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Separating Sets for the Alternating and Dihedral Groups

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Abstract

This thesis presents the results of an investigation into the representation theory of the alternating and dihedral groups and explores how their irreducible representations can be distinguished with the use of class sums.

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Chapter 1

Introduction

Given a finite group G and a $\mathbb{C}G$-module V, we seek to decompose an element $f \in V$ into components that will yield information about f. V has a canonical, *isotypic* decomposition into invariant submodules, and projecting f onto these invariant submodules yields information about f. One of the most well-known applications of such projections concerns voting theory. Computing the isotypic projections of a vector $f \in \mathbb{C}S_n$ is of particular interest, for these projections can be used to analyze patterns in voting ([13] is a good reference). We seek optimal methods for accomplishing isotypic projections. For the symmetric group, much is known (see, for example, [3], [9], or [12]). In this research, we have focused upon the alternating and dihedral groups.

Chapter 2 contains the basic theory and definitions necessary to understand this thesis. Chapter 3 defines the concept of a *separating set* for a group and describes how class sums can be used to construct separating sets. We describe a collection of minimal separating sets for the dihedral group. A theorem regarding separating sets for direct products of groups is presented as well. Chapter 4 is an introduction to the representation theory of the symmetric group, which is necessary in order to study the representation theory of the alternating group. In Chapter 5, we describe the irreducible representations of the alternating group and give branching rules for the restriction of irreducible representations from A_n to A_{n-1}. We also present the results of our research concerning minimal separating sets for the alternating group.

Chapter 2

Definitions

2.1 Modules

Before we proceed, we will require some definitions to understand the representation theory of a finite group G. A good reference is [4].

Definition 2.1. *A left module* M over a ring R is an additive abelian group upon which a left action of R is defined. We additionally require the following conditions: Let r, $s \in R$ and m, $n \in M$. Then

(a) $(r+s)m = rm + sm$

(b) $(rs)m = r(sm)$

(c) $r(m+n) = rm + rn$

(d) If $1 \in R$ we require $1m = m$.

For simplicity, we will refer to left modules as *modules*.

Definition 2.2. *A submodule* of M is a subgroup N of M that is closed under the action of R.

Definition 2.3. We say that N is *irreducible* if the only submodules of N are 0 and N.

Definition 2.4. Given a finite group G, we define the *group ring* $\mathbb{C}G$ to be the set of all formal sums $a_1g_1 + \cdots a_ng_n$, where $a_i \in \mathbb{C}$, $g_i \in G$. We treat the elements of G as a basis for the group ring as a vector space over $\mathbb{C}$. We define addition and multiplication as follows: Let $a_i, b_i \in \mathbb{C}$, $g_i \in G$. Then

$$\sum_{i=1}^{n} a_i g_i + \sum_{i=1}^{n} b_i g_i = \sum_{i=1}^{n} (a_i + b_i) g_i,$$
$$\sum_{i=1}^{n} a_i g_i \times \sum_{i=1}^{n} b_i g_i = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i b_j)(g_i g_j).$$

2.2 Representation Theory

We will now introduce the basic concepts that are fundamental to this research.

Definition 2.5. Given a $\mathbb{C}G$-module V with basis $\{v_1, \cdots, v_n\}$, the corresponding *representation* of G is the homomorphism $\Phi : G \to GL_n(\mathbb{C})$, determined by $\Phi(g) = [gv_1, \cdots, gv_n]$.

Definition 2.6. We say that two representations ψ and Φ are *equivalent* or *isomorphic* if there is an invertible $n \times n$ matrix T such that $T\psi(g)T^{-1} = \Phi(g)$ for all $g \in G$.

Definition 2.5 relied upon the chosen basis of V. However, suppose we had selected a different basis $\{w_1, \cdots, w_n\}$ of V. This choice of basis would yield a representation τ of G. We can construct the change-of-basis matrix P and thus we have that $P\Phi(g)P^{-1} = \tau(g)$ for all $g \in G$. So we see that any $\mathbb{C}G$-module yields a unique representation of G, up to isomorphism.

Similarly, suppose M and N are isomorphic $\mathbb{C}G$-modules. If we fix bases of M and N, and consider the corresponding representations ψ and Φ, then let $S : M \to N$ be an isomorphism with respect to these bases. It follows that $S\psi(g)S^{-1} = \Phi(g)$ for all $g \in G$ and thus we see that *each isomorphism class of $\mathbb{C}G$ modules gives rise to a unique isomorphism class of representations of G* [4].

Likewise, a representation Φ gives an action of G on a vector space V that is independent of the chosen basis of V. We can extend this action linearly to obtain an action of $\mathbb{C}G$ on V, thus turning V into a $\mathbb{C}G$-module:

$$\left(\sum_{i=1}^{n} \alpha_i g_i\right)(v) = \sum_{i=1}^{n} \alpha_i \Phi(g_i)(v).$$

In this way, a representation of G uniquely determines a $\mathbb{C}G$-module. Thus we have a direct correspondence between $\mathbb{C}G$-modules and representations of G [4]. We will refer to representations of G and $\mathbb{C}G$-modules interchangably, depending upon the context.

We now consider the irreducibility of $\mathbb{C}G$-modules. Even if a $\mathbb{C}G$-module is not irreducible, we can still describe it in terms of irreducible $\mathbb{C}G$-modules.

Theorem 2.7. *Any $\mathbb{C}G$-module is the direct sum of irreducible $\mathbb{C}G$-modules.*

Definition 2.8. Given a representation Φ of G corresponding to the $\mathbb{C}G$-module V, we say that Φ is *irreducible* if V is irreducible.

Based upon our definition, we see that each representation of G is a direct sum of irreducible representations of G.

Remark 2.9. We will often refer to an irreducible representation as an *irrep*, which is standard in much of the literature.

Example 2.10. An important representation of G is obtained by considering $\mathbb{C}G$ as a $\mathbb{C}G$-module; in other words, $\mathbb{C}G$ acts on $\mathbb{C}G$ by left multiplication. Let the elements of G be our basis for $\mathbb{C}G$. Then the representation given by this module is called the *regular representation* of G.

Example 2.11. A simple example of a representation is the *trivial representation*. Let G be any finite group, and define a one-dimensional representation of G by $\Phi(g) = (1)$ for all elements of G. If we consider the regular representation of $\mathbb{C}G$, the element $\sum_{g\in G} g$ will generate the module corresponding to the trivial representation of G.

We will now consider all of the irreducible modules of G. There are only finitely many isomorphism types of irreducible modules of G. Consider $\mathbb{C}G$ as a $\mathbb{C}G$-module. Given an irreducible $\mathbb{C}G$-module M, we take the sum of all irreducible modules isomorphic to M that are contained in $\mathbb{C}G$ to obtain the *isotypic subspace* of $\mathbb{C}G$ corresponding to M. We can now consider the *isotypic decomposition* of $\mathbb{C}G$:

Theorem 2.12. *(Wedderburn) Let G be a finite group with h conjugacy classes. Then*

$$\mathbb{C}G \cong W_1 \oplus \cdots \oplus W_h$$

where h is the number of conjugacy classes of G and the W_i are the isotypic subspaces of $\mathbb{C}G$. This decomposition is unique.

This result leads to an even more useful and specific description of this decompostion (see [4]):

Theorem 2.13. *Let G be a finite group with h conjugacy classes.*

1. *$\mathbb{C}G$ has h distinct isomorphism types of irreducible modules, say of dimensions $n_1, \cdots, n_h$.*
2. *$\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_h}(\mathbb{C})$, where $M_{n_i}(C)$ is the matrix ring of $n_i \times n_i$ matrices with coefficients in $\mathbb{C}$.*
3. *$\sum_{i=1}^{h} n_i^2 = \mid G \mid$.*

This canonical decompositon of $\mathbb{C}G$ is useful because a decomposition of $\mathbb{C}G$ into irreducible submodules is not unique in general. For every irreducible $\mathbb{C}G$-module M of dimension greater than 1, $\mathbb{C}G$ contains infinitely many isomorphic copies of the module, and if W_i is the isotypic space corresponding to M then we can write W_i as the direct sum of n_i copies of M in infinitely many ways.

An isomorphism $\Phi : \mathbb{C}G \to M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_h}(\mathbb{C})$ specifies a decomposition of each isotypic subspace into a direct sum of irreducible modules, where the irreducible modules correspond to the columns of the matrices, which are invariant under the left action of matrix multiplication.

Definition 2.14. Given a representation Φ of G, we define the *character* χ of Φ by $\chi(g) := \text{tr}(\Phi(g))$, where tr($\Phi(g)$) denotes the trace of $\Phi(g)$.

The character is a *class function*; it assigns the same value to any element of a given conjugacy class of G.

Definition 2.15. We say a character is *irreducible* if its corresponding representation is irreducible.

We will often refer to a representation by its character, since *two representations are equivalent if and only if they have the same character* [4].
We can give the dimension of a representation M in terms of its character χ, since dim(M) = $\chi(1)$.

Definition 2.16. With knowledge of all the non-isomorphic irreducible representations of G, one can construct the *character table* of G. This table will have h rows and h columns, where h is the number of conjugacy classes of G. Each row corresponds to a different irreducible representation of G. Each column corresponds to a different conjugacy class of G. The (i,j)-th entry is the character that the ith irreducible representation of G assigns to a representative from the jth conjugacy class of G. We will list a representative from each conjugacy class at the top of each column.

Example 2.17. Let $G = \mathbb{Z}/2\mathbb{Z} = \{1, g\}$. Then there are only two irreducible submodules of $\mathbb{C}G$, generated by $(1+g)$ and $(1-g)$. Both are one-dimensional. $(1+g)$ corresponds to the trivial representation of G, so the value of its character on each element is 1.
Let Φ denote the representation corresponding to $(1-g)$. Now, $1(1-g) = (1-g)$ and $g(1-g) = -(1-g)$ so we know that $\Phi(1) = (1)$ and $\Phi(g) = (-1)$. Thus tr$(\Phi(1)) = 1$ and tr$(\Phi(g)) = -1$. Now we can construct the character table:

	1	g
χ_1	1	1
χ_2	1	-1

Example 2.18. The following is the character table of $\mathbb{Z}/3\mathbb{Z}$, where ω is a primitive third root of unity.

	1	g	g^2
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

Example 2.19. Let $G = Q_8$, the quaternion group of order 8. The character table of Q_8 is as follows:

	1	-1	i	j	k
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

It is worthy of note that the character tables of D_8 and Q_8 are the same, but that the two groups are non-isomorphic. Thus we cannot in general distinguish a group by its character table.

