$ and $v_2 \in \text{Span}\{\eta\}$ such that $u_2 \perp u_1$, $v_2 \perp v_1$, and $< u_2, v_2 >= \cos\alpha_2$ maximized. This defines the second angle. Continuing this process we get a unique sequence of three angles $\alpha_1 \geq \alpha_2 \geq \alpha_3$ which characterize the separation of the 3-planes.

Many areas of calibrated geometry focus on determining the allowed angles between minimizing planes. This is the first step toward determining the allowed angles of minimizing cones. It is also useful in the study of singularities. For our purposes it will be interesting to discover the allowed angles between Cayley planes. In this chapter we derive conditions on the allowed angles between Cayley planes. doing so will require some extensive cases by case analysis.

4.1 Choosing a Standard Position

We begin by considering two Cayley planes, ξ and η separated by the unique angles $\theta_1, \theta_2, \theta_3, \theta_4$, and noting that since $\text{Spin}_7 \subset SO(8)$ it preserves angles. Hence we are free to apply elements of Spin_7 to change to a more convenient basis. Thus we may take $\xi = 1 \wedge \hat{\imath} \wedge \hat{\jmath} \wedge \hat{k} = \mathbb{H} \subset \mathbb{O}$, and still retain the freedom to apply elements of the isotropy group of ξ, namely $K \cong HU(1) \times HU(1) \times HU(1)/\mathbf{Z}_2$.

From the discussion above it is clear that in most cases we have $\eta = y_1 \wedge y_2 \wedge y_3 \wedge y_4$, with,

$$y_1 = u_1 \cos\theta_1 + v_1 \hat{e} \sin\theta_1, \tag{4.1}$$

$$y_2 = u_2 \cos\theta_2 + v_2 \hat{e} \sin\theta_2, \tag{4.2}$$

$$y_3 = u_3 \cos\theta_3 + v_3 \hat{e} \sin\theta_3, \tag{4.3}$$

$$y_4 = u_4 \cos\theta_4 + v_4 \hat{e} \sin\theta_4, \tag{4.4}$$

where $u_i, v_i \in \mathbb{H}$ and we have made use of the Cayley multiplication. There are only two places where the previous discussion needs modification. First of all, if one or more of the angles is zero then the corresponding vector v_i is no longer well defined. However, if that is the case then we know that Span$\{\xi, \eta\}$ falls short of filling out all of $\mathbb{O}$. In fact, each zero angle adds a dimension to the space of vectors orthogonal to both ξ and η. Thus in such a situation we should choose v_i from this space, subject to being orthogonal to the previous v_is[1] A similar ambiguity occurs if a particular angle is $\pi/2$. In this case, u_i is not well defined. However, it forces the existence of a subspace of ξ that is orthogonal to η and of exactly the desired dimension. Choosing u_i from this space, we can continue as before. These choices ensure that $\{u_i\}$ and $\{v_i\}$ are orthonormal sets whose span is $\mathbb{H}$.

Secondly, in order to preserve the desired orientations, we must choose to relax the convention $0 \leq \theta_4 \leq \pi/2$, consequently we also sacrifice the monotonicity of the angles. In particular we choose $u_1 \wedge u_2 \wedge u_3 \wedge u_4 = 1 \wedge \hat{i} \wedge \hat{j} \wedge \hat{k}$.

We can further simplify η by carefully applying an element from K. Recall that $(q_1, q_2, q_3) \in K$ acts on $a + b\hat{e} \in \mathbb{H} \oplus \mathbb{H}\hat{e}$ by

$$a + b\hat{e} \mapsto (q_3 a \overline{q}_1) + (q_2 b \overline{q}_1)\hat{e}. \tag{4.5}$$

Restricted to just $\mathbb{H}$, K becomes $HU(1) \times HU(1)/\mathbf{Z}_2 \cong SO(4)$. Thus we may apply an element of K with q_1 and q_3 chosen so that $u_1 = 1, u_2 = \hat{i}, u_3 = \hat{j}$, and $u_4 = \hat{k}$. We can choose $q_2 = v_1^{-1} q_1$, so that after applying the element of K, $v_1 \mapsto 1$. Finally, since $\{1, v_2, v_3, v_4\}$ forms an orthonormal basis for $\mathbb{H}$, we have $v_4 = \pm v_2 v_3$. Having relaxed the conditions on the range of θ_4, we can, without loss of generality, select

[1] This vector still may not be well defined, however this convention will ensure the correctness of what follows.

the positive case. This yields the final basis simplification,

$$
\begin{aligned}
y_1 &= 1\cos\theta_1 + \hat{e}\sin\theta_1, && (4.6)\\
y_2 &= \hat{i}\cos\theta_2 + v_2\hat{e}\sin\theta_2, && (4.7)\\
y_3 &= \hat{j}\cos\theta_3 + v_3\hat{e}\sin\theta_3, && (4.8)\\
y_4 &= \hat{k}\cos\theta_4 + (v_2v_3)\hat{e}\sin\theta_4. && (4.9)
\end{aligned}
$$

Up to now, we have not made use of the fact that η is Cayley. This final requirement will put restrictions on the allowed values of the θ_i. Recall that $\eta = y_1 \wedge y_2 \wedge y_3 \wedge y_4$ is Cayley if and only if the following two conditions hold,

$$
\begin{aligned}
\mathrm{Im}\,(y_1 \times y_2 \times y_3 \times y_4) &= 0, && (4.10)\\
\mathrm{Re}\,(y_1 \times y_2 \times y_3 \times y_4) &= 1. && (4.11)
\end{aligned}
$$

See equation (3.5). Using Lemma B.10 from Harvey and Lawson, [18], we can evaluate these. [2]

Since the y_i are orthonormal the second of the above equations follows from the

[2]Notice, however that Lemma B.10 of Harvey and Lawson needs the following modification:

$$
\begin{aligned}
x \times y \times z \times w \;=\; & \langle a, b \times c \times d\rangle + \langle \alpha, \beta \times \gamma \times \delta\rangle \\
& -(\alpha \times \beta)(c \times d) + (\alpha \times \gamma)(b \times d) - (\alpha \times \delta)(b \times c) \\
& -(\beta \times \gamma)(a \times d) + (\beta \times \delta)(a \times c) - (\gamma \times \delta)(a \times b) \\
& +\hat{e} \text{ terms.}
\end{aligned}
$$

first. Thus the fact that η is Cayley is equivalent to the following the requirement,

$$
\begin{aligned}
1 &= (y_1 \times y_2 \times y_3 \times y_4) && (4.12)\\
&= 1[\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4 + \sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4]\\
&\quad -(v_2\bar{\hat{i}})[\sin\theta_1\sin\theta_2\cos\theta_3\cos\theta_4 + \cos\theta_1\cos\theta_2\sin\theta_3\sin\theta_4]\\
&\quad -(v_3\bar{\hat{j}})[\sin\theta_1\cos\theta_2\sin\theta_3\cos\theta_4 + \cos\theta_1\sin\theta_2\sin\theta_3\cos\theta_4]\\
&\quad -(v_2v_3\bar{\hat{k}})[\sin\theta_1\cos\theta_2\cos\theta_3\sin\theta_4 + \cos\theta_1\sin\theta_2\sin\theta_3\cos\theta_4]\\
&\quad -\hat{e}[\cos\theta_1\sin\theta_2\sin\theta_3\sin\theta_4 - \sin\theta_1\cos\theta_2\cos\theta_3\cos\theta_4]\\
&\quad -(v_2\bar{\hat{i}})\hat{e}[\sin\theta_1\cos\theta_2\sin\theta_3\sin\theta_4 - \cos\theta_1\sin\theta_2\cos\theta_3\cos\theta_4]\\
&\quad -(v_3\bar{\hat{j}})\hat{e}[\sin\theta_1\sin\theta_2\cos\theta_3\sin\theta_4 - \cos\theta_1\cos\theta_2\sin\theta_3\cos\theta_4]\\
&\quad -(v_2v_3\bar{\hat{k}})\hat{e}[\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4 - \cos\theta_1\cos\theta_2\cos\theta_3\sin\theta_4]. && (4.13)\\
&= 1C_1 + (v_2\bar{\hat{i}})C_2 + (v_3\bar{\hat{j}})C_3 + (v_2v_3\bar{\hat{k}})C_4\\
&\quad -\hat{e}S_1 - (v_2\bar{\hat{i}})\hat{e}S_2 - (v_3\bar{\hat{j}})\hat{e}S_3 - (v_2v_3\bar{\hat{k}})\hat{e}S_4 && (4.14)
\end{aligned}
$$

where we have defined the following terms for simplification,

$$
\begin{aligned}
C_1 &= \frac{1}{4}[\cos(\theta_1+\theta_2+\theta_3+\theta_4) + \cos(\theta_1+\theta_2-\theta_3-\theta_4)\\
&\quad + \cos(\theta_1-\theta_2+\theta_3-\theta_4) + \cos(\theta_1-\theta_2-\theta_3+\theta_4)] && (4.15)\\
C_2 &= \frac{1}{4}[\cos(\theta_1+\theta_2+\theta_3+\theta_4) + \cos(\theta_1+\theta_2-\theta_3-\theta_4)\\
&\quad - \cos(\theta_1-\theta_2+\theta_3-\theta_4) - \cos(\theta_1-\theta_2-\theta_3+\theta_4)] && (4.16)\\
C_3 &= \frac{1}{4}[\cos(\theta_1+\theta_2+\theta_3+\theta_4) - \cos(\theta_1+\theta_2-\theta_3-\theta_4)\\
&\quad + \cos(\theta_1-\theta_2+\theta_3-\theta_4) - \cos(\theta_1-\theta_2-\theta_3+\theta_4)] && (4.17)\\
C_4 &= \frac{1}{4}[\cos(\theta_1+\theta_2+\theta_3+\theta_4) - \cos(\theta_1+\theta_2-\theta_3-\theta_4)\\
&\quad - \cos(\theta_1-\theta_2+\theta_3-\theta_4) + \cos(\theta_1-\theta_2-\theta_3+\theta_4)] && (4.18)
\end{aligned}
$$

$$
\begin{aligned}
S_1 &= \frac{1}{4}[\sin(-\theta_1+\theta_2+\theta_3-\theta_4)+\sin(-\theta_1+\theta_2-\theta_3+\theta_4) && (4.19)\\
&\quad -\sin(\theta_1+\theta_2+\theta_3+\theta_4)-\sin(\theta_1+\theta_2-\theta_3-\theta_4)] && (4.20)\\
S_2 &= \frac{1}{4}[\sin(+\theta_1-\theta_2+\theta_3-\theta_4)+\sin(+\theta_1-\theta_2-\theta_3+\theta_4)\\
&\quad -\sin(\theta_1+\theta_2+\theta_3+\theta_4)-\sin(\theta_1+\theta_2-\theta_3-\theta_4)] && (4.21)\\
S_3 &= \frac{1}{4}[\sin(\theta_1+\theta_2-\theta_3-\theta_4)+\sin(-\theta_1+\theta_2-\theta_3+\theta_4)\\
&\quad -\sin(\theta_1+\theta_2+\theta_3+\theta_4)-\sin(-\theta_1+\theta_2+\theta_3-\theta_4)] && (4.22)\\
S_4 &= \frac{1}{4}[\sin(-\theta_1+\theta_2+\theta_3-\theta_4)+\sin(+\theta_1+\theta_2-\theta_3-\theta_4)\\
&\quad -\sin(\theta_1+\theta_2+\theta_3+\theta_4)-\sin(-\theta_1+\theta_2-\theta_3+\theta_4)] && (4.23)
\end{aligned}
$$

Immediately we can see that the nature of the conditions on the θ_i depends on how the basis $1, v_2, v_3, v_2v_3$ compares to $1, \hat{\imath}, \hat{\jmath}, \hat{k}$. If they were to coincide, then the $\mathbb{H}\hat{e}$ portion of equation (4.14) reduces to $S_1+S_2+S_3+S_4 = -\sin(\theta_1+\theta_2+\theta_3+\theta_4) = 0$. This forces the sum of the θ_i to be an integer multiple of pi. However, because of orientation considerations we can further reduce this to even integer multiples of pi. There are further equations from the remaining portions of (4.14), which we will discuss below. For now it suffices to show that we must choose a representation of the coordinates of v_2 and v_3 that brings to light how this basis compares to the standard basis for $\mathbb{H}$.

Furthermore, the C_i and S_i, while defined for convenience, demonstrate and enormous amount of symmetry. This comes through when we look at various sums and differences among the C_i and S_i,

$$
\begin{aligned}
C_1 + C_2 + C_3 + C_4 &= \cos(\theta_1 + \theta_2 + \theta_3 + \theta_4), && (4.24)\\
C_1 + C_2 - C_3 - C_4 &= \cos(\theta_1 + \theta_2 - \theta_3 - \theta_4), && (4.25)\\
C_1 - C_2 - C_3 + C_4 &= \cos(\theta_1 - \theta_2 - \theta_3 + \theta_4), && (4.26)\\
C_1 - C_2 + C_3 - C_4 &= \cos(\theta_1 - \theta_2 + \theta_3 - \theta_4), && (4.27)\\
S_1 + S_2 + S_3 + S_4 &= -\sin(\theta_1 + \theta_2 + \theta_3 + \theta_4), && (4.28)\\
S_1 + S_2 - S_3 - S_4 &= -\sin(\theta_1 + \theta_2 - \theta_3 - \theta_4), && (4.29)\\
S_1 - S_2 - S_3 + S_4 &= -\sin(\theta_1 - \theta_2 - \theta_3 + \theta_4), && (4.30)\\
S_1 - S_2 + S_3 - S_4 &= -\sin(\theta_1 - \theta_2 + \theta_3 - \theta_4). && (4.31)
\end{aligned}
$$

These formulas will be very helpful in analyzing equation (4.14).

