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Bringing van Hiele and Piaget Together: A Case for Topology in Early Mathematics Learning

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Abstract

Topological concepts arise naturally in young children's spatial reasoning yet topology is not part of our current K-12 education. In this paper, we extend the van Hiele model of spatial reasoning using Piaget's theory of representational space. By merging these two existing theories, we are able to place topological concepts in a context that is age appropriate.

1. Introduction

For a few months in 2012, a problem was circulating social media sites popular among bloggers. Given a platform for sharing ideas, many people posted solutions and comments. While many of the solutions are interesting to read, the comments proved to be more interesting. Here is one wording of this problem followed by a variety of comments from different blogs [17].

Example. *This problem can be solved by preschool children in 5-10 minutes, by a programmer in 1 hour, and by people with higher education...well check it out for yourself! If*

$8809 = 6,$	$3333 = 0,$	$7111 = 0,$	$5555 = 0,$
$2172 = 0,$	$8193 = 3,$	$6666 = 4,$	$8096 = 5,$
$1111 = 0,$	$7777 = 0,$	$3213 = 0,$	$9999 = 4,$
$7662 = 2,$	$7756 = 1,$	$9313 = 1,$	$6855 = 3,$
$0000 = 4,$	$9881 = 5,$	$2222 = 0,$	$5531 = 0,$

then $2581 = ?$

Comments:

1. Alix: *My preschooler got it in 3 seconds. I tried to have him explain how he got it. So confused. But I'm thinking that the question mark looks like a 2 so it must be 2* [8].
2. Micah: *Did the preschooler have complete information about the problem? It's obviously meant to be confusing* [8].
3. El Guapo: *Nope. didn't even come close. Brain too highly developed, I s'pose...* [12].
4. Midas: *This problem is bollocks. I didn't solve it, and yet I have no trouble solving Mensa problems. I don't think they mean anything either to be fair* [12].
5. (Response to Midas) *I am a member of Mensa. The problem is valid and has an answer and very logical reasoning. You just have to move out of an adult frame of mind and into a child's frame of mind. They think more visually at a young age. Preschoolers cannot add, and counting is limited. But what can they do? See consistent shapes because that's what they've been doing... squares, triangles, CIRCLES...* [12].

Here is a spoiler: The solution to the problem is obtained by counting the number of holes in each set of numbers. For instance, 8 has two holes, 4 has one hole, and 2 has zero holes. Therefore, 842 is 3.[1] The problem wording implies that the more mature and more educated you are, the less likely you are to figure out this relationship. The comments also support this idea. Yet, counting holes is fundamental in topology, which is a non-trivial subfield of mathematics. Why are more sophisticated thinkers less likely to access topological intuition? Compare the task of counting holes needed to solve the problem to counting angles in a regular polygon. Many adults differentiate a triangle from a rectangle by counting the number of angles. If topological ideas were taught in K-12 mathematics education, would counting holes of shapes be as natural as counting angles of a regular polygon?

Topology forms a general foundation for mathematics and promotes its unity [4]. It is fundamental to many areas of mathematics. In many cases, however, the first topology course students can take is an upper-division college course. Yet many researchers, such as Piaget, Inhelder, Laurendeau and Pinard, believe that topological knowledge precedes geometric knowledge [7, 11].

[1] We will not comment on the abuse of notation involving the "=" sign.

So, what exactly is the difference between topological knowledge and geometric knowledge? One difference is that geometry is an area that uses a lot of measuring while topology does not involve measuring in the traditional sense. In topology, a rope of length 5 ft is equivalent to a rope of length 6 inches. But, ropes of two different lengths would be considered geometrically different. In topology, it is helpful to imagine objects made out of clay. As long as you do not rip, tear, or puncture the clay, you can turn one object into another. For topologists, a donut is the same as a coffee cup![2]

Many topological concepts are observable and could be explored by curious young minds. While it would be incorrect to assume that all material in an upper-division topology course is appropriate material for our schools, we are missing out on a lot of opportunities to teach a fundamental area of mathematics if we stay away from topology altogether. Here is a sample of missed opportunities for emphasizing topological concepts within basic activities. When a student bends a pipe cleaner from a shape of a triangle to a circle, they could be told this is a mathematical (topological) operation that allows us to think of a circle as a triangle. When a child cuts a strip of paper into two pieces, they could explore the connectedness properties of the paper: the paper was at first connected and after the cut it became disconnected. Lastly, consider when children play with clay. They often make shapes (balls, cylinders, cubes, etc) and then "smash" the clay objects. This is a great opportunity to explore the similarities and differences between balls, cylinders, and disks (flattened balls). It turns out that a ball is topologically equivalent to a disk but not to a cylinder because a cylinder has a "hole" that goes through it while balls and disks do not have holes. (Figure 1).