Definition 2.20. Given a representation of H with character χ, we can define the *induced character* of χ on G to be

$$(\chi \uparrow G)(g) = \frac{1}{\mid H \mid} \sum_{x \in G} \chi(x^{-1}gx)$$

where

$$\chi(x^{-1}gx) = \begin{cases} \chi(x^{-1}gx) & \text{if } x^{-1}gx \in H, \\ 0 & \text{if } x^{-1}gx \notin H. \end{cases}$$

Restricting a representation from G to H is much more straightforward.

Definition 2.21. If χ is a character of G and $H \leq G$, then the *restriction* of χ to H is defined by $(\chi \downarrow H)(h) = \chi(h)$ for all $h \in H$.

Example 2.22. Consider the alternating representation of S_3, which corresponds to the one-dimensional $\mathbb{C}S_3$-module generated by $1 - (12) - (13) - (23) + (123) + (132)$. Each element of S_3 is assigned the sign of its permutation as a character. When we restrict to A_3, then every element has sign 1 and thus the representation assigns 1 to every element of the group. Thus the restriction of this representation to A_3 is actually equal to the trivial representation of A_3.

An irreducible character χ of G will not necessarily restrict to another irreducible character of H, but $\chi \downarrow H = \sum \alpha_i \psi_i$, where $\alpha_i \in \mathbb{N}$ and the ψ_i are irreducible characters of H. Suppose we would like to know if a certain irreducible character γ of H is contained in the restriction of χ to H. We can make use of a formula that indicates how many copies of γ are contained in this restriction.

Definition 2.23. The *inner product* in G of two characters χ and ψ of G is given by the following formula:

$$\langle \chi, \psi \rangle_G = \frac{1}{\mid G \mid} \sum_{g \in G} \chi(g)\overline{\psi(g)}.$$

Note that $\overline{\psi(g)} = \psi(g^{-1})$ ([4]), so the above formula becomes

$$\langle \chi, \psi \rangle_G = \frac{1}{|\, G \,|} \sum_{g \in G} \chi(g)\psi(g^{-1}).$$

It is shown in [4] that the irreducible characters of G form an orthonormal basis for the space of all class functions: If $\{\chi_1, \cdots, \chi_h\}$ is a complete set of irreducible characters of G, then $\langle \chi_i, \chi_j \rangle_G = \delta_{ij}$. Each character of G can be uniquely expressed as a linear combination $\sum_{i=1}^{h} a_i \chi_i$, where $a_i \in \mathbb{N}_0$. If ψ_1 and ψ_2 are any two characters of G, then we know that $\psi_1 = \sum_{i=1}^{h} d_i \chi_i$ and $\psi_2 = \sum_{i=1}^{h} e_i \chi_i$, where $d_i, e_i \in \mathbb{N}_0$. It follows, as shown in [8], that

$$\langle \psi_1, \psi_2 \rangle_G = \sum_{i=1}^{h} d_i e_i.$$

Chapter 3

Separating Sets

Given a $\mathbb{C}G$-module V and a vector $f \in V$, we would like to compute the projections of f onto the isotypic subspaces of V. Some of the most well-known applications of such projections occur in the case of the symmetric group (see, for example, [13]). In this section, we will consider how to compute these isotypic projections. We can calculate the projections directly in $\mathbb{C}G$ with the use of a formula, as shown in [4].

Proposition 3.1. *Let W be an isotypic subspace of $\mathbb{C}G$, and let χ be the character of the irreducible submodule corresponding to W. Define*

$$z := \frac{\chi(1)}{\mid G \mid} \sum_{g \in G} \chi(g^{-1})g.$$

The isotypic projection of f onto W is then given by zf.

Computing these projections directly can be very time-inefficient, since to compute the projections we will need to sum up a potentially tremendous number of terms if G is of large order. But the whole process often becomes much easier if we have a separating set for G with respect to the module V.

Definition 3.2. *A separating set* for G with respect to V is a set of simultaneously diagonalizable linear operators $\{T_1, \cdots, T_s\}$ of $\mathbb{C}G$ that distinguish the isotypic subspaces of V with their eigenspaces. Each isotypic subspace will equal exactly one intersection of eigenspaces $E_1 \cap \cdots \cap E_s$, where E_i is an eigenspace of T_i. This means that if W_i and W_j are distinct isotypic subspaces of V, then some T_k assigns a different eigenvalue to each of them.

Example 3.3. Recall Example 2.17 in which we examined the irreducible representations of $\mathbb{Z}/2\mathbb{Z}$. The irreps are also the isotypic subspaces in this case. The class sum g assigns an eigenvalue of 1 to χ_1 and an eigenvalue of -1 to χ_2. Thus the two eigenspaces of g, both of which are one-dimensional, are exactly the isotypic spaces of $\mathbb{C}\mathbb{Z}/2\mathbb{Z}$.

This is extremely useful to us because if we want to compute the isotypic projections of $f \in V$, we can accomplish this by projecting f onto the eigenspaces of T_1, then projecting the results onto the eigenspaces of T_2, and so on. When we finish projecting onto the eigenspaces of T_s, each eigenspace projection will be a different isotypic projection of f.

3.1 Class Sums

The class sums of G can be used to construct separating sets. Recall that, given a conjugacy class $K \in G$, the corresponding class sum is the following element of the group ring:

$$C^K = \sum_{g \in K} g.$$

We would now like to determine the *eigenvalue* $\lambda_W(C^K)$ assigned by C^K to a d-dimensional irrep W of G. Since the trace of a linear transformation is the sum of its eigenvalues, $\mathrm{tr}(C^K) = d\lambda_W(C^K)$. But we also know that

$$\mathrm{tr}(C^K) = \sum_{g \in K} tr(g) = \mid K \mid \chi_W(g)$$

where g is any element of K. As discussed in [11], this leads to the following formula for $\lambda_W(C^K)$:

Proposition 3.4. *Let W be an irreducible module of G with corresponding character χ. Then the class sum of K will assign the eigenvalue*

$$\lambda_W(C^K) = \mid K \mid \frac{\chi(g)}{dim(W)}$$

to W, where g is any element of K.

Example 3.5. Consider the trivial representation of $\mathbb{C}S_3$, which in the regular representation of $\mathbb{C}S_3$ corresponds to the module generated by $(id + (12) + (13) + (23) + (123) + (132))$. The class sum of transpositions is $(12) + (13) + (23)$, and it will assign an eigenvalue of 3 to any element of this one-dimensional irreducible submodule:

$$\begin{aligned}&((12) + (13) + (23))\,(id + (12) + (13) + (23) + (123) + (132))\\&= 3\,(id + (12) + (13) + (23) + (123) + (132))\,.\end{aligned}$$

Definition 3.6. Given the character table of a group G with h conjugacy classes $K_1, \cdots, K_h$, we can construct the *modified character table of G* by scaling the (i,j)-th entry, $i, j \in \{1, 2, \cdots, h\}$, by $\frac{|K_j|}{\chi_i(1)}$. We then obtain a table specifying the eigenvalue that each class sum assigns to each irreducible representation.

Definition 3.7. The *semi-modified* character table of G is obtained from the character table by dividing the ith row, $1 \leq i \leq h$, by $\chi_i(1)$, but not scaling the columns by the sizes of the conjugacy classes.

This table is more convenient to construct than the modified character table. It is equivalent to the modified character table for the purpose of finding a separating set because if χ, ψ are distinct characters and C^K is the class sum of a conjugacy class K, then we see that

$$\lambda_\chi(C^K) = \lambda_\psi(C^K) \Leftrightarrow \frac{\mid C \mid \chi(K)}{\chi(1)} = \frac{\mid C \mid \psi(K)}{\psi(1)} \Leftrightarrow \frac{\chi(K)}{\chi(1)} = \frac{\psi(K)}{\psi(1)}.$$

Even though the semi-modified character table will not actually contain the eigenvalues that each class sum assigns to each irrep, it will still allow us to determine a set of class sums that will distinguish all of the irreps of G and will be easier to compute from the original character table.

Example 3.8. For $G = \mathbb{Z}/2\mathbb{Z}$, the character table is actually equal to the modified character table, because each conjugacy class has size 1 and each irreducible module has dimension 1. So the modified character table of G is the same as the character table, which was given in Example 2.17:

	1	g
χ_1	1	1
χ_2	1	-1

Example 3.9. For $G = Q_8$, the modified character table is obtained from the character table (given in Example 2.19) with the additional knowledge that the conjugacy classes are $\{1\}, \{-1\}, \{i, i^{-1}\}, \{j, j^{-1}\}, \{k, k^{-1}\}$ and of corresponding sizes 1, 1, 2, 2, and 2.

	1	-1	i	j	k
χ_1	1	1	2	2	2
χ_2	1	1	-2	2	-2
χ_3	1	1	2	-2	-2
χ_4	1	1	-2	-2	2
χ_5	1	-1	0	0	0

It turns out that the set of all class sums of G will always suffice as a separating set [12]. But sometimes we can successfully form a separating set with fewer class sums. A smaller separating set means that we have fewer eigenspace projections to compute, and this can lead to faster computation of the isotypic projections, although this is not always the case [5]. Finding a separating set of minimal size with the use of the modified character table is NP-complete, as shown in [5]. However, with knowledge of the structure of the representation theory of a group, we have other tools besides the character table with which to seek small separating sets.

Katriel [9] has examined the use of class sums as separating sets for the symmetric group. In particular, he has examined how many single-cycle classes must be employed to decompose $\mathbb{C}S_n$ into its isotypic components. The order on the single-cycle classes is given by first considering the transpositions, then the three-cycles, then the four-cycles, and so on.

Katriel builds a separating set out of class sums taken in this order. He has shown that the transpositions suffice as a separating set for S_n when $n \leq 5$ and that the first four class sums suffice as long as $n \leq 41$. This is remarkable, considering the size of S_{41}. What follows is the data that appears at the end of [9].

Number of Class Sums Needed to Decompose $\mathbb{C}S_n$	Maximum S_n:
1	S_5
2	S_{14}
3	S_{23}
4	S_{41}

This is an amazing result, when one considers that there are 44,583 irreps of S_{41}.

We seek similar results for other groups. We would like to know the minimal size of a separating set for a group and which class sums to use to accomplish this. Before presenting our results for direct products and the dihedral and alternating groups, we give a simple example in order to illustrate how separating sets can be found.

