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Convergence of Planar Domains and of Harmonic Measure Distribution Functions

Ariel Barton
Harvey Mudd College

Conditions on Harmonic Measure Distribution Functions of Planar Domains

Ariel Barton

Lesley Ward, Advisor

Henry Krieger, Reader

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HARVEY MUDD COLLEGE

Department of Mathematics

Abstract

Consider a region Ω in the plane and a point z_0 in Ω. If a particle which travels randomly, by Brownian motion, is released from z_0, then it will eventually cross the boundary of Ω somewhere. We define the harmonic measure distribution function, or h-function h_Ω, in the following way. For each $r > 0$, let $h_\Omega(r)$ be the probability that the point on the boundary where the particle first exits the region is at a distance at most r from z_0. We would like to know, given a function f, whether there is some region Ω such that f is the h-function of that region.

We investigate this question using convergence properties of domains and of h-functions. We show that any well-behaved sequence of regions must have a convergent subsequence. This, together with previous results, implies that any function f that can be written as the limit of the h-functions h_{Ω_n} of a sufficiently well-behaved sequence $\{\Omega_n\}$ of regions is the h-function of some region.

We also make some progress towards finding sequences $\{\Omega_n\}$ of regions whose h-functions converge to some predetermined function f, and which are sufficiently well-behaved for our results to apply. Thus, we make some progress towards showing that certain functions f are in fact the h-function of some region.

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Acknowledgments

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Chapter 1

Introduction

1.1 Previous work

In 1989, D. A. Brannan and W. K. Hayman published a paper [3] summarizing the current state of some problems in complex analysis, and listing some new ones.

One problem, proposed by Ken Stephenson, was as follows. Given a region in the complex plane, it is possible to construct a certain function g from that region based on the spatial distribution of its boundary. Two questions immediately arise. First, what functions can be constructed as the g-function of some region? Second, what can be determined about a region based on its g-function?

In [16], [19], and [20], Byron Walden, Lesley Ward, and Marie Snipes investigated these questions for the h-functions of regions. The h-functions are similar to the g-functions; see Definition 3, below. They have found some answers. For example, in [19], it was established that any function which violated a certain lower bound could only arise as the h-function of a multiply connected region, never as the h-function of a simply connected region. As another example, in [16], it was established that any step function which meets certain simple conditions is the h-function of some region.

In this paper, we investigate the following question in more detail.

Question. For what functions f does there exist a region Ω such that $f = h_\Omega$?

We work towards finding sufficient conditions for a function f to be equal to the h-function of some simply connected, bounded region Ω at points of continuity of f.

1.2 Definitions

First, we will define some notation and a few basic ideas from complex analysis. We let $\mathbb{C}$ denote the complex plane, $\mathbb{U}$ denote the open upper half-plane, and $\mathbb{D}$ denote the open unit disc centered at 0. We let $B(z_0, r)$ denote the open disk centered at z_0 with radius r. In particular, $B(0, 1) = \mathbb{D}$.

We define a *domain* in the complex plane to be a connected open subset of the complex plane. We use an overline to denote the closure of a domain. For example, $\overline{\mathbb{D}}$ and $\overline{B(0,1)}$ both denote the closed unit disk.

Also, for some domains, we will need a special map, the Riemann map, defined as follows.

Definition 1 *Given a simply connected domain Ω, where Ω is contained within but not equal to the complex plane, and a point $z_0 \in \Omega$, the* Riemann map *of Ω is the one-to-one, analytic map Φ which maps the unit disk $\mathbb{D}$ onto Ω, normalized such that $\Phi(0) = z_0$ and $\Phi'(0)$ is real and positive.*

The famous Riemann Mapping Theorem [2, p. 230] states that the Riemann map, as defined above, exists and is unique for every simply connected domain Ω and every point $z_0 \in \Omega$, except when $\Omega = \mathbb{C}$.

Next, we define the functions we will investigate and some specific domains we will need.

Definition 2 *The* harmonic measure *of a subset E of the boundary of a domain Ω in the complex plane, measured from a point z_0 in Ω, is the probability that a particle traveling by Brownian motion, released from z_0, first reaches the boundary of Ω somewhere in E. It is denoted by $\omega(z_0, E, \Omega)$.*

For example, by symmetry, if E is an arc in the boundary of the unit disk $\mathbb{D}$, and $z_0 = 0$, then $\omega(z_0, E, \mathbb{D})$ is the length of E divided by 2π.

In the plane, if the boundary has nonzero capacity—for example, if it contains any continuum—then, as shown in [8, Theorem 6], it is guaranteed that the particle will hit the boundary eventually.

In higher dimensions, this is not true. For example, by [8, p. 906], in all higher dimensions there is a nonzero probability that a Brownian particle released from outside a unit sphere never enters that unit sphere.

As Kakutani showed in [8, Theorem 1], harmonic measure can also be calculated using the solution u to this specific instance of Dirichlet problem:

$$\begin{cases} \Delta u(z) = 0, & z \in \Omega, \\ u(z) = 1, & z \in E, \\ u(z) = 0, & z \in \partial\Omega \backslash E. \end{cases} \tag{1.1}$$

Then $\omega(z_0, E, \Omega) = u(z_0)$.

Definition 3 *The* harmonic measure distribution function *is denoted $h_\Omega(r; z_0)$, and is defined by*

$$h_\Omega(r; z_0) = \omega(z_0, \overline{B(z_0, r)} \cap \partial\Omega, \Omega).$$

If it is clear from context, this may be rewritten as $h_\Omega(r)$ or even $h(r)$.

For convenience, we often refer to the harmonic measure distribution function of a region as its h-function. This is the h-function investigated by Ward and Walden in [19] and [20]. Stephenson's g-function, which inspired this problem, is defined similarly, as

$$g_\Omega(r; z_0) = \omega(z_0, \overline{B(z_0, r)} \cap \partial\Omega, \Omega \cap B(z_0, r)).$$

We will work with two important classes of domains, circle domains and blocked circle domains. See Fig. 1.1 for examples.

Definition 4 *A* circle domain *is a connected domain D such that for some positive number r, $D \subset B(0, r)$ and ∂D consists of $\partial B(0, r)$ together with a finite number of closed arcs of circles centered at* 0*.*

We define the term *arc length* as follows.

Definition 5 *The* arc length *of an arc of a circle, such as the boundary arcs of circle domains, is defined to be the angle subtended at the center of the circle by the endpoints of that arc.*

So, for example, if $r > 0$, the arc length of the arc from r to ir in $\partial B(0, r)$ is $\pi/2$, regardless of the value of r. The arc length is not the actual length of an arc; that can be obtained by multiplying the arc length by the radius of the circle containing the arc.

Definition 6 *A* blocked circle domain *is a simply connected domain D such that for some r, $D \subset B(0, r)$, and ∂D consists of $\partial B(0, r)$, a finite number of closed arcs of circles centered at* 0*, and a finite number of line segments, called* gates*, which lie on rays from* 0 *and whose endpoints lie on adjacent arcs.*

The ends of the arcs which protrude past both of the adjoining gates are referred to as *spikes*.

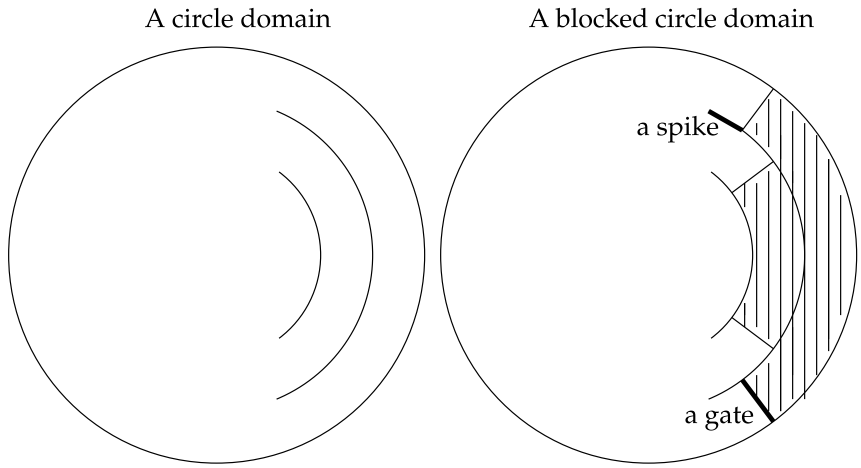

Figure 1.1: A circle domain and a blocked circle domain.

1.3 Snipes' theorems

In [16], Marie Snipes proved two important theorems. The first, Theorem 2 in [16], is as follows.

Theorem 1 (Snipes) *Let Ω and Ω_n, $n \geq 1$, be simply connected domains containing the point 0, with harmonic measure distribution functions h and h_n respectively. Suppose that the normalized Riemann mappings $\Phi : \mathbb{D} \to \Omega$, $\Phi_n : \mathbb{D} \to \Omega_n$, with $\Phi(0) = \Phi_n(0) = 0$ and $\Phi'(0) > 0$, ${\Phi_n}'(0) > 0$, have continuous extensions to the closed unit disk $\overline{\mathbb{D}}$, and that $\Phi_n \to \Phi$ uniformly on $\overline{\mathbb{D}}$. Then $h_n \to h$ pointwise at all points of continuity of h.*

If a sequence of simply connected domains converges as described in Theorem 1, that is, their Riemann maps converge uniformly on the closed unit disk, we say that those domains *converge in the sense of Fréchet.*

Note the following corollary.

Corollary 2 *If any subsequence of the sequence $\{\Omega_n\}$ of simply connected domains converges in the sense of Fréchet to some domain Ω, and the harmonic measure distribution functions $h_{\Omega_n}(r)$ converge pointwise to some function $f(r)$ at points of continuity of f, then $f(r) = h_\Omega(r)$ at all points of continuity of h_Ω and f.*

Consider the following partial converse to Theorem 1.

Conjecture 1 *For any sufficiently well-behaved sequence $\{\Omega_n\}$ of simply connected domains such that $h_{\Omega_n} \to h$ pointwise at all points of continuity of h, some subsequence of $\{\Omega_n\}$ converges in the sense of Fréchet.*

We prove a version of Conjecture 1 in Chapter 3. This means that, given any function $f(r)$ for which there exists a sufficiently well-behaved sequence $\{\Omega_n\}$ of simply connected domains, such that

$$f(r) = \lim_{n\to\infty} h_{\Omega_n}(r)$$

at all points of continuity of f, there must be some simply connected domain Ω such that $f(r) = h_\Omega(r)$ at all points of continuity of both f and h_Ω.

This would answer the question we investigate, of which functions arise as h-functions, for all functions f which met the condition above.

Note that the preceding analysis shows that $f(r) = h_\Omega(r)$ only at points of continuity of h_Ω and f. The following theorem lets us know where the points of continuity of h_Ω are.

Theorem 3 *If h and f are monotonic functions and $f(r) = h(r)$ at all points of continuity of h and f, then $f(r) = h(r)$ at all points of continuity of f.*

Proof Suppose that r is not a point of continuity of h. Assume that h is monotonically increasing. (The case where h is monotonically decreasing is similar, and is irrelevant to our problem.) Then $h^+ = \lim_{x\to r^-} h(x)$ and $h^- = \lim_{x\to r^+} h(x)$ both exist, by [15, Theorem 4.29]. Furthermore, by [15, Theorem 4.30], h and f have at most countably many discontinuities. So in any interval (y, z) in the real line, h and f must have some common point of continuity s.

Fix $\delta > 0$. Then there is some point $s \in (r, r+\delta)$ such that s is a point of continuity of h and f. Now, by monotonicity and continuity,

$$f(s) = h(s) \geq h^+.$$

Similarly, there is some $t \in (r, r-\delta)$ such that $f(t) = h(t) \leq h^-$. But $h^+ - h^- > 0$ since r is a point of discontinuity of h and h is monotonic. Let $\varepsilon = (h^+ - h^-)/2$. Then for every $\delta > 0$, there is some x, where $x = s$ or $s = t$, with $|x - r| < \delta$ such that $|f(x) - f(r)| \geq \varepsilon$.

Therefore, r is a point of discontinuity of f. Hence, every point of continuity of f is a point of continuity of h, and so $f(r) = h(r)$ at every point of continuity of f. ■

Theorem 3 allows us to rewrite Corollary 2 as follows, removing the last $h_\Omega(r)$.

Corollary 4 *If any subsequence of the sequence $\{\Omega_n\}$ of domains converges in the sense of Fréchet to some domain Ω, and the h-functions $h_{\Omega_n}(r)$ converge pointwise to some function $f(r)$ at points of continuity of f, then $f(r) = h_\Omega(r)$ at all points of continuity of $f(r)$.*

In [16, Theorem 3], Marie Snipes also proved the following theorem about step functions and circle domains.

Theorem 5 (Snipes) *Let $f(r)$ be a right-continuous step function, increasing from 0 to 1, with its discontinuities at $r_1, r_2, \ldots, r_{n+1}$, with $0 < r_1 < r_2 < \cdots < r_{n+1}$. Then there exists a circle domain X with n arcs whose harmonic measure distribution function $h_X(r)$ is equal to $f(r)$. The radii of the n arcs and of the boundary circle in X are given by $r_1, r_2, \ldots, r_n$ and by r_{n+1} respectively.*

1.4 Statement of results and outline of thesis

In this chapter, we have defined our terms and summarized some previous results, in particular Snipes' convergence theorem and Snipes' circle domain theorem.

In Chapter 2, we give counterexamples to the full converse of Snipes' convergence theorem, state a weaker converse for simply connected domains, and show an example which suggests that the weaker converse might not hold for multiply connected domains. This weaker converse is the following theorem. Note that it is a version of Conjecture 1 in which we specify some necessary conditions on the sequences of domains $\{\Omega_n\}$ for which it holds.

Theorem 6 *Suppose that $\{\Omega_n\}$ is a sequence of simply connected, uniformly bounded domains containing 0, and that their normalized Riemann maps Φ_n are equicontinuous. Suppose further that for some $\rho > 0$, $B(0, \rho) \subset \Omega_n$ for all n.*

Then there is some subsequence $\{\Phi_{n_k}\}$ of the Riemann maps that converges uniformly to a function Φ that is itself the Riemann map of some domain Ω.

In Chapter 3, we prove Theorem 6. Also, we rewrite Corollary 4, to use the conditions of Theorem 6 rather than the condition that the sequence $\{\Omega_n\}$ of domains converges.