4.2 Choosing the Basis Representation

With the observations of the last section we can now proceed to write the $1, v_2, v_3, v_2v_3$ orthonormal basis in a useful manner. First observe that we can write $v_2 = \hat{\imath}\cos\phi + v_2'\sin\phi$, where v_2' is an imaginary unit quaternion orthogonal to $\hat{\imath}$. This provides us with a useful representation, with v_2 automatically normalized, with $\phi = 0$ corresponding to $v_2 = \hat{\imath}$. Now it is useful to fully specify v_2', hence we let $v_2' = \hat{\jmath}\cos\lambda + \hat{k}\sin\lambda$. Since we require v_3 to be orthogonal to v_2 this limits our choices. Let $w_1 = -\hat{\imath}\sin\phi + v_2'\cos\phi$ and $w_2 = -\hat{\jmath}\sin\lambda + \hat{k}\cos\lambda$. Thus, w_1 is orthogonal to v_2, and w_2 is orthogonal to $\hat{\imath}$ and v_2', hence also orthogonal to v_2. Since w_1 is in the span of $\hat{\imath}$ and v_2', w_1 and w_2 for an orthonormal basis for the imaginary quaternions orthogonal to v_2. Hence, we can define a new angle ψ such that $v_3 = w_1\cos\psi + w_2\sin\psi$, so that v_2 and v_3, (and hence their product), can be fully defined in terms of three

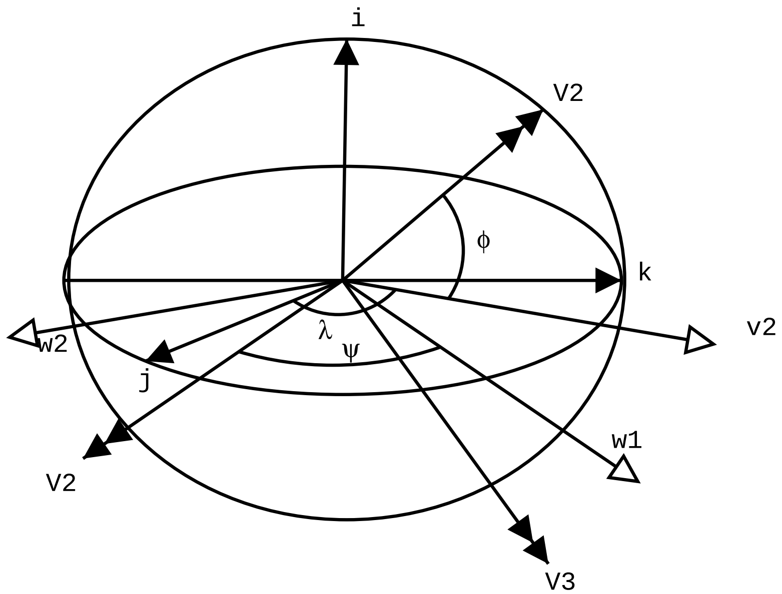


Figure 4.2: An Arbitrary Basis

angles ϕ, λ, ψ as follows:

$$v_2 = \hat{\imath}\cos\phi + \hat{\jmath}\sin\phi\cos\lambda + \hat{k}\sin\phi\sin\lambda, \tag{4.32}$$

$$\begin{aligned} v_3 = {} & \hat{\imath}(-\sin\phi\cos\psi) + \hat{\jmath}(\cos\phi\cos\lambda\cos\psi - \sin\lambda\sin\psi) \\ & +\hat{k}(\cos\phi\sin\lambda\cos\psi + \cos\lambda\sin\psi), \end{aligned} \tag{4.33}$$

$$\begin{aligned} v_2v_3 = {} & \hat{\imath}\sin\phi\sin\psi + \hat{\jmath}(-\sin\lambda\cos\psi - \cos\phi\cos\lambda\sin\psi) \\ & +\hat{k}(\cos\lambda\cos\psi - \cos\phi\sin\lambda\sin\psi). \end{aligned} \tag{4.34}$$

Notice that if $\phi = \lambda = \psi = 0$, we have the standard $\hat{\imath}, \hat{\jmath}, \hat{k}$ basis. Also, if $\sin\phi = 0$ then λ is not fully specified. In such a situation we may, without loss of generality, take $\lambda = 0$.

Using these expressions for v_2 and v_3 we can calculate the $-v_2\hat{\imath}$, $-v_3\hat{\jmath}$ and $-v_2v_3\hat{k}$ terms of equation (4.14). Separating out the linearly independent pieces yields the following eight equations involving the C_i and S_i terms,

$$
\begin{aligned}
1 &= C_1 + C_2\cos\phi + C_3(\cos\phi\cos\lambda\cos\psi - \sin\lambda\sin\psi) \\
&\quad + C_4(\cos\lambda\cos\psi - \cos\phi\sin\lambda\sin\psi), && (4.35) \\
0 &= C_2\sin\phi\sin\lambda - C_4\sin\phi\sin\psi && (4.36) \\
0 &= C_2\sin\phi\sin\lambda - C_4\sin\phi\sin\psi && (4.37) \\
0 &= C_3(\cos\phi\sin\lambda\cos\psi + \cos\lambda\sin\psi) + C_4(\sin\lambda\cos\psi + \cos\phi\cos\lambda\sin\psi) && (4.38) \\
0 &= S_1 + S_2\cos\phi + S_3(\cos\phi\cos\lambda\cos\psi - \sin\lambda\sin\psi) \\
&\quad + S_4(\cos\lambda\cos\psi - \cos\phi\sin\lambda\sin\psi), && (4.39) \\
0 &= S_2\sin\phi\sin\lambda - S_4\sin\phi\sin\psi && (4.40) \\
0 &= S_2\sin\phi\sin\lambda - S_4\sin\phi\sin\psi && (4.41) \\
0 &= S_3(\cos\phi\sin\lambda\cos\psi + \cos\lambda\sin\psi) + S_4(\sin\lambda\cos\psi + \cos\phi\cos\lambda\sin\psi) && (4.42)
\end{aligned}
$$

We can now begin a case by case analysis based on the three angles, ϕ, λ, ψ. We choose to work from the more general cases to the more specific cases, and we will first consider $\sin\phi \neq 0$. This will lead to numerous subcases which we will analyze in turn, after which we will return to the $\sin\phi = 0$ cases.

4.3 Case by Case Analysis

The assumption $\sin\phi \neq 0$ allows us to simplify the above equations a great deal. In particular we can define the column vectors $\vec{c} = (C_1, C_2, C_3, C_4),\ \vec{s} = (S_1, S_2, S_3, S_4)$, and the following array,

$$
A = \begin{pmatrix} \cos\lambda & 0 & 0 & \cos\psi \\ \sin\lambda & 0 & -\sin\psi & 0 \\ 0 & -\sin\lambda & 0 & \sin\psi \\ 0 & \cos\lambda & \cos\psi & 0 \end{pmatrix}. \tag{4.43}
$$

Then after some manipulation equations (4.35)-(4.42) can be reduced to,

$$A\vec{c} = \begin{pmatrix} \cos\lambda \\ \sin\lambda \\ 0 \\ 0 \end{pmatrix}, \qquad A\vec{s} = \vec{0}. \tag{4.44}$$

This leads us to consider $detA = \cos^2\psi - \cos^2\lambda$. The only way for $\vec{s}$ to be non-trivial is for A to be singular, thus in the most general cases A is non-singular and $\vec{s} = \vec{0}$. Let us focus on the case where $\vec{s} = \vec{0}$ and return later to the case where A is singular. Thus we have $S_1 = S_2 = S_3 = S_4 = 0$, which quickly leads to the following equations,

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = n_1\pi, \tag{4.45}$$

$$\theta_1 + \theta_2 - \theta_3 - \theta_4 = n_2\pi, \tag{4.46}$$

$$\theta_1 - \theta_2 + \theta_3 - \theta_4 = n_3\pi, \tag{4.47}$$

$$\theta_1 - \theta_2 - \theta_3 + \theta_4 = n_4\pi. \tag{4.48}$$

where $n_i \in \mathbb{Z}$. This permits us to make progress with the remaining equations. Define the function $ev : \mathbb{Z} \to \{\pm 1\}$ to be $+1$ if the integer is even and -1 otherwise. Then we have,

$$C_1 = \frac{1}{4}[ev(n_1) + ev(n_2) + ev(n_3) + ev(n_4)], \tag{4.49}$$

$$C_2 = \frac{1}{4}[ev(n_1) + ev(n_2) - ev(n_3) - ev(n_4)], \tag{4.50}$$

$$C_3 = \frac{1}{4}[ev(n_1) - ev(n_2) + ev(n_3) - ev(n_4)], \tag{4.51}$$

$$C_4 = \frac{1}{4}[ev(n_1) - ev(n_2) - ev(n_3) + ev(n_4)], \tag{4.52}$$

forcing $C_i = 1, \frac{1}{2}, 0, -\frac{1}{2}, \text{ or } -1$. Using this we can return to equation (4.44), and reduce it to three cases. The four $ev(n_i)$ can always be divided into two groups, based on their value. The three cases we will consider are the three ways those groups can be realized: $(4,0),(3,1)$, and $(2,2)$. We can use these groupings to determine the possible values of the C_i, and then make use of equations (4.44) to determine which are allowed and which are not. This in turn will determine the angles θ_i. It is important to keep in mind our assumption that $\cos\psi \neq \pm\cos\lambda$, as most of the following cases will contradict it (we could equivalently use sines).

$(\mathbf{2},\mathbf{2})$: Since the same number of $ev(n_i)$ are positive and negative, we immediately have $C_1 = 0$. The remaining C_i are related to C_1 by changing the effective sign of two of the $ev(n_i)$. Inspection shows that at least one of the C_i must be zero and one ± 1. The remaining C_i can have the other value of ± 1 or be zero. It is useful to recall the relevant equations from (4.44),

$$C_1 \cos\lambda + C_4 \cos\psi = \cos\lambda, \tag{4.53}$$

$$C_1 \sin\lambda - C_3 \sin\psi = \sin\lambda, \tag{4.54}$$

$$-C_2 \sin\lambda + C_4 \sin\psi = 0, \tag{4.55}$$

$$C_2 \cos\lambda + C_3 \cos\psi = 0. \tag{4.56}$$

Notice that because of the first two equations, since $C_1 = 0$, if either of C_3 or C_4 is ± 1, then our assumption that $\cos\lambda \neq \pm\cos\psi$ is violated. Thus we must take both C_3 and C_4 equal to zero. However, with this choice the last two equations force $C_2 = 0$ which is in violation of our earlier analysis. Thus there are no cases where the angles are grouped as $(2,2)$.

$(\mathbf{3},\mathbf{1})$: Here again we can determine the values of $C_1 = \pm\frac{1}{2}$ right away. The rest of the analysis is slightly more complicated then the $(2,2)$ case. Notice that the remaining C_is are identical to C_1, except that they have two $ev(n_i)$ with their signs changed, $i \neq 1$. Thus which group $ev(n_1)$ belongs to becomes important. We have two cases. First suppose that $ev(n_1)$ belongs to the group of one. Then a simple calculation shows that $C_1 = -C_2 = -C_3 = -C_4 = \pm\frac{1}{2}$. However, the second pair of equations then imply that $\cos\lambda = \pm\cos\psi$ in violation of our assumption. Thus we must take $ev(n_1)$ to belong to the group of three. In this case all the C_is will have absolute value $\frac{1}{2}$, however while two of the C_is will have the same sign as C_1, the remaining one will have the opposite sign. Regardless of the choice of these signs, if we look at the second pair of equations we find that again we are in violation of our assumption $\cos\lambda = \pm\cos\psi$. We are thus forced to conclude that this grouping is also not permitted.

$(\mathbf{4}, \mathbf{0})$: From this grouping we immediately find that $C_2 = C_3 = C_4 = 0$ and that $C_1 = \pm 1$. Examining the first two of the above equations shows that only the positive case is a solution. However it is a solution. Thus we have found that the only configuration that works is when all the n_i are even integers, $n_i = 2m_i$. Thus we can solve for the allowed angles,

$$\theta_1 = \pi\frac{m_1 + m_2 + m_3 + m_4}{2}, \tag{4.57}$$

$$\theta_2 = \pi\frac{m_1 + m_2 - m_3 - m_4}{2}, \tag{4.58}$$

$$\theta_3 = \pi\frac{m_1 - m_2 + m_3 - m_4}{2}, \tag{4.59}$$

$$\theta_4 = \pi\frac{m_1 - m_2 - m_3 + m_4}{2}, \tag{4.60}$$

where $m_i \in \mathbb{Z}$ are any four integers. This shows us that in most instances the only allowed angles are multiples of $\frac{\pi}{2}$. The only way to get more interesting angles is to have the $\mathbb{H}$ and $\mathbb{H}\hat{e}$ bases aligned in a sense yet to be made precise.