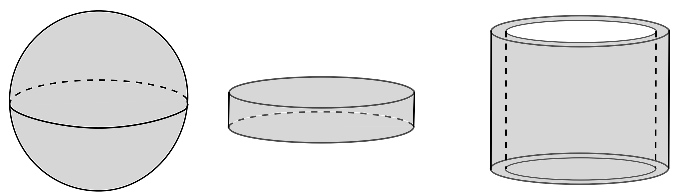

Figure 1: From left to right: A ball, disk, and cylinder.

[2]For a visual proof of this fact, see https://www.youtube.com/watch?v=4iHjt2Ovqag or https://en.wikipedia.org/wiki/File:Mug_and_Torus_morph.gif, both accessed on January 24, 2017.

The goal of this paper is threefold. The first goal is to show that introducing topological terms and teaching topological concepts in our schools could be beneficial. We could thus offer our students a framework for ideas and concepts that we observe in everyday life. The second goal is to extend the van Hiele model of geometric thinking using the topological framework given by Piaget and Inhelder in *The Child's Conception of Space* [11]. Lastly, I would like to convince the reader that there is a need to do research to see the effects of teaching topological concepts to young children. While researchers such as Piaget and Inhelder have done studies that suggest that children understand topological concepts, I have not heard of a complete early education curriculum which explicitly *teaches* topological concepts, though also see [5] for the case study of a seven-year-old girl who participated in an eighteen-week teaching experiment aiming to model the development of her intuitive and informal topological ideas. Educators who work with young children often observe misunderstandings of certain concepts. Perhaps by introducing topological concepts and giving both educators and students new tools and techniques for working with shapes, these misconceptions could be addressed using a topological viewpoint. This, in turn, could lead to new teaching methods.

2. Spatial thinking in early development

In this section, we review two preexisting theories for spatial thinking: The van Hiele model of geometric thinking and Piaget's model for the development of children's representational thinking about the nature of space. One of the inherent differences between these two theories is that progression using the van Hiele model is instruction-based while progression through Piaget's stages is developmental. Nevertheless, there are similarities between the theories in terms of how they describe these stages. In the next section, we will use Piaget's theory to extend the van Hiele model. This will allow us to place topological concepts within an early stage of a spatial framework.

2.1. The van Hiele model

The van Hiele model of geometric thinking was developed in 1957 by two Dutch mathematics educators, P.M. van Hiele and Dina van Hiele-Geldof, see a relatively recent exposition of the model by van Hiele in [15]. The model became well-known in the US after Izaak Wirszup presented the model to the National Council of Teachers of Mathematics in 1974; for an overview of this history see [16]. The original van Hiele model consists of five levels of learning and each level has five sequential stages of instruction. The levels can be described as naturally inductive [6, page 206]. The advancement from one level to the next is more dependent on instruction than biology [2]. So, while age is not a factor for determining at which level a student is performing, it is nonetheless used to some degree to align the van Hiele model to K-12 teaching.

There are five ordered levels in the van Hiele Model of Geometric Thinking: (1) visualization, (2) analysis, (3) informal deduction, (4) formal deduction, and (5) rigor. At Level 1, children can name and recognize shapes, but specific properties may not be identified. They might recognize characteristics, but these characteristics are not used for recognizing the shapes. At Level 2, children start to use vocabulary relating to properties. Size and orientation become irrelevant as they begin to focus on specific properties of a shape. At Level 3, children recognize relationships between shapes and are able to reason about relationships. At Level 4, children begin to understand deduction, postulates, theorems, and proofs. At Level 5, children begin to understand how to work with axiomatic systems.

Since the Van Hiele progression in geometric thinking is not age-based, developing a geometric grade-based curriculum using the van Hiele model can be challenging, but some attempts have been made. For example, students in primary grades are assumed to be at Level 2 and many preschoolers are at Level 1, which is described below:

> Level 1[3]: Children learn to recognize geometric figures such as squares and circles by their holistic physical appearance. For example, a given figure is a circle because it looks like a clock. Children at this level do not think about the attributes or properties of shapes [1].

According to the Pennsylvania State Team of the Mid-Atlantic Eisenhower Consortium for Mathematics and Science Education at *Research for Better Schools*, a typical high school geometry class is appropriate for students at

[3] The original van Hiele model has Levels 0–4. Today these are often relabeled as Levels 1–5. Copley uses the 0–4 scaling which I have adjusted to match the 1–5 scaling.

the van Hiele Level 4 and a typical college class is appropriate for students at Level 5 [18]. But, this might be a misalignment of levels with age groups. A study by Usiskin showed that 70% of high school graduates were at the first, second, or third level of van Hiele thinking, with about 50% of these high school graduates performing at the first level of thinking [13].