In Chapter 4, we fix a function f, and use Theorem 5 to construct a sequence of domains X_n whose h-functions approach f. We then find conditions on X_n and f that ensure that f is equal to the h-function of some simply connected domain, using Theorem 6 and a sequence $\{\Omega_n\}$ of well-behaved, simply connected domains similar to $\{X_n\}$. These conditions are summarized in the following theorem.

Theorem 7 *Let f be a function that is 0 on $(0, \mu)$, strictly increasing on $[\mu, M]$, and equal to 1 on (M, ∞). Using Theorem 5, construct a sequence $\{X_n\}$ of circle domains, where X_n has $n + 1$ boundary arcs including the outermost circle, whose h-functions approach f. Let $\psi_{n,j}$ be half the arc length of the jth boundary arc in X_n. Assume that for every $E > 0$ there exists an $N_E > 0$, and for every $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$, such that $N_E, E, \delta_\varepsilon$, and ε satisfy the following conditions.*

(a) If $n \geq N_E$, then $n\psi_{n,k} > E$.

(b) If $|r_{n,k} - r_{n,j}| < \delta_\varepsilon$, then $\min(\psi_{n,j}, \psi_{n,k}) - \psi_{n,i} < \varepsilon$ for all $j < i < k$.

Then f is equal to the h-function of some simply connected domain Ω at all points of continuity of f.

Roughly speaking, Condition (a) implies that the arcs of $\{X_n\}$ cannot be too short. The arc lengths of boundary arcs in X_n can decrease to zero as n increases, but they must decrease slowly. For example, the condition would be satisfied if $\psi_{n,k} = 1/\sqrt{n}$ for all k, but would not be satisfied if $\psi_{n,k} = 1/n^2$ for all k.

Condition (b) implies that there cannot be short arcs of X_n between nearby long arcs of X_n.

In Chapter 5, we discuss possible future work.

In Appendix A, we prove that harmonic measure is a conformal invariant, is additive, and is monotonic. In Appendix B, we calculate harmonic measure in several specific domains, and in Appendix C, we investigate circle domains and present some estimates of their harmonic measure.

Chapter 2

Counterexamples

2.1 Counterexample to the full converse

The statement of the full converse of Snipes' convergence theorem (Theorem 1) is as follows. If, for a set of domains Ω_n with harmonic measure distribution functions h_n such that $h_n \to h$ pointwise for some function h, then $\Omega_n \to \Omega$ in some sense.

If we restrict ourselves to sequences of simply connected domains, then we can have $\Omega_n \to \Omega$ in the sense of Fréchet, as used in Theorem 1. If we want to look at multiply connected domains, we need some other definition of convergence, such as Carathéodory convergence, defined later in this chapter (Definition 7).

The converse is false, as shown in the next theorem.

Theorem 8 *There exists a sequence $\{\Omega_n\}$ of simply connected domains, with harmonic measure distribution functions h_n, such that $h_n = h_m$ for all n and m, but $\{\Omega_n\}$ does not converge.*

Proof Define the sequence $\{\Omega_n\}_{n=1}^{\infty}$ as follows. For $k = 1, 2, 3, \ldots$, let

$$\Omega_{2k} = \left\{ a + bi \;\middle|\; b > -1 \right\},$$

$$\Omega_{2k+1} = C \setminus \left\{ a \;\middle|\; a \in R, |a| \geq 1 \right\}.$$

In both cases, let the basepoint be 0. Then by [19, Example 2], $h_n = h_m$ for all n and m.

However, $\{\Omega_n\}$ has two subsequences $\{\Omega_{2k}\}$ and $\{\Omega_{2k+1}\}$ which, by any reasonable definition of convergence, clearly converge to different domains. Hence $\{\Omega_n\}$ cannot converge. ■

2.2 A conjectured weaker converse

The full converse is thus false. Is some sort of partial converse true? We conjecture that the converse is true if we replace sequences with subsequences.

Conjecture 2 *If the harmonic measure distribution functions h_{Ω_n} of some sequence of domains $\{\Omega_n\}$ converge to some function f, then there is a subsequence Ω_{n_k} such that Ω_{n_k} converges to some region Ω.*

Note that this conjecture generalizes Conjecture 1 from simply connected domains to multiply connected domains.

As we will show in Chapter 3, the converse holds for all sequences $\{\Omega_n\}$ of domains that are simply connected, uniformly bounded, all contain some neighborhood of the basepoint, and whose complements are uniformly locally connected. (See Theorem 6 in Section 1.4.)

As we will show in Section 2.3, however, if the converse holds for arbitrary multiply connected domains, then either we must use an unusual definition of convergence, or at least one sequence of domains must converge to a domain with the wrong h-function.

From Theorem 1, we know that for a certain (very strong) kind of convergence of domains, namely Fréchet convergence of simply connected domains, if there is a convergent subsequence $\{\Omega_{n_k}\}$, then it must converge to a domain Ω with the h-function we expect. That is, $h_\Omega = \lim_{n\to\infty} h_{\Omega_n}$. It is unknown whether a sequence of domains $\{\Omega_n\}$ that converges in some other sense (e.g. the sense of Carathéodory) must converge to something with the appropriate h-function.

2.3 A cautionary example, using circle domains

In Chapter 3, we prove Conjecture 2 for the special case where the domains Ω_n are simply connected and satisfy a few other conditions. For simply connected domains, we can use Fréchet convergence, as used in Theorem 1.

It is tempting to try to prove Conjecture 2 for the more general case of multiply connected domains, using some other definition of convergence, such as Carathéodory convergence, defined below. However, it is possible to construct a sequence of domains which, if they converge in the sense of Carathéodory, must converge to something with the wrong h-function. This implies one of three possibilities.

- Either Carathéodory convergence is the wrong type of convergence to use, but Conjecture 2 is true if we use some other definition of convergence,
- or Conjecture 2 is true, but some subsequences converge to domains with the wrong h-function,
- or Conjecture 2 simply does not hold in general for multiply connected domains.

Our example is constructed as follows. We take the definition of Carathéodory convergence from [11, p. 13].

Definition 7 (Pommerenke) *A sequence of domains Ω_n converges in the sense of* kernel convergence *(or the sense of* Carathéodory*) to a domain Ω with respect to some $w_0 \in \Omega$ if*

1. *either $\Omega = \{w_0\}$, or $\Omega \neq \mathbb{C}$, $w_0 \in \Omega$, and for every $w \in \Omega$, there is some neighborhood $B(w, r)$ of w and some integer M such that $B(w, r) \subset \Omega_n$ for all $n \geq M$, and*
2. *for each $w \in \partial\Omega$ there is some sequence $\{w_n\}$ such that $w_n \in \partial\Omega_n$ and $w_n \to w$ as $n \to \infty$.*

Note that a neighborhood, in this section, is what we elsewhere refer to as a disk.

Recall that sufficiently well-behaved step functions always arise as the harmonic measure distribution functions of a class of planar domains.

Define the piecewise-linear function f as follows.

$$f(r) = \begin{cases} 0, & r < 1, \\ (r-1)/6, & 1 \leq r \leq 7, \\ 1, & 7 < r. \end{cases}$$

Then f is a piecewise-linear function that starts out at 0, increases from 0 to 1 linearly, and then remains at 1 over the rest of the real line. Define the step functions h_n to approximate f as follows.

$$h_n(r) = \begin{cases} 0, & r < 1, \\ \frac{j}{n}, & 6\frac{j-1}{n} + 1 \leq r < 6\frac{j}{n} + 1,\ j \in \mathbb{Z},\ 0 < j \leq n, \\ 1, & 7 \leq r. \end{cases}$$

So $\{h_n\}$ is a sequence of step functions that approaches f in the limit. See Fig. 2.1.

Furthermore, h_n is a right-continuous step function that increases monotonically from 0 to 1 in a finite number of steps, so we can use Theorem 5.

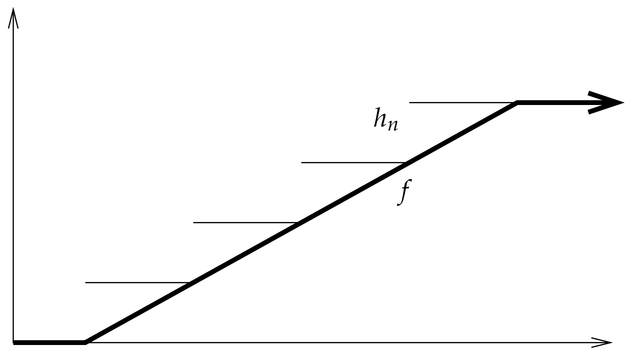

Figure 2.1: The function f used in Section 2.3, and an approximating step function h_n.

For $n = 1, 2, 3, \ldots$, let Ω_n be a circle domain, with all arcs centered on the positive real axis, such that $h_{\Omega_n}(r) = h_n(r)$. Since for each h_n, there is a jump at $1 + 6j/n$, all of the boundary arcs in Ω_n must have positive length. Also, for each n, $\Omega_n \subset B(0, 7)$.

Clearly, $h_{\Omega_n}(r) \to f(r)$ as $n \to \infty$, for each $r > 0$.

Assume some subsequence $\{\Omega_{n_k}\}$ converges to some region Ω in the sense of Carathéodory. I claim that $h_\Omega(r) \neq f(r)$.

Let $|w| \geq 7$. Then for all Ω_{n_k}, $w \notin \Omega_{n_k}$. So $w \notin \Omega$.

Suppose that w is a positive real number with $1 \leq w \leq 7$. Let $B(w, \varepsilon)$ be the neighborhood of w with radius ε. Fix an integer M. For any subsequence $\{\Omega_{n_k}\}$, there is some $n_k > M$ with $n_k > 6/\varepsilon$. Then there is some j such that $1 + 6j/n_k \in B(w, \varepsilon)$. Since $6j/n_k \notin \Omega_{n_k}$, $B(w, \varepsilon) \not\subset \Omega_{n_k}$. Thus, there is no neighborhood $B(w, \varepsilon)$ and integer M such that if $n_k > M$, $B(w, \varepsilon) \subset \Omega_{n_k}$. Thus, by the definition of Carathéodory convergence, $w \notin \Omega$.

Thus, $\Omega \subset B\,(0, 7) \setminus [1, 7]$. See Fig. 2.2.

Then

$$
\begin{aligned}
h_\Omega(r) &= \omega(0, \partial\Omega \cap \overline{B(0, r)}, \Omega) \\
&\geq \omega(0, \partial\Omega \cap \overline{B(0, r)}, \Omega \cap B(0, r)) \\
&\geq \omega\,(0, [1, r]\,, B(0, r) \setminus [1, r])
\end{aligned}
$$

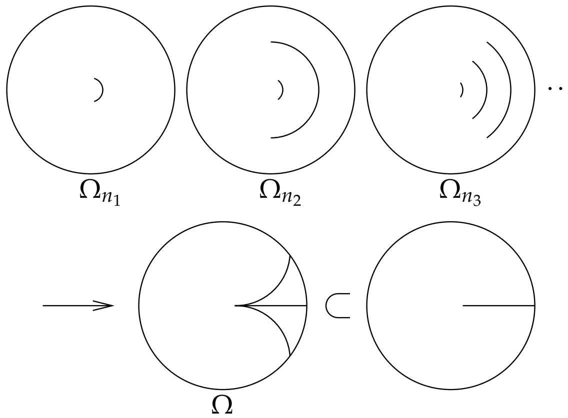

Figure 2.2: The subsequence of domains Ω_{n_k}, which give rise to the step functions h_{n_k}, and converge to a hypothetical domain Ω which must be contained in the domain $B(0,7) \setminus [1,7]$.

by Lemma 36 in Appendix A.

By [19, Equation (5)], $\omega(0,[1,r], B(0,r) \setminus [1,r]) = 1 - \frac{4}{\pi} \arctan \sqrt{1/r}$. But on the interval $(1,3)$, $f(r) < 1 - \frac{4}{\pi} \arctan \sqrt{1/r}$. Thus, $h_\Omega(r) \neq f(r)$ on the interval (1,3).

Therefore, one of the three possibilities listed above must be true.

Chapter 3

Convergent subsequences of Riemann maps

3.1 Conditions on the Riemann maps and our goal

In this chapter, we will prove that any sequence of domains $\{\Omega_n\}$ which satisfies certain conditions must contain a subsequence $\{\Omega_{n_k}\}$ which converges to some limiting domain Ω.

We will need several conditions on $\{\Omega_n\}$. First, we will require that all domains be simply connected. Thus, we define convergence of domains in the sense of Fréchet, that is, uniform convergence of Riemann maps on the closed unit disk $\overline{\mathbb{D}}$. This will conveniently allow us to use Corollary 4, to show that the limit of the h-functions $\{h_{\Omega_n}\}$ is itself an h-function.

In this chapter, we assume that all Riemann maps Φ are normalized such that $\Phi(0) = 0$ and $\Phi'(0) > 0$.

Another condition we will need is that the sequence of complements to the domains in our original sequence be uniformly locally connected.

Definition 8 *A closed set A is* locally connected *if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that any two points $b, c \in A$ with $|b - c| < \delta$ can be joined by a continuum $B \subset A$ of diameter at most ε.*

A sequence of closed sets $\{A_n\}_{n=1}^{\infty}$ is uniformly locally connected *if, for every $\varepsilon > 0$, there exists a $\delta > 0$ independent of n such that any two points $b, c \in A_n$ with $|b - c| < \delta$ can be joined by continua $B_n \subset A_n$ of diameter at most ε.*

See Fig. 3.1 for examples. Note that any finite collection of locally connected sets is uniformly locally connected.

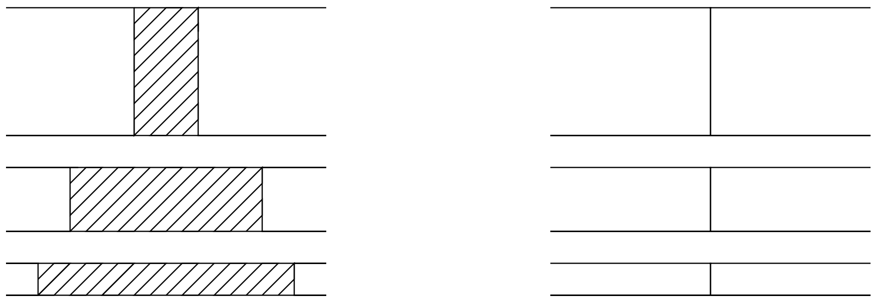

Figure 3.1: The first three sets in each of two infinite sequences of closed sets. The sequence represented on the left is uniformly locally connected, but the sequence represented on the right is not.