3.1.1 Example: The Quaternion Group

For the quaternion group of order 8, there are four non-trivial class sums, but we only need to use two of them, $i+i^{-1}$ and $j+j^{-1}$. Recall the modified character table of Q_8:

	1	-1	i	j	k
χ_1	1	1	2	2	2
χ_2	1	1	-2	2	-2
χ_3	1	1	2	-2	-2
χ_4	1	1	-2	-2	2
χ_5	1	-1	0	0	0

We can immediately see that the two class sums specified above do, in fact, assign a different list of eigenvalues to each irreducible representation: χ_1 receives $\{2,2\}$, χ_2 receives $\{-2,2\}$, χ_3 receives $\{2,-2\}$, χ_4 receives $\{-2,-2\}$, and χ_5 receives $\{0,0\}$. It can further be observed that this is a separating set of minimal size, for no single column of the modified character table lacks a repeated eigenvalue. In other words, given any single class sum of $\mathbb{C}Q_8$, there are two distinct irreducible modules of Q_8 to which it assigns the same eigenvalue. $\{j+j^{-1}, k+k^{-1}\}$ and $\{i+i^{-1}, k+k^{-1}\}$ are also separating sets of size 2 for Q_8.

3.2 Separating Sets for Direct Products of Groups

If G and H are two finite groups for which the irreducible representations are known, then we can construct all of the irreducible representations of $G \times H$, as shown in [8].

Theorem 3.10. *Let G and H be two finite groups and let $\{\chi_1, \cdots, \chi_s\}$ and $\{\psi_1, \cdots, \psi_t\}$ be the complete sets of irreducible characters of G and H, respectively. Then a complete set of irreducible characters of $G \times H$ is given by*

$$\{\chi_i\psi_j\}_{\substack{1\leq i\leq s\\ 1\leq j\leq t}},$$

and if $g \in G$ and $h \in H$ we have $\chi_i\psi_j(g,h) = \chi_i(g)\psi_i(h)$.

Given two groups G and H for which separating sets (of class sums) of minimal size are known, we can construct a separating set for the group $G \times H$ with the use of the separating sets for the original groups. The set we construct, interestingly enough, will not always be of minimal size.

Theorem 3.11. *Given a minimal separating set $\{R_1, \cdots, R_k\}$ for G and a minimal separating set $\{T_1, \cdots, T_m\}$ for H, we can construct a separating set for $G \times H$ of size $k + m$.*

Proof. The character table of $G \times H$ is the tensor product of the character tables of G and H [8]. Let $A_{f \times f} = (a_{ij})$ and $B_{g \times g} = (b_{ij})$ be the character tables of G and H respectively. For simplicity, we will consider the semi-modified character table T of $G \times H$. To compute T, we divide each row of $A \otimes B$ by the first entry of that row.
But since $A \otimes B$ is an $fg \times fg$ matrix of the form

$$\begin{pmatrix} a_{11}B & \cdots & a_{1f}B \\ \vdots & & \vdots \\ a_{f1}B & \cdots & a_{ff}B \end{pmatrix}$$

we can see that the semi-modified character table of $G \times H$ is equivalent to the tensor product of the semi-modified character tables of G and H. If B_s denotes the semi-modified character table of H, then the semi-modified character table T of $G \times H$ will be

$$\begin{pmatrix} B_s & \frac{a_{12}}{a_{11}}B_s & \cdots & \frac{a_{1f}}{a_{11}}B_s \\ \vdots & \vdots & & \vdots \\ B_s & \frac{a_{f2}}{a_{f1}}B_s & \cdots & \frac{a_{ff}}{a_{f1}}B_s \end{pmatrix}.$$

Since the conjugacy classes of $G \times H$ are of the form (C,D) where C is a conjugacy class of G and D is a conjugacy class of H, consider the following classes of $G \times H$:

$$\{(R_1, 1), \cdots, (R_k, 1), (1, T_1), \cdots, (1, T_m)\}.$$

Note that if χ is a character of G and ψ is a character of H, then

$$\frac{\chi\psi(R_i, 1)}{\chi\psi(1, 1)} = \frac{\chi(R_i)\psi(1)}{\chi(1)\psi(1)} = \frac{\chi(R_i)}{\chi(1)},$$

and similarly

$$\frac{\chi\psi(1, T_j)}{\chi\psi(1, 1)} = \frac{\psi(T_j)}{\psi(1)}.$$

This is assumed below.

Since the $\{R_i\}$ distinguish the irreps of G, the $\{(R_i, 1)\}$ will serve to distinguish the set of irreps corresponding to rows $\{a_{i1}B \cdots a_{if}B\}$ from the set of irreps corresponding to rows $\{a_{j1}B \cdots a_{jf}B\}$, $i \neq j$. For each $(R_i, 1)$, the corresponding class sum will assign the same eigenvalue to the first g irreps of $G \times H$, and the same eigenvalue to the next g irreps, and so on. Thus, by the separating set property of $\{R_i\}$, any two sets of irreps corresponding to rows $\{a_{i1}B \cdots a_{if}B\}$ and $\{a_{j1}B \cdots a_{jf}B\}$ are distinguished by the class sum corresponding to some $(R_k, 1)$.

It remains to distinguish all the irreps corresponding to the rows $\{a_{i1}B \cdots a_{if}B\}$, for a fixed i. $\{(1, T_j)\}$ will accomplish this, since the $\{T_j\}$ distinguish the irreps of H with their eigenvalues. Consider a block of the form $\frac{a_{i1}}{a_{11}}B_s$. Multiplying B_s by a scalar has no effect upon whether a particular subset of columns of B_s suffices to distinguish all of the rows of B_s. Thus $\{(1, T_j)\}$ distinguishes the first g irreps of $G \times H$ from one another, and the next g irreps from one another, and so on. Therefore we have distinguished all of the irreps of $G \times H$ with the given set of class sums, which is thus a separating set for $G \times H$ of size $k + m$. □

Theorem 3.12. *The separating set given in the proof of Theorem 3.11 is not necessarily of minimal size.*

Proof. In the case of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, each group in the direct product has a minimal separating set of size 1, consisting of the generator of the group. Our construction would yield a separating set of size 2. But consider the semi-modified character table of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, as shown below. The character table is obtained from tensoring the character tables of $\mathbb{Z}/2\mathbb{Z}$ (see Example 2.17) and $\mathbb{Z}/3\mathbb{Z}$ (see Example 2.18) together, and is equal to the semi-modified character table because all of the irreps are one-dimensional [8]. In the following table, ω is a primitive third root of unity.

	(1,1)	(1,ω)	(1,ω^2)	(-1,1)	(-1,ω)	(-1,ω^2)
χ_1	1	1	1	1	1	1
χ_2	1	ω	ω^2	1	ω	ω^2
χ_3	1	ω^2	ω	1	ω^2	ω
χ_4	1	1	1	-1	-1	-1
χ_5	1	ω	ω^2	-1	$-\omega$	$-\omega^2$
χ_6	1	ω^2	ω	-1	$-\omega^2$	$-\omega$

Since the class sum corresponding to either $(-1, \omega)$ or $(-1, \omega^2)$ assigns different eigenvalues to all six irreps, it serves as a separating set of size 1.

This completes the proof. □

3.3 Separating Sets for the Dihedral Group

There is a very natural and simple separating set of size two that will work for any dihedral group. Recall that the dihedral group of order $2n$ is given by the following presentation:

$$D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle.$$

The separating set consists of the class sums corresponding to the classes of r and s. We will denote these class sums by C^r and C^s, respectively. To see why this always works, we will need the character table of the dihedral group D_{2n}. The character table depends upon the parity of n. The tables employed in the proof are derived in [8].

Theorem 3.13. *$\{C^r, C^s\}$ form a separating set of minimal size for the dihedral group D_{2n}.*

Proof. Case 1: n is odd. Then the character table of D_{2n} is as follows:

	1	r^a $(1 \leq a \leq \frac{n-1}{2})$	s
χ_1	1	1	1
χ_2	1	1	-1
ψ_j $(1 \leq j \leq \frac{n-1}{2})$	2	$2 \cdot \cos\left(\frac{2\pi}{n} ja\right)$	0

We can construct the semi-modified character table easily:

	1	r^a $(1 \leq a \leq \frac{n-1}{2})$	s
χ_1	1	1	1
χ_2	1	1	-1
ψ_j $(1 \leq j \leq \frac{n-1}{2})$	1	$\cos\left(\frac{2\pi}{n} ja\right)$	0

Now we can see that the class sum corresponding to s serves to distinguish χ_1 and χ_2 from each other and from all the ψ_j. Now we show that the class sum corresponding to r will distinguish all of the ψ_j from one another. Suppose C^r assigns the same eigenvalue to ψ_x and ψ_y. Then this implies that

$$\cos\left(\frac{2\pi x}{n}\right) = \cos\left(\frac{2\pi y}{n}\right).$$

But since $j \in \{1, \cdots, \frac{n-1}{2}\}$ we know that

$$0 < \frac{2\pi j}{n} \leq \frac{\pi(n-1)}{n} < \pi.$$

Two distinct numbers in the open interval $(0, \pi)$ have distinct cosines, so it must be the case that $\psi_x = \psi_y$. Thus $\{C^r, C^s\}$ form a separating set for D_{2n} when n is odd.

Case 2: n is even. Then the character table of D_{2n} is:

	1	$r^{\frac{n}{2}}$	r^a $(1 \leq a \leq \frac{n}{2} - 1)$	s	rs
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$-1^{\frac{n}{2}}$	-1^a	1	-1
χ_4	1	$-1^{\frac{n}{2}}$	-1^a	-1	1
ψ_j $(1 \leq j \leq \frac{n}{2} - 1)$	2	$2(-1^j)$	$2 \cdot \cos\left(\frac{2\pi}{n} ja\right)$	0	0

and the semi-modified character table is as follows:

	1	$r^{\frac{n}{2}}$	r^a $(1 \leq a \leq \frac{n}{2} - 1)$	s	rs
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$-1^{\frac{n}{2}}$	-1^a	1	-1
χ_4	1	$-1^{\frac{n}{2}}$	-1^a	-1	1
ψ_j	1	-1^j	$\cos\left(\frac{2\pi}{n} ja\right)$	0	0

First we will see how C^r and C^s distinguish the χ_i from one another and from each ψ_j. C^s assigns each ψ_j an eigenvalue of 0 and assigns χ_i a nonzero eigenvalue. The two class sums assign distinct lists of eigenvalues to each χ_i: χ_1 receives the list $\{1, 1\}$, χ_2 receives the list $\{1, -1\}$, χ_3 receives $\{-1, 1\}$, and χ_4 receives $\{-1, -1\}$. Now we need to show that C^r suffices to distinguish all of the ψ_j from one another. But this is virtually the same situation as in Case 1, where

$$0 < \frac{2\pi j}{n} \leq \frac{\pi(n-2)}{n} < \pi$$

and since two distinct numbers in $(0, \pi)$ have different cosines, all of the ψ_j get different eigenvalues. Thus we see that $\{C^r, C^s\}$ is a separating set for D_{2n} when n is even.