Recall that any three orthogonal vectors in $\mathbb{O}$ determine a unique Cayley plane containing them. Thus we suspect that the above prescription is over determined. Indeed, since the m_i are only determined up to mod 4, we can eliminate one of them. This can be made more apparent if we use vector notation:

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix} = m_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + m_2 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + m_3 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + m_4 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = m_1\vec{t_1} + m_2\vec{t_2} + m_3\vec{t_3} + m_4\vec{t_4}. \tag{4.61}$$

$\sin\phi \neq 0$, $\cos\psi \neq \pm\cos\lambda$ Case.

Angles		
θ_1	$=$	$\pi\frac{m_2+m_3+m_4}{2}$
θ_2	$=$	$\pi\frac{m_2-m_3-m_4}{2}$
θ_3	$=$	$\pi\frac{-m_2+m_3-m_4}{2}$
θ_4	$=$	$\pi\frac{-m_2-m_3+m_4}{2}$

Table 4.1: Summary of the solutions to the $\sin\phi \neq 0$, $\cos\psi \neq \pm\cos\lambda$ case. Here $m_i \in \mathbb{Z}$ mod 4. Note that the form of these solutions is not unique.

Now the four vectors $\vec{t_1}, \vec{t_2}, \vec{t_3}, \vec{t_4}$, while linearly independent under $\mathbb{Z}$, are not linearly independent under $\mathbb{Z}\ mod\ 4\mathbb{Z}$. For example a choice of $m_2 = m_3 = m_4 = 3n,\ m_1 = 0$ is equivalent to the choice $m_1 = n$ with $m_i = 0,\ i = 2,3,4$. Thus without loss of generality we have,

$$\theta_1 = \pi\frac{m_2+m_3+m_4}{2}, \tag{4.62}$$

$$\theta_2 = \pi\frac{m_2-m_3-m_4}{2}, \tag{4.63}$$

$$\theta_3 = \pi\frac{-m_2+m_3-m_4}{2}, \tag{4.64}$$

$$\theta_4 = \pi\frac{-m_2-m_3+m_4}{2}, \tag{4.65}$$

for any $m_i \in \mathbb{Z}\ mod\ 4$, $i = 2,3,4$.

For now, we will return to a case that was previously put aside. Namely we will consider when the matrix A is singular. This is equivalent to $\cos\lambda = \pm\cos\psi$. Since A is singular, we cannot immediately jump to the conclusion that $\vec{s} = \vec{0}$, however we can still simplify the equation (4.44). First of all we can fully describe all the singular cases with the angle ϕ and two $\pm$ prescriptions: $\cos\psi = (\pm)_\alpha \cos\lambda$ and $-\sin\psi = (\pm)_\beta \sin\lambda$. Assuming that $\psi \neq 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, the results of the simplifications are the following,

$$C_1 + C_2(\pm)_\alpha C_3(\pm)_\alpha C_4 = 1, \tag{4.66}$$

$$C_1 - C_2(\mp)_\alpha C_3(\pm)_\alpha C_4 = 1, \tag{4.67}$$

$$C_1 + C_2(\pm)_\beta C_3(\pm)_\beta C_4 = 1, \tag{4.68}$$

$$C_1 - C_2(\pm)_\beta C_3(\mp)_\beta C_4 = 1, \tag{4.69}$$

$$S_1 + S_2(\pm)_\alpha S_3(\pm)_\alpha S_4 = 0, \tag{4.70}$$

$$S_1 - S_2(\mp)_\alpha S_3(\pm)_\alpha S_4 = 0, \tag{4.71}$$

$$S_1 + S_2(\pm)_\beta S_3(\pm)_\beta S_4 = 0, \tag{4.72}$$

$$S_1 - S_2(\pm)_\beta S_3(\mp)_\beta S_4 = 0. \tag{4.73}$$

This naturally results in four cases. In each case the eight equations result in redundancies that reduce the number of equations to six. These six equations can be solved to find conditions on the four angles dependent on one free parameter.

For example, consider $(+)_\alpha, (+)_\beta$. The resulting equations are:

$$C_1 + C_2 + C_3 + C_4 = 1, \tag{4.74}$$

$$C_1 - C_2 - C_3 + C_4 = 1, \tag{4.75}$$

$$C_1 - C_2 + C_3 - C_4 = 1, \tag{4.76}$$

$$S_1 + S_2 + S_3 + S_4 = 0, \tag{4.77}$$

$$S_1 - S_2 - S_3 + S_4 = 0, \tag{4.78}$$

$$S_1 - S_2 + S_3 - S_4 = 0. \tag{4.79}$$

Because of the symmetry of the C_i and S_i these equations actually simplify a great deal. Discarding the C_i and S_i notation we get,

$$\cos(\theta_1+\theta_2+\theta_3+\theta_4) = 1, \quad (4.80)$$

$$\cos(\theta_1-\theta_2-\theta_3+\theta_4) = 1, \quad (4.81)$$

$$\cos(\theta_1-\theta_2+\theta_3-\theta_4) = 1, \quad (4.82)$$

$$\sin(\theta_1+\theta_2+\theta_3+\theta_4) = 0, \quad (4.83)$$

$$\sin(\theta_1-\theta_2-\theta_3+\theta_4) = 0, \quad (4.84)$$

$$\sin(\theta_1-\theta_2+\theta_3-\theta_4) = 0. \quad (4.85)$$

The first of these forces $\theta_1+\theta_2+\theta_3+\theta_4 = 2\pi m$ for some $m \in \mathbb{Z}$, and similarly for the other two sums of angles. However, in this case the angle $\theta_1+\theta_2-\theta_3-\theta_4 = \alpha$ remains undetermined. Nonetheless, we can solve for the angles exactly as before,

$$\theta_1 = \pi\frac{m_1+x+m_3+m_4}{2}, \quad (4.86)$$

$$\theta_2 = \pi\frac{m_1+x-m_3-m_4}{2}, \quad (4.87)$$

$$\theta_3 = \pi\frac{m_1-x+m_3-m_4}{2}, \quad (4.88)$$

$$\theta_4 = \pi\frac{m_1-x-m_3+m_4}{2}, \quad (4.89)$$

where $m_i \in \mathbb{Z}$ and x is free. Notice that just as before these equations are over determined. We have m_i m_i+4 and x $x+4$. This allows us to eliminate on of the integers. Without loss of generality we choose to eliminate m_1, yielding,

$$\theta_1 = \delta+\pi\frac{m_3+m_4}{2}, \quad (4.90)$$

$$\theta_2 = \delta+\pi\frac{-m_3-m_4}{2}, \quad (4.91)$$

$$\theta_3 = -\delta+\pi\frac{m_3-m_4}{2}, \quad (4.92)$$

$$\theta_4 = -\delta+\pi\frac{-m_3+m_4}{2}, \quad (4.93)$$

where we have made the substitution $\delta = \pi x/2$. The remaining cases are similar and we have tabulated them for reference in Table 4.2.

$\sin\phi \neq 0, \sin\psi = \pm\sin\lambda,$

$\cos\psi = \pm\cos\lambda$ Non-Boundary Cases

Cases	Angles		
$(+)_\alpha, (+)_\beta$	θ_1	$=$	$\delta + \pi\frac{m_3+m_4}{2}$
	θ_2	$=$	$\delta + \pi\frac{-m_3-m_4}{2}$
	θ_3	$=$	$-\delta + \pi\frac{m_3-m_4}{2}$
	θ_4	$=$	$-\delta + \pi\frac{-m_3+m_4}{2}$
$(-)_\alpha, (+)_\beta$	θ_1	$=$	$\delta + \pi\frac{m_2+m_3}{2}$
	θ_2	$=$	$-\delta + \pi\frac{m_2-m_3}{2}$
	θ_3	$=$	$-\delta + \pi\frac{-m_2+m_3}{2}$
	θ_4	$=$	$\delta + \pi\frac{-m_2-m_3}{2}$
$(+)_\alpha, (-)_\beta$	θ_1	$=$	$\delta + \pi\frac{m_2+m_4}{2}$
	θ_2	$=$	$-\delta + \pi\frac{m_2-m_4}{2}$
	θ_3	$=$	$\delta + \pi\frac{-m_2-m_4}{2}$
	θ_4	$=$	$-\delta + \pi\frac{-m_2+m_4}{2}$
$(-)_\alpha, (-)_\beta$	θ_1	$=$	$\delta + \pi\frac{m_2+m_3}{2}$
	θ_2	$=$	$\delta + \pi\frac{m_2-m_3}{2}$
	θ_3	$=$	$\delta + \pi\frac{-m_2+m_3}{2}$
	θ_4	$=$	$\delta + \pi\frac{-m_2-m_3}{2}$

Table 4.2: Summary of the solutions to the $\sin\phi \neq 0$, $\cos\psi = (\pm)_\alpha \cos\lambda \neq 0$, $-\sin\psi = (\pm)_\beta \sin\lambda \neq 0$ cases. Here δ is a free angle and $m_i \in \mathbb{Z} \bmod 4$. Note that the form of these solutions is not unique.

The boundary cases are somewhat more interesting. If either $\sin\psi$ or $\cos\psi$ is zero, then the rank of A will be reduced to two. This results in fewer equations and would seem to permit greater freedom for the angles θ_i. In the course of our calculations, we will see that indeed this is the case.

Suppose that $\sin\psi = \sin\lambda = 0$. We have two subcases, $\cos\psi = (\pm)\cos\lambda$. In the following derivations we will use notation that reflects this choice. The upper sign in the parentheses will correspond to the positive choice, while the lower one to the negative. This reduces equations (4.44) to the following,

$$C_1(\pm)C_4 = 1, \tag{4.94}$$

$$C_2(\pm)C_3 = 0, \tag{4.95}$$

$$S_1(\pm)S_4 = 0, \tag{4.96}$$

$$S_2(\pm)S_3 = 0. \tag{4.97}$$

This simple set of equations can be reduced to give equations for the allowed θ_i.

In the (+)-case, the above equations reduce to the following equations,

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 2\pi m_1, \tag{4.98}$$

$$\theta_1 - \theta_2 - \theta_3 + \theta_4 = 2\pi m_4. \tag{4.99}$$

Now, instead of a single free parameter, we have two, which can be used to solve for the angles. However, just as before, we have a redundancy, which once removed, yields the following equations,

$$\theta_1 = \delta_2 + \delta_3 + \pi\frac{m_4}{2}, \tag{4.100}$$

$$\theta_2 = \delta_2 + -\delta_3\pi\frac{-m_4}{2}, \tag{4.101}$$

$$\theta_3 = -\delta_2 + \delta_3 + \pi\frac{-m_4}{2}, \tag{4.102}$$

$$\theta_4 = -\delta_2 - \delta_3 + \pi\frac{m_4}{2}. \tag{4.103}$$

By now it should be clear that as we move to cases more restrictive on the angles ϕ, ψ, and λ, we gain more freedom in the θ_is, exchanging the half-integer multiples of pi for free parameters. The remaining three boundary cases are similar with the results listed in Table 4.3.

Up until now, we have only considered cases where $\sin\phi \neq 0$. These have covered most of the cases, yet the results have been rather limited, with only a few discrete angles permitted. Now we turn to the case where $\sin\phi = 0$. In this case, as mentioned above, we can set $\lambda = 0$ without loss of generality. Note that $\cos\phi = \pm 1$. The equations (4.35) - (4.42) now become,

$$\begin{aligned} 1 &= C_1 + C_2\cos\phi + C_3(\cos\phi\cos\psi) \\ &\quad + C_4(\cos\psi), \end{aligned} \tag{4.104}$$

$$0 = C_3(\sin\psi) + C_4(\cos\phi\sin\psi) \tag{4.105}$$

$$\begin{aligned} 0 &= S_1 + S_2\cos\phi + S_3(\cos\phi\cos\psi) \\ &\quad + S_4(\cos\psi), \end{aligned} \tag{4.106}$$

$$0 = S_3(\sin\psi) + S_4(\cos\phi\sin\psi) \tag{4.107}$$

If $\sin\psi \neq 0$, then these quickly reduce to a case nearly identical to the recently examined boundary cases, depending on the sign of $\cos\phi = \pm 1$. Namely we get the following equations,

$$\theta_1 + \cos\phi\theta_2 + \theta_3 + \cos\phi\theta_4 = 2\pi n_1, \tag{4.108}$$

$$\theta_1 + \cos\phi\theta_2 - \theta_3 - \cos\phi\theta_4 = 2\pi n_2, \tag{4.109}$$

where $n_i \in \mathbb{Z}$. These of course brake up into two cases, both of which we can solve for the angles θ_i. The results are listed in Table 4.4.