We will also, as an intermediate step, need our sequence of Riemann maps to be equicontinuous on the closed unit disk $\overline{\mathbb{D}}$.

Definition 9 *A sequence of maps $\{\Phi_n\}$ on some metric space D is equicontinuous if, for all $\varepsilon > 0$, there exists a $\delta > 0$, depending only on ε, such that if $x, y \in D$ and $|x - y| < \delta$, then $|\Phi_n(x) - \Phi_n(y)| < \varepsilon$ for all n.*

In this chapter, we prove the following theorem, stated as Theorem 6 in Section 1.4 above.

Theorem 9 *Suppose that $\{\Omega_n\}_{n=1}^{\infty}$ is a sequence of domains in the complex plane such that the following conditions hold.*

- *Ω_n is simply connected.*
- *There exists a positive number $\rho > 0$ such that for all n, $B(0, \rho) \subset \Omega_n$.*
- *There exists a positive number $R > 0$ such that for all n, $\Omega_n \subset B(0, R)$.*
- *$\{\mathbb{C} \backslash \Omega_n\}_{n=1}^{\infty}$ is uniformly locally connected.*

Then the sequence of Riemann maps $\{\Phi_n\}_{n=1}^{\infty}$ of the domains Ω_n contains a subsequence $\{\Phi_{n_k}\}_{k=1}^{\infty}$ which converges uniformly on $\overline{\mathbb{D}}$ to some map Φ which is itself the Riemann map of some domain Ω.

The key point is that Theorem 9 produces a domain Ω, or at least shows that it exists. If the sequence of domains h_{Ω_n} converge to some function f, then by Theorem 1, f must be the h-function of the domain Ω. This allows us to rewrite Corollary 4 as follows.

Corollary 10 *If a sequence of functions $\{h_n\}$ converges pointwise to some function f at points of continuity of f, and there exist simply connected domains Ω_n such that*

- $h_n(r) = h_{\Omega_n}(r; 0)$,
- $\{\mathbb{C}\backslash\Omega_n\}$ *is uniformly locally connected, and*
- *there exist numbers $\rho, R > 0$ such that for all n, $B(0,\rho) \subset \Omega_n \subset B(0,R)$,*

then there exists some domain Ω such that $h_\Omega(r;0) = f(r)$ at all points of continuity of f.

3.2 Finding a convergent subsequence of equicontinuous Riemann maps

In this section, we prove that all uniformly bounded, equicontinuous sequences of Riemann maps contain convergent subsequences. After proving this theorem, I discovered that it was a known result, the Arzelá-Ascoli Theorem.

Notation. Because these results will eventually be applied to the complex plane, we use $|a-b|$, rather than $d(a,b)$, to denote the distance between the points a and b in a general metric space.

First, we will show that if a sequence of maps $\{\varphi_n\}$ takes a *countable* set S into a compact space C, then $\{\varphi_n\}$ has a convergent subsequence. This is easier than our goal-namely, to show the same thing on an uncountable set, $\overline{\mathbb{D}}$. We follow the treatment given in [13, p. 170] for real numbers.

Theorem 11 *Let $S = \{s_1, s_2, \ldots\}$ be a countable set, and let $\varphi_n : S \to C$ be a countable sequence of maps from S into some compact metric space C. Then there is some subsequence φ_{n_k} such that $\{\varphi_{n_k}(s_j)\}_{k=1}^{\infty}$ converges for all values of j.*

Proof Because C is compact, Theorems 2.41 and 3.2 in [15] imply that the sequence $\{\varphi_n(s_1)\}_{n=1}^{\infty}$ has a convergent subsequence $\{\varphi_{n_{1,k}}(s_1)\}_{k=1}^{\infty}$. This defines the sequence $\{\varphi_{n_{1,k}}\}_{k=1}^{\infty}$.

We define the sequences $\{\varphi_{n_{j,k}}\}_{k=1}^{\infty}$ recursively as follows. For each $j \in \mathbb{Z}^+$, we know that the sequence $\{\varphi_{n_{j,k}}(s_{j+1})\}_{k=1}^{\infty}$ has a convergent subsequence. Let this subsequence be $\{\varphi_{n_{j+1,k}}(s_{j+1})\}_{k=1}^{\infty}$, and from this define $\{\varphi_{n_{j+1,k}}\}_{k=1}^{\infty}$.

Then $\{\varphi_{n_{j+1,k}}\}_{k=1}^{\infty} \subset \{\varphi_{n_{j,k}}\}_{k=1}^{\infty}$. Let $\varphi_{n_k} = \varphi_{n_{k,k}}$. Then $n_{k+1} > n_k$, and so $\{\varphi_{n_k}\}_{k=j}^{\infty}$ is a subsequence of $\varphi_{n_{j,k}}$ for all j. Thus,

$$\lim_{k\to\infty} \varphi_{n_k}(s_j) = \lim_{k\to\infty} \varphi_{n_{j,k}}(s_j)$$

exists for all j. ∎

Unfortunately, the closed unit disk is uncountable. Fortunately, it has countable dense subsets. For example, $\{qe^{ip\pi}\}$, where q and p are rational and $|q| \leq 1$, is a countable dense subset of $\overline{\mathbb{D}}$. We would like to show that if a sequence of continuous functions $\{\varphi_n\}$ converges on a dense subset D of a compact space S, then that sequence $\{\varphi_n\}$ converges on all of S. We will need a fairly strong condition: that the sequence $\{\varphi_n\}$ be equicontinuous.

Theorem 12 *Suppose that S is a compact metric space, and $\varphi_n : S \to C$ is an equicontinuous sequence of functions on a dense set $D \subset S$, that converges pointwise to the function φ. (Note that φ is defined only on D!) Then $\{\varphi_n\}$ is uniformly convergent on S.*

Proof Fix $\varepsilon > 0$.

For every $d \in D$, there is some N such that $|\varphi(d) - \varphi_n(d)| < \varepsilon/6$ for all $n \geq N$.

Let

$$D_N = \left\{ d \in D \;\middle|\; \text{for all } n, m \geq N, |\varphi_n(d) - \varphi_m(d)| < \varepsilon/3 \right\}.$$

Then every $d \in D$ is in some D_N.

Since $\{\varphi_n\}$ is equicontinuous, there is some $\delta > 0$ such that if $|s - t| < \delta$, then $|\varphi_n(s) - \varphi_n(t)| < \varepsilon/3$ for all n. Also, because D is dense, for every $s \in S$, there is some $d \in D$ with $|s - d| < \delta$.

Let

$$E_N = \bigcup_{d\in D_N} B(d, \delta).$$

Then by the above remarks, $\{E_N\}$ is an open cover of S. Hence it has a finite subcover $\{E_{N_k}\}_{k=1}^{m}$. Note that $E_N \subset E_{N+1}$. Let $M = N_m$.

Then for every $s \in S$, if $n, m \geq M$, then there is some $d \in D$ such that $|s - d| < \delta$ and $|\varphi_m(d) - \varphi_n(d)| \leq \varepsilon/3$. Thus,

$$|\varphi_n(s) - \varphi_m(s)| \leq |\varphi_n(s) - \varphi_n(d)| + |\varphi_n(d) - \varphi_m(d)| + |\varphi_m(d) - \varphi_m(s)| < \varepsilon.$$

Thus, by Theorem 7.8 in [15], $\{\varphi_n\}$ is uniformly convergent on S. ∎

It follows from [15, Theorem 7.12] that φ, the limit function of $\{\varphi_n\}$, is continuous.

These two theorems can be summarized as follows.

Theorem 13 *Let S and C be two compact metric spaces with C complete, and assume that there exists a countable dense subset of S. If $\{\varphi_n\}$ is an equicontinuous sequence of functions $\varphi_n : S \to C$, then there is some subsequence φ_{n_k} which converges uniformly to some continuous function φ.*

In particular, if $\{\Phi_n\}$ is a uniformly bounded, equicontinuous sequence of Riemann maps, then there is some subsequence $\{\Phi_{n_k}\}$ which converges uniformly to a continuous function $\Phi : \overline{\mathbb{D}} \to \mathbb{C}$.

3.3 Showing that the limit function is a Riemann map

Returning to the special case of the complex plane, recall that we wish to show that any sufficiently well-behaved sequence of Riemann maps contains a subsequence which converges uniformly to a Riemann map.

We have shown that any uniformly bounded, equicontinuous sequence $\{\Phi_n\}$ of Riemann maps contains a subsequence which converges uniformly to some continuous function Φ on $\overline{\mathbb{D}}$. We would like to show that the limit function Φ is a Riemann map. Thus, we must show the following.

- Φ is analytic in $\mathbb{D}$.
- Φ is one-to-one.
- $\Phi(0) = 0$.
- $\Phi'(0) > 0$.

Note the following theorem from Ahlfors' book, [2, p. 176].

Theorem 14 *Suppose Φ_n is analytic in $\mathbb{D}$, and that the sequence $\{\Phi_n\}$ converges to Φ in $\mathbb{D}$, uniformly on every compact subset of $\mathbb{D}$. Then Φ is analytic in $\mathbb{D}$. Moreover, $\Phi_n'(z)$ converges uniformly to $\Phi'(z)$ on every compact subset of $\mathbb{D}$.*

Thus, we know that Φ is analytic in $\mathbb{D}$. Clearly, $\Phi(0) = \lim_{n\to\infty} \Phi_n(0) = 0$. Also, $\lim_{n\to\infty} \Phi_n'(0) \geq 0$ because $\Phi_n'(0) > 0$; hence, $\Phi'(0) \geq 0$ and we need only show that $\Phi'(0) \neq 0$. This will be easier to do once we have shown that Φ is one-to-one.

To show that Φ is one-to-one, note another theorem from [2, p. 178].

Theorem 15 *If the functions $f_n(z)$ are analytic and nonzero in a region Ω, and if $f_n(z)$ converges to $f(z)$, uniformly on every compact subset of Ω, then $f(z)$ is either identically zero or never equal to zero in Ω.*

This has an immediately useful corollary:

Corollary 16 *If the functions $\Phi_n(z)$ are analytic and one-to-one on $\mathbb{D}$, and converge uniformly on $\overline{\mathbb{D}}$ to Φ, then Φ is either a constant or is one-to-one.*

Proof Fix some $z_0 \in \mathbb{D}$, and let $G = \mathbb{D} - \{z_0\}$. Let $f_n(z) = \Phi_n(z) - \Phi_n(z_0)$. Then $f_n(z)$ is analytic and nonzero in G. Further, $f_n(z)$ converges to $\Phi(z) - \Phi(z_0)$ uniformly. Then either $\Phi(z) - \Phi(z_0) = 0$ for all z, implying Φ is constant, or $\Phi(z) \neq \Phi(z_0)$ for all z. Since z_0 was arbitrary, this means that Φ is one-to-one. ∎

Recall that we assumed that $B(0, \rho) \subset \Omega_n$ for all n. This implies that $\{\Phi_n\}$ does not converge to a constant.

Lemma 17 *If for all n, $B(0, \rho) \subset \Phi_n(\mathbb{D})$ and Φ_n converges uniformly to to Φ, then Φ is not a constant.*

Proof Because Φ_n converges uniformly, there exists a $N > 0$ such that for all $z \in \mathbb{D}$, $|\Phi(z) - \Phi_N(z)| < \rho/4$. Thus, for all $z, w \in \mathbb{D}$,

$$|\Phi(z) - \Phi(w)| \geq |\Phi_N(z) - \Phi_N(w)| - \rho/2.$$

In particular, if $\Phi_N(z) = \rho/2$ and $\Phi_N(w) = -\rho/2$, then

$$|\Phi(z) - \Phi(w)| \geq \rho - \rho/2 = \rho/2 > 0.$$

We know that some such z and w exist, and so Φ is not a constant. ∎

Finally, to show that $\Phi'(0) > 0$, we use another theorem from [2, p. 131].

Theorem 18 *Suppose $f(z)$ is analytic at 0, and f has a zero of order n at 0. If $\varepsilon > 0$ is sufficiently small, then there exists a $\delta > 0$ such that if $|a| < \delta$, the equation $f(z) = a$ has exactly n roots in the disk $|z| < \varepsilon$.*

This has as a corollary the following important result.

Corollary 19 *Suppose Φ is analytic and one-to-one in $\mathbb{D}$. Then $\Phi'(0) \neq 0$.*

Proof Suppose $f(z) = \Phi(z) - \Phi(0)$ has a zero of order n at 0. Let ε be sufficiently small to apply Theorem 18. Assume without loss of generality that $\varepsilon < 1$, and let $a = \delta/2$, where δ is given by Theorem 18. Then the equation $f(z) = a$ has exactly n roots. Since Φ is one-to-one, f is one-to-one, and so $n \leq 1$. Thus, $f(z)$ has a zero of order 1. This implies that $f'(0) \neq 0$, by [2, p. 127]. Since $\Phi' = f'$, this implies $\Phi'(0) \neq 0$. ∎

Also, note that by [15, Theorem 7.12], because $\Phi_n \to \Phi$ uniformly on $\overline{\mathbb{D}}$ and because Φ_n is continuous on $\overline{\mathbb{D}}$ for all n, Φ has a continuous extension to $\overline{\mathbb{D}}$.

We summarize the results of this chapter in the following theorem.

Theorem 20 *Suppose that $\{\Omega_n\}$ is a sequence of simply connected, uniformly bounded domains containing 0, and that their Riemann maps Φ_n are equicontinuous on $\overline{\mathbb{D}}$. Suppose further that for some $\rho > 0$, $B(0, \rho) \subset \Omega_n$ for all n.*

Then there is some subsequence $\{\Phi_{n_k}\}$ of the Riemann maps that converges uniformly to a function Φ that is itself the Riemann map of some domain Ω, and which is continuous on $\overline{\mathbb{D}}$.

Furthermore, by Corollary 4, if the harmonic measure distribution functions h_{Ω_n} converge to some function f, then $f = h_\Omega$ at all points of continuity of f.

In the remainder of this chapter, we investigate which sets of domains have equicontinuous Riemann maps, and state some sufficient conditions from Pommerenke's book [11].

3.4 Three examples of non-equicontinuous Riemann maps

We present several examples of sequences of functions on $\overline{\mathbb{D}}$ which are not equicontinuous, and which do not contain any subsequences which converge to Riemann maps. Because of these examples, we do not pursue any less restrictive conditions than equicontinuity.