Thus $\{C^r, C^s\}$ suffices as a separating set for the dihedral group. Furthermore, this is a separating set of minimal size, because no single class sum assigns a different eigenvalue to each irrep of D_{2n}, as is clear from the character table in each case. $\square$

The above proof relied upon the fact that the sequence

$$\left\{\cos\left(\frac{2\pi}{n}\right), \cdots, \cos\left(\frac{\pi(n-1)}{n}\right)\right\}$$

does not repeat. It follows naturally from these observations that perhaps if gcd(q,n)=1 then $\{C^{r^q}, C^s\}$ will suffice as a separating set.

Theorem 3.14. *For any number q relatively prime to n, $\{C^{r^q}, C^s\}$ will form a separating set for the dihedral group D_{2n}.*

Proof. First we will show that all the ψ_j are distinguished from one another by these class sums. From the proof of Theorem 3.13, we know that it suffices to show that if $1 \leq \{j,k\} \leq \frac{n-1}{2}$, then $\cos\left(\frac{2\pi jq}{n}\right) = \cos\left(\frac{2\pi kq}{n}\right)$ implies $j = k$. If the former is true, then there are two cases to consider.

Case 1: $\frac{2\pi jq}{n} = \frac{2\pi kq}{n} + 2\pi m$ for some $m \in \mathbb{Z}$. Then $(j-k)q = mn$ and since n and q are relatively prime we have that $n \mid (j-k)$. However, $j,k \in \{1, \cdots, \frac{n-1}{2}\}$ and thus their difference is bounded above by $\frac{n-1}{2}$.

Case 2: $\frac{2\pi jq}{n} = -\frac{2\pi kq}{n} + 2\pi m$. Then we have that $(j+k)q = mn$ and thus, since gcd(n,q)=1, $n \mid (j+k)$. But $j + k \leq \frac{n-1}{2} + \frac{n-1}{2} = n-1$ so this is impossible.

Thus C^{r^q} assigns a different eigenvalue to each irrep ψ_j, regardless of whether n is even or odd. In the case where n is odd, we still have χ_1 and χ_2 distinguished from each other and from all the ψ_j as before, so the parity of q is irrelevant. In the case where n is even, it can be seen from the table that we must have q odd; otherwise, we will not distinguish χ_1 from χ_3 or χ_2 from χ_4 with the two class sums. But this we already know to be true, because n is even and gcd(q,n)=1. Thus $\{C^{r^q}, C^s\}$ suffices as a separating set for the dihedral group when gcd(n,q)=1. $\square$

These results are interesting because we now have a group for which the class sums corresponding to a minimal set of generators form a separating set. This raises the following question:

Question 3.15. For which finite groups G do the class sums corresponding to a minimal set of generators of G form a separating set?

This question is interesting because there is a pattern in the cases we have seen. For the dihedral group, we found a collection of minimal sets of generators that correspond to minimal separating sets. In the case of abelian groups, the class sum corresponding to a generator is the generator itself and forms a separating set of size 1 [5]. For the quaternion group, i and j are a minimal set of generators and their corresponding class sums form a separating set of minimal size. However, there are symmetric groups, such as S_8, for which this does not happen, so the answer to the question is likely to be a quite complex one.

Chapter 4

The Representation Theory of the Symmetric Group

In this chapter we consider the representation theory of S_n. The irreducible representations of S_n are indexed by the partitions of n. The majority of the results of this thesis concern the alternating group, for which the irreducible representations can be derived in a very natural way from those of the symmetric group. Thus we must first have working knowledge of the irreducible representations of the symmetric group in order to proceed. Much of the following terminology and theorems are taken from [7], which provides a comprehensive treatment of the subject. We begin by considering all partitions of n.

Definition 4.1. We define a partition α of n to be a sequence of non-increasing, positive integers $(\alpha_1, \cdots, \alpha_k)$ such that $\alpha_1 + \cdots + \alpha_k = n$. The a_i are the *parts* of α.

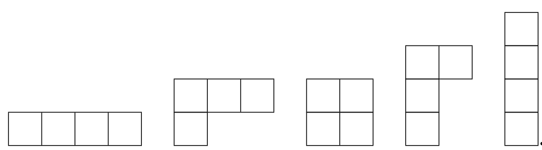
Each partition will correspond to an irreducible representation of S_n, as shown below, but first we will need some definitions and terminology.

Definition 4.2. We will place an order on the partitions of n.
If $\alpha = (\alpha_1, \cdots, \alpha_s)$ and $\beta = (\beta_1, \cdots, \beta_t)$ are two partitions of n, we will say that $\alpha < \beta$ if for some index i, $\alpha_1 = \beta_1, \cdots, \alpha_{i-1} = \beta i - 1$ and $\alpha_i < \beta_i$.

Definition 4.3. We will represent each partition as an object called a *Young diagram*. The diagram is a set of n boxes, placed into k rows so that there are a_1 boxes in the first row, a_2 boxes in the second row, and so on. The leftmost boxes of each row line up in a single column. The *content* of the box in the ith row and jth column is $j - i$. Given a box b in the ith row and jth column of a Young diagram, the *hook rooted at* b is defined to be the set of boxes to the right of b in the ith row and below b in the jth column, together with b itself. The hook's *arm* is the set of boxes in the hook to the right of b, and its *leg* is the set of boxes in the hook below b. The *diagonal hooks* of α are given by the set of hooks rooted at the (i,i)-th boxes of the diagram.

Definition 4.4. Given a partition α and its corresponding Young diagram, the *conjugate* of α, denoted $\alpha^{'}$, is the partition obtained by considering the number of boxes in each column of the Young diagram of α. If $\alpha = \alpha^{'}$, then we say that α is *self-conjugate*.

Example 4.5. Consider the case $n = 4$. Then the set of partitions of n is {4, 3+1, 2+2, 2+1+1, 1+1+1+1}. Thus we have five possible Young diagrams, and thus five irreducible representations of S_4. These Young diagrams are as follows:



Note that there are two pairs of conjugate partitions of n, and one self-conjugate partition of n.

We are almost ready to describe the construction of an irreducible representation of S_n from a partition α of n, but first we require another key idea.

Definition 4.6. Given a partition α of n, the *Young Subgroup* S_α of S_n is given by the direct product $S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times S_{\alpha_k}$. The Young subgroup S_α is isomorphic to the subgroup of S_n that permutes certain subsets of $\{1, 2, \cdots, n\}$ among themselves; in particular, S_α can be thought of as the set of $\pi \in S_n$ such that

$$\begin{aligned}
&\pi : \{1, 2, \cdots, \alpha_1\} \to \{1, 2, \cdots, \alpha_1\} \\
&\pi : \{\alpha_1 + 1, \cdots, \alpha_1 + \alpha_2\} \to \{\alpha_1 + 1, \cdots, \alpha_1 + \alpha_2\} \\
&\vdots \\
&\pi : \{n - \alpha_k + 1, \cdots, n\} \to \{n - \alpha_k + 1, \cdots, n\}.
\end{aligned}$$

Now consider the trivial and alternating representations of S_α:

$$IS_\alpha : \pi \vdash 1$$
$$AS_\alpha : \pi \vdash \text{ sgn}(\pi)$$

In other words, the trivial representation assigns character 1 to every element, and the alternating representation assigns the sign of the permutation as its character.
The following theorem, which is proved in [7], utilizes these concepts to provide an irreducible representation of S_n corresponding to α.

Theorem 4.7. *Let α be a partition of n. Then*

$$[\alpha] := IS_\alpha \uparrow S_n \cap AS_{\alpha'} \uparrow S_n$$

is an irreducible representation of S_n. Furthermore, the set

$$\{[\alpha] : \ \alpha \textit{ is a partition of } n\}$$

is a complete set of equivalence classes of irreducible representations of S_n.

It is natural to ask how the irreducible representations of S_{n-1} relate to those of S_n. What follows is a statement of the branching theorem for $S_n \downarrow S_{n-1}$ in terms of the notation that we have developed. This theorem is proved in [7].

Theorem 4.8. *(Branching Theorem) If $[\alpha]$ is an irreducible representation of S_n, where $\alpha = (\alpha_1, \cdots, \alpha_k)$ is a partition of n, then upon restriction to S_{n-1} we have*

$$[(\alpha_1, \cdots \alpha_k)] \downarrow S_{n-1} = [(\alpha_1 - 1, \alpha_2, \cdots \alpha_k)] + \cdots + [(\alpha_1, \cdots \alpha_{k-1}, \alpha_k - 1)].$$

Any term in this sum that does not correspond to a valid partition of $n-1$ is taken to be 0.

Chapter 5

The Alternating Group

In this chapter we discuss methods to decompose $\mathbb{C}A_n$ into invariant subspaces with the use of class sums. Section 1 describes the irreducible representations of A_n and presents the branching rules for irreps of A_n when restricted to A_{n-1}. Section 2 presents results regarding separating sets for A_n. For a useful source on this topic, see [7].

Theorem 5.1. *Given a conjugacy class P of S_n of cycle type $(p_1)(p_2)\cdots(p_k)$, it is either a conjugacy class in A_n, or it splits into two conjugacy classes P^+ and P^-. The latter occurs exactly when $p_1 + \cdots + p_k$ is a partition of n into distinct odd parts. We fix the notation by declaring that*

$$U = (1, \cdots, p_1)(p_1 + 1, \cdots, p_1 + p_2) \cdots (n - p_k + 1, \cdots, n) \in P^+.$$

Example 5.2. (123) is a representative of the conjugacy class

$$K = \{(123), (132)\}$$

of S_3. But in A_3, these elements are divided into two conjugacy classes, $K^+ = \{(123)\}$ and $K^- = \{(132)\}$.

In the following sections, we will view S_{n-1} as the subgroup of S_n consisting of permutations that fix n. Our embedding of A_{n-1} in A_n follows directly from this.