On the other hand, if $\sin\psi = 0$, then $\cos\psi = \pm 1$ and we get only the one equation,

$$\theta_1 + \theta_2\cos\phi + \theta_3\cos\phi\cos\psi + \theta_4\cos\psi = 2\pi n, \tag{4.110}$$

where $n \in \mathbb{Z}$. There are of course four cases, corresponding to the four possible choices for $\cos\psi = \pm 1, \cos\phi = \pm 1$. These break down nicely into different conditions on the angles θ_i, which can in turned into solutions for the θ_i in terms of three

$\sin\phi \neq 0, \sin\psi = \pm\sin\lambda,$

$\cos\psi = \pm\cos\lambda$ Boundary Cases

Case	Angles
$\sin\psi = \sin\lambda = 0$	$\theta_1 = \delta_2 + \delta_3 + \pi\frac{m_4}{2}$
$\cos\psi = +\cos\lambda$	$\theta_2 = \delta_2 - \delta_3 + \pi\frac{-m_4}{2}$
	$\theta_3 = -\delta_2 + \delta_3 + \pi\frac{-m_4}{2}$
	$\theta_4 = -\delta_2 - \delta_3 + \pi\frac{m_4}{2}$
$\sin\psi = \sin\lambda = 0$	$\theta_1 = \delta_1 + \delta_4 + \pi\frac{m_2}{2}$
$\cos\psi = -\cos\lambda$	$\theta_2 = \delta_1 - \delta_4 + \pi\frac{m_2}{2}$
	$\theta_3 = \delta_1 - \delta_4 + \pi\frac{-m_2}{2}$
	$\theta_4 = \delta_1 + \delta_4 + \pi\frac{-m_2}{2}$
$\sin\psi = +\sin\lambda$	$\theta_1 = \delta_1 + \delta_3 + \pi\frac{m_2}{2}$
$\cos\psi = \cos\lambda = 0$	$\theta_2 = \delta_1 - \delta_3 + \pi\frac{m_2}{2}$
	$\theta_3 = \delta_1 + \delta_3 + \pi\frac{-m_2}{2}$
	$\theta_4 = \delta_1 - \delta_3 + \pi\frac{-m_2}{2}$
$\sin\psi = -\sin\lambda$	$\theta_1 = \delta_2 + \delta_4 + \pi\frac{m_3}{2}$
$\cos\psi = \cos\lambda = 0$	$\theta_2 = \delta_2 - \delta_4\pi\frac{-m_3}{2}$
	$\theta_3 = -\delta_2 - \delta_4 + \pi\frac{m_3}{2}$
	$\theta_4 = -\delta_2 + \delta_4 + \pi\frac{-m_3}{2}$

Table 4.3: Summary of the solutions to the $\sin\phi \neq 0$, $\cos\psi = (\pm)_\alpha \cos\lambda \neq 0$, $-\sin\psi = (\pm)_\beta \sin\lambda \neq 0$ cases. Here δ is a free angle and $m_i \in \mathbb{Z}$ mod 4. Note that the form of these solutions is not unique.

free parameters. However, the symmetry is brought out more if left in the conditions form. These last remaining cases are tabulated in Table 4.4. Finally, these results yield the following theorem.

Theorem 4.3.1 *Let $\eta = y_1 \wedge y_2 \wedge y_3 \wedge y_4$ be a Cayley plane in the standard position with respect to $\mathbb{H}$, i.e. with,*

$$y_1 = 1\cos\theta_1 + \hat{e}\sin\theta_1, \tag{4.111}$$

$$y_2 = \hat{\imath}\cos\theta_2 + v_2\hat{e}\sin\theta_2, \tag{4.112}$$

$$y_3 = \hat{\jmath}\cos\theta_3 + v_3\hat{e}\sin\theta_3, \tag{4.113}$$

$$y_4 = \hat{k}\cos\theta_4 + (v_2v_3)\hat{e}\sin\theta_4, \tag{4.114}$$

$$v_2 = \hat{\imath}\cos\phi + \hat{\jmath}\sin\phi\cos\lambda + \hat{k}\sin\phi\sin\lambda, \tag{4.115}$$

$$\begin{aligned} v_3 &= \hat{\imath}(-\sin\phi\cos\psi) + \hat{\jmath}(\cos\phi\cos\lambda\cos\psi - \sin\lambda\sin\psi) \\ &\quad + \hat{k}(\cos\phi\sin\lambda\cos\psi + \cos\lambda\sin\psi), \end{aligned} \tag{4.116}$$

$$\begin{aligned} v_2v_3 &= \hat{\imath}\sin\phi\sin\psi + \hat{\jmath}(-\sin\lambda\cos\psi - \cos\phi\cos\lambda\sin\psi) \\ &\quad + \hat{k}(\cos\lambda\cos\psi - \cos\phi\sin\lambda\sin\psi). \end{aligned} \tag{4.117}$$

Then the angles $\theta_i, \phi, \psi, \lambda$ must be as designated in Tables 4.1, 4.2, 4.3, and 4.4.

$\sin\phi = 0$ Cases

Case	Angles or Conditions
$\sin\psi \neq 0$ $\cos\phi = +1$	$\theta_1 = \delta_3 + \delta_4 + \pi\frac{m_2}{2}$ $\theta_2 = -\delta_3 - \delta_4 + \pi\frac{m_2}{2}$ $\theta_3 = \delta_3 - \delta_4 + \pi\frac{-m_2}{2}$ $\theta_4 = -\delta_3 + \delta_4 + \pi\frac{-m_2}{2}$
$\sin\psi \neq 0$ $\cos\phi = -1$	$\theta_1 = \delta_1 + \delta_2 + \pi\frac{m_4}{2}$ $\theta_2 = \delta_1 + \delta_2 + \pi\frac{-m_4}{2}$ $\theta_3 = \delta_1 - \delta_2 + \pi\frac{-m_4}{2}$ $\theta_4 = \delta_1 - \delta_2 + \pi\frac{m_4}{2}$
$\sin\psi = 0$ $\cos\phi = +1, \cos\psi = +1$	$2\pi n = \theta_1 + \theta_2 + \theta_3 + \theta_4$
$\sin\psi = 0$ $\cos\phi = -1, \cos\psi = +1$	$2\pi n = \theta_1 - \theta_2 - \theta_3 + \theta_4$
$\sin\psi = 0$ $\cos\phi = +1, \cos\psi = -1$	$2\pi n = \theta_1 + \theta_2 - \theta_3 - \theta_4$
$\sin\psi = 0$ $\cos\phi = -1, \cos\psi = -1$	$2\pi n = \theta_1 - \theta_2 + \theta_3 - \theta_4$

Table 4.4: Summary of the solutions to the $\sin\phi = 0$, cases. Here δ_i is a free angle and $m_i \in \mathbb{Z}$ mod 4 and $n \in \mathbb{Z}$. Note that the form of these solutions is not unique.

Chapter 5

Reflection Groups and Polytopes

Ever since the time of the ancient Greeks, mathematicians have been fascinated with polygons and polyhedra, particularly when they possess some sort of symmetry. While many things have been known about these shapes in two and three dimensions we have really only begun to understand the higher dimensional cases in the past 150 years. Indeed there are still many questions still left unanswered to this day.

The study of polyhedra (and in higher dimensions, polytopes), begins with the study of reflection groups, as demonstared by the life work of H. S. M. Coxeter and others. The vast majority of regular polytopes have symmetry groups that are generated by reflections. Indeed one of the principal tools in studying polytopes is to study their symmetry groups. We will find that the reflection groups play a double role in the search for Cayley polytopes, and so we breifly this subject.

Recently there have been generalizations of the notion of a reflection to include complex and quaternionic reflections. Not unexpectedly, these lead to corresponding complex and quaternionic polytopes. Many of the constructions used in the real case generalize to the complex case as well. These ideas factor into our search for Cayley polytopes, but as we will see the Cayley setting helps provide a unifying framework for understanding them.

5.1 The Real Reflection Groups

Recall that a reflection in $(V, \langle , \rangle) \cong \mathbb{R}^n$ through a fixed vector v can be explicitly defined using the standard inner product.

$$\mathrm{Ref}_v(u) = u - \frac{\langle v, u \rangle}{\langle v, v \rangle}, \tag{5.1}$$

where $u \in V$. In this context v is called a *root vector* of the reflection. Clearly this map leaves invariant the subspace orthogonal to v while sending the line containing v to its antipode. Thus if G is a finite reflection group generated by such reflections as the above one, we can associate with G a collection of lines in V corresponding to all the reflections contained in G. Clearly G must permute these lines into themselves. Thus the set of unit vectors $\{\pm v_1, \pm v_2, \ldots, \pm v_m\}$ contained in these lines are permuted and hence stable under the action of G.

The above observations help motivate the following definition of a *root system*, see [19]. This definition is more general as it permits non-unit length vectors. The important thing is that for each reflection group we can construct such a root system.

Definition 5.1.1 *A root system in a finite dimensional vector space V is a finite collection of vectors Θ that satisfies the following two properties:*

$$\Theta \cap \mathbb{R}v = \{\pm v\}, \textit{ for all } v \in \Theta, \tag{5.2}$$

$$Ref_v \Theta = \Theta, \textit{ for all } v \in \Theta. \tag{5.3}$$

Moreover, starting with a root system, Θ we can associate a finite reflection group generated by reflections in the roots of Θ. Thus studying root systems in conjunction with finite reflection groups is very natural. However as Humphreys points out in [19], the problem with root systems is that they can be extremely large compared with the dimension of V, e.g. the dihedral group can have a root system in $\mathbb{R}^2$ with as many elements as the group itself.

Fortunately, although it is not at all obvious, given a root system, Θ, with associated group G, we can find a distinguished subset, $\Delta \subset \Theta$, unique up to conjugation by G, such that (1) Δ is a basis for the $\mathbb{R}$-span of Θ in V, (2) the group G associated with Θ is generated by reflections in the vectors of Δ, and (3) every $v \in \Theta$ is a linear combination of Δ with coefficients all of the same sign. Such a set is called a *simple system*. For a proof of their existence and motivation see [19].

Since it is a basis, the number of vectors in Δ is equal to the dimension of the span of Θ. This is known as the *rank* of Θ or G. Any two vectors in such a simple system will have negative inner product, ie $\langle v_1, v_2 \rangle = \cos\theta_{12} \leq 0$ for all $v_1, v_2 \in \Delta$. Let $m(v_1, v_2)$ denote the order of $\text{Ref}_{v_1}\text{Ref}_{v_2} \in G$. Recall that in two dimensions, (the span of $\{v_1, v_2\}$), the composition of two reflections through roots with angle $\theta = \pi - \phi,\ 0 \leq \phi \leq \pi/2$ results in a rotation in the plane of angle $2\theta = 2\pi - 2\phi \equiv -2\phi$. Since in our case this element has order $m(v_1, v_2)$, we have that $2\phi = 2\pi/m(v_1, v_2)$ or $\phi = \pi/m(v_1, v_2)$.

Remarkably, we can associate more objects with our growing collection, G, Θ, Δ. Given a simple system we can construct a labeled graph, Γ, (known as a *Coxeter graph*) that contains all the information of the simple system. For each root in the simple system we have a node, and between any two nodes we place a labeled edge with label $m(v_i, v_j)$ where the roots v_i, v_j are those associated with the edge. Since the simple systems are conjugate under G, these graphs are unique for each finite reflection group. Note however that not all graphs are permissible. Only certain ones will actually be realized as corresponding to finite reflection groups. See Figure 5.1. In general we omit edges with label 2 and leave edges that would be marked with the label 3 as unmarked, since these are so common.

These graphs provide us with a useful representation of the group. We can often detect many of the outer automorphisms of the group by examining the symmetries of its graph. Moreover we will see that these graphs can be used to systematically construct uniform polytopes that have the symmetry groups which

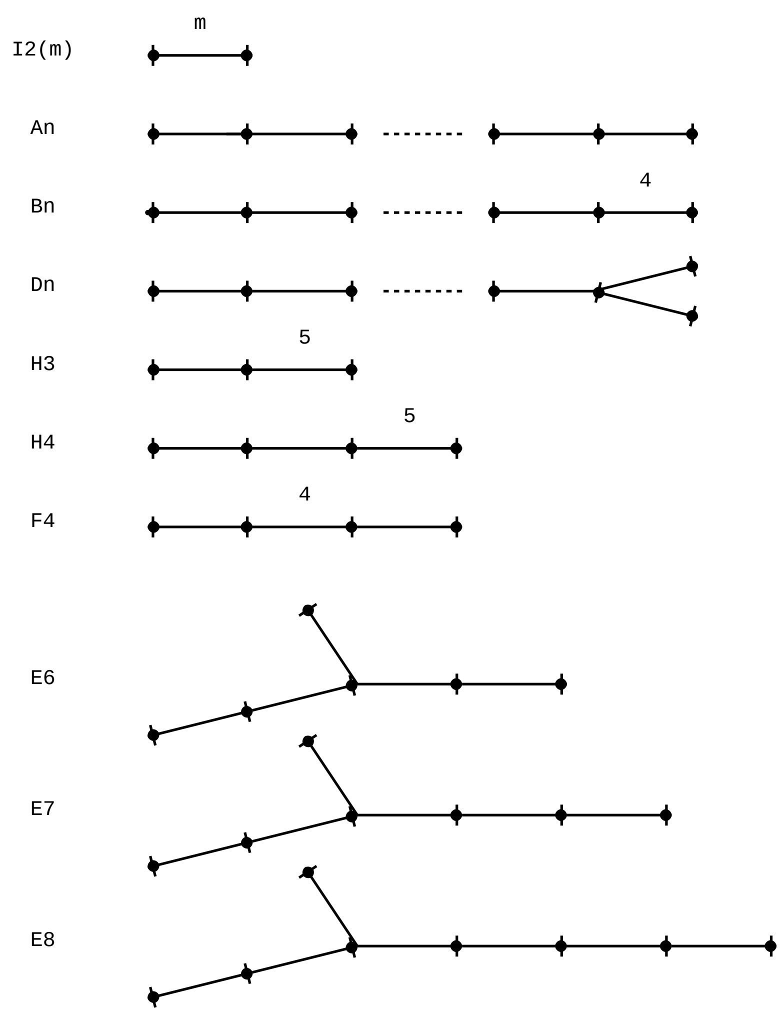


Figure 5.1: Graphs Corresponding to the Real reflection groups

are reflection groups. Then certain symmetries of the graphs correspond to certain relationships between the polytopes. This will be made more precise later, but for example, the graph for the group A_3 is symmetric with respect to its outer nodes. This is the reflection group of the tetrahedron. The symmetry of the graph corresponds to the fact that the tetrahedron is dual to itself. Most of these groups also arise as the reflection symmetries of certain lattices in different vector spaces.