One example is $\varphi_n(z) = nz$. Here the domains $\Omega_n = \phi_n(\mathbb{D})$ are not uniformly bounded.

Another example is $\varphi_n(z) = z^n$. Here, the functions ϕ_n are not one-to-one, and so they are not the Riemann maps of any domains.

Consider

$$\varphi_n(z) = z\frac{2 - 1/n}{(n-1)z + n}.$$

This Möbius transformation takes $\mathbb{D}$ to the unit disk centered at $-1 + 1/(2n)$. It is one-to-one, uniformly bounded, and analytic; furthermore, $\varphi_n(0) = 0$, and $\varphi_n'(0) > 0$. Thus, $\{\varphi_n\}$ is a sequence of uniformly bounded Riemann maps. So far, it comes very close to satisfying our conditions.

Now

$$\varphi_n' = \frac{2-1/n}{(n-1)z+n} - z\frac{(2-1/n)(n-1)}{((n-1)z+n)^2} = \frac{2n-1}{((n-1)z+n)^2}.$$

For any $\delta > 0$, let $n > 2/\delta$. Using Maple, we compute that

$$\begin{aligned} |\varphi_n(1/n-1) - \varphi_n(1/2n-1)| &= \left| \frac{(1/n-1)(2-1/n)}{(n-1)(1/n-1)+n} - \frac{(1/2n-1)(2-1/n)}{(n-1)(1/2n-1)+n} \right| \\ &= \left| \frac{n}{3n-1} \right| > 1/3. \end{aligned}$$

Thus, $\{\varphi_n\}$ is not equicontinuous.

Note that $\lim_{n\to\infty} \varphi_n(z) = 0$ for all $z \in \overline{\mathbb{D}}$ (except for $z = -1$). Thus, this sequence of Riemann maps does not converge to a Riemann map.

Note that ϕ_n maps 0 to itself. Also, note that the boundary of $\mathbb{D}$ is mapped arbitrarily close to 0, because ϕ_n maps $\mathbb{D}$ to the unit disk centered at $-1 + 1/(2n)$. Thus, the limiting domain, if it is open, cannot contain $\phi_n(0)$.

3.5 When are the Riemann maps of a sequence of domains equicontinuous?

This question has fortunately been studied. We have this theorem from Pommerenke's book, [11, p. 22].

Theorem 21 *Let Φ_n map $\mathbb{D}$ conformally onto Ω_n with $\Phi_n(0) = 0$. Suppose that*

$$B(0,\rho) \subset \Omega_n \subset B(0,R)$$

for all n. If $\{\mathbb{C}\backslash\Omega_n\}$ is uniformly locally connected, then $\{\Phi_n\}_{n=1}^{\infty}$ is equicontinuous on $\overline{\mathbb{D}}$.

This completes the proof of Theorem 9.

The following lemma and its corollaries give a useful sufficient condition for identifying sequences of domains with path-connected or uniformly locally connected complements.

Lemma 22 *Let Ω be a domain. Suppose that for some $\varepsilon > 0$ there is a $\delta > 0$ such that for every $\zeta, \omega \in \partial\Omega$ with $|\zeta - \omega| < \delta$, there is a continuum in $\mathbb{C}\backslash\Omega$ connecting ζ and ω of diameter at most ε. Then for every $z, w \in \mathbb{C}\backslash\Omega$ with $|z - w| < \delta$, there is a continuum in $\mathbb{C}\backslash\Omega$ connecting z and w of diameter at most 2ε.*

Proof Assume without loss of generality that $\delta < \varepsilon$. Let $z, w \in \mathbb{C}\backslash\Omega$ with $|z - w| < \delta$. We wish to find a continuum of diameter at most 2ε in $\mathbb{C}\backslash\Omega$ connecting z and w.

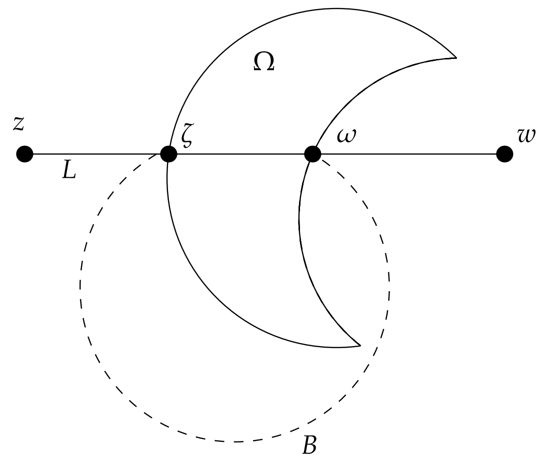

Figure 3.2: If we can connect any two points ζ and ω in $\partial\Omega$ by a continuum of small diameter in $\mathbb{C}\backslash\Omega$, then we can connect any two points z and w in $\mathbb{C}\backslash\Omega$ by a continuum of small diameter.

Let L be the straight-line contour from z to w, so $\operatorname{diam}(L) < \delta < \varepsilon$. Then if $L \subset \mathbb{C}\backslash\Omega$, we are done. Otherwise, let $\zeta, \omega \in L \cap \partial\Omega$ be such that there are no points $\xi \in L \cap \partial\Omega$ with either $|z - \xi| < |z - \zeta|$ or $|w - \xi| < |w - \omega|$. See Fig. 3.2. Then there is a continuum B of diameter at most ε connecting ζ and ω, and lying entirely in $\mathbb{C}\backslash\Omega$.

By appending the pieces of L lying between z and ζ, and between ω and w, we get a continuum, lying entirely inside $\mathbb{C}\backslash\Omega$, that connects z and w. Furthermore, this continuum lies entirely within $L \cup B$, and so because $L \cap B$ is nonempty, its diameter is at most $\operatorname{diam}(L) + \operatorname{diam}(B) < 2\varepsilon$. Hence there is a continuum of diameter at most 2ε in $\mathbb{C}\backslash\Omega$ connecting z and w, as desired. ■

This has two useful corollaries.

Corollary 23 *Let Ω be a domain. If for every $z, w \in \partial\Omega$ there is a continuum in $\mathbb{C}\backslash\Omega$ connecting z and w, then $\mathbb{C}\backslash\Omega$ is path-connected.*

Corollary 24 *Let $\{\Omega_n\}$ be a sequence of domains. If for every $\varepsilon > 0$ there is a $\delta > 0$ such that, if $z, w \in \partial\Omega_n$ and $|z - w| < \delta$, there is a continuum in $\mathbb{C}\backslash\Omega_n$*

connecting z and w of diameter at most ε, then $\{\mathbb{C}\backslash\Omega_n\}$ is a uniformly locally connected sequence.

Thus, to check to see if the complements $\{\mathbb{C}\backslash\Omega_n\}$ of a sequence $\{\Omega_n\}$ of domains are uniformly locally connected, it suffices to check only the points in $\partial\Omega_n$, rather than all of the points in $\mathbb{C}\backslash\Omega_n$.

Thus, we know that for any sequence of simply connected domains Ω_n, if $\partial\Omega_n$ is uniformly locally connected and Ω_n is uniformly bounded, then there is a subsequence Ω_{n_k} with uniformly convergent Riemann maps.

Chapter 4

Showing that some functions are h-functions

4.1 Previous theorems and the goal of this chapter

Given any function f, the main question we wish to answer is: when is f the h-function of some domain Ω? In this chapter, we look at the more restrictive question: when is f the h-function of some domain Ω that may be written as the limit of a sequence $\{\Omega_n\}$ of domains that satisfy certain conditions?

We establish some sufficient conditions on f and the sequence $\{\Omega_n\}$, which are summarized in Theorem 33 in Section 4.9 below.

Throughout this chapter, we assume that all h-functions are taken from the base point 0, and that all circle domains and blocked circle domains are centered at the base point 0.

Suppose there is some sequence $\{\Omega_n\}$ of domains, with h-functions h_n, such that $h_n \to f$ at all points of continuity of f. We can conclude the following, based on previous chapters.

1. If Ω_n is simply connected, then by [2, p. 230] there exists a Riemann map $\Phi_n : \mathbb{D} \mapsto \Omega_n$ such that $\Phi_n(0) = 0$ and $\Phi_n'(0) > 0$.

2. By Theorem 9, if there exist positive real numbers R and ρ such that for all n, $B(0,\rho) \subset \Omega_n$ and $\Omega_n \subset B(0,R)$, and if the sequence $\{\mathbb{C}\backslash\Omega_n\}$ is uniformly locally connected, then there is some subsequence of the Riemann maps $\{\Phi_{n_k}\}$ that converges uniformly on $\overline{\mathbb{D}}$ to a function Φ that is the Riemann map of some domain Ω. (Note that there may be

several such subsequences, and that they may converge to different domains. However, this will not affect the remaining conclusions.)

3. By Theorem 1, because $\Phi_{n_k} \to \Phi$ uniformly on $\overline{\mathbb{D}}$, $h_{n_k} \to h_\Omega$ pointwise at points of continuity of h_Ω.

4. Therefore, $f = h_\Omega$ at points of continuity of h_Ω and of f. By Theorem 3, if f is monotonic, all points of continuity of f are points of continuity of h_Ω. Thus, f is equal to the h-function of the domain Ω at all points of continuity of f.

Thus, if f is monotonic and $f(r) = \lim_{n\to\infty} h_{\Omega_n}(r)$ at points of continuity of f, where the sequence $\{\Omega_n\}$ satisfies the conditions given in (1) and (2) above, then $f(r) = h_\Omega(r)$ for some Ω (except perhaps at points of discontinuity of f.)

We will break the task of finding such a sequence $\{\Omega_n\}$ into two steps: finding some sequence $\{X_n\}$ of circle domains such that h_{X_n} converges pointwise to f, and finding a sequence $\{\Omega_n\}$ of blocked circle domains which satisfy the conditions in (1) and (2) above, such that $h_{X_n} - h_{\Omega_n}$ converges pointwise to 0.

Note that our program will not construct all possible h-functions. It will not construct the h-function of a domain Ω unless Ω may be written as the limit of a sequence $\{\Omega_n\}$ of domains that satisfy the conditions in (1) and (2). This means that Ω must be bounded, simply connected, and by Lemma 25, must have a locally connected and path-connected complement.

Thus, our program will fail for many functions f that are, in fact, the h-functions of some domain Ω.

Lemma 25 *Suppose that for a simply connected domain Ω with Riemann map Φ there is a sequence $\{\Omega_n\}$ of domains with Riemann maps $\{\Phi_n\}$ that are equicontinuous on the closed unit disk $\overline{\mathbb{D}}$, and that $\Phi_n \to \Phi$ uniformly on $\overline{\mathbb{D}}$.*

Then $\mathbb{C}\backslash\Omega$ is path-connected and locally connected.

Proof By Theorem 20, we know that Φ has a continuous extension to $\overline{\mathbb{D}}$.

So by [11, Theorem 2.1], $\mathbb{C}\backslash\Omega$ is locally connected, and $\partial\Omega$ is path-connected. Hence by Corollary 23, $\mathbb{C}\backslash\Omega$ is path-connected. ■

4.2 Necessary conditions

It is possible to eliminate, a priori, some functions f from consideration.

Any function f which is not monotonically increasing, is ever negative, or which is ever greater than 1, cannot arise as the h-function of any domain. There are other known conditions on the h-functions of bounded, simply connected domains with path-connected complements, which are summarized in this section.

Given a function f which increases monotonically from 0 to 1, we define the constants μ and M as follows:

$$M = \sup\left\{r \mid f(r) < 1\right\}, \quad \mu = \inf\left\{r \mid f(r) > 0\right\}.$$

If f is the h-function of any domain Ω, measured from some basepoint z_0, then because domains are open, there must be some $\rho > 0$ where $B(z_0, \rho) \subset \Omega$; consequently, we must have that $\mu \geq \rho > 0$. Furthermore, if Ω is a bounded domain, we must have that M is finite.

It was shown in [19, Equation (5)] that if f arises as the h-function of a simply connected domain, then

$$f(r) \geq 1 - \frac{4}{\pi}\arctan\sqrt{\frac{\mu}{r}}. \tag{4.1}$$

Also, consider this theorem.

Theorem 26 *If $0 < h_\Omega(r_1) = h_\Omega(r_2) < 1$ for some $r_1 < r_2$, then $\mathbb{C}\backslash\Omega$ is not path-connected.*

Proof Since $0 < h_\Omega(r_1)$, there is some point $z \in \partial\Omega$ with $|z| \leq r_1$. Since $h_\Omega(r_2) < 1$, there is some point $w \in \partial\Omega$ with $|w| > r_2$.

Since $h_\Omega(r_1) = h_\Omega(r_2)$, we have that

$$\begin{aligned} \omega(0, \left\{\zeta \in \partial\Omega \mid r_1 < |\zeta| \leq r_2\right\}, \Omega) &= \omega(0, \partial\Omega \cap \overline{B(0, r_2)}, \Omega) \\ &\quad - \omega(0, \partial\Omega \cap \overline{B(0, r_1)}, \Omega) \\ &= h_\Omega(r_2) - h_\Omega(r_1) = 0. \end{aligned}$$

So in particular, the annulus $\left\{\zeta \in \partial\Omega \mid r_1 < |\zeta| \leq r_2\right\}$ contains no continuum of $\partial\Omega$. Thus, there is no path connecting z and w, for if there were, that path would have to go through that annulus. So $\mathbb{C}\backslash\Omega$ is not path-connected. Hence, by assumption, $\mathbb{C}\backslash\Omega$ is not connected; hence Ω is not simply connected. ■

Therefore, by Lemma 25, we only wish to consider functions f that are strictly increasing on some range $[\mu, M]$, where $0 < \mu < M < \infty$, satisfy the lower bound in Equation (4.1), and that equal 0 on $(0, \mu)$ and 1 on (M, ∞).

4.3 A sequence $\{X_n\}$ of domains such that $h_{X_n} \to f$.

Recall Theorem 5, restated here.

Theorem 27 (Snipes) *Let $f(r)$ be a right-continuous step function, increasing from 0 to 1, with its discontinuities at $r_1, r_2, \ldots, r_{n+1}$, with $0 < r_1 < r_2 < \cdots < r_{n+1}$. Then there exists a circle domain X with n arcs whose harmonic measure distribution function $h_X(r)$ is equal to $f(r)$. The radii of the n arcs and of the boundary circle in X are given by $r_1, r_2, \ldots, r_n$ and by r_{n+1} respectively.*

Let $\{f_n\}$ be a sequence of right-continuous step functions, increasing from 0 to 1, such that $\lim_{n\to\infty} f_n(r) = f(r)$ at points of continuity of f. We can do this by defining f_n as follows. Let floor(x) be the largest integer less than or equal to x.