5.1 Representation Theory

The representation theory of the alternating group is strongly connected to that of the symmetric group. Knowledge of the symmetric group makes it much simpler to understand the workings of the irreducible submodules of $\mathbb{C}A_n$. We can understand the representation theory of A_n by knowing how its irreducible submodules are obtained from those of S_n. Given a partition α of n, there are two possibilities, depending on whether α is self-conjugate.

If $\alpha \neq \alpha^{'}$, then the two corresponding irreducible representations of S_n are actually isomorphic irreducible representations of A_n. In terms of modules, the irreducible $\mathbb{C}S_n$-modules corresponding to $[\alpha]$ and $[\alpha^{'}]$ are irreducible, isomorphic $\mathbb{C}A_n$-modules. Without loss of generality, assume $\alpha > \alpha^{'}$ with respect to Definition 4.2, and denote the corresponding irreducible representation of A_n by $[\alpha]$.

If $\alpha = \alpha^{'}$, then the corresponding irreducible representation $[\alpha]$ of S_n splits into two *conjugate* irreducible representations, $[\alpha]^+$ and $[\alpha]^-$, upon restriction to A_n. Each of these representations is half the dimension of the original.

Every irreducible representation of A_n is associated with a partition of n. Every irrep of A_n of the form $[\alpha]$ is associated with α and not $\alpha^{'}$, because we assumed $\alpha > \alpha^{'}$. Every irrep of A_n of the form $[\alpha]^+$ or $[\alpha]^-$ is associated with the partition $\alpha = \alpha^{'}$ of n.

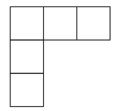
We now consider how the character table of A_n is obtained from the character table of S_n. Any irrep of A_n of the form $[\alpha]$ is actually an irrep of S_n, and thus assigns the same character to each conjugacy class of A_n that it assigned to the corresponding class of S_n. If a conjugacy class of S_n splits in A_n, then $[\alpha]$ assigns the same character to each resulting class sum in A_n that the original class in S_n received.

There is a very natural bijection between partitions of n into distinct odd parts and irreps of S_n that split upon restriction to A_n. Consider a self-conjugate Young diagram of a partition α of n. Consider all the diagonal hooks $h_{11}, \cdots, h_{kk}$ of α. Each such hook has leg length equal to arm length and no two such hooks can have the same length. Thus if we let p_i denote the length of h_{ii}, we see that $p_1 + \cdots + p_k$ corresponds to a partition of n into distinct odd parts.

Definition 5.3. If α and $\{p_i\}$ are as above, denote the conjugacy class of S_n of cycle type $(p_1)\cdots(p_k)$ by C_α. In A_n, this class splits into two classes, denoted C_α^+ and C_α^-.

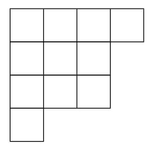
Example 5.4. For A_5, the partition of 5 given by $3+1+1$ has corresponding

Young diagram



and we can see that there is exactly one diagonal hook, which is of length $p_1 = 5$. Thus there will be two conjugacy classes of 5-cycles in A_5.

Example 5.5. The partition of 11 given by $4+3+3+1$ has corresponding Young diagram



and the diagonal hook lengths are given by $p_1 = 7$, $p_2 = 3$, and $p_3 = 1$. Thus there will be two conjugacy classes of cycle type $(7)(3)(1)$ in A_{11}.

An irrep $[\alpha]^+$ or $[\alpha]^-$ of A_n assigns half of the original character of $[\alpha]$ to each class, with the exception of C_α^+ and C_α^-. If C_α is of type $(p_1)\cdots(p_k)$, then $[\alpha]^+$ and $[\alpha]^-$ assign different characters to C_α^+ and C_α^-, as follows:

$$\chi_{\alpha^\pm}(C_\alpha^\pm) = \frac{1}{2}\left((-1)^{\frac{n-k}{2}} + \sqrt{(-1)^{\frac{n-k}{2}} p_1 \cdots p_k}\right)$$
$$\chi_{\alpha^\pm}(C_{\alpha^\mp}) = \frac{1}{2}\left((-1)^{\frac{n-k}{2}} - \sqrt{(-1)^{\frac{n-k}{2}} p_1 \cdots p_k}\right).$$

Example 5.6. We will see how the character table of A_3 is obtained from that of S_3. As seen before, the alternating representation of S_3 restricts to the trivial representation of A_3, and thus the trivial and alternating representations of S_3 both restrict to the trivial representation of A_3. The two-dimensional irrep χ_3 of S_3 corresponds to the self-conjugate partition $2+1$ and thus splits into two one-dimensional irreps χ_3^+ and χ_3^- upon restriction to A_3. The class of 3-cycles splits into two classes, $(3)^+$ and $(3)^-$, upon restriction to A_3. Before presenting the character table of A_3, we present a sample calculation of one of the new characters. Here $n = 3, k = 1$, and $p_1 = 3$.

$$\chi_3^+\left((3)^+\right) = \frac{1}{2}\left((-1)^{\frac{3-1}{2}} + \sqrt{(-1)^{\frac{3-1}{2}} 3}\right) = \frac{1}{2}\left(-1 + \sqrt{-3}\right).$$

We will denote this character by ω and its conjugate by $\overline{\omega}$. Note that ω is a primitive third root of unity. Following are the character tables of S_3 and of A_3, for comparison.

Character Table of S_3			
	id	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Character Table of A_3			
	id	(123)	(132)
ψ_1	1	1	1
ψ_2	1	ω	$\overline{\omega}$
ψ_3	1	$\overline{\omega}$	ω

This character table should come as no surprise; A_3 is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ and we know the character table of the latter from Example 2.18.

As we will see, some of the results in this section require us to know exactly when an element of A_n and its inverse are in the same conjugacy class.

Definition 5.7. We say that a conjugacy class K of a group G is *ambivalent* if for all $g \in G$, $g \in K$ if and only if $g^{-1} \in K$.

We know that S_n is ambivalent because the conjugacy classes are determined by cycle type. It is known exactly when a class of A_n is ambivalent [7].

Theorem 5.8. *Suppose K is a conjugacy class of A_n of cycle type $(p_1)\cdots(p_k)$, where $p_1+\cdots+p_k$ is a partition of n into distinct odd parts. Then K is ambivalent if and only if the number of p_i that are congruent to 3 modulo 4 is even.*

A simpler way to determine if a class of A_n is ambivalent can be derived from Theorem 5.8.

Corollary 5.9. *Let α be a self-conjugate partition of n, and let C_α of cycle type $(p_1)\cdots(p_k)$ be defined as in Definition 5.3. Then C_α^+ and C_α^- are ambivalent if and only if $\frac{n-k}{2}$ is even.*

Proof. If the classes $C_\alpha^\pm$ are ambivalent, then by Theorem 5.8 there are an even number of p_i congruent to 3 modulo 4. If $p_i \equiv 3$ modulo 4, then $p_i - 1 \equiv 2$ modulo 4 and so $\frac{p_i-1}{2}$ is odd. If $p_i \equiv 1$ modulo 4, $\frac{p_i-1}{2}$ is even. Thinking in terms of the Young diagram of α, we see that since there are k diagonal hooks, $\frac{n-k}{2}$ is the sum of the leg lengths (or the arm lengths) of

the diagonal hooks of the diagram. But this sum is given by $\sum_{j=1}^{k} \frac{p_j-1}{2}$, and thus since there are an even number of parts congruent to 3 modulo 4, we see that this sum is even and so $\frac{n-k}{2}$ is even.
Conversely, if $\frac{n-k}{2}$ is even, there must be an even number of parts p_i congruent to 3 modulo 4, and thus the classes $C_\alpha^\pm$ are ambivalent. □

Now we will develop the branching rules for $A_n \downarrow A_{n-1}$. Similar rules are given without proof in [10], but they appear to be incomplete and do not cover certain cases. Thus we will here present a more complete derivation of the rules. We will use the following notation. Given an element $g \in S_n$ and an irrep $[\alpha]$ (or $[\alpha]^\pm$), we will denote the character of g on the irrep by $[\alpha](g)$ (or $[\alpha]^\pm(g)$). We will also need to make use of a theorem from [7]:

Theorem 5.10. *Given a self-conjugate partition α, if C_α is defined as in Definition 5.3 then $[\alpha](g) = (-1)^{\frac{n-k}{2}}$ for all $g \in C_\alpha$.*

Now we are ready to state the branching rules for the alternating group.

Theorem 5.11. *(Branching Theorem for A_n) Let Φ be an irreducible representation of A_n. Consider the partition $\alpha = (\alpha_1, \cdots, \alpha_m)$ associated with Φ, where the diagonal hook lengths of α are $\{p_1, \cdots, p_k\}$ with $p_i \geq p_{i+1}$. In the following, if γ is not a valid partition, then we take $[\gamma]$ to be 0.*
Case 1: α is not self-conjugate and $\Phi = [\alpha]$. Then we have that

$$[\alpha] \downarrow A_{n-1} = [(\alpha_1 - 1, \alpha_2, \cdots \alpha_m)] + \cdots + [(\alpha_1, \cdots \alpha_{m-1}, \alpha_m - 1)]$$

where each term $[\beta]$ in the sum is taken to be $[\beta]^+ + [\beta]^-$ if β is self-conjugate.
Case 2: α is self-conjugate and all diagonal hook lengths of α are greater than 1. Then $\Phi = [\alpha]^+$ or $[\alpha]^-$ and we have

$$[\alpha]^\pm \downarrow A_{n-1} = \frac{1}{2}\left([(\alpha_1 - 1, \alpha_2, \cdots, \alpha_m)] + \cdots + [(\alpha_1, \cdots \alpha_{m-1}, \alpha_m - 1)]\right).$$

Case 3: α is self-conjugate and one of the diagonal hook lengths of α is 1. If that diagonal hook of length 1 occurs in the kth row of α, then if $\Phi = [\alpha]^+$ we have

$$\begin{aligned}[\alpha]^\pm \downarrow A_{n-1} = {} & [\alpha_1, \cdots, \alpha_{k-1}, \alpha_k - 1, \alpha_{k+1}, \cdots \alpha_m]^\pm \\ & + \frac{1}{2} \sum_{\substack{1 \leq i \leq m \\ i \neq k}} [\alpha_1, \cdots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1}, \cdots, \alpha_m].\end{aligned}$$

Proof. Case 1 follows from the fact that Φ is an irrep of S_n, and thus we can compute the restriction of Φ to A_{n-1} by the sequence of restrictions

$\Phi \downarrow S_{n-1} \downarrow A_{n-1}$. Our known branching rule for the symmetric group gives this restriction. This agrees with the rule given in [10].