Recall that a *lattice* in a vector space V is the integer span of a basis Λ for V. We call an orthogonal subgroup $G \subset O(V)$ crystallographic if it stabilizes such a lattice. One particularly interesting kind of lattice is when it is the span of a root system in V such that the reflections in the root vectors stabilize the lattice. In this case there will be an associated crystallographic reflection group which will be exactly the group corresponding to root system. However not all root systems yield lattices in this way. It can be shown that a necessary and sufficient condition is that the integers $m(v_1, v_2)$ be $2, 3, 4$ or 6, see [19]. Additionally the root vectors can have two different lengths, which are referred to as long or short root vectors. Thus sometimes for the same group, we can get different lattices. A summary of the relevant results can be found in Table 5.1. Notice that the group B_n has two associated lattices. The first is referred to as the B_n lattice, while the second is the C_n lattice.

The sporadic groups E_n have a very nice lattice representation in terms of the Cayley numbers. If we identify $\mathbb{R}^8 \cong \mathbb{O}$ then the following Cayley numbers are the

Group	A_n	B_n	C_n (B_n)
Order	$(n+1)!$	$2^n n!$	$2^n n!$
Space	$\mathbb{R}^{n+1}$, $\sum$ coords $= 0$	$\mathbb{R}^n$	$\mathbb{R}^n$
Long Roots	$\epsilon_i - \epsilon_j$,	$\pm\epsilon_i \pm \epsilon_j$	$\pm 2\epsilon_i$
Short Roots	$1 \leq i \neq j \leq n+1$	$\pm\epsilon_i$	$\pm\epsilon_i \pm \epsilon_j$
Num. Roots Long : Short	$n(n+1)$	$2n(n-1) : 2n$	$2n : 2n(n-1)$

Group	F_4	D_n
Order	1152	$2^{n-1}n!$
Space	$\mathbb{R}^4$	$\mathbb{R}^n$
Long Roots	$\pm\epsilon_i \pm \epsilon_j$	$\pm\epsilon_i \pm \epsilon_j$
Short Roots	$\pm\epsilon_i,\ \frac{1}{2}(\pm\epsilon_i \pm \epsilon_j \pm \epsilon_k \pm \epsilon_l)$	
Num. Roots Long : Short	$24 : 24$	$2n(n-1)$

Group	E_8	E_6	E_7
Order	$696,729,600$ $= 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	$2,903,040$ $= 2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$51,840$ $= 2^7 \cdot 3^4 \cdot 5$
Space	$\mathbb{R}^8 \cong \mathbb{O}$	$\mathbb{R}^7 \cong \operatorname{Im}\mathbb{O}$	$\mathbb{R}^6 \cong \{v \in \operatorname{Im}\mathbb{O} \mid v \perp \hat{i}\}$
Long Roots Short Roots	See Below	See Below	See Below
Num. Roots Long : Short	240	126	72

Table 5.1: The Finite Crystallographic Groups and their Root Lattices

240 root vectors of the E_8 Lattice.

$$\begin{gathered}
\pm 1, \pm\hat{i}, \pm\hat{j}, \pm\hat{k}, \pm\hat{e}, \pm\hat{ie}, \pm\hat{je}, \pm\hat{ke}, \\
\frac{1}{2}(\pm 1 \pm \hat{j} \pm \hat{k} \pm \hat{ie}), \frac{1}{2}(\pm\hat{i} \pm \hat{e} \pm \hat{je} \pm \hat{ke}), \\
\frac{1}{2}(\pm 1 \pm \hat{k} \pm \hat{i} \pm \hat{je}), \frac{1}{2}(\pm\hat{j} \pm \hat{e} \pm \hat{ke} \pm \hat{ie}), \\
\frac{1}{2}(\pm 1 \pm \hat{i} \pm \hat{j} \pm \hat{ke}), \frac{1}{2}(\pm\hat{k} \pm \hat{e} \pm \hat{ie} \pm \hat{je}), \\
\frac{1}{2}(\pm 1 \pm \hat{ie} \pm \hat{je} \pm \hat{ke}), \frac{1}{2}(\pm\hat{i} \pm \hat{j} \pm \hat{k} \pm \hat{e}), \\
\frac{1}{2}(\pm 1 \pm \hat{i} \pm \hat{e} \pm \hat{ie}), \frac{1}{2}(\pm\hat{j} \pm \hat{k} \pm \hat{je} \pm \hat{ke}), \\
\frac{1}{2}(\pm 1 \pm \hat{j} \pm \hat{e} \pm \hat{je}), \frac{1}{2}(\pm\hat{k} \pm \hat{i} \pm \hat{ke} \pm \hat{ie}), \\
\frac{1}{2}(\pm 1 \pm \hat{k} \pm \hat{e} \pm \hat{ke}), \frac{1}{2}(\pm\hat{i} \pm \hat{j} \pm \hat{ie} \pm \hat{je}).
\end{gathered}$$

By taking just the imaginary (i.e. orthogonal to $1 \in \mathbb{O}$) portion we recover an E_7 lattice. Taking the portion of this orthogonal to $\hat{i}$ yields an E_6 lattice. Remarkably the E_8 lattice is closed under Cayley multiplication (see [6], [9]).

5.2 Real Regular Polytopes

Abstractly, we can view a polytope in $V = \mathbb{R}^n$ as a collection of vertices Π_0, edges Π_1, faces Π_2, ..., and $n-1$-cells Π_{n-1}. We require that these collections satisfy certain requirements to be consistent with out notion of a polytope, e.g. each vertex belongs to an edge. Such a definition is very general and includes all manner of strange and unusual polytopes. In order to get a handle on this we will have to impose certain restrictions. In particular we will concentrate on *regular* polytopes.

Definition 5.2.1 *A polytope* $(\Pi_0, \ldots, \Pi_{n-1})$ *is uniform if all its subpolytopes are regular as well, and there is a subgroup of* $O(n)$ *that is transitive on the vertices of the polytope. This subgroup will be known as the symmetry group. If the symmetry group is additionally transitive on the m-cells of the polytope, then the polytope is regular.*

In fact this definition is too general as well. In low dimensions all uniform polytopes have symmetries that can be realized as reflection groups. However in higher dimensions there are polytopes that have symmetries that are not reflection groups. See [9].

Nevertheless the polytopes that have symmetries that are reflection groups are well known and have a nice construction that makes use of the graph of the symmetry group. Consider the group B_3. This group is generated by three reflections in three "mirrors", $\mathrm{Ref}_{(1,-1,0)},\ \mathrm{Ref}_{(0,1,-1)},\ \mathrm{Ref}_{(0,0,1)}$. See Figure 5.2. Notice that that origin lies on the intersection of all three mirrors, and that hence this point is fixed by the entire reflection group. However if we instead consider a point that lies in the first two mirrors but off the third mirror, then the situation is more rich. Let's designate this by putting a circle around the third node as in 5.2. First of all the orbit of this point consists of the eight vertices of the cube. Also, the isotropy group of this point is the group $I_2(3)$, as can easily be seen by removing the circled point from the graph. If we look at subgraphs that contain this point we find two: $I_2(4)$ and $A_1 \times A_1$. Using this same method it is easy to see that the first corresponds to a square while the second a line segment. These are the various types of subpolytopes that are contained in this polytope. If instead we had decide to start with a vertex on the last two mirrors, but off the firsts, it is straight forward to check that the resulting polytope is the octahedron, the dual of the cube. Moreover all of the same observations carry over, the interesting subgraph with the circled point now being $I_2(3)$ corresponding to the triangular faces.

The above construction can easily be generalized in several ways. We can choose an interior point or more then one point (provided it is equidistant from the circled mirrors). Moreover all these sorts of constructions result in regular polytopes of one kind or another. This is known as Wythoff's construction and is described in [7], [9], [6], [8]. When the circled vertex is an outer vertex we can determine the type of faces that the polytope will have by removing one of the

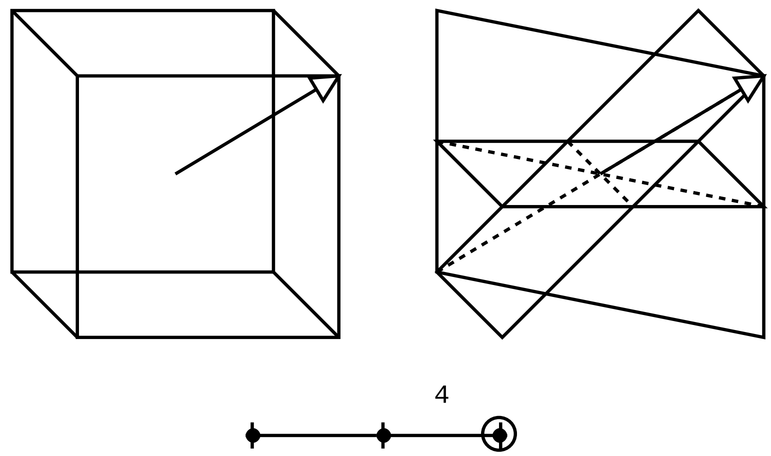


Figure 5.2: The Mirrors of the Reflection Group B_3

other outer vertices. It is clear that if the graph is linear, then there will only be one remaining outer vertex. Thus the polytope will have only one type of "face". Moreover it can be shown that in the linear case the symmetry group is transitive on these faces. By induction this means that only linear graphs result in regular polytopes. In low dimensions the only graphs are linear so there is only one possibility; in higher dimensions there are other types of reflection groups and hence the difference between uniform and regular polytopes is necessary. It can further be shown that any regular real polytope admits a reflection group for symmetry group, which is necessarily linear, see [8]. Thus we recover the classic result:

Theorem 5.2.1 *The only regular polytopes in dimension* ≥ 5 *are the self-dual regular n-simplex, the n-cube and its dual the n-cross polytope, corresponding to the reflection groups* A_n *and* B_n *respectively.*

When the circled point is not an exterior point or when there are more then one circled point the result is a star- or truncated- or snub-polytope. See Figure 5.3. Also if all the points are circled then there a different polytope will result if only the index 2 rotation subgroup is used. This can be designated by using all white

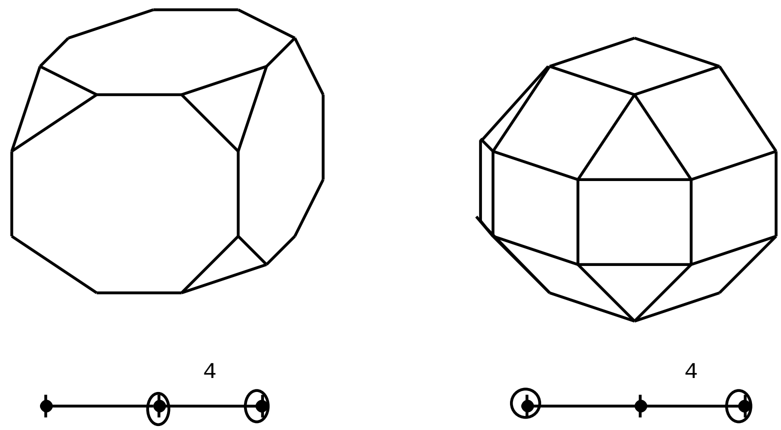


Figure 5.3: Some exotic regular polytopes with symmetry B_3.

dots. For the most part we will be focusing on the polytopes where only a single outer vertex is circled. These are tabulated in Table 5.2.

This construction helps us put the polyhedral dualities that we are familiar with in a broader context. Recall that every regular polygon and regular polyhedra has a dual which can be obtained, in the polyhedral case, by putting a vertex in the center of each face and connecting each of these vertices where their faces share and edge. Since the faces are regular, the resulting polyhedra will also be regular. The new faces thus correspond to the old vertices, with symmetry group corresponding to the vertices' isotropy group. In this way we see that for a polytope resulting from an external node of a linear graph, the dual polytope results from the same graph with other external vertex circled.

In higher dimensions the story can be more complicated. The graphs of the groups D_n, E_6, E_7, and E_8 all contain three external vertices. With these groups the corresponding polytopes undergo a kind of triality similar to the duality of the linear case, see [9].

5.3 Complex Reflection Groups and Polytopes

The notion of reflection can be generalized to complex vector spaces as well. Many of the properties of real reflections will be carried over, but some new ones will emerge. First of all we can define a complex line to be the $\mathbb{C}$-span of a fixed vector, $v \in \mathbb{C}^n$. We require that any complex reflection fix the orthogonal complement of its reflection line. Moreover we require the reflection to be unitary, so that the standard hermitian inner-product is preserved. This allows more freedom than in the real case. Complex reflections differ from real reflections in that now we have a phase factor that can be any root of unity. Requiring the reflections to have finite order leads to the following definition:

Definition 5.3.1 *A complex reflection of order p through the complex vector v is the unitary map defined as follows,*

$$Ref_{\mathbb{C},v,p}(u) = u + (e^{i\frac{2\pi}{p}} - 1)v\frac{\langle u, v\rangle}{\langle v, v\rangle}, \tag{5.4}$$

where $\langle , \rangle$ is the standard hermitian inner product.