Define $f_n(r)$ as

$$f_n(r) = \begin{cases} 0, & r < \mu \\ f\left(\frac{M-\mu}{n}\,\mathrm{floor}\left(\frac{n(r-\mu)}{M-\mu}\right) + \mu\right), & \mu \le r < M \\ 1, & M \le r \end{cases}$$

See Fig. 4.1.

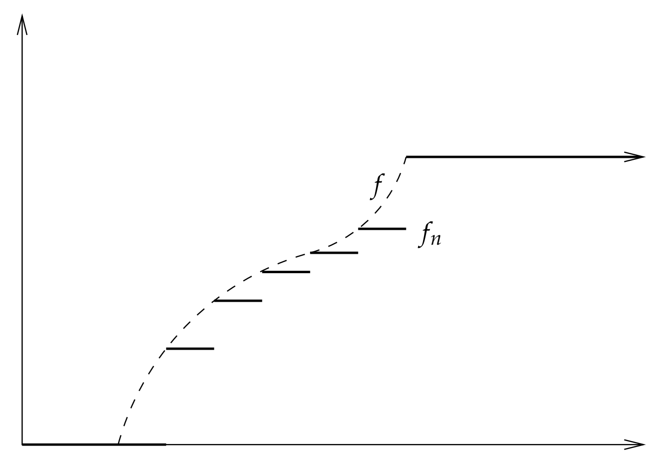

Figure 4.1: An arbitrary, strictly increasing function f, and an approximating step function f_n.

Then f_n is a right-continuous step function, and for $k = 0, 1, \ldots, n$, $f_n(r) = f(r)$ when $r = \mu + k(M-\mu)/n$. Furthermore, the discontinuities of $f_n(r)$ occur at the points $r = \mu + k(M-\mu)/n$. Thus, f_n is a step function, with $n+1$ discontinuities, such that $\lim_{n\to\infty} f_n(r) = f(r)$ at all points of continuity of f. By Theorem 5, the function $f_n(r)$ arises as the h-function of some circle domain X_n.

Unfortunately, the functions $f_n(r)$ do not arise as the h-functions of simply connected domains; they arise as the h-functions of multiply connected circle domains. We refer to these circle domains as X_n, and for convenience, assume that all boundary arcs in X_n are centered on the positive real axis. We intend to modify the circle domains X_n so that they are simply connected and so that their complements are uniformly locally connected.

Notation. We denote the radius of the kth boundary arc in X_n by $r_{n,k}$, and let the angle subtended by the kth boundary arc in X_n be $2\psi_{n,k}$. See Fig. 4.2. Thus, the kth boundary arc in X_n is the set of points

$$\left\{ r_{n,k} e^{i\theta} \mid |\theta| \le \psi_{n,k} \right\}.$$

Thus,

$$X_n = B(0,M) \setminus \bigcup_{k=1}^{n} \left\{ r_{n,k} e^{i\theta} \mid |\theta| \le \psi_{n,k} \right\}.$$

4.4 A well-behaved sequence $\{\Omega_n\}$ that is similar to $\{X_n\}$

We wish to construct a sequence of blocked circle domains $\{\Omega_n\}$ from the circle domains $\{X_n\}$ such that $\{\mathbb{C}\setminus\Omega_n\}$ is uniformly locally connected and $h_{X_n} - h_{\Omega_n} \to 0$.

Note that all blocked circle domains are simply connected, and that if the domains Ω_n have their boundary arcs in the same places as the arcs of X_n, then we will automatically have that $B(0,\mu) \subset \Omega_n \subset B(0,M)$. Thus, to show that f is the h-function of some simply connected domain, we need only show that

- $h_{X_n} - h_{\Omega_n} \to 0$, and
- $\{\mathbb{C}\setminus\Omega_n\}$ is uniformly locally connected.

Throughout the rest of this chapter, we will develop constraints on the arc lengths of arcs in X_n and on the positions of the gates in the domains Ω_n such that those conditions will hold.

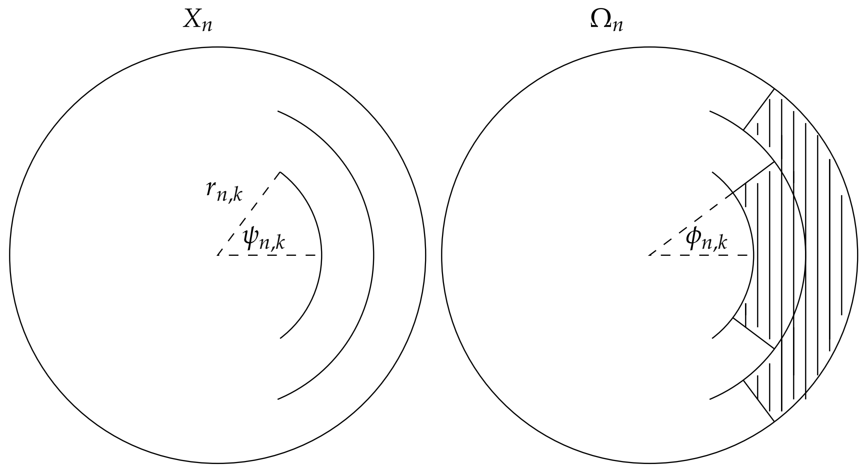

Figure 4.2: A multiply connected circle domain X_n, and the blocked circle domain Ω_n constructed from X_n by blocking off sectors between the boundary arcs. Here $\psi_{n,k}$ is half the angle subtended at 0 by the kth boundary arc in X_n, and $\phi_{n,k}$ is half the angle subtended at 0 by the sector blocked off by gates between the kth and $k+1$th boundary arcs in Ω_n.

We construct the blocked circle domains as follows. Let $\{\phi_{n,k}\}_{n=1}^{\infty}$ be a doubly indexed sequence of nonnegative real numbers, where $k \le n+1$ and $\phi_{n,k} < \min(\psi_{n,k}, \psi_{n,k+1})$. Let the sequence $\{\Omega_n\}$ be given by

$$\Omega_n = X_n \backslash \bigcup_{k=1}^{n} \left\{ re^{i\theta} \mid r_{n,k} < r < r_{n,k+1}, |\theta| \le \phi_{n,k} \right\}.$$

That is, from the circle domain X_n, we remove sectors of annuli, bounded by the circular arcs of X_n and by straight line segments (gates) which lie along rays through 0. These blocked sectors each span an angle of $2\phi_{n,k} < 2\min(\psi_{n,k}, \psi_{n,k+1})$; we wish the sectors to be shorter than the neighboring arcs of the original circle domain. See Fig. 4.2.

Note that $r_{n,k} - r_{n,k-1} = (M - \mu)/n$.

We would like to find sufficient conditions on the numbers $\psi_{n,k}$ and $\phi_{n,k}$ such that $\{\mathbb{C} \backslash \Omega_n\}$ is uniformly locally connected, and such that $|h_{X_n} - h_{\Omega_n}| \to 0$.

4.5 When is $\{\mathbb{C}\backslash\Omega_n\}$ uniformly locally connected?

First, we prove a theorem that will establish sufficient conditions for $\{\mathbb{C}\backslash\Omega_n\}$ to be uniformly locally connected, that is, for the conditions of Corollary 24 to hold.

Note that nearby points on boundary arcs of Ω_n can take on three different configurations, as shown in Fig. 4.3. They can be nearby on one side of the positive real axis (Possibility 1), nearby and across the positive real axis from each other (Possibility 2), or (if the boundary arcs are very long) nearby and across the negative real axis from each other (Possibility 3).

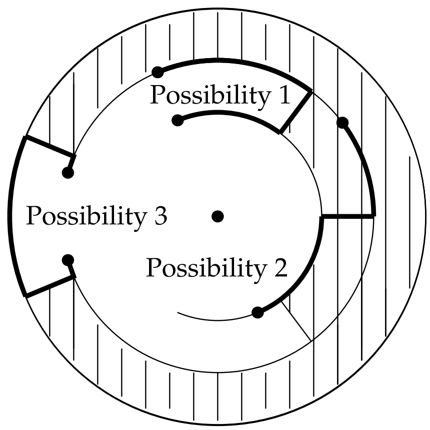

Figure 4.3: The three possibilities for close points z and w on boundary arcs of Ω_n, and the continua connecting them.

Theorem 28 *Let* $\theta_{n,j,k} = \min(\psi_{n,j}, \psi_{n,k}) - \min_{j \leq i < k} \phi_{n,i}$.

Suppose that for all $\varepsilon > 0$ *there exist numbers* δ_ε, δ'_ε *with* $\delta_\varepsilon > \delta'_\varepsilon > 0$, *such that the following conditions hold.*

(a) If $|r_{n,k} - r_{n,j}| < \delta_\varepsilon$ *then* $\theta_{n,j,k} < \varepsilon$, *and*

(b) If $\mu \sin(\pi - \psi_{n,k}) < \delta'_\varepsilon$ *and* $\psi_{n,k} \geq \pi/2$, *then* $M - r_{n,k} < \delta_\varepsilon$ *and for all* $i \geq k$, $\pi - \phi_i < \varepsilon$.

Then $\{\mathbb{C}\backslash\Omega_n\}$ *is uniformly locally connected.*

See Fig. 4.4 for an illustration of the angle $\theta_{n,j,k}$.

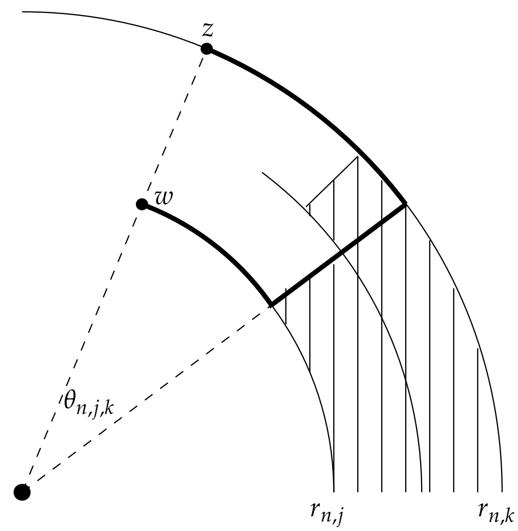

Figure 4.4: The angle $\theta_{n,j,k}$ is the maximum subtended angle which a path in $\mathbb{C}\backslash\Omega$ must traverse if it is to connect a point w on the jth boundary arc to a point z lying along the same ray through 0 on the kth boundary arc.

Condition (a) ensures that nearby points, arranged as shown in Possibility 1 in Fig. 4.3, will have a connecting path of small diameter. Condition (b) ensures that nearby points arranged as in Possibility 3 will have a connecting path of small diameter. Points arranged as in Possibility 2 will automatically have a small connecting path if they are near each other.

Proof Fix some $\varepsilon > 0$, and assume without loss of generality that $\varepsilon < M$. We wish to find some $\delta > 0$ such that if $z, w \notin \Omega_n$ and $|z - w| < \delta$, then there is a continuum of diameter at most ε contained in $\mathbb{C}\backslash\Omega_n$ connecting z and w. By Corollary 23, we need only look at the case where $z, w \in \partial\Omega_n$.

Then there are three cases:

1. Both z and w lie on arcs in X_n.
2. z lies on an arc in X_n, but w does not. (This implies that w lies on a gate.)
3. Neither z nor w lies on an arc in X_n.

See Fig. 4.5.

Suppose that z and w lie on the arcs of radii $r_{n,k}$ and $r_{n,j}$ with $j \leq k$.

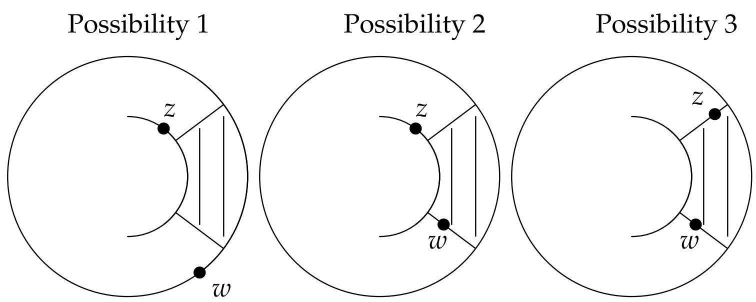

Figure 4.5: The three possible arrangements of two boundary points of a blocked circle domain.

Let $e = \varepsilon/3$, and let

$$d = \min\left(\mu, \delta'_E, \frac{e}{\sqrt{2}}, \frac{5\mu e}{11\sqrt{2}M}\right),$$

where $E = e/2\sqrt{2}M$. So in particular, Conditions (a) and (b) hold for E and δ'_E, δ_E. Thus, for example, if $|r_{n,k} - r_{n,j}| < d$, then $\theta_{n,j,k} < E$.

We wish to show that if $|z - w| < d$, then there is a continuum of diameter at most e lying entirely in $\mathbb{C}\backslash\Omega_n$ that connects z and w.

Either z and w both lie on the same side of the real axis, or they do not.

If they do, then they are connected by a continuum that lies entirely in a sector of an annulus of inner radius $r_{n,j}$, outer radius $r_{n,k}$, and which subtends an angle $2\alpha = \theta_{n,j,k} + \eta$. We define η to be the maximum angle subtended at 0 between z and w, subject only to the constraints that z and w lie on arcs of radii $r_{n,j}$ and $r_{n,k}$ and that $|z - w| \leq d$. See Fig. 4.6 for an illustration of η and a continuum connecting z and w. Then the continuum connecting z and w subtends an angle at most $\eta + \theta_{n,j,k}$ by definition of η and $\theta_{n,j,k}$. Let this sector of the annulus be A. See Fig. 4.7 for an illustration of A. Thus, as shown in Fig. 4.6, η is an angle in the triangle of side-lengths $r_{n,j}$, $r_{n,k}$, and d. By the law of cosines, $d^2 = r_{n,k}^2 + r_{n,j}^2 - 2r_{n,j}r_{n,k}\cos\eta$.