One of the characteristics of the van Hiele model is that instruction at a higher level than the level of the student can inhibit learning and can lead to frustration [16, 14]. Therefore, if a high school geometry class is aimed at Level 4 students but the students taking the class are at Levels 1–3, then their learning may be inhibited. It is important that students are taught at their level. Memorization of definitions and algorithms does not progress a student through the van Hiele levels of thinking [14].

2.2. Piaget's Theory of Spatial Development

We now turn our attention to Jean Piaget and Bärbel Inhelder. Piaget and Inhelder conducted a study aimed at understanding how representational space develops from perception in children publishing their findings in *The Child's Conception of Space* [11]. Here, *perception* is the knowledge of objects from direct contact (sensorimotor engagement) while *representation* is the knowledge of objects in their absence (mental representations) [11].

For the Piaget-Inhelder experiments, children ranging from 2-7 years of age performed three tasks. The first task had younger children hold familiar objects under a table to see if they could recognize the given object. In the second task, children were given cardboard cut-outs with their hands hidden and various ways of identifying the object (by name, by picture, etc). The cutouts increased in complexity in each section, starting with simple and symmetric objects, followed by more complicated symmetric objects, then asymmetric objects with straight lines, and lastly "purely topological" objects (irregular surfaces pierced by one or two holes, open or closed rings, two intertwined rings, etc.; see Figure 2).[4] The third task had children touch hidden objects and draw what they touched.

[4]It should be noted that all objects are topological. Therefore, when Piaget and Inhelder describe an object as *purely topological*, they are describing shapes that are distinguished by holes and linkages.

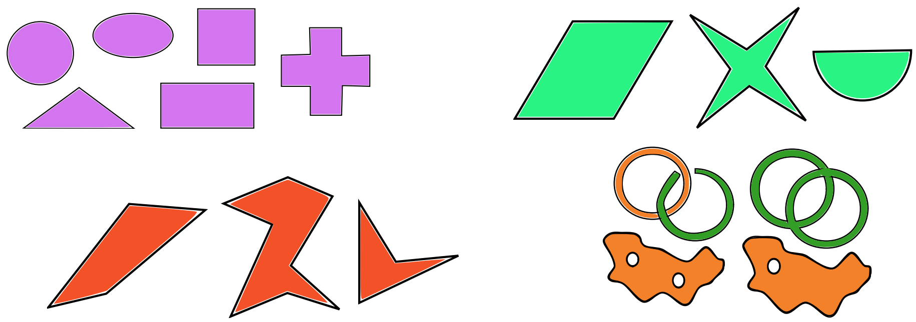

Figure 2: Upper left: Shapes used in Task 2 Section 1 (simple and symmetric). Upper right: Shapes used in Task 2 Section 2 (more complicated and symmetric). Lower Left: Shapes used in Task 2 Section 3 (asymmetric). Lower right: Shapes used in Task 2 Section 4 (topological).

From this experiment, Piaget and Inhelder developed a theory about the development of spatial sense.[5] They theorized that the representational space that develops first is topological in nature as opposed to Euclidean. Here, topological properties for Piaget and Inhelder are characterized as *proximity*, *separation*, *order*, and *continuity* [10].

The Piaget-Inhelder theory consists of four stages. Stage 0 (age 0–2) is when children develop *perceptual space*; this is how one perceives an object in its presence. In Stages 1-3 (age 2.5–12 years), children develop *representational space*; that is, the capacity to hold on to a mental representation of an object in its absence. At the beginning of Stage 1 (1A), children can identify familiar objects such as clocks or houses, but not idealized or abstract shapes such as circles or triangles. Children later progress to Stage 1B where they begin to identify shapes that are topological in nature, as opposed to Euclidean in nature. In other words, children begin to view a cardboard ring as different from a cardboard solid circle. During Stage 2, children start to have a crude capacity to differentiate rectilinear and curvilinear (2A) which later progresses to complete differentiation of the classes of rectilinear and curvilinear objects (2B). Lastly in Stage 3 (age 7–12), children can distinguish complex forms and can construct mental images by operational activities.

[5]We must point out that Piaget and Inhelder's theory is not merely about spatial sense but describes cognitive development in a broader sense. In the following we are exclusively interested in the parts of their work that involve spatial reasoning.