With this definition many results from the real case carry over. In particular, we can associate a labeled graph with each reflection group. The only modification is that now each reflection in a given root vector may have an order not equal to two. To convey this information we label each node of the graph with a number, p, omitting the label 2 since it is so common.

The definition of a complex-polytope is modeled after the real polytopes. Instead of edges which are segments of real-lines, in a complex polytope the "edges" are "segments" of complex lines. Hence they will be real polygons. Despite this complication, much of the machinery used in the case of real polytopes carries over to the case of complex polytopes. The regular complex polytopes are classified in [25] and [8]. Wythoff's construction carries over in the natural way, and is very useful in Coxeter's classification.

Many interesting phenomena regarding complex reflection groups and polyhedra can be explained in the context of $\mathrm{Spin}_6 \cong SU(4)$, however, it will be unnecessary to delve into the details. In particular, there are many more complex reflection groups than in the real case, and their subgroups are more complicated. The quaternionic analog, described below, conveys the same insight, yet offers a more reasonable number of examples. Moreover the quaternionic case more readily lends itself to useful generalizations in the Cayley manifold case.

5.4 Quaternionic Reflection Groups and Polytopes

Just as we could extend the real reflections to complex reflection, we can also extend this to quaternionic reflections. Moreover, we can similarly define quaternionic polytopes. This will be helpful as it will provide a model for generalization to the Cayley case. Recall that a quaternionic vector space can be given an associated inner product $\langle x, y\rangle = \sum x_i \overline{y}_i$, that is preserved by left unit quaternion multiplication, and hence corresponds to a right quaternion structure. We can define a quaternionic reflection that preserves the above inner product by,

$$\mathrm{Ref}_{\mathbb{H},v,\xi}(u) = u + (\xi - 1)v\frac{\langle u, v\rangle}{\langle v, v\rangle}, \tag{5.5}$$

where $\xi \in \mathbb{H}$ is a unit quaternion of finite order, usually belonging to a particular finite subgroup of $HU(1)$. Notice that analogous to the complex case, this leaves the orthogonal compliment of the right-quaternion span of v fixed, pointwise.

In 1978 Cohen, [5], classified the finite quaternionic reflection groups. His classification involved splitting the various groups up into several cases, and employing some deep group theoretical results. Moreover his classification depends on the classification of complex reflection groups. In particular he examines the irreducible quaternionic reflection groups according to the separate categories: imprimitive, primitive with imprimitive complexification, and primitive with primitive complexification. In two dimensions these yield 22, 4, and 6 quaternionic reflection groups respectively. The first category yields groups of the following form:

Definition 5.4.1 *The quaternionic reflection group $G(K, H, K/H, \alpha)$, originally defined in [5], where $K, H \in HU(1)$ are finite with $H \trianglelefteq K$ and $\alpha \in Aut(K/H)$ of order less then or equal to 2, is the following set of elements in $HU(2)$,*

$$\cup_{m=1,2} \cup_{x \in K} \begin{pmatrix} xH & 0 \\ 0 & \alpha(xH) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^m . \tag{5.6}$$

The last category relies on analyzing particular root systems and associating groups with them. Cohen defines six root systems $O_1, O_2, O_3, P_1, P_2, P_3$ in $\mathbb{H}^2$ with their associated groups $W(\Sigma)$.

In 1995 Cuypers, [11], expanded on this to produce a classification of the regular quaternionic polytopes. Just as with the real and complex case, the symmetry groups of regular quaternionic polytopes can be associated with quaternionic reflection groups, which in turn can be associated with a linear graph. This is novel because unlike the real case, the classification of quaternionic reflection groups did not make use of graphs. It was conjectured by Cuypers the they could be used to provide a more direct classification, [11]. Nevertheless, Cuypers was able to generalize the associated graphs to the particular cases needed for classifying quaternionic polytopes. The only difference between the complex graphs and his graphs originates in the fact that for $S^1 = U(1)$ a single number p is sufficient for uniquely determining a finite subgroup, whereas in $S^3 = HU(1)$ it is better to label the nodes of the graph by the desired group. The non-starry purely quaternionic polytopes in $\mathbb{H}^2$ are listed in Table 5.4. There are also non-pure polytopes which are quaternionic extensions of 2-dimensional real or complex polytopes.

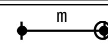
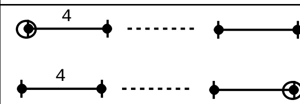
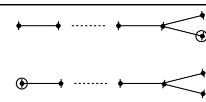
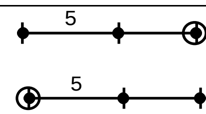
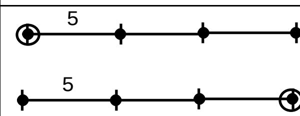
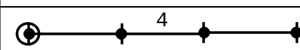
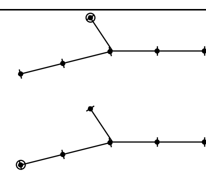
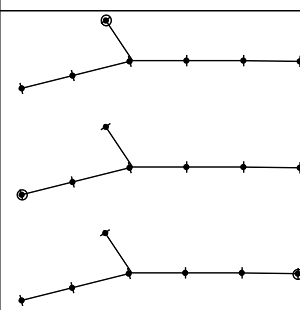
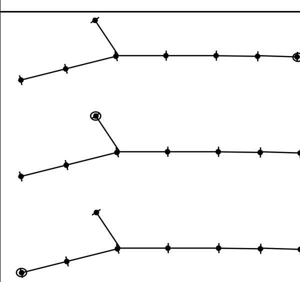
Reflection Group	Polytope Name and Symbol	Graph
$I_2(m)$	m-gon	
A_n	Simplex, α_n	
B_n	n-Cube, γ_n n-Cross Polytope, β_n	
D_n	Half-Measure Polytope, $h\gamma_n$ β_n	
H_3	Icosahedron Dodecahedron	
H_4	120-Cell 600-Cell	
F_4	24-Cell	
E_6	1_{22} 2_{21}	
E_7	1_{32} 2_{31} 3_{21}	
E_8	4_{21} 1_{42} 2_{41}	

Table 5.2: The uniform polytopes with associated reflection groups

Polytope	Ref. Group	Num. Vertices	Num. $\mathbb{H}$-Edges
$\mathcal{O}_1 = \overline{\mathcal{O}_1}$	$W(O_1)$	40	40
$\mathcal{P}_1 = \overline{\mathcal{P}_1}$	$W(P_1)$	80	80
$\mathcal{P}_2 = \overline{\mathcal{P}_2}$	$W(P_2)$	240	240
$\mathcal{P}_3,\ (\overline{\mathcal{P}_3})$	$W(P_3)$	$1920,\ (480)$	$480,\ (1920)$
$\mathcal{C}_H,\ (\overline{\mathcal{C}_H})$	$G(H,H,1,1)$	$2\lvert H\rvert,\ (\lvert H\rvert^2)$	$\lvert H\rvert^2,\ (2\lvert H\rvert)$

Table 5.3: The Quaternionic Polygons and Corresponding Reflection Groups

Chapter 6

Constructing Cayley Honeycombs

We are interested in construction lattice periodic arrangements of Cayley planes such that the union of these arrangements remain volume minimizing. One way to approach this is to realize that each of these Cayley Honeycombs will have an associated finite subgroup of Spin_7, which acts transitively on one of the cells of the Honeycomb. Thus understanding the finite subgroups of Spin_7 helps us understand the Cayley Honeycombs.

Remarkably, there is a very natural way in which the relationship of the Spin and Pin groups to the orthogonal and special orthogonal groups allows us to use the classification of reflection groups to make this understanding more clear. Moreover, this context helps provide a cohesive framework in which to study the Unitary and Quaternionic reflection groups and polytopes. Overall this will prove to be a very productive journey.

6.1 Constructing Finite Spin Groups

In odd dimension, there is an element of the orthogonal group which is singled out, namely $-I$. This is the unique element not contained in this special orthogonal subgroup, which commutes with all other elements of the orthogonal group. With the identity it forms a $\mathbf{Z}_2$ subgroup which is the center of $O(n)$. Moreover, every element of $O(n)\backslash SO(n)$ can be uniquely written as $-I\cdot A = A\cdot -I$ where $A \in SO(n)$. We can define a homomorphism $\phi : O(n) \to SO(n)$ which simply sends $-I$ to the identity.

Continuing in the odd dimensional case, if $G \subset O(n)$ is a subgroup, then we can uniquely partition G into two sets. Let G^+ denote the index 2 pure subgroup of G, i.e. $G^+ = G \cap SO(n)$. Then the remaining portion of G can be written as $-\tilde{G}^+$ where $\tilde{G}$ is a subset of $SO(n)$. The map $\phi : G = G^+ \cup -\tilde{G}^+ \to G^+ \cup \tilde{G}^+$. It may be the case that G contains $-I$. In this case $G^+ = \tilde{G}^+$, and the map ϕ is a $2:1$ homomorphism. Otherwise it is an isomorphism between G and its image.

In even dimensions we do not have this luxury. However, we can still make some headway. Each subgroup $G \subset O(n)$ can be embedded in $SO(n+1)$ in a very natural way. Namely $g \mapsto (\pm 1, g)$ where the sign on the 1 is determined by the determinant of $g \in O(n)$. Such an embedding is isomorphic to the original group, and will be an index 2 subgroup of $A_1 \times G$. We will denote this group as $(G)^+_{n+1}$, e.g. the dihedral group $I_2(m)$ can be embedded in $SO(3)$ and is denoted $(I_2(m))^+_3$.

In odd dimensions the standard volume element for $Cl(n)$, λ, is odd and commutes with the entire algebra. Hence $\tilde{\lambda} = -\lambda$ and we have $\tilde{Ad}_\lambda(x) = \tilde{\lambda} x \lambda = -\lambda^2 x$. In the dimensions we are interested in, namely 3 and 7 we additionally have $\lambda^2 = 1$ so that the $\pm\lambda$ map to the unique $-I \in O(3 \text{ or } 7)$. Thus we can lift the map ϕ to the level of Spin and Pin groups. Namely, since $\mathbf{Z}_2 = \{1, \lambda\}$ is normal in $\text{Pin}_{7 \text{ or } 3}$ we have that $\phi : \text{Pin}_{3 \text{ or } 7} \to \text{Spin}_{3 \text{ or } 7}$ mapping λ to the identity is a well defined homomorphism and that the following diagram commutes,

$$\begin{array}{ccc} \text{Pin}_{7 \text{ or } 3} & \xrightarrow{\tilde{Ad}} & O(7 \text{ or } 3) \\ \downarrow \phi & & \downarrow \phi \\ \text{Spin}_{7 \text{ or } 3} & \xrightarrow{\tilde{Ad}} & SO(7 \text{ or } 3). \end{array}$$

In dimension five we cannot lift the map to the Pin level so easily, nonetheless, we can still use ϕ to get groups in $SO(5)$ and then lift those at the Spin level.

It is useful to examine the specific case of dimension seven in more detail. Recall the model that the Cayley numbers provide for studying $Cl(7)$. Spin_7 is generated by right Cayley multiplication, the positive and negative representations being equal while λ has a simple representation,

$$\text{Spin}_7 \text{ generated by } \begin{pmatrix} R_u & 0 \\ 0 & R_u \end{pmatrix}, \ u \in \text{Im}\,\mathbb{O}; \qquad \lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.1}$$

Consider one of the generators of Spin_7. $\chi_{R_u}(v) = R_u(R_{\overline{u}}(1) \cdot v) = -(uv)u$. Since any two elements generate at most a subalgebra isomorphic to $\mathbb{H}$, we have that

$$\chi_{R_u}(v) = \begin{Bmatrix} -v, & v \perp u \\ v, & \text{otherwise} \end{Bmatrix}. \tag{6.2}$$

In other words, $\chi_{R_u} = -Ref_u$. Consider now the case where we start with a reflection group in $O(7)$ that contains $-I$. Such a group is generated by a number of reflections along vectors $u_1, \ldots, u_7$ in Im $\mathbb{O}$. It's double cover in Pin_7 will then be generated by elements of the form $\lambda R_{u_1}, \ldots, \lambda R_{u_7}$. Since the original group contained $-I$ the Pin_7 group will contain λ. Thus its projection under ϕ into Spin_7 will be generated by $R_{u_1}, \ldots, R_{u_7}$, and we have obtained a finite Spin group from one of the well know reflection groups. The following well known example will make this clear.

6.2 An Instructive Example, Spin_3

It is useful to examine the three dimensional case in some detail as it is well understood and provides a model for the more complicated seven dimensional case. Using the ideas outlined above we should be able to move from the three dimensional finite reflection groups to finite subgroups of $\text{Spin}_3 \cong HU(1)$. Fortunately these are both well known and give us a suitable context in which to evaluate this method.

There are a variety of finite reflection groups in three dimensions. The irreducible ones are A_3, B_3, and H_3 corresponding to the reflection symmetry groups of the tetrahedron, cube or octahedron, and icosahedron or dodecahedron respectively. Of these groups, A_3 and B_3 are crystallographic, while H_3 is not, and both B_3 and H_3 contain $-I_3$ while A_3 does not. The orders of these groups are $|A_3| = 4! = 24$, $|B_3| = 2^3 3! = 48$, and $|H_3| = 120$. As we will see, it is convenient to know the number of roots of the following four dimensional groups, D_4 with 24 roots, F_4 with 48 roots, and H_4 with 120 roots.