Since $d \leq \mu \leq r_{n,j}$, we must have that $\eta \leq \pi/3$. Using the Taylor series expansion for $\cos(x)$, we know that $1 - x^2/2! + x^4/4! \geq \cos x$ for all $0 \leq x \leq \sqrt{30}$. Thus, we can find an upper bound on η that depends only

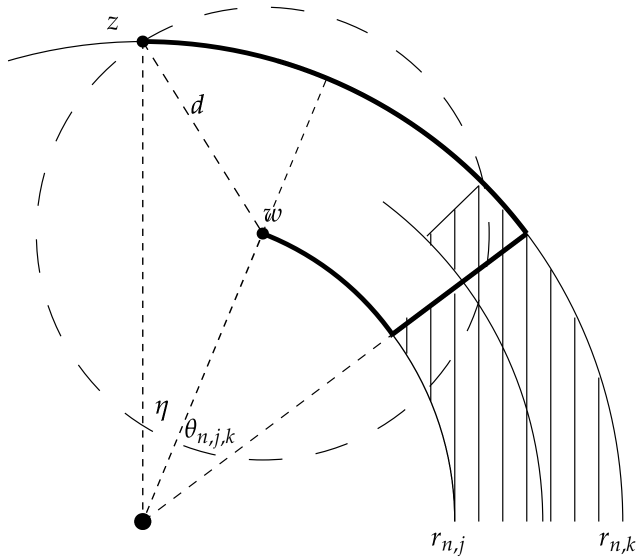

Figure 4.6: A continuum in $\mathbb{C}\backslash\Omega_n$ connecting z and w, and an illustration of the maximal angular distance η between z and w.

on d and μ.

$$\begin{aligned} (1-\eta^2/2+\eta^4/24) &\geq \cos\eta = \frac{r_{n,k}^2 + r_{n,j}^2 - d^2}{2r_{n,j}r_{n,k}} \\ &= \frac{2r_{n,k}r_{n,j} + (r_{n,k}-r_{n,j})^2 - d^2}{2r_{n,j}r_{n,k}} \\ &\geq 1 - \frac{d^2}{2\mu^2}. \end{aligned}$$

This implies that $\eta^2(10/11) < \eta^2(1-\pi^2/108) \leq \eta^2 - \eta^4/12 \leq (d/\mu)^2$ since $\eta \leq \pi/3$. So

$$\eta < (11/10)d/\mu.$$

There are two possibilities for diam(A), labeled l_1 and l_2 in Fig. 4.7. Because A lies entirely on one side of the real axis, it subtends an angle of at most π. So the diameter, defined as the maximum distance between two points in A, clearly connects two corners of the annular sector. They cannot both be inner corners; thus, either the diameter is the distance between the two outer corners, or the distance from an inner corner to an outer corner.

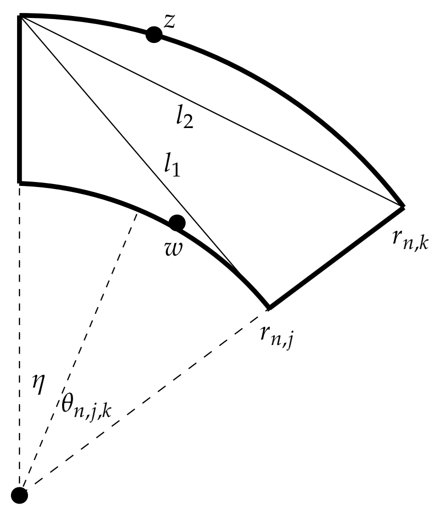

Figure 4.7: The annular sector A containing z, w, and a connecting path in $\mathbb{C}\backslash\Omega_n$, and the two possibilities, l_1 and l_2, for its diameter.

It is possible to estimate both of these distances using $r_{n,j}$, $r_{n,k}$, and $2\alpha = \theta_{n,j,k} + \eta$.

$$\begin{aligned} l_1 &= 2r_{n,k}\sin\alpha \leq 2M\alpha = M\theta_{n,j,k} + M\eta < ME + (11/10)Md/\mu \\ &\leq M(e/2\sqrt{2}M) + (11/10)M(5e\mu/11\sqrt{2}M)/\mu \\ &= e/\sqrt{2}. \end{aligned}$$

Here we have used the fact that $d \leq \delta_E$, implying that by Condition (a) $\theta_{n,j,k} < E$. We have also used the fact that $d \leq 5e\mu/11\sqrt{2}M$.

We use the law of cosines again to estimate the other possible value of the diameter of A, that is, the length l_2:

$$\begin{aligned}
l_2 &= \sqrt{r_{n,j}^2 + r_{n,k}^2 - 2r_{n,j}r_{n,k}\cos(\theta_{n,j,k}+\eta)} \\
&= \sqrt{(r_{n,j}-r_{n,k})^2 + 2r_{n,j}r_{n,k}(1-\cos(\theta_{n,j,k}+\eta))} \\
&\leq \sqrt{d^2 + 2M^2(1-\cos(\theta_{n,j,k}+\eta))} \\
&\leq \sqrt{d^2 + M^2(\theta_{n,j,k}+\eta)^2} \\
&< \sqrt{d^2 + (ME + (11/10)dM/\mu)^2} \\
&\leq \sqrt{e^2/2 + (e/\sqrt{2})^2} = e.
\end{aligned}$$

On the fourth line, we have used the Taylor series for $\cos(x)$. On the fifth line, we have used the fact that $|r_{n,k} - r_{n,j}| < d \leq \delta_E' \leq \delta_E$, implying that $\theta_{n,j,k} < E$, and the bound on η. On the final line, we have used the facts that $d \leq e/\sqrt{2}$ and that $d \leq (e/\sqrt{2})(5\mu/(11M))$.

So in either case, z and w are connected by a continuum of diameter at most e.

If z and w lie on different sides of the real axis, then since $d \leq \mu$, z and w must lie on the same side of the imaginary axis. If they lie on the right hand side, then they can be connected by a continuum in $\mathbb{C}\backslash\Omega_n$ of diameter at most $|z-w| < d \leq e/\sqrt{2}$. (See Fig. 4.3. This is Possibility 2.)

Otherwise, z and w lie on the left hand side of the imaginary axis, and so we know that $\psi_{n,j}$ and $\psi_{n,k}$ must be at least $\pi/2$. Therefore,

$$\mu\sin(\pi - \psi_{n,k}) \leq d \leq \delta_E', \quad \mu\sin(\pi-\psi_{n,j}) \leq d \leq \delta_E'.$$

So $M - r_{n,k} < \delta_E$ and $M - r_{n,j} < \delta_E$, by Condition (b). Furthermore, for all $i \geq k$, $\pi - \phi_i < E$, also by Condition (b). So z and w can be connected by a continuum that lies entirely in a sector B of an annulus of inner radius $r_{n,j}$, outer radius M, and that subtends an angle $2E$. (See Fig. 4.8.)

We can estimate the diameter of B in exactly the same way we estimated the diameter of A. We arrive at the conclusion that $\operatorname{diam}(B) \leq e$. (Note that this continuum will also connect all points in the gates above z and w.)

Therefore, for any two points z and w on arcs in ∂X_n, if $|z-w| < d$, then z and w are connected by a continuum of diameter less than e.

Recall that $e = \varepsilon/3$, and that we seek a δ such that if $z, w \in \partial\Omega_n$ with $|z-w| < \delta$, then z and w lie on a continuum of diameter less than ε in $\mathbb{C}\backslash\Omega_n$.

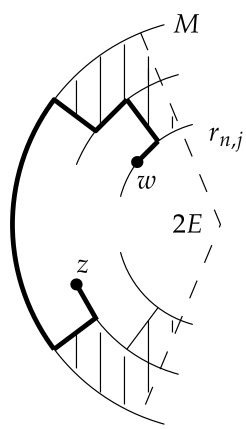

Figure 4.8: A continuum connecting two nearby boundary points of Ω_n that lie to the left of the imaginary axis.

Let $\delta = d/3$. So $\delta = d/3 \leq e/\sqrt{2} = \varepsilon/3\sqrt{2}$. Then if z and w lie on arcs in ∂X_n, and $|z - w| < 3\delta$, then z and w are connected by a continuum of diameter less than $\varepsilon/3$ in $\mathbb{C}\backslash\Omega_n$.

Consider any two points z and w in $\partial\Omega_n$ with $|z - w| < \delta$. Then if z, w lie on arcs of ∂X_n, they can be connected by a continuum of diameter at most $\varepsilon/3$. Suppose, then, that z and w lie in $\partial\Omega_n$ and that w lies on a gate, and z lies on either an arc or a gate.

If both z and w lie on the same gate, they can be connected by a straight-line continuum of length less than $\delta \leq \varepsilon$.

If z and w lie on the pair of gates connecting two adjacent arcs, they may be connected by a continuum of small diameter if they both lie to the right of the imaginary axis. This continuum is very similar to the one connecting the points on arcs in Fig. 4.3, Possibility 2.

If z and w lie to the left of the imaginary axis, on a pair of gates connecting two adjacent arcs, let ζ and ω be points at the ends of the gates, as shown in Fig. 4.9. Then $|\zeta - \omega| < |z - w| < \delta$. Hence there is a continuum connecting ζ and ω of diameter less than ε, as shown in Fig. 4.9. This continuum also connects z and w.

Finally, consider the case where z lies on either an arc, or a gate that is farther from 0 than the gate on which w lies. Let $\zeta = z$ if z lies on an arc, and let ζ be an endpoint of the straight-line segment containing z, if z lies on a line segment. Similarly, let ω be an endpoint of the line containing w. Choose ζ and ω such that $|z| \geq |\zeta| \geq |\omega| > |w|$. See Fig. 4.10.

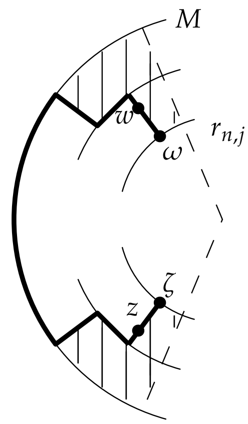

Figure 4.9: Two points z and w on gates to the left of the imaginary axis, two nearby points ζ and ω on arcs, and a continuum of small diameter connecting ζ and ω, and therefore z and w.

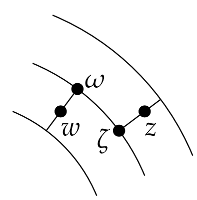

Figure 4.10: Two points in $\partial\Omega_n$ and nearby points on boundary arcs.

Then $|z-\zeta| \le |z-w| < \delta$ and $|w-\omega| < |w-z| < \delta$, and so $|\zeta-\omega| < 3\delta$. Hence, there is a continuum of diameter less than $\varepsilon/3$ connecting ζ and ω; hence, this continuum can be extended by straight-line segments to a continuum of diameter less than $\varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon$ connecting z and w.

Thus, for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $|z-w| < \delta$ and $z, w \in \partial\Omega_n$, then there is a continuum of diameter at most ε contained entirely in $\mathbb{C}\backslash\Omega_n$ connecting z and w. Hence, by Corollary 24, $\{\mathbb{C}\backslash\Omega_n\}$ is uniformly locally connected. ■

4.6 When does $h_{X_n} - h_{\Omega_n} \to 0$?

We wish to find conditions such that $h_{X_n} - h_{\Omega_n} \to 0$. We follow the development originally given in [16], but adapted to fit our (more general) situation.

We will use the following lemma, cited in [6, Theorem H.8].

Lemma 29 *Let D be a Jordan domain and let $z_0 \in D$. Let $b > x_0 = \operatorname{Re}(z_0)$ and suppose $F \subset \left\{z \in \overline{D} \;\middle|\; \operatorname{Re}(z) \geq b\right\}$ is a finite union of arcs. Assume that for $x_0 < x < b$, there exists $I_x \subset D \cap \{\operatorname{Re}(z) = x\}$ such that I_x separates z_0 from F and $\theta(x)$, the length of I_x, is measurable. Then*

$$\omega(z_0, F, D) \leq \frac{8}{\pi} \exp\left(-\pi \int_{x_0}^{b} \frac{dx}{\theta(x)}\right).$$

See Fig. 4.11 for an example.

Roughly speaking, the probability that a Brownian particle released from z_0 reaches F before it hits the side of the channel increases with the width of the channel and decreases exponentially with the length of the channel.

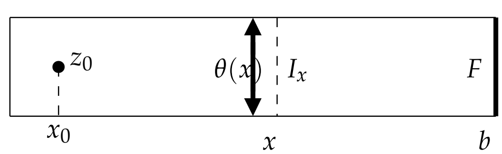

Figure 4.11: An illustration of Lemma 29: F is the right-hand edge of the rectangular domain D. If D is long and thin, the harmonic measure of F from z_0 is small.

We wish to find sufficient conditions for h_{X_n} to approach h_{Ω_n}. To this end, we first prove the following theorem.

Theorem 30 *Consider the domain R shown in Fig. 4.12. Let A_1 and A_2 be the curved arcs, and let l_1, l_2 be the straight-line segments (gates). Let χ be the angle subtended by the shorter of the two spikes lying outside the region given by the straight-line segments, and let r be the radius of the inner arc A_1, $r + w$ be the radius of the outer arc A_2. Then if $w \leq r$ and $\chi \leq \pi/4$,*

$$\omega(z_0, l_1 \cup l_2, R) \leq \frac{16}{\pi} e^{-\pi r \sin(2\chi)/3w}. \tag{4.2}$$

(If $\chi \geq \pi/4$, we can replace χ in the above equation by $\pi/4$.) Furthermore, the quantity on the right-hand side of Equation (4.2) is also an upper bound on the probability that a particle starting out on l_1 or l_2 in $\mathbb{C}\backslash(A_1 \cup A_2)$ escapes the channel.

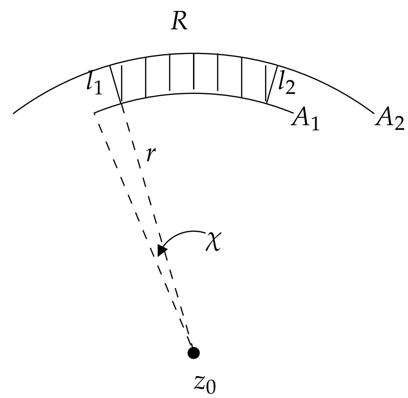

Figure 4.12: The domain R in Theorem 30: R is the domain exterior to two circular arcs A_1 and A_2, forming a channel, and two short lines, l_1 and l_2, blocking that channel.

