Research Proposal: Extensions of Fibonacci identities to arbitrary linear recurrence relations

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1 Introduction

The Fibonacci numbers, given by \( f_0 = 1, \ f_1 = 1, \) and

\[ f_n = f_{n-1} + f_{n-2}, \]

for \( n \geq 2 \) have many interesting properties. Many of these interesting properties can be easily proven combinatorially by interpreting the Fibonacci numbers as the number of ways, to tile a 1xn board with squares (1x1 tiles) and dominoes (1x2 tiles).

Also, this interpretation has been generalized to interpret recurrence relations with non-negative integer coefficients and initial conditions. Specifically,

\[ g_n = \sum_{i=1}^{k} a_i g_{n-i}, \]

with the \( a_i \)'s non-negative integers can be interpreted as the number of ways to tile a board of length \( n \) with tiles of length 1 through \( k \), where tiles of length \( i \) are given one of \( a_i \) colors, and the first tile is given a certain number of phases depending on the \( a_i \), the initial conditions, and the length.

Many known identities have been proven more simply using this approach and several new identities have been found. However, at this time, no general techniques are available for interpreting recurrence relations with negative coefficients, or recurrence relations that are not linear.

For example, the recurrence relations

\[ a_1 = 3, \ a_2 = 8, \ a_n = 3a_{n-1} - a_{n-2}, \ n \geq 2, \]

and

\[ b_1 = 1, \ b_n = \sum_{i=1}^{n-1} b_i, \ n \geq 2, \]

cannot be interpreted using the above techniques.
2 Proposed Research

This coming year, I will work on developing general techniques for interpreting linear recurrence relations with negative coefficients and initial conditions, as well as recurrence relations which are not linear. Also, I am interested in developing a combinatorial interpretation for the negative indices of recurrence relations. If I am successful in finding general techniques for interpreting such sequences, I hope to prove combinatorially that the same identities hold in a more general context.

The partitions function $p(n)$ gives the number of ways to write the integer $n$ as the sum of integers (where the order of the summation is irrelevant). For example, $p(4) = 5$, since

\[
4 = 4 \\
= 3 + 1 \\
= 2 + 2 \\
= 2 + 1 + 1 \\
= 1 + 1 + 1 + 1.
\]

The sum-of-the-divisors function $\sigma(n)$ gives the sum of the divisors of a positive integer $n$. For example $\sigma(4) = 7$, since 1, 2, and 4 are the divisors of 4, and $1 + 2 + 4 = 7$.

A well-known result of Euler, known as the Pentagonal Number Theorem, states that $p(n)$ and $\sigma(n)$ (with different conditions) satisfy the non-linear recurrence relation

\[
c_n = c_{n-1} + c_{n-2} - c_{n-5} - c_{n-7} + c_{n-12} + c_{n-15} - c_{n-22} - c_{n-26} + \cdots,
\]

where $1, 2, 5, 7, 12, 15, \ldots$ is the sequence of pentagonal numbers, i.e. those of the form $(3n^2 \pm n)/2$.

I hope to work on developing a combinatorial interpretation of the partition function and the sum-of-divisors function, and if I am successful, I will attempt to prove the pentagonal number theorem using my interpretation, as well as investigate any new results that can be obtained about the partition and sum-of-divisors functions. In particular, I am interested in finding combinatorial proofs of Ramanujan’s results that

\[
p(5n + 4) \equiv 0 \pmod{5} \\
p(7n + 5) \equiv 0 \pmod{7}.
\]
3 Prior Research

Work related to the problem of a combinatorial interpretation for negative indices was studied by Propp [2], and Barcucci, et. al. have done work developing an interpretation for negative coefficients in [1]

References