From our previous discussion, we can immediately generate two groups in Spin_3 based on the groups B_3 and H_3. These groups are formed by taking the pre-image in Pin_3 and projecting down into Spin_3 by ϕ. Since the original groups contain $-I_3$, the resulting groups in Spin_3 will have the same order as these original groups, namely 48 and 120. We can find an explicit construction of these groups by looking at where the generators of the original groups are sent in Spin_3.

Choosing an arbitrary, but useful, representation of B_3, we find that it contains reflections in the following unit vectors: (using the $\hat{\imath}, \hat{\jmath}, \hat{k}$ basis)

$$\pm\hat{\imath}, \pm\hat{\jmath}, \pm\hat{k}, \frac{1}{\sqrt{2}}(\pm\hat{\imath} \pm \hat{\jmath}), \frac{1}{\sqrt{2}}(\pm\hat{\imath} \pm \hat{k}), \frac{1}{\sqrt{2}}(\pm\hat{\jmath} \pm \hat{k}).$$

These reflections map to elements of Spin_3 corresponding to right quaternion multiplication by that imaginary unit vector. Since the non-zero quaternions form a group, we can view right quaternion multiplication as isomorphic to the quaternions themselves ($R_u \leftrightarrow \overline{u}$ since we are using *right* quaternion multiplication instead of left). Thus we see that we generate the additional elements,

$$\pm 1, \frac{1}{\sqrt{2}}(\pm 1 \pm \hat{\imath}), \frac{1}{\sqrt{2}}(\pm 1 \pm \hat{\jmath}), \frac{1}{\sqrt{2}}(\pm 1 \pm \hat{k}), \frac{1}{2}(\pm 1 \pm \hat{\imath} \pm \hat{\jmath} \pm \hat{k}).$$

It is easy to check that these elements are the root vectors (renormalized to unit length) for the exceptional lattice F_4. They form a subgroup of $\mathbb{H}$ known as the binary-octahedral group (it is easy to check that this group is the Spin double cover of the rotation symmetry group of a regular octahedron). Moreover it is easy to check that the unnormalized lattice is a subalgebra of $\mathbb{H}$.

A similar analysis on H_3 shows that it corresponds to a group in $\mathbb{H}$ known as the binary-icosahedral group (which is the double cover of the icosahedron's rotation symmetry group). The elements of this group likewise correspond to a root system in $\mathbb{H} \cong \mathbb{R}^4$ of order 120. It is easy to verify that this is the root system for the group H_4, see [13] and [17]. Unfortunately, this root system is not crystallographic and so does not suit our needs. Nevertheless this correspondence is illuminating and helps show the relationship of the icosahedral symmetries to the structure of the H_4 lattice. Let $\tau = (\sqrt{5}+1)/2 = 2\cos(\pi/5)$, the golden ratio. Then the group corresponding to H_3 has the following elements,

$$\pm 1, \pm\hat{\imath}, \pm\hat{\jmath}, \pm\hat{k}, \frac{1}{2}(\pm 1 \pm \hat{\imath} \pm \hat{\jmath} \pm \hat{k}), \text{ all even permutations of } \frac{1}{2}(\pm\tau \pm \hat{\imath} \pm \tau^{-1}\hat{\jmath}).$$

Even more illuminating observations can be made if we return to the F_4 lattice above. It is easy to verify based on the root lattice constructions that for all $n \geq 4$ D_n is a subgroup of B_n, hence, even though D_7 doesn't contain $-I_7$, we can find a group corresponding to it as a subgroup of the group corresponding to B_7. The group A_n is more tricky. We know that A_{n-1} is a subgroup of B_n. However examining the orders of $|A_n| = (n+1)!$ and $|B_n| = 2^n n!$ we see that the order of A_n only divides the order of B_n when $n = 3, 7, \ldots, 2^m - 1$. It is remarkable that in dimension 3 A_3 is indeed realized as a subgroup of B_3. It is worth discussing this in more detail.

Let A_3^+ be the index 2 pure subgroup of A_3, namely the tetrahedron's rotation symmetry group. Let $A_3^c = A_3^+ \cup -A_3^+ \neq A_3$ be the group generated by A_3^+ and the central inversion. Similarly let B_3^+ be the index 2 pure subgroup of B_3. Then it can be shown that A_3^+ forms an index 4 subgroup of B_3. We can thus construct three additional subgroups of B_3 which contain A_3^+, namely B_3^+, A_3, and A_3^c. See Du Val, [13], for an extensive discussion of the finite three dimensional orthogonal groups. Of these three new subgroups, only A_3^c contains the central reflection. Applying our standard process to this group results in a Spin group that is the double cover of A_3^+, known as the binary-tetrahedral group (just as this process yields double covers of B_3^+ and H_3^+). Following Du Val's discussion we know that the maximal

Group	$SO(3)$ Image
$\mathcal{C}_n$	$(I_2^+(m))_3^+$
$\mathcal{D}_n$	$(I_2(m))_3^+$
$\mathcal{T}$	A_3^+
$\mathcal{O}$	$B_3^+ = \phi(A_3)$
$\mathcal{I}$	H_3^+

Table 6.1: The finite Spin_3 subgroups and their associated $SO(3)$ images

common subgroup of H_3 and B_3 is the group A_3^c. Hence, to find its corresponding group in $\mathbb{H}$, we only need to intersect the two groups corresponding to B_3 and H_3. The resulting group has the following 24 elements,

$$\pm 1, \pm \hat{i}, \pm \hat{j}, \pm \hat{k}, \frac{1}{2}(\pm 1 \pm \hat{i} \pm \hat{j} \pm \hat{k}).$$

This root lattice corresponds to the lattice of D_4. Lastly, $\phi(A_3) = B_3^+$ so that no new group is achieved this way.

It is remarkable that the structure of the F_4 and D_4 lattices, and the H_4 root system as well as their relationships and group structure when viewed as subsets of the quaternions can be explained in terms of the simpler groups B_3, A_3, and H_3 and through the Spin and Pin maps.

There are a few more examples of subgroups of Spin_3 which we will include for completeness. Consider the two dimensional dihedral reflection groups $I_2(m)$ and their even cyclic subgroups $I_2^+(m)$. From the discussion above we can isomorphically embed the dihedral group into $SO(3)$ and then lift this group into Spin_3. It will of course have an associated index 2 subgroup which is the lift of the cyclic subgroup. All of these results are listed in Table 6.2.

Lemma 6.2.1 *Any closed finite set of units of any subalgebra (resp. subgroup) G of even order in* $\mathbb{O}$ *(resp.* $\mathbb{H}$*) is a root system when regarded as a subset of* $\mathbb{R}^8$ *(resp.* $\mathbb{R}^4$*).*

Proof: This is modified from a similar lemma in [17] which deals with only the quaternion case. Since both $\mathbb{O}$ and $\mathbb{H}$ are normed algebras every such element must have unit norm (since the norm maps the invertible elements of $\mathbb{O}$ and $\mathbb{H}$ homomorphically into $\mathbb{R}^+$ whose only finite group is the identity). Now, since G is closed under inverses, G is also closed under conjugation. It is well known that the only element of order 2 in $\mathbb{H}$ is -1. Because of Theorem 1.1.1, the only element of order 2 in $\mathbb{O}$ is also -1. Since every such set of even order contains an element of order 2, G must contain -1. Thus if $u \in G$, then $-u \in G$. Recall that in $\mathbb{H}$ reflection through $u \in HU(1)$ can be written $v \mapsto -u\bar{v}u$. Again due Theorem 1.1.1, reflection through unit $u \in \mathbb{O}$ can also be written $v \mapsto -u\bar{v}u$. Thus since G is closed under multiplication and conjugation, G is closed under reflection through its elements. Hence G satisfies the requirements of a root system.

In the quaternion case it was very useful that the invertible elements of $\mathbb{H}$ form a group. This caused right quaternion multiplication by u to be identified with $\bar{u} \in \mathbb{H}$. In the Cayley numbers, we don't have this luxury. Consequently elements such as $R_{\hat{i}}R_{\hat{j}}$ and $R_{\hat{k}}$ are not equal in Spin_7, although they both correspond to the element $\hat{k} \in \mathbb{O}$. Thus we cannot use the trick of identifying the group order and the number of root vectors in the corresponding higher dimensional lattice.

6.3 The Higher Dimensional Cases

The octonion model for Spin_7 also provides a useful model for understanding finite subgroups of the lower dimensional cases of $\text{Spin}_6 \cong SU(4)$ and $\text{Spin}_5 \cong HU(2)$ as well. Recall that these spin groups have a particularly nice representation using the map χ. $\text{Spin}_6 = \{g \in \text{Spin}_7 \mid \chi_g(\hat{i}) = \hat{i}\}$ so that the complex structure $R_{\hat{i}}$ is preserved and $\text{Spin}_5 = \{g \in \text{Spin}_6 \mid \chi_g(\hat{j}) = \hat{j}\}$ so that the complex structures $R_{\hat{i}}$,

$R_{\hat{j}}$ and $R_{\hat{i}}R_{\hat{j}}$ are preserved. Thus to understand some of the finite subgroups of these groups, we can employ our techniques from Spin_7 to reflection groups that leave various elements of $\mathbb{R}^7 \cong \mathrm{Im}\,\mathbb{O}$ invariant.

Beginning with the irreducible seven dimensional reflection groups A_7, D_7, B_7, and E_7, we see that only B_7 and E_7 contain the central inversion $-I_7$.

The root system of B_7 is as follows,

$$\begin{aligned}
&\pm\hat{i}, \pm\hat{j}, \pm\hat{k}, \pm\hat{e}, \pm\hat{ie}, \pm\hat{je}, \pm\hat{ke},\\
\frac{1}{\sqrt{2}}(\pm\hat{i}\pm\hat{j}), \frac{1}{\sqrt{2}}(\pm\hat{i}\pm\hat{k}), &\frac{1}{\sqrt{2}}(\pm\hat{i}\pm\hat{e}), \frac{1}{\sqrt{2}}(\pm\hat{i}\pm\hat{ie}), \frac{1}{\sqrt{2}}(\pm\hat{i}\pm\hat{je}), \frac{1}{\sqrt{2}}(\pm\hat{i}\pm\hat{ke}),\\
&\frac{1}{\sqrt{2}}(\pm\hat{j}\pm\hat{k}), \frac{1}{\sqrt{2}}(\pm\hat{j}\pm\hat{e}), \frac{1}{\sqrt{2}}(\pm\hat{j}\pm\hat{ie}), \frac{1}{\sqrt{2}}(\pm\hat{j}\pm\hat{je}), \frac{1}{\sqrt{2}}(\pm\hat{j}\pm\hat{ke}),\\
&\frac{1}{\sqrt{2}}(\pm\hat{k}\pm\hat{e}), \frac{1}{\sqrt{2}}(\pm\hat{k}\pm\hat{ie}), \frac{1}{\sqrt{2}}(\pm\hat{k}\pm\hat{je}), \frac{1}{\sqrt{2}}(\pm\hat{k}\pm\hat{ke}),\\
&\frac{1}{\sqrt{2}}(\pm\hat{e}\pm\hat{ie}), \frac{1}{\sqrt{2}}(\pm\hat{e}\pm\hat{je}), \frac{1}{\sqrt{2}}(\pm\hat{e}\pm\hat{ke}),\\
&\frac{1}{\sqrt{2}}(\pm\hat{ie}\pm\hat{je}), \frac{1}{\sqrt{2}}(\pm\hat{ie}\pm\hat{ke}),\\
&\frac{1}{\sqrt{2}}(\pm\hat{je}\pm\hat{ke}).
\end{aligned}$$

The generators of the group corresponding to B_7 (i.e. the double cover of B_7^+) are right Cayley multiplication by these root elements. A simple examination allows us to see that the orbit of $1 \in \mathbb{O}$ will contain, in addition to the above elements, ones of the form

$$\pm 1, \frac{1}{\sqrt{2}}(\pm 1 \pm \varepsilon_i), \frac{1}{2}(\pm\varepsilon_i \pm \varepsilon_j \pm \varepsilon_k \pm \varepsilon_l), \tag{6.3}$$

$$\frac{1}{2\sqrt{2}}(\pm 1 \pm \hat{i} \pm \hat{j} \pm \hat{k} \pm \hat{e} \pm \hat{ie} \pm \hat{je} \pm \hat{ke}), \text{ etc.} \tag{6.4}$$

After careful inspection, however, we see that if the "long roots" of B_7 are left with square norm 2, then we simply recover the lattice belonging to B_8 and moreover that this lattice is closed under Cayley multiplication. Alternately we could use the lattice of C_8, and recover a similar result.