For Case 2, consider the classes C_α^+ and C_α^- of A_n, as defined in Definition 5.3. Since all the cycles are of length greater than one, we lose these conjugacy classes upon restriction to A_{n-1}. However, these were the only classes upon which the two irreps $[\alpha]^+$ and $[\alpha]^-$ of A_n disagreed, and so we must have

$$[\alpha]^+ \downarrow A_{n-1} = [\alpha]^- \downarrow A_{n-1}.$$

Now consider $[\alpha] \downarrow S_{n-1} = \sum[\beta_y]$, where the $[\beta_y]$ are obtained by considering all valid Young diagrams obtained by deleting a box from a row of α's Young diagram, and then taking their corresponding irreps. None of the β_y can be self-conjugate, because the only way to delete a box from a self-conjugate Young diagram and obtain another self-conjugate Young diagram is by deleting the (k,k)-th box of the diagram. But we assumed that $p_k > 1$, so in particular the hook rooted at (k,k) has arm length greater than one and so we cannot delete this box. Furthermore, the set of $\{\beta_y\}$ is a set of conjugate pairs of partitions of $n-1$, because α is self-conjugate. Thus we have

$$\sum[\beta_y] \downarrow A_{n-1} = 2\sum[\gamma_z]$$

where the γ_z are obtained by considering all pairs $\{\beta_y, \beta_y^{'}\}$ and taking the larger partition from each pair.
Since $[\alpha] \downarrow A_n = [\alpha]^+ + [\alpha]^-$ and $[\alpha]^+ \downarrow A_{n-1} = [\alpha]^- \downarrow A_{n-1}$, we now have that

$$[\alpha]^\pm \downarrow A_{n-1} = \sum[\gamma_z]$$

or, in terms of the partition α itself, that

$$[\alpha]^\pm \downarrow A_{n-1} = \frac{1}{2}\left([(\alpha_1 - 1, \alpha_2, \cdots \alpha_m)] + \cdots + [(\alpha_1, \cdots \alpha_{m-1}, \alpha_m - 1)]\right),$$

as desired.

For Case 3, we will denote $[\alpha_1, \cdots, \alpha_{k-1}, \alpha_k - 1, \alpha_{k+1}, \cdots \alpha_m]^\pm$ by $[\delta]^\pm$. We will now make a few crucial observations.
First of all, note that $C_\delta^+ \subset C_\alpha^+$ and similarly $C_\delta^- \subset C_\alpha^-$. Recall the defining element U of C_δ^+ as given in Theorem 5.1. Note that since $p_k = 1$, U is of the form $(1, \cdots, p_1)(p_1 + 1, \cdots, p_1 + p_2) \cdots (n)$. This means that $U \in A_{n-1}$ is the defining element of C_δ^+ as well. This will be assumed below.

Furthermore, note that the character of $[\alpha]^+$ on $C_\alpha^\pm$ is equal to the character of $[\delta]^+$ on $C_\delta^\pm$, by the following observation: Since $p_k = 1$, we have that

$$\frac{1}{2}\left((-1)^{\frac{n-k}{2}} \pm \sqrt{(-1)^{\frac{n-k}{2}} p_1 \cdots p_k}\right)$$
$$= \frac{1}{2}\left((-1)^{\frac{(n-1)-(k-1)}{2}} \pm \sqrt{(-1)^{\frac{(n-1)-(k-1)}{2}} p_1 \cdots p_{k-1}}\right)$$

and thus the characters are equal. Similarly, the characters of $[\alpha]^-$ on $C_\alpha^\pm$ and of $[\delta]^-$ on $C_\delta^\pm$ are equal.

We can calculate the size of the class C_α:

$$| C_\alpha | = \binom{n}{p_1}(p_1-1)!\binom{n-p_1}{p_2}(p_2-1)!\cdots\binom{n-p_1-\cdots-p_{k-1}}{p_k}(p_k-1)!$$
$$= \frac{n!}{p_1\cdots p_k}.$$

Thus each of C_α^+ and C_α^- are half this size. Similarly, C_δ is of size $\frac{(n-1)!}{p_1\cdots p_{k-1}}$ and $C_\delta+$ and C_δ^- are each of size $\frac{(n-1)!}{2p_1\cdots p_{k-1}}$.

Since α is self-conjugate, the restriction $[\alpha] \downarrow S_{n-1}$ is of the form $[\delta] + \sum[\beta] + [\beta']$, where the sum is over all conjugate pairs of partitions $\{\beta, \beta'\}$ that appear in the set $\{(\alpha_1 - 1, \cdots, \alpha_m), \cdots, (\alpha_1, \cdots, \alpha_m - 1)\}$. Thus the restriction $[\alpha] \downarrow A_{n-1}$ is of the form $[\delta]^+ + [\delta]^- + 2\sum[\gamma]$, where from each pair of partitions $\{\beta, \beta'\}$ we take the larger partition γ. This implies that $[\alpha]^+ + [\alpha]^- \downarrow A_{n-1} = [\delta]^+ + [\delta]^- + 2\sum[\gamma]$.

We will show the branching rules for this case in two steps. First we will show that each $[\gamma]$ is contained in the restriction $[\alpha]^\pm \downarrow A_{n-1}$ with multiplicity 1. Then we will show that $[\delta]^\pm$ is contained in the restriction $[\alpha]^\pm \downarrow A_{n-1}$ with multiplicity 1 and in the restriction $[\alpha]^\mp \downarrow A_{n-1}$ with multiplicity 0.

To show this, we will make use of the inner product formula as given in Definition 2.23. Let h represent an arbitrary element of $C_\delta^\pm$.

$$\langle[\alpha]^+ \downarrow A_{n-1}, [\gamma]\rangle_{A_{n-1}} = \frac{2}{(n-1)!}\sum_{g\in A_{n-1}}[\alpha]^+(g)[\gamma](g^{-1})$$
$$= \frac{2}{(n-1)!}\left(\sum_{g\in C_\delta^+}[\alpha]^+(g)[\gamma](g^{-1}) + \sum_{g\in C_\delta^-}[\alpha]^+(g)[\gamma](g^{-1}) + \sum_{g\notin C_\delta^\pm}[\alpha]^+(g)[\gamma](g^{-1})\right).$$

We know the character of $[\alpha]^+$ on g when $g \in C_\delta^\pm$, so we can use that fact here. Furthermore, we know that when $g \notin C_\delta^\pm$, the character of $[\alpha]^+(g)$ is half the value of $[\alpha](g)$. So the above expression becomes

$$\frac{2}{(n-1)!} \mid C_\delta^+ \mid \frac{1}{2}\left((-1)^{\frac{n-k}{2}} + \sqrt{(-1)^{\frac{n-k}{2}} p_1 \cdots p_k}\right)[\gamma](h^{-1})$$
$$+ \frac{2}{(n-1)!} \mid C_\delta^- \mid \frac{1}{2}\left((-1)^{\frac{n-k}{2}} - \sqrt{(-1)^{\frac{n-k}{2}} p_1 \cdots p_k}\right)[\gamma](h^{-1})$$
$$+ \frac{2}{(n-1)!} \sum_{g \notin C_\delta^\pm} \frac{1}{2}[\alpha](g)[\gamma](g^{-1}).$$

Now, taking the sum over all $g \notin C_\delta^\pm$ is equivalent to taking the sum over all $g \in A_{n-1}$ and then subtracting off the sum over all $g \in C_\delta^\pm$. But this sum is now in terms of characters of S_{n-1} so we can write $C_\delta^\pm$ as C_δ. Now we have that

$$\langle [\alpha]^+ \downarrow A_{n-1}, [\gamma] \rangle_{A_{n-1}}$$
$$= \frac{2}{(n-1)!}(-1)^{\frac{n-k}{2}} \mid C_\delta^+ \mid [\gamma](h^{-1}) + \frac{1}{(n-1)!} \sum_{g \in A_{n-1}} [\alpha](g)[\gamma](g^{-1})$$
$$- \frac{1}{(n-1)!} \sum_{g \in C_\delta} [\alpha](g)[\gamma](g^{-1})$$
$$= \frac{1}{(n-1)!} \mid C_\delta \mid [\alpha](h)[\gamma](h^{-1}) - \frac{1}{(n-1)!} \mid C_\delta \mid [\alpha](h)[\gamma](h^{-1})$$
$$+ \frac{1}{2}\langle [\alpha] \downarrow A_{n-1}, [\gamma] \rangle_{A_{n-1}}$$
$$= \frac{1}{2}(2) = 1,$$

because there are two copies of the irrep $[\gamma]$ in the restriction $[\alpha] \downarrow A_{n-1}$.

The calculations to show

$$\langle [\alpha]^- \downarrow A_{n-1}, [\gamma] \rangle_{A_{n-1}} = 1$$

proceed identically, but we do not even need to perform them because we now know that $\langle [\alpha] \downarrow A_{n-1}, [\gamma] \rangle_{A_{n-1}} = 2$, $[\alpha] \downarrow A_n = [\alpha]^+ + [\alpha]^-$, and $\langle [\alpha]^+ \downarrow A_{n-1}, [\gamma] \rangle_{A_{n-1}} = 1$. Thus we have that

$$\langle [\alpha]^\pm \downarrow A_{n-1}, [\gamma] \rangle_{A_{n-1}} = 1.$$

Now we will show $\langle [\alpha]^\pm \downarrow A_{n-1}, [\delta]^\pm \rangle_{A_{n-1}} = 1$ and $\langle [\alpha]^\pm \downarrow A_{n-1}, [\delta]^\mp \rangle_{A_{n-1}} = 0$. There are two cases to consider, depending upon whether the classes $C_\delta^\pm$ are ambivalent. Note that $C_\alpha^\pm$ is ambivalent exactly when $C_\delta^\pm$ is ambivalent, because $C_\delta^\pm \subset C_\alpha^\pm$. In the following calculations, when the classes $C_\delta^\pm$ are ambivalent we will replace $(-1)^{\frac{n-k}{2}}$ with 1, since by Corollary 5.9 we know that $\frac{n-k}{2}$ is even. Similarly, when the classes $C_\delta^\pm$ are not ambivalent, we will replace $(-1)^{\frac{n-k}{2}}$ with -1, since the exponent in this case is odd. We first proceed with the ambivalent case.