It should be noted that the Piaget-Inhelder experiments were conducted by other researchers such as Lovell [9] and Martin [10] resulting in different conclusions. While these researchers observed some understanding of topological notions in the children, these children also demonstrated some competence with geometric concepts, casting doubt on Piaget and Inhelder's claim that topological concepts precede geometric concepts. Nevertheless, what is important for this article is that in these follow-up studies, too, children displayed some understanding of topological concepts at an early age.

3. An Extension of Van Hiele's Model of Geometric Thinking

In this section, I extend the van Hiele model using Piaget's theory of spatial development. While Piaget's theory is age-based and the van Hiele model is instruction-based, there is a way to merge the two theories into a larger and more comprehensive theory of spatial thinking. I use Hoffer's 1993 description of the van Hiele levels [6]. In particular, at level n, the objects studied are now the statements that were explicitly made at level $n-1$ as well as explicit statements that were only implicit at level $n-1$. In effect, the objects at level n consist of extensions at level $n-1$ [6, page 208].

The van Hiele model of geometric thinking is a commonly used method to describe how children develop geometric and spatial sense [1]. At the first level of the van Hiele model, children can differentiate between a rectangle and a square [16], but this is not how spatial thinking begins. To advance into the van Hiele Level 1, a child needs to have successfully transitioned through the van Hiele phases of earlier levels. So, what comes before Level 1? For instance, a four-year-old cannot differentiate a rectangle and a square [11, page 21], and therefore is not yet at Level 1. We use Copley's description of Level 1[6], where children recognize a circle because it looks like a clock [1, page 107], and we note that this agrees with the abilities of the children labeled at Stage 2A by Piaget. Three examples of children at Piaget's Stage 2A are; Eri (age 4 years, 4 months) who is given a circle and describes it as a wheel [11, page 32], Char (5 years, 2 months) who describes a rhombus as two roofs [11, page 32], and Ast (3 years, 2 months) who describes a notched circle as a rake [11, page 31]. From these descriptions, we can conclude that it is reasonable to associate van Hiele's Level 1 with Piaget's Stage 2A.

[6]Recall once again that Copley uses the scaling the parallel scaling 0–4 for the levels.

We can verify this association by examining van Hiele's Level 2 and Piaget's Stage 3. At this level, children begin to learn isolated characteristics or attributes of the forms, such as "a square has four equal sides" [1]. During Piaget's Stage 3, children are able to draw stars with the correct number of points, shapes with right angles, and they have a sense of orientation [11, page 36]. They also can differentiate objects by side lengths [11, page 37]. In particular, a square would be differentiated from a rectangle and a child in this stage would be able to recognize that a square has four equal sides.

After verifying that the van Hiele Level 1 and Level 2 align with Piaget's Stage 2 and Stage 3, respectively, it is natural to define a preceding Level 0 for the van Hiele's model using Piaget's Stage 1. We do so by defining Level 0 as the stage where children can differentiate between topologically distinct objects such as a ring (not filled in) and a circle (filled in), but cannot differentiate geometrically distinct objects such as a circle from a square. It is here that we are able to place topological notions into an early level of a geometric/spatial framework. Here are three examples from Piaget's study that would be placed into this Level 0:

1. Ani (age 3 years 5 months) is sure that a square is the same as a circle but can differentiate a ring (not filled in) from a circle (filled in);

2. Don (age 3 years, 6 months) can distinguish a ring, a two holed surface, and a half-opened ring, but cannot distinguish a circle and an ellipse;

3. Sim (4 years) cannot distinguish a triangle from a circle or square, but correctly identifies an open ring, two intertwined rings, and an irregular surface with one hole [11, page 26].

In each one of these examples, the children distinguish between topologically distinct objects and associate topologically equivalent objects. Therefore, part of the transition from our Level 0 to Level 1 in the van Hiele model is where children develop a sense of basic geometric properties such as straight or curved.

4. Concluding Remarks

Outside of the math community, topology is mostly an unfamiliar subject. But it does not have to be. From a very early age, children have a topological understanding that our K-12 curriculum currently does not nourish. By finding a place for topological notions within the van Hiele model, we can begin a conversation about engaging students with topological ideas alongside Euclidean ideas at an early age. In particular, given the discussion above, we can justifyably argue that certain topological concepts are appropriate topics for preschoolers.

The extension of the van Hiele model described here offers a coherent and comprehensive framework to describe the development of geometric and topological intuition and understanding in young learners. My hope is that it may help educators encourage and nurture the development of spatial reasoning and understanding in students. We can, for instance, use the van Hiele phases of learning at Level 0 to help students transition successfully to Level 1. Furthermore, by building a strong Level 0 basis, students will be prepared for Level 1 since the objects at Level 1 are extensions from Level 0.