Along similar lines we can consider the exceptional reflection group $E_7 = E_7^+ \cup -E_7^+$. This also results in an eight-dimensional lattice that is closed under Cayley multiplication. This is, up to conjugation, the E_8 lattice with the following 240 elements,

$$
\begin{array}{c}
\pm 1, \pm\hat{i}, \pm\hat{j}, \pm\hat{k}, \pm\hat{e}, \pm\hat{ie}, \pm\hat{je}, \pm\hat{ke}, \\
\frac{1}{2}(\pm 1 \pm \hat{j} \pm \hat{k} \pm \hat{ie}), \frac{1}{2}(\pm\hat{i} \pm \hat{e} \pm \hat{je} \pm \hat{ke}), \\
\frac{1}{2}(\pm 1 \pm \hat{k} \pm \hat{i} \pm \hat{je}), \frac{1}{2}(\pm\hat{j} \pm \hat{e} \pm \hat{ke} \pm \hat{ie}), \\
\frac{1}{2}(\pm 1 \pm \hat{i} \pm \hat{j} \pm \hat{ke}), \frac{1}{2}(\pm\hat{k} \pm \hat{e} \pm \hat{ie} \pm \hat{je}), \\
\frac{1}{2}(\pm 1 \pm \hat{ie} \pm \hat{je} \pm \hat{ke}), \frac{1}{2}(\pm\hat{i} \pm \hat{j} \pm \hat{k} \pm \hat{e}), \\
\frac{1}{2}(\pm 1 \pm \hat{i} \pm \hat{e} \pm \hat{ie}), \frac{1}{2}(\pm\hat{j} \pm \hat{k} \pm \hat{je} \pm \hat{ke}), \\
\frac{1}{2}(\pm 1 \pm \hat{j} \pm \hat{e} \pm \hat{je}), \frac{1}{2}(\pm\hat{k} \pm \hat{i} \pm \hat{ke} \pm \hat{ie}), \\
\frac{1}{2}(\pm 1 \pm \hat{k} \pm \hat{e} \pm \hat{ke}), \frac{1}{2}(\pm\hat{i} \pm \hat{j} \pm \hat{ie} \pm \hat{je}).
\end{array}
\tag{6.5}
$$

Coxeter has written about this remarkable lattice, see [6], [9]. In fact regarding this set of integral Cayley numbers he comments, "It seems somewhat paradoxical that the cyclic permutation $(\hat{e}\ \hat{i}\ \hat{j}\ \hat{ie}\ \hat{ke}\ \hat{k}\ \hat{je})$ which preserves the integral domain is not an automorphism of the whole ring of octaves: it transforms the associative triad $\hat{i}\ \hat{j}\ \hat{k}$ into the anti-associative triad $\hat{j}\ \hat{ie}\ \hat{je}$."

6.4 The Spin$_5$ Case

In the Spin$_5$ case we consider the subgroup $SO(5) \subset SO(7)$ that leave $\hat{i}$ and $\hat{j}$ fixed. Thus when analyzing these subgroups the map ϕ is not so useful, as none of the elements we are considering will contain the central inversion. However, we can define a new homomorphism ϕ_5 which sends the central inversion in $O(5)$ to the identity.

Viewing $\mathbb{O}$ as $\mathbb{H}^2 = \mathbb{H} \oplus \mathbb{H}\hat{e}$, it is natural to split elements into there "e-part" and non "e-part," as follows $a + b\hat{e} = (a, b)$ with $a, b \in \mathbb{H}$[1]. Consider the action of Spin_5 elements R_{qe} where $q \in \mathbb{H}$. $R_{qe}(a, b) = (\overline{q}b, -qa)$, so that these elements swap a and b, with some additional quaternionic action. On the other hand the element R_k does not swap these elements. This is a natural splitting in this context. Right Cayley multiplication by an element that is not of the form qe nor $\pm k$, cannot be decomposed into either simple swap or no-swap. Hence, recalling Cohen's definition of the groups $G(K, H, K/H, \alpha)$, 5.4.1, we see that all of them must split in this way, and hence, in the vector representation, correspond to subgroups of $S(O(1) \times O(4))$. This allows us to easily rederive Cohen's $G(K, H, K/H, \alpha)$ groups in the Spin context. It is instructive to work some examples.

First we need to recall some basic facts about the relevant reflection groups. H_4 and H_3 contain the central inversion in their dimensions. Hence $H_3 = H_3^+ \cup -H_3^+$ and $-I_4 \in H_4^+$. The maximal reflection group in $O(5)$ containing H_4 is $A_1 \times H_4$ acting on orthogonal subspace. However, since $-I_4 \in H_4^+$, all elements of the form $(\pm 1, \pm h_1)$ map to elements of the form $(1, h_2)$, where $h_1, h_2 \in H_4^+$. Hence there are really only two subgroups in Spin_5 that contains a double cover of H_4^+, one of which is the double cover of $1 \times \mathbb{H}_4^+$. This is clearly Cohen's $G(\mathcal{I}, \mathcal{I}, 1, 1)$, see [5]. Similarly, the maximal reflection groups containing H_3 and not H_4 are of the forms $I_2(m) \times H_3$ or $A_1 \times A_1 \times H_3$. Various subgroups of this last one (using the nontrivial outer automorphism of H_3) account for the remaining $G(\mathcal{I}, \cdot, \cdot)$ groups discussed in [5]. Presumably the groups obtained by this method that are not discussed in [5] are not quaternionic reflection groups.

Similar analysis holds for the remaining groups discussed by Cohen. Moreover

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[1]This is not entirely accurate. The quaternionic structures we are using to define $\mathbb{H}^2$ are $R_{\hat{\imath}}$, $R_{\hat{\jmath}}$, and $R_{\hat{\imath}}R_{\hat{\jmath}}$. By examining the Cayley multiplication we see that the element $q_1 + x\hat{e} + y\hat{\imath e} + z\hat{\jmath e} + w\hat{k e}$ actually corresponds to the element $(q_1, x - y\hat{\imath} - z\hat{\jmath} + w\hat{k}) \in \mathbb{H}^2$, so that in the e-part we have $-\hat{\imath}, -\hat{\jmath}, \hat{k}$ as the quaternionic structure. However, we can still choose to split $a + b\hat{e}$ as (a, b), and all of the above analysis holds.

this technique produces many additional subgroups which are not discussed in [5]. While it will not be necessary to have a detailed discussion of all the quaternionic cases, it is remarkable that the reflection groups can be reconstructed through this method. For convenience I have listed some Cohen's groups along with their images in $SO(5)$ in Table 6.4.

In order to classify the quaternionic reflection groups with primitive complexification, Cohen had to employ the use of root systems and engage in a substantial analysis of them. All of these can be understood in the context of Spin_5 quite easily. In the vector representation they correspond to rotation subgroups of irreducible five-dimensional reflection groups. Now the analysis presented here is not complete. Finding all the finite $HU(2)$ subgroups requires a detailed knowledge of all the finite $SO(5)$ subgroups. However these are far easier to work with then $HU(2)$ directly. In Cohen's analysis he had to employ many additional technical and non-trivial results, while also relying on his earlier classification of the finite complex reflection groups. He also had to separate out the various groups in different categories, and do separate analysis in order to show that these were the only finite reflection groups. It is interesting to notice how his groups can be understood more easily and in a more unified fashion in the context of Spin_5. The techniques used in the Spin_5 context also yield many additional subgroups, which were not discussed in [5].

6.5 The Cayley Polytopes

First of all the quaternionic polytopes are examples of Cayley polytopes.

Theorem 6.5.1 *All quaternionic polytopes in $\mathbb{H}^2$ are Cayley polytopes.*

Proof: $HU(2) \cong \text{Spin}_5 \subset \text{Spin}_7$ and all such quaternionic polytopes have $\mathbb{H} \subset \mathbb{O}$ as an edge. Thus all the edges are Cayley planes and the polytope is Cayley. More-

$HU(2)$ Subgroup	Order	Image in $SO(5)$
$G(\mathcal{T},\mathcal{T},1,1)$	1152	$1\times F_4^+$?
$G(\mathcal{O},\mathcal{O},1,1)$	4608	$(G_{(\mathcal{O},\mathcal{O})})_5^+$
$G(\mathcal{O},\mathcal{T},\mathcal{C}_2,1)$	2304	$\phi_5(A_1\times F_4)=(F_4)_5^+$
$G(\mathcal{O},\mathcal{D}_2,A_3,1)$	768	$\phi_5(A_1\times B_4)=(B_4)_5^+$
$G(\mathcal{I},\mathcal{I},1,1)$	14400	$1\times H_4^+$
$W(O_2)$	720	A_5^+
$W(O_3)$	1440	$\phi_5(A_5)$
$W(P_2)$	1920	D_5^+
$W(P_3)$	3840	B_5^+

The group $G_{(\mathcal{O},\mathcal{O})}$ is a subgroup of $O(4)$ of order 2304. It is not a reflection group, but is generated by elements of the form $g : a \mapsto q_1\overline{aq}_2$ where $a \in \mathbb{H} \cong \mathbb{R}^4$ and $q_1, q_2 \in \mathcal{O} \subset HU(1)$.

Table 6.2: The correspondence between some irreducible $\text{Spin}_5 = HU(2)$ reflection groups and $SO(5)$ subgroups

over, these polytopes have associated root systems which make them attractive for us.

For the more general case of pure Spin_7 Cayley polytopes we need to make some generalizations. It will be crucial in generalizing to the Cayley case to understand how to move from a given finite symmetry group to the associated polytope. Cuypers, [11], makes use of Wythoff's construction on the graphs of certain quaternionic reflection groups. Unfortunately that is not available to us. However since we are not concerned with making our polytopes regular, we need only consider the orbit of $\mathbb{H} \subset \mathbb{O}$ under the action of the associated group. Recall that each of the groups we considered had an associated lattice in $\mathbb{R}^8$. If we take a vector from this lattice $v \perp \mathbb{H}$, we can consider the edge $v + \mathbb{H}$ (this is a Cayley four plane). The orbit of this edge will yield the polytope.

If we had chosen a vector that was not from the lattice, then the resulting polytope would not be uniform. Recall that we can make various real polytopes by choosing a vector that is orthogonal to multiple root vectors, yet equidistant from them. The resulting polytope is usually exotic and non-regular. If instead we choose a vector that is not equidistant from the orthogonal root vectors, we still get a polytope, however it is usually not uniform. See [24]. Choosing a non-lattice vector is analogous in the Cayley case.

In computing specifics about the polytopes it will be useful to know the size of the resulting isotropy subgroups. Recall the the isotropy subgroup of $\mathbb{H} \subset \mathbb{O}$ is K defined in Theorem 3.2.2. Elements of K have the following action on $\mathbb{O}$,

$$g : a + b\hat{e} \mapsto q_3 a \overline{q}_1 + (q_2 b \overline{q}_1)\hat{e}, \tag{6.6}$$

where $q_i \in \mathbb{H}$. In the vector representation such an element has the following representation,

$$\begin{aligned}\chi_{(q_1,q_2,q_3)}(a+b\hat{e}) &= g(g^{-1}(1)\cdot(a+b\hat{e})) \\ &= g(\overline{q_3}q_1\cdot(a+b\hat{e})) \\ &= g((\overline{q_3}q_1a)+(b\overline{q_3}q_1)\hat{e}) \\ &= q_1a\overline{q}_1+(q_2b\overline{q_3})\hat{e}. \end{aligned} \quad (6.7)$$

Hence the preimage of the isotropy subgroup K is $SO(3)\times SO(4)\subset SO(7)$. This fact allows us to analyze the various polytopes we wish to construct.

For example, starting with B_7^+, we see that the isotropy subgroup of $\mathbb{H}$, in the vector representation, is $B_7^+\cap SO(3)\times SO(4) = B_3^+\times B_4^+$ of order 2^3 $3!$ 2^4 $4! = 2^{11}3^2$. Thus the isotropy subgroup is an index 35 subgroup. Any element of the symmetry group that leaves the edge $v+\mathbb{H}$ invariant must at least fix the subspace $\mathbb{H}$ and hence must belong to the double cover of $B_3^+\times B_4^+$. Moreover this group must leave fixed the vector v which we can without loss of generality take as $(0,1)\in\mathbb{H}^2$. Hence elements of the isotropy subgroup of this edge must be of the form $(q,p,q)\in K$, and belong to the subgroup $B_3^+\times B_3^+\subset B_3^+\times B_4^+$. This subgroup is of index 8 in $B_3^+\times B_4^+$, hence there are $35\cdot 8 = 280$ edges in this Cayley Polytope, with 8 vertices per edge. Moreover $B_3^+\times B_3^+$ is an index 280 subgroup in B_7^+, yielding 280 vertices and hence 8 edges per vertex. Since the lattice associated to this group is the B_8 lattice, and it is manifestly invariant under the action of this group it is trivial to extend this to a Cayley Honeycomb. However notice that the action of B_7^+s double cover on the B_8 lattice must be transitive on the long and short roots separately. Hence, we get the following,

Theorem 6.5.2 *Let $G\subset Spin_7$ be the double cover of B_7^+. Let Λ be its associated B_8 lattice in $\mathbb{O}$. Let Λ_s and Λ_l be the integer spans of the short and long lattice vectors, respectively.*

Then the following arrangements of Cayley planes are a lattice periodic Cayley Honeycomb,

$$B_s = \{v + g(\mathbb{H}) \mid v \in \Lambda_s,\ g \in G\}\,, \tag{6.8}$$

$$B_l = \{v + g(\mathbb{H}) \mid v \in \Lambda_l,\ g \in G\}\,. \tag{6.9}$$

Then given any of the many sublattices $\Lambda' \subset \Lambda$, we are assured the existence of two novel Cayley cycle in $T^8 = \mathbb{R}^8/\Lambda'$. B_s and B_l will be the lifts of these Cayley cycles to $\mathbb{R}^8$. The Cayley polytopes B_s and B_l are dual to each other in a manner analogous to the more familiar duality of the cross polytope and the cube.

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