Proof Suppose we rotate R as shown in Fig. 4.13. To reach l_1, a Brownian particle starting at z_0 must go through the end of the channel g. Hence $\omega(z_0, l_1, R_2) \leq \max_{\zeta \in g} \omega(\zeta, l_1, R_2)$. Also, to escape from the channel in $\mathbb{C}\backslash(A_a \cup A_2)$, a a particle starting on l_1 must either reach g, or reach l_2 and then escape through the other end of the channel.

Let I_x be the vertical line segment, a distance x from the nearest point in g, connecting two points on the arcs. Let $y(x) = r \sin \chi - x$ be the distance from I_x to the line through z_0 and l_1.

Then $\theta(x)$, the length of I_x, can be computed by applying the Pythagorean theorem to triangles with vertices at z_0, the endpoints of I_x, and points on the line through z_0 and l_1, a distance $y(x)$ from the endpoints of I_x. Thus,

$$\theta(x) = \sqrt{(r+w)^2 - y(x)^2} - \sqrt{r^2 - y(x)^2}.$$

So for any $\zeta \in g$, Lemma 29 implies that

$$\omega(\zeta, l_1, R) \leq \frac{8}{\pi} \exp\left(-\pi \int_0^{r \sin \chi} \frac{dx}{\sqrt{(r+w)^2 - y(x)^2} - \sqrt{r^2 - y(x)^2}} \right).$$

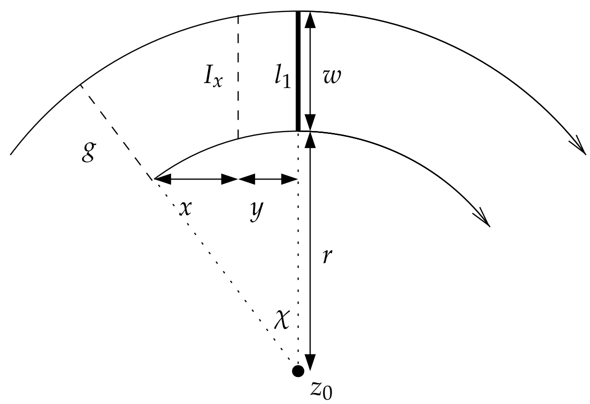

Figure 4.13: The end of the channel, rotated to let us use Lemma 29.

Using the facts that $y(x) = r\sin\chi - x$ (which implies that $dy = -dx$), $r \geq w$, and for all $\chi > 0$, $\chi > \sin\chi \geq \sin\chi\cos\chi$, we can derive a bound on $\omega(\zeta, l_1, R)$ in terms of r, w, and χ:

$$\begin{aligned}
-\frac{1}{\pi}\log\left(\frac{\pi}{8}\,\omega(\zeta, l_1, R)\right) &\geq \int_0^{r\sin\chi} \frac{dx}{\sqrt{(r+w)^2 - y(x)^2} - \sqrt{r^2 - y(x)^2}} \\
&= \int_0^{r\sin\chi} \frac{\sqrt{(r+w)^2 - y^2} + \sqrt{r^2 - y^2}}{((r+w)^2 - y^2) - (r^2 - y^2)}\,dy \\
&= \frac{1}{2rw + w^2}\int_0^{r\sin\chi} \sqrt{(r+w)^2 - y^2} + \sqrt{r^2 - y^2}\,dy \\
&\geq \frac{1}{3rw}\int_0^{r\sin\chi} 2\sqrt{r^2 - y^2}\,dy \\
&= \frac{r}{3w}[\chi + \cos\chi\sin\chi] \geq \frac{r}{3w}2\cos\chi\sin\chi \\
&= \frac{r\sin 2\chi}{3w}.
\end{aligned}$$

In the second line, we have made the change of variables from x to $y = r\sin\chi - x$. Note that this does not change the limits of integration.

Therefore, we may conclude that

$$\omega(\zeta, l_1, R) \leq \frac{8}{\pi}\exp\left(-\pi\frac{r\sin 2\chi}{3w}\right).$$

The same construction puts the same bound on $\omega(z_0, l_2, R)$, and so

$$\omega(z_0, l_1 \cup l_2, R) \leq \frac{16}{\pi} \exp\left(-\pi \frac{r \sin 2\chi}{3w}\right).$$

The same construction puts the same bound on the probability of a Brownian particle reaching g from any point in l_1, provided that it first leaves the channel through g. Because the particle can also escape the channel through the other end, we double this to get an upper bound on the probability of escaping the channel. Thus, the probability of escaping the channel is also at most

$$\frac{16}{\pi} \exp\left(-\pi \frac{r \sin 2\chi}{3w}\right).$$

■

We now consider h_{X_n} and h_{Ω_n}.

Theorem 31 *Let* $\chi_{n,k} = \min(\psi_{n,k}, \psi_{n,k+1}) - \phi_{n,k}$ *and let* $\chi_n = \min_{1 \leq k \leq n} \chi_{n,k}$. *That is,* $\chi_{n,k}$ *is the subtended angle of the shorter of the spikes at the ends of the kth channel;* χ_n *is the shortest subtended angle of any spike in* Ω_n. *If n is large enough that* $(M - \mu)/n < \mu$, *then for all* $r \in [\mu, M]$,

$$|h_{X_n}(r) - h_{\Omega_n}(r)| \leq \frac{32}{\pi} \exp\left(-n\pi \frac{\mu \sin 2\chi_{n,k}}{3(M-\mu)}\right).$$

Proof Let $E = \partial X_n \cap \overline{B(0,r)}$, and $F = \partial X_n \backslash E$. Then E is a collection of boundary arcs centered on 0.

Let

$$p_{n,k} = (16/\pi) \exp\left(-\pi \frac{r_{n,k} \sin 2\chi_{n,k}}{3(M-\mu)/n}\right).$$

Since a given channel in $\Omega_{n,k}$ has width $(M - \mu)/n$, by Theorem 30, $p_{n,k}$ is an upper bound on the probability of escaping from the kth channel, given that the particle reaches a radial segment in that channel.

There are four relevant subsets of the gates in Ω_n:

- l_E, the gates connecting two arcs in E;
- l_F, the gates connecting two arcs in F;
- l_e, the part of the gates connecting the outermost arc in E to the innermost arc in F that lies within r of 0;

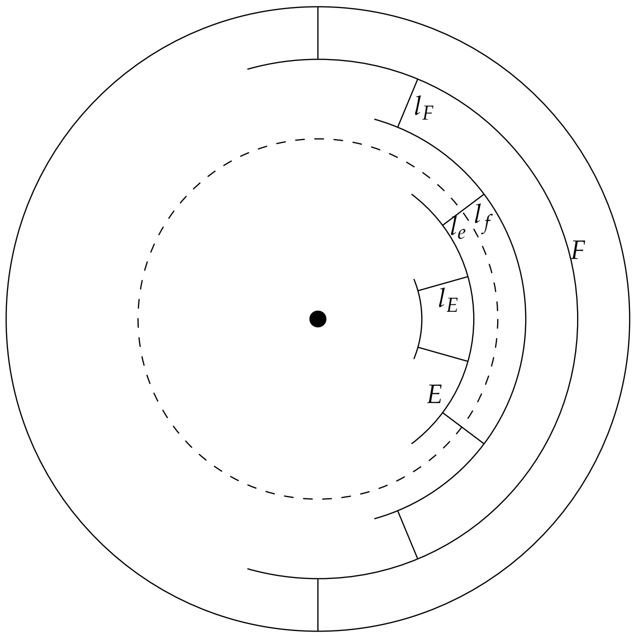

Figure 4.14: Ω_n and the subsets of $\partial\Omega_n$ used in Theorem 31 to show that $|h_{X_n}(r) - h_{\Omega_n}(r)|$ is small.

- l_f, the other parts of those gates.

See Fig. 4.14.

Then $\partial\Omega_n = E \cup F \cup l_e \cup l_f \cup l_E \cup l_F$, and by definition of the h-function, $h_{X_n}(r) = \omega(0, E, X_n)$, and $h_{\Omega_n}(r) = \omega(0, E \cup l_E \cup l_e, \Omega_n)$.

For a subset A of ∂X_n, let $P(l_E, A)$ be the probability that a Brownian particle in X_n that is initially released from 0 first hits ∂X_n in A, and first crosses $\partial\Omega_n$ in l_E. Define $P(l_F, A)$, $P(l_e, A)$, and $P(l_f, A)$ similarly.

So

$$\omega(0, A, X_n) = \omega(0, A, \Omega_n) + \sum_{j=e,f,E,F} P(l_j, A).$$

Also, note that

$$P(l_j, \partial X_n) = \omega(0, l_j, \Omega_n),$$

and so

$$\omega(0, l_j, \Omega_n) - P(l_j, A) = P(l_j, \partial X_n \backslash A).$$

Recall that the probability of escaping from the kth channel, given that the particle crosses a radial segment in that channel, is at most $p_{n,k}$. Therefore, $P(l_F, E) + P(l_E, F)$ is at most the maximal probability of escaping a channel from l_E or l_F; thus, it must be at most $\max_k p_{n,k}$.

Then, because the probability of getting to $l_e \cup l_f$ at all is at most $\max_k p_{n,k}$, and the probability of escaping the channel from any point in $l_E \cup l_F$ is also at most $\max_k p_{n,k}$, we may deduce the following inequality:

$$\begin{aligned}
|h_{X_n}(r) - h_{\Omega_n}(r)| &= |\omega(0, E, X_n) - \omega(0, E \cup l_E \cup l_e, \Omega_n)| \\
&= |P(l_E, E) + P(l_F, E) + P(l_e, E) + P(l_f, E) + \omega(0, E, \Omega_n) \\
&\quad -\omega(0, E, \Omega_n) - \omega(0, l_E, \Omega_n) - \omega(0, l_e, \Omega_n)| \\
&= |P(l_F, E) - [\omega(0, l_E, \Omega_n) - P(l_E, E)] \\
&\quad +P(l_f, E) - [\omega(0, l_e, \Omega_n) - P(l_e, E)]| \\
&= |P(l_F, E) + P(l_f, E) - P(l_E, F) - P(l_e, F)| \\
&\leq [P(l_F, E) + P(l_E, F)] + [P(l_f, E) + P(l_e, F)] \\
&\leq 2 \max_k p_{n,k}.
\end{aligned}$$

However,

$$p_{n,k} = (16/\pi) \exp\left(-n\pi \frac{r_{n,k} \sin 2\chi_{n,k}}{3(M-\mu)}\right) \leq (16/\pi) \exp\left(-n\pi \frac{\mu \sin 2\chi_n}{3(M-\mu)}\right).$$

So, as desired, we have that

$$|h_{X_n}(r) - h_{\Omega_n}(r)| \leq (32/\pi) \exp\left(-n\pi \frac{\mu \sin 2\chi_n}{3(M-\mu)}\right).$$

■

Therefore, if $n \sin 2\chi_n$ gets arbitrarily large, then

$$\lim_{n\to\infty} h_{\Omega_n} = \lim_{n\to\infty} h_{X_n} = f$$

at points of continuity of f, as desired.

4.7 Rephrasing our prior conditions as conditions on $\{X_n\}$

To ensure that $h_{X_n} \to h_{\Omega_n}$ and that $\Omega_{n_k} \to \Omega$ for some $\{n_k\}_{k=1}^{\infty}$ and some Ω, we have shown that it suffices that for all $E > 0$ there is some $N_E > 0$, and for all $\varepsilon > 0$ there are $\delta'_\varepsilon, \delta_\varepsilon > 0$ with $\delta'_\varepsilon < \delta_\varepsilon$ that satisfy the following three conditions.

- If $n \geq N_E$, $n \sin 2\chi_n > E$.
- If $|r_{n,k} - r_{n,j}| < \delta_\varepsilon$ then $\theta_{n,j,k} < \varepsilon$.
- If $\mu \sin(\pi - \psi_{n,k}) < \delta'_\varepsilon$ and $\psi_{n,k} \geq \pi/2$, then $M - r_{n,k} < \delta_\varepsilon$ and for all $i \geq k$, $\pi - \phi_i < \varepsilon$.

If we choose $\{\chi_n\}$ to be a sequence that approaches zero but for which $n\chi_n$ gets arbitrarily large, such as $\chi_n = \sqrt{1/n}$, the following conditions on X_n suffice.

- If $n \geq N_E$, then $n\psi_{n,k} > E$.
- If $|r_{n,k} - r_{n,j}| < \delta_\varepsilon$ then $\min(\psi_{n,j}, \psi_{n,k}) - \psi_{n,i} < \varepsilon$ for all $j < i < k$.
- If $\mu \sin(\pi - \psi_{n,k}) < \delta'_\varepsilon$ and $\psi_{n,k} \geq \pi/2$, then $M - r_{n,k} < \delta_\varepsilon$.

We have eliminated half of the last condition by combining it with the second. That is, since very long arcs must be very near the outer edge, we can consider the boundary circle of X_n to be the $n+1$th boundary arc of X_n, and define $\psi_{n,n+1} = \pi$. Then the second condition will imply the missing half of the third.

4.8 Rephrasing some of the conditions on $\{X_n\}$ as conditions on f

Recall that we required that f be strictly increasing. We use the following theorem to eliminate the third conditions on X_n.

Theorem 32 *If f is a strictly increasing function on $[\mu, M]$, and $0 \leq f(\mu) < f(M) \leq 1$, and X_n is the circle domain defined in Section 4.3, then for all $0 < \varepsilon < 1$ there exists a δ such that if $\pi - \psi_{n,k} < \delta$, then $M - r_{n,k} < \varepsilon$.*

Proof Fix $\varepsilon > 0$, and let $\delta = \pi - \pi f(M - \varepsilon)$.

Suppose that $\pi - \psi_{n,k} < \delta$. Let $E = \left\{z \in \partial X_n \;\middle|\; M - |z| > r_{n,k}\right\}$. Then

$$\omega(0, E, X_n) = 1 - f(r_{n,k})$$

by definition of X_n.

But the probability that a Brownian particle reaches E is clearly at most the probability that it first reaches the circle $|z| = r_{n,k}$ in the "gap" at the end of the arc. That probability is in turn bounded by the harmonic measure of that "gap" in $B(0, r_{n,k})$. See Fig. 4.15.