I believe the next step is to investigate further how topological concepts are understood by children in early education and how they relate to geometric concepts. We should also research how nurturing a preexisting topological intuition might support and advance the geometric knowledge of students. If the K-12 mathematics curriculum begins to incorporate some topological ideas, we might possibly observe many beneficial consequences. For one, students' geometric intuition could be strengthened if we routinely encourage them to compare and contrast topological and geometrical ideas. Also, if introduced to topology at an earlier stage, students might develop familiarity with an area of mathematics that has in modern times penetrated almost all other branches of mathematics [4]. This might help students make mathematical connections earlier in their education between areas of mathematics that might not otherwise be made, using topology as the common thread.

References

[1] Juanita Copley, *Geometry and Spatial Sense in the Early Childhood Curriculum*, National Association for the Education of Young Children, Washington DC, 2000.

[2] T. L. Corley, *Students' Levels of Thinking as Related to Achievement in Geometry*, Ed.D. dissertation, Arizona State University, Tempe AZ, 1990.

[3] Mary L. Crowley, "The van Hiele Model of Development of Geometric Thought", *Learning and Teaching Geometry, K-12.*, edited by Mary Montgomery Lindquist (National Council of Teachers of Mathematics, Reston VA, 1987), pages 1–16.

[4] General topology, *Encyclopedia of Mathematics*, adapted from an original article by A.V. Arkhangelśkii (originator), which appeared in *Encyclopedia of Mathematics*, edited by Michiel Hazewinkel (Kluwer Academic Publishers, 2002), available at http://www.encyclopediaofmath.org/index.php?title=General_topology&oldid=35422, accessed on January 23, 2017.

[5] Steven Greenstein, "Making sense of qualitative geometry: The case of Amanda", *Journal of Mathematical Behavior*, Volume **36** (2014), pages 73–94.

[6] Alan Hoffer, "van Hiele Based Research", in *Acquisition of mathematical concepts and processes*, edited by Richard Lesh and Marsha Landau (Academic Press, New York,1983), pages 207–228.

[7] Monique Laurendeau and Adrien Pinard, *The Development of the Concept of Space in the Child*, International Universities Press, New York, 1970.

[8] J. D. Long, "Solving Easy Problems the Hard Way", blog post at *Cerebral Mastication*, posted on March 17, 2012, available at http://www.cerebralmastication.com/2012/03/solving-easy-problems-the-hard-way/, accessed on January 23, 2017.

[9] K. Lovell, "A Follow-up study of some aspects of the work of Piaget and Inhelder on the Child's Conception of Space", *British Journal of Educational Psychology*, Volume **29** Issue 2 (1959), pages 104–117.

[10] J. Larry Martin, "An Analysis of Some of Piaget's Topological Tasks from a Mathematical Point of View", *Journal for Research in Mathematics Education*, Volume **7** Number 1 (1976), pages 8–24.

[11] Jean Piaget and Bärbel Inhelder, *The Child's Conception of Space* (translated by F. J. Langdon and J. L. Lunzer), Compton Printing Limited London and Aylesbury, Great Britain, 1971

[12] Michelle, "Can You Solve This Problem?", online news entry at *Motley News*, posted on March 24, 2012, available at http://motleynews.net/2012/03/24/can-you-solve-this-problem/, accessed on January 23, 2017.

[13] Zalman Usiskin, *van Hiele Levels of Achievement in Secondary School Geometry*, University of Chicago, Chicago, 1982.

[14] John A. Van de Walle, *Elementary and Middle School Mathematics: Teaching Developmentally*, 4th edition, Addison Wesley Longman, New York, pages 310–311.

[15] Pierre van Hiele, *Structure and Insight: A Theory of Mathematics Education*, Academic Press, Orlando FL, 1986.

[16] Der-Bang Wu, *A study of the use of the van Hiele model in the teaching of non-Euclidean geometry to perspective elementary school teachers in Taiwan, the Republic of China*, Ph.D. dissertation, University of Northern Colorado, Colorado, 1994.

[17] Colleen Young, "Can You Solve this Problem?", online post at *Mathematics for Students*, posted on March 31, 2012, available at https://mathematicsforstudents.wordpress.com/2012/03/31/can-you-solve-this-problem/, accessed on January 23, 2017.

[18] Improving Measurement and Geometry in Elementary Schools, "Van Hiele Levels of Geometric Reasoning", web resource at http://images.rbs.org/cognitive/van_hiele.shtml, currently archived at http://web.archive.org/web/20150310132745/http://images.rbs.org/cognitive/van_hiele.shtml, accessed on January 23, 2017.