$$\langle [\alpha]^+ \downarrow A_{n-1}, [\delta]+\rangle = \frac{2}{(n-1)!}\sum_{g\in A_{n-1}} [\delta]^+(g)[\alpha]^+(g^{-1})$$
$$= \frac{2}{(n-1)!}\sum_{g\in C_\delta^+} [\delta]^+(g)[\alpha]^+(g^{-1}) + \frac{2}{(n-1)!}\sum_{g\in C_\delta^-} [\delta]^+(g)[\alpha]^+(g^{-1})$$
$$+ \frac{2}{(n-1)!}\sum_{g\notin C_\delta^\pm} [\delta]^+(g)[\alpha]^+(g^{-1}).$$

Since the conjugacy class C_δ^+ is ambivalent, we can without loss of generality replace g^{-1} with g in the above expression. Now, note that $[\delta]^+(g) = [\alpha]^+(g) = \frac{1}{2}\left((-1)^{\frac{n-k}{2}} \pm \sqrt{(-1)^{\frac{n-k}{2}} p_1\cdots p_k}\right)$ when $g \in C_\delta^\pm$ and thus we can make that substitution and continue the calculation:

$$= \frac{2}{(n-1)!} \mid C_\delta^+ \mid \frac{1}{4}\left((-1)^{\frac{n-k}{2}} + \sqrt{(-1)^{\frac{n-k}{2}} p_1\cdots p_k}\right)^2$$
$$+ \frac{2}{(n-1)!} \mid C_\delta^- \mid \frac{1}{4}\left((-1)^{\frac{n-k}{2}} - \sqrt{(-1)^{\frac{n-k}{2}} p_1\cdots p_k}\right)^2$$
$$+ \frac{2}{(n-1)!}\left(\sum_{g\notin C_\delta^\pm} \frac{1}{4}[\delta](g)[\alpha](g^{-1})\right).$$

Once again, the last term of the expansion is now in terms of characters of S_n, so we apply the same trick as before, summing over all the elements of A_{n-1} and subtracting off the terms in C_δ. Then the above expression becomes

$$\frac{1+p_1\cdots p_k}{2p_1\cdots p_k} + \frac{1}{2(n-1)!}\left(\sum_{g\in A_{n-1}} [\delta](g)[\alpha](g^{-1}) - \sum_{g\in C_\delta}[\delta](g)[\alpha](g^{-1})\right)$$
$$= \frac{1}{2p_1\cdots p_k} + \frac{1}{2} + \left(\frac{1}{4}\langle [\delta]\downarrow A_{n-1}, [\alpha]\downarrow A_{n-1}\rangle_{A_{n-1}}\right) - \frac{(n-1)!(-1)^{n-k}}{2(n-1)!p_1\cdots p_k}.$$

We know that

$$\begin{aligned}
&\langle [\delta] \downarrow A_{n-1}, [\alpha] \downarrow A_{n-1} \rangle_{A_{n-1}} \\
&= \langle [\delta]^+ + [\delta]^-, [\alpha] \downarrow A_{n-1} \rangle_{A_{n-1}} \\
&= \langle [\delta]^+, [\alpha] \downarrow A_{n-1} \rangle_{A_{n-1}} + \langle [\delta]^-, [\alpha] \downarrow A_{n-1} \rangle_{A_{n-1}} \\
&= 1 + 1 = 2,
\end{aligned}$$

and thus the above expression for $\langle [\alpha]^+ \downarrow A_{n-1}, [\delta]+ \rangle_{A_{n-1}}$ becomes

$$\frac{1}{2p_1 \cdots p_k} - \frac{1}{2p_1 \cdots p_k} + 1 = 1.$$

Thus we have that $[\delta]^+$ is contained in the restriction of $[\alpha]^+$ to A_{n-1} with multiplicity 1 when $C_\alpha^\pm$ are ambivalent classes of A_n.

Now we will show $\langle [\alpha]^+ \downarrow A_{n-1}, [\delta]^+ \rangle_{A_{n-1}} = 1$ in the case where $C_\alpha^\pm$ are non-ambivalent classes of A_n. In this case, we have that $[\delta]^\pm(g) = [\delta]^\mp(g^{-1})$.

$$\begin{aligned}
&\langle [\alpha]^+ \downarrow A_{n-1}, [\delta]^+ \rangle_{A_{n-1}} = \frac{2}{(n-1)!} \sum_{g \in A_{n-1}} [\alpha]^+(g)[\delta]^+(g^{-1}) \\
&= \frac{2}{(n-1)!} \sum_{g \in C_\delta^+} [\alpha]^+(g)[\delta]^+(g^{-1}) + \frac{2}{(n-1)!} \sum_{g \in C_\delta^-} [\alpha]^+(g)[\delta]^+(g^{-1}) \\
&+ \frac{2}{(n-1)!} \sum_{g \notin C_\delta^\pm} [\alpha]^+(g)[\delta]^+(g^{-1}) \\
&= \frac{2}{(n-1)!} \mid C_\delta^+ \mid (\frac{1}{2})^2 \left((-1)^{\frac{n-k}{2}} + (-1)^{\frac{n-k}{4}} \sqrt{p_1 \cdots p_k} \right) \left((-1)^{\frac{n-k}{2}} - (-1)^{\frac{n-k}{4}} \sqrt{p_1 \cdots p_k} \right) \\
&+ \frac{2}{(n-1)!} \mid C_\delta^- \mid (\frac{1}{2})^2 \left((-1)^{\frac{n-k}{2}} + (-1)^{\frac{n-k}{4}} \sqrt{p_1 \cdots p_k} \right) \left((-1)^{\frac{n-k}{2}} - (-1)^{\frac{n-k}{4}} \sqrt{p_1 \cdots p_k} \right) \\
&+ \frac{2}{(n-1)!} \sum_{g \in A_{n-1}} \frac{1}{4} [\alpha](g)[\delta](g^{-1}) - \frac{2}{(n-1)!} \sum_{g \in C_\delta} [\alpha](g)[\delta](g^{-1}) \\
&= \left(\frac{2(n-1)!}{2(n-1)! p_1 \cdots p_k} \frac{1}{4} ((-1)^{n-k} - (-1)^{\frac{n-k}{2}} p_1 \cdots p_k) \right) \\
&+ \left(\frac{2(n-1)!}{2(n-1)! p_1 \cdots p_k} \frac{1}{4} ((-1)^{n-k} - (-1)^{\frac{n-k}{2}} p_1 \cdots p_k) \right) \\
&+ \left(\frac{2}{(n-1)!} \sum_{g \in A_{n-1}} \frac{1}{4} [\delta](g)[\alpha](g^{-1}) - \frac{2}{(n-1)!} \sum_{g \in C_\delta} \frac{1}{4} [\delta](g)[\alpha](g^{-1}) \right).
\end{aligned}$$

Now we apply Theorem 5.10 to determine the character of $[\delta](g)$ when $g \in C_\delta$. This reduces the above calculation to

$$\frac{1 + p_1 \cdots p_k}{2p_1 \cdots p_k} + \frac{1}{4}\langle[\delta] \downarrow A_{n-1}, [\alpha] \downarrow A_{n-1}\rangle_{A_{n-1}} - \frac{2(n-1)!(-1)^{n-k}}{4(n-1)!p_1 \cdots p_k}$$
$$= \frac{1}{2p_1 \cdots p_k} - \frac{1}{2p_1 \cdots p_k} + 1 = 1.$$

We have now shown in all cases that $\langle[\alpha]^+ \downarrow A_{n-1}, [\delta]^+\rangle_{A_{n-1}} = 1$. In order to avoid repetition, we will not present here the calculations showing $\langle[\alpha]^+ \downarrow A_{n-1}, [\delta]^-\rangle_{A_{n-1}} = 0$. Instead, we will point out that the calculation of $\langle[\alpha]^+ \downarrow A_{n-1}, [\delta]^-\rangle_{A_{n-1}}$ in the non-ambivalent case is almost identical to the calculation of $\langle[\alpha]^+ \downarrow A_{n-1}, [\delta]^+\rangle_{A_{n-1}}$ in the ambivalent case, except for the fact that now we have that $\frac{n-k}{2}$ is odd.

Similarly, the calculation of $\langle[\alpha]^+ \downarrow A_{n-1}, [\delta]^-\rangle_{A_{n-1}}$ in the ambivalent case closely parallels the calculation of $\langle[\alpha]^+ \downarrow A_{n-1}, [\delta]^+\rangle_{A_{n-1}}$ in the non-ambivalent case, except for the parity of $\frac{n-k}{2}$. This is due to the fact that, for example, the quantities $[\alpha]^+(g)[\delta]^+(g^{-1})$ ($g \in C_\alpha^\pm$, $C_\alpha^\pm$ ambivalent) and $[\alpha]^+(g)[\delta]^-(g^{-1})$ ($g \in C_\alpha^\pm$, $C_\alpha^\pm$ not ambivalent) are equal. It can be observed in the corresponding calculations that changing the parity of $\frac{n-k}{2}$ yields that $\langle[\alpha]^+ \downarrow A_{n-1}, [\delta]^-\rangle_{A_{n-1}} = 0$ whether or not the classes $C_\alpha^\pm$ are ambivalent. Since the multiplicities of $[\delta]^+$ and $[\delta]^-$ in the restriction $[\alpha] \downarrow A_{n-1}$ are both 1, we have now completely described the restrictions $[\alpha]^+ \downarrow A_{n-1}$ and $[\alpha]^- \downarrow A_{n-1}$. This completes the proof. □

Corollary 5.12. *The restriction $A_n \downarrow A_{n-1}$ is multiplicity-free.*

Proof. Let all symbols and notation be as in the proof of Theorem 5.11. In the case where the irrep Φ is of the form $[\alpha]$, it suffices to show that given a non self-conjugate Young diagram A, we cannot obtain both a Young diagram and its conjugate by deleting a box from two different rows of A. Since A is not self-conjugate, there exists an index i such that the ith row is longer than the ith column, without loss of generality. Suppose B and C are two Young diagrams obtained from deleting a box from A, such that B and C are a pair of conjugates. Since we deleted a different box from A to obtain each, without loss of generality we can say that we did not delete a box from the ith row of A to obtain B. Thus the ith row of B is the same length as the ith row of A. However, the ith column of C is either the same or shorter than the ith column of A. But for B and C to be conjugates, we would need to have the ith row of B the same length as the ith column of C, which is clearly impossible.