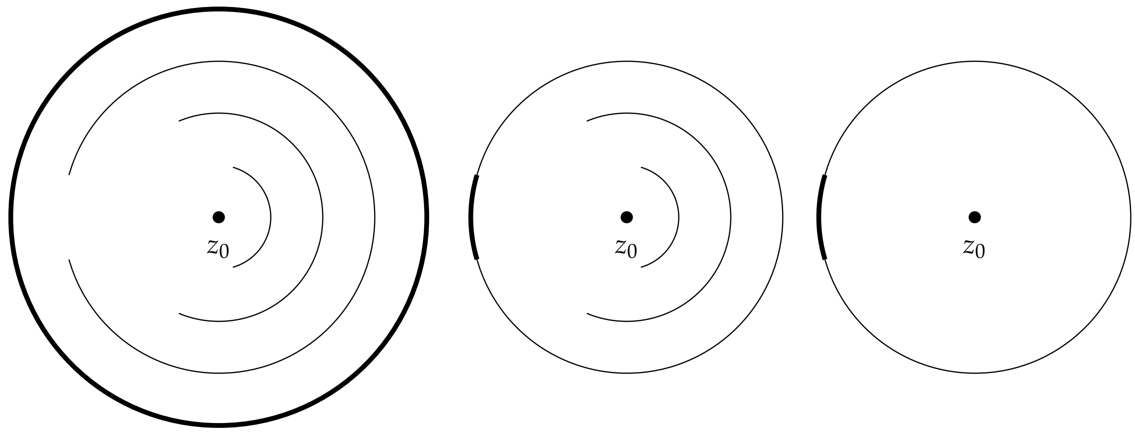

Figure 4.15: A bound on the harmonic measure of E: the harmonic measure of E, the bold set in the left-hand domain, is at most the harmonic measure of the bold set in the middle domain, which in turn is at most the harmonic measure of the bold set in the right-hand domain.

Thus,

$$\omega(0, E, X_n) \leq 2(\pi - \psi_{n,k})/(2\pi) < \delta/\pi = 1 - f(M - \varepsilon).$$

So $1 - f(r_{n,k}) < 1 - f(M - \varepsilon)$. Since f is strictly increasing, this implies that $M - r_{n,k} < \varepsilon$, as desired. ∎

This allows us to eliminate the third condition on X_n by replacing it with the condition that f be monotonically increasing.

4.9 Summary

In this chapter, we have proven the following theorem.

Theorem 33 *Let f be a function that is 0 on $[0, \mu)$, strictly increasing on $[\mu, M]$, and equal to 1 on (M, ∞). Using Theorem 5, construct circle domains X_n, each with $n + 1$ boundary arcs, including the outermost circle, whose h-functions approach f pointwise at points of continuity of f. Let $\psi_{n,j}$ be half the arc length of the jth boundary arc in X_n. Assume that for every $E > 0$ there exists some $N_E > 0$, and that for every $\varepsilon > 0$ there exists some $\delta_\varepsilon > 0$ which satisfy the following conditions.*

- *If* $n \geq N_E$*, then* $n\psi_{n,k} > E$*.*
- *If* $|r_{n,k} - r_{n,j}| < \delta_\varepsilon$ *then* $\min(\psi_{n,j}, \psi_{n,k}) - \psi_{n,i} < \varepsilon$ *for all* $j < i < k$*.*

Then f *is equal to the h-function of some simply connected domain* Ω *at all points of continuity of* f*.*

Chapter 5

Future Work

There are several possible avenues of future work.

First, in Chapter 4, we showed that a function f must be the h-function of some simply connected domain Ω, provided that f met some conditions, and that a sequence of circle domains $\{X_n\}$, whose h-functions converged to f, met some other conditions. We would like to be able to rephrase the conditions on X_n as conditions on f.

Second, in Chapter 2, we showed that the full converse of Theorem 1 is false, and we showed that the obvious partial converse for multiply connected domains does not hold. We would like to find out when some version of that partial converse holds. Also, we would like to find out whether Theorem 1 holds for multiply connected domains.

Finally, in Appendix C, we investigated the behavior of circle domains. We would like to show that circle domains with arcs centered on nearby points, with similar h-functions, must have arcs of similar lengths. We would also like to be able to determine whether arcs get longer or shorter when they move apart.

Appendix A

General theorems about harmonic measure

Recall Equation (1.1). Let u be a solution to the Dirichlet problem on Ω:

$$\begin{cases} \Delta u(z) &= 0, \quad z \in \Omega; \\ u(z) &= 1, \quad z \in E; \\ u(z) &= 0, \quad z \in \partial\Omega \backslash E. \end{cases}$$

Then $\omega(z_0, E, \Omega) = u(z_0)$.

We will use this equation to prove a number of theorems about harmonic measure. All of these theorems are probably already known; however, we have found it more convenient to prove them here than to cite proofs written elsewhere.

First, we show that harmonic measure is a conformal invariant.

Theorem 34 *If $S(z)$ is a conformal mapping on Ω, $z_0 \in \Omega$, and $E \subset \partial\Omega$, then*

$$\omega(z_0, E, \Omega) = \omega(S(z_0), S(E), S(\Omega)).$$

Proof Let u be harmonic in Ω with $u(z) = 1$ if $z \in E$ and $u(z) = 0$ if $z \in \partial\Omega\backslash E$. So $u(z_0) = \omega(z_0, E, \Omega)$. Let $v(S(z)) = u(z)$. Then v is harmonic in $S(\Omega)$ because S is conformal, $v(z) = 1$ if $z \in S(E)$ and $v(z) = 0$ if $z \in S(\partial\Omega\backslash E)$.

Note that since S is continuous, $S(E) \subset \partial S(\Omega)$ and $\partial S(\Omega)\backslash S(E) = S(\partial\Omega\backslash E)$; hence $v(S(z_0)) = \omega(S(z_0), S(E), S(\Omega))$ by Equation (1.1). But $v(S(z_0)) = u(z_0) = \omega(z_0, E, \Omega)$. This proves the theorem. ∎

Next, we establish a form of additivity of harmonic measure.

Theorem 35 *If E and F are disjoint subsets of $\partial\Omega$, and $z_0 \in \Omega$, then*

$$\omega(z_0, E \cup F, \Omega) = \omega(z_0, E, \Omega) + \omega(z_0, F, \Omega).$$

See Fig. A.1.

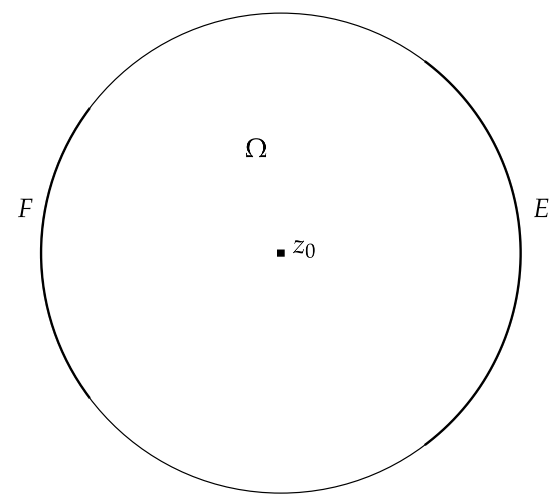

Figure A.1: A domain Ω and two disjoint subsets E and F of its boundary. The harmonic measure of the union of E and F is the sum of the individual harmonic measures.

Proof Let $f(z)$ and $g(z)$ solve the Dirichlet problems

$$\begin{cases} \Delta f(z) &= 0, \quad z \in \Omega; \\ f(z) &= 1, \quad z \in E; \\ f(z) &= 0, \quad z \in \partial\Omega \setminus E; \end{cases}$$

and

$$\begin{cases} \Delta g(z) &= 0, \quad z \in \Omega; \\ g(z) &= 1, \quad z \in F; \\ g(z) &= 0, \quad z \in \partial\Omega \setminus F. \end{cases}$$

By Equation (1.1), this implies that $f(z_0) = \omega(z_0, E, \Omega)$ and $g(z_0) = \omega(z_0, F, \Omega)$.

Then $h(z) = f(z) + g(z)$ is also harmonic, and solves

$$\begin{cases} \Delta h(z) &= 0, \quad z \in \Omega; \\ h(z) &= 1, \quad z \in E \cup F; \\ h(z) &= 0, \quad z \in \partial\Omega \setminus (E \cup F). \end{cases}$$

Thus, again by Equation (1.1),

$$\omega(z_0, E \cup F, \Omega) = h(z_0) = f(z_0) + g(z_0) = \omega(z_0, E, \Omega) + \omega(z_0, F, \Omega).$$

■

Finally, we establish the monotonicity of harmonic measure.

Lemma 36 *Let Ω and Ω' be two domains with $\Omega \subset \Omega'$. Let $F \subset \partial\Omega'$, and assume that $F \cap \Omega = \emptyset$. Let $E = \partial\Omega \backslash F$, $E' = \partial\Omega' \backslash F$. Then*

$$\omega(z_0, E', \Omega') \leq \omega(z_0, E, \Omega)$$

for all $z_0 \in \Omega$, with equality holding only if F or $E \backslash E'$ is a set of measure 0.

See Fig. A.2.
Proof Let u be harmonic on Ω, $u = 1$ on E, $u = 0$ on F, and so by Equation (1.1), $\omega(z_0, E, \Omega) = u(z_0)$. Define u' similarly on Ω'.

Then $v = u - u'$ is harmonic on Ω, and has boundary values of 0 on F and $E \cap E'$, and of $1 - u'$ on $E \backslash E'$. If F is not a set of measure 0, then $u' < 1$ on $E \backslash E'$. Therefore, if $E \backslash E'$ is not a set of measure 0, then for all $z \in \Omega$, $0 < v(z)$. Otherwise, $v = 0$ in Ω. Thus,

$$0 \leq v(z_0) = \omega(z_0, E, \Omega) - \omega(z_0, E', \Omega')$$

with equality holding only if F or $E' \backslash E$ is a set of measure 0, as desired.■

As a special case of the above lemma, consider the case when $\Omega' \backslash \Omega$ is nothing but boundary, and when this added boundary lies entirely in E', as shown in Fig. A.2. This special case implies, for example, that in a circle domain, if the length of a single arc is increased, then the harmonic measure of that arc also increases.

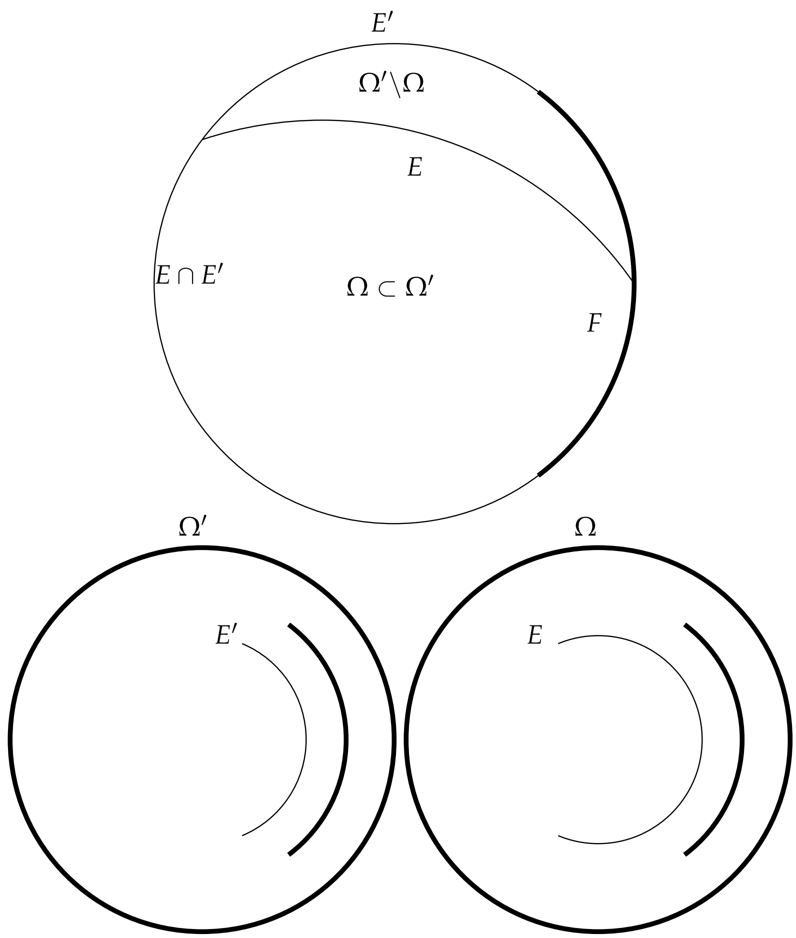

Figure A.2: As in Lemma 36, making a domain larger can decrease the harmonic measure of boundary components. The general case is shown on top; a special case, particularly applicable to circle domains, is shown below.

Appendix B

Harmonic measure of some specific domains

B.1 Harmonic measure of an interval in the upper half plane

Theorem 37 *The harmonic measure of an interval* $[a,b]$ *in the upper half plane* $\mathbb{U}$ *from a point* z_0 *is the angle of sight* $\arg(z_0-b)-\arg(z_0-a)$*, normalized. Thus,*

$$\omega(z_0,[a,b],\mathbb{U}) = \frac{\arg(z_0-b)-\arg(z_0-a)}{\pi}.$$

In particular,

$$\omega(i,[a,b],\mathbb{U}) = \frac{1}{\pi}\arctan\left(\frac{b-a}{1+ba}\right)$$

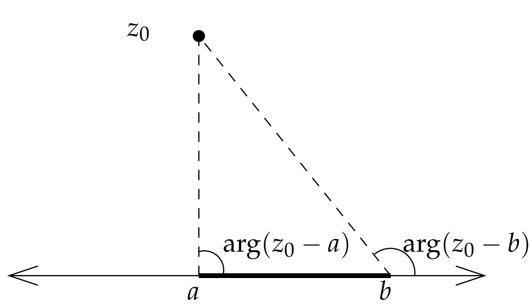

Figure B.1: The upper half-plane: the domain in Section B.1.

and

$$\omega(i, [-b, b], \mathbb{U}) = \frac{2\arctan(b)}{\pi}.$$

Also, the harmonic measure distribution function in $\mathbb{U}$, measured from the point i, is

$$h(r) = \frac{2\arctan(\sqrt{r^2-1})}{\pi}$$

for $r > 1$.

Proof If $f(z)$ is a bounded function that is harmonic on $\mathbb{U}$, and takes on values of 0 on $\partial\mathbb{U}\backslash[a, b]$ and 1 on $[a, b]$, then $f(z) = \omega(z, [a, b], \mathbb