Next we consider the case where the irrep Φ is of the form $[\alpha]^\pm$ and the

Young diagram corresponding to α has no diagonal hook of length 1. We know that the restriction is multiplicity-free because for every pair of irreps $[\beta]$, $[\beta^{'}]$ that resulted from the restriction $[\alpha] \downarrow S_{n-1}$, we showed in the proof that there was exactly one copy of the irrep $[\beta] = [\beta^{'}]$ in the restriction $[\alpha]^{\pm} \downarrow A_{n-1}$.

Similarly, if Φ is of the form $[\alpha]^{\pm}$ and the Young diagram of α contains a diagonal hook of length 1, we showed in the proof of Theorem 5.11 that for all pairs of irreps $[\beta] = [\beta]^{'}$ of A_{n-1} that result from the restriction $[\alpha] \downarrow A_{n-1}$, there will be exactly one copy of $[\beta]$ in the restriction $[\alpha]^{\pm} \downarrow A_{n-1}$. And we have already seen that if $[\alpha] \downarrow S_{n-1}$ contains an irrep corresponding to a self-conjugate partition δ, then $[\alpha]^{\pm}$ contains one copy of $[\delta]^{\pm}$ and no copy of $[\delta]^{\mp}$ in its restriction to A_{n-1}. Thus the restriction is always multiplicity-free. □

5.2 Separating Sets: Lower Bounds and a Conjecture

In this section, we will consider how to find separating sets of minimal size for A_n. We began this component of our research by employing a C++ computer program from [5] to simply see how many class sums were needed in a separating set for each alternating group. The number needed seemed to grow much faster with n than in the case of the symmetric group; for example, four class sums are already needed for A_9, compared with the two that are needed for S_9. But with knowledge of the representation theory of A_n, a statement can be made immediately.

Lemma 5.13. *For every partition of n into distinct odd parts, one of the two corresponding class sums must be employed in any separating set for A_n.*

Proof. Each such partition of n corresponds to the hooks of a conjugate pair of irreducible representations of A_n. Both representations in that conjugate pair are assigned the same character by every other class sum. Since the two representations are of the same dimension, they are then assigned the same eigenvalue by every other class sum as well. Thus we need at least one of the class sums mentioned above to distinguish the two irreps. □

Corollary 5.14. *The minimal size of a separating set for A_n is bounded below by the number of partitions of n into distinct odd parts.*

This lower bound is only an exact one for $n = 3, 4$. There appears to be a deeper underlying structure. Certain class sums seem to only distinguish a few irreps, while other class sums distinguish the vast majority of the irreps from one another in the cases we have examined, which include up to $n = 25$. For example, for $n = 3$ and $n = 4$ either of the class sums corresponding to a 3-cycle suffices as a separating set, and for $n \geq 5$, the class sum corresponding to a 3-cycle seems to always play a vital role in any separating set.

Conjecture 5.15. *The class sum of A_n corresponding to a single 3-cycle is necessary in any separating set for A_n.*

We have examined the irreducible representations of A_n, $n \leq 25$, computationally with the use of GAP [6]. The results of this examination strongly suggest the following conjecture:

Conjecture 5.16. *Consider the sequence of conjugacy classes*

$$(3), (2)(2), (5), (2)(4), (7), (2)(6), \cdots (k), (2)(k-1), \cdots .$$

The class sums corresponding to the first $\lceil \frac{n}{8} \rceil$ terms of this sequence, along with

$$\{C_\alpha^+ \mid \alpha = \alpha'\},$$

where C_α^+ is as in Definition 5.3, will suffice as a separating set for A_n.

The C++ program mentioned above verifies the conjecture for $n \leq 16$, but for $n = 17$ it was unable to handle the immense character table. Thus a similar Java program (see Appendix A) was used that is more specific to the task at hand. The program verifies the conjecture for $n \leq 25$:

Class Sums Used (in addition to the splitting classes)	Maximum A_n
(3)	A_8
(3),(2)(2)	A_{16}
(3),(2)(2),(5)	A_{24}
(3),(2)(2),(5),(2)(4)	A_{25}

It is worthy of note that A_{25} has 997 irreps. Verifying the conjecture computationally any further is not currently feasible. GAP appears unable to compute the character table of A_{28} in a realistic amount of time, and the Java program is unable to handle the large amounts of memory needed to verify the conjecture for A_{26} and A_{27}.

Chapter 6

Conclusion and Future Work

We have examined separating sets for several different groups, including direct products, the dihedral group, and the alternating group. In Chapter 3, we saw separating sets of minimal size for the dihedral group that correspond to minimal sets of generators for the group. We have also seen how the structure of $\mathbb{C}A_n$ is closely connected to that of $\mathbb{C}S_n$ and examined various methods for achieving a decomposition with the use of separating sets. Conjecture 5.16 seems likely to hold for all n, but remains a challenge to verify. Additionally, it is suspected that the separating set given by the conjecture is of minimal size. Attempting to prove this conjecture would be an intriguing and challenging future direction of research.

Another interesting future line of work would be to examine which groups have minimal sets of generators for which the corresponding class sums form separating sets. In the cases we have seen, this happens for the quaternion group, abelian groups, and dihedral groups. This is a broad question and would likely be a fascinating and complex topic to explore.

Appendix A

Java Program Used to Computationally Verify The Conjecture For A_n

This is the Java program used to verify Conjecture 5.16 computationally for $n \leq 25$. This program reads in a text file that contains the entries of a character table, row by row. The place in the code where this file is specified is indicated. Then the program asks the user to specify which columns are to be checked to see if they form a separating set. The program calculates the semi-modified character table and then checks to see if all pairs of irreps are distinguished by some column in the set and prints the answer.

```
import java.math.BigInteger; import java.io.*; import java.util.*; import java.lang.*;

public class TrivialApplication

{

public static BigInteger gcd(BigInteger v, BigInteger w) {

   if ( w.intValue() == 0 ) {
             return v;
                         }
   else                 {
             return v.gcd( v.mod(w));
                        }
}
 public static BigInteger lcm(BigInteger a, BigInteger b) {

 BigInteger temp1 = a.gcd(b);

 BigInteger temp2 = a.multiply(b);

 return temp2.divide(temp1);

}

public static BigInteger get_lcm (BigInteger[][] table, int reps) {

        String tempStringHere = Integer.toString(1);
```

```
        BigInteger k = new BigInteger(tempStringHere);
        for (int i = 0; i < reps; i++) {
                k = lcm(k,table[i][0]);
    }
    return k;
}

public static void make_eigenvalues (BigInteger[][] table, int reps, int classes) {
//this function creates the semi-modified character table
  BigInteger k = get_lcm(table, reps);

  for (int i = 0; i < reps; i++) {
        BigInteger dim = table[i][0];
        for (int j = 0; j < classes; j++) {
                table[i][j] = table[i][j].multiply( k.divide(dim) );
        }
  }
}

public static void check_these_cols (BigInteger[][] table,int reps,int classes) throws Exception

{
       InputStreamReader incoming = new InputStreamReader (System.in);
        BufferedReader lineInput = new BufferedReader (incoming);
        PrintWriter outgoing = new PrintWriter (System.out, true);
        int evensize = 1;
        System.out.println("How many classes in the separating set?");
        evensize = Integer.parseInt(lineInput.readLine());
        int[] evencheck = new int[evensize];
        System.out.println("Now enter the column numbers.");
        for (int xyzzy = 0; xyzzy < evensize; xyzzy++) {
                evencheck[xyzzy] = Integer.parseInt(lineInput.readLine());
                evencheck[xyzzy] = evencheck[xyzzy] - 1;
        }

int num_pairs = (reps)*(reps-1)/2;

int[][] Pairs = new int[num_pairs][2];
//contains all pairs that still need to be distinguished

int counter = 0; for (int whee = 0; whee < reps; whee++) {
        for (int waa = whee+1; waa < reps; waa++) {
                 if ( counter < Pairs.length ) {
                 Pairs[counter][0] = whee;
                 Pairs[counter][1] = waa;
                 counter++;
                 }
        }
}

for (int xy = 0; xy < evencheck.length; xy++) {

        int[][] tempPairs = new int[0][0];

        for (int plength = 0; plength < Pairs.length; plength++) {

        //the tempPairs array will contain the pairs that were not
        //distinguished by the current column in
        //question.  We then transfer this array to the Pairs array at
        //the end of each iteration, to see
        //which pairs of irreps still need to be distinguished.

        int[][] temptempPairs = new int[tempPairs.length + 1][2];
                //temptempPairs is used to shorten the length of
                //the Pairs array, if necessary.
        int col = evencheck[xy];
        int check1 = Pairs[plength][0];
        int check2 = Pairs[plength][1];
        if (table[check1][col].intValue() == table[check2][col].intValue()) {

                 if (tempPairs.length > 0) {
                     for (int e = 0; e < tempPairs.length; e++) {

                              temptempPairs[e][0] = tempPairs[e][0];
                              temptempPairs[e][1] = tempPairs[e][1];
                                    }
```

```
                        }
                temptempPairs[tempPairs.length][0] = check1;
                temptempPairs[tempPairs.length][1] = check2;
                tempPairs = temptempPairs;
                }
        }
        Pairs = tempPairs;
    }
//now Pairs tells us how many pairs of irreps were not distinguished
//by any of the class sums we used

if (Pairs.length == 0) {
        System.out.println("That’s a separating set!");
        }

if (Pairs.length != 0) {
        System.out.println("That is NOT a separating set!");
        }
 }

 public static void main(String[] args) throws Exception
 {

 InputStreamReader incoming = new InputStreamReader (System.in);
 BufferedReader lineInput = new BufferedReader (incoming);
 PrintWriter outgoing = new PrintWriter (System.out, true);

 InputStreamReader gimme = new InputStreamReader (new FileInputStream("a26.txt"));
 //this is where the user specifies the
//text file containing the character table
 BufferedReader dataInput = new BufferedReader (gimme);

 System.out.println("Enter the number of irreps.");
 int reps = Integer.parseInt(lineInput.readLine());
 System.out.println("Enter the number of classes.");
 int classes = Integer.parseInt(lineInput.readLine());

BigInteger[][] table = new BigInteger[reps][classes]; String uh = dataInput.readLine(); StringTokenizer st = new
StringTokenizer(uh);

for (int i = 0; i < reps; i++) {
    for (int j = 0; j < classes; j++) {
        if (!st.hasMoreTokens()) {
            while (!st.hasMoreTokens()) {
                uh = dataInput.readLine();
                st = new StringTokenizer(uh);
            }
        }
        table[i][j] = new BigInteger(st.nextToken());
    }
} make_eigenvalues(table, reps, classes); check_these_cols(table, reps, classes);
        }
}
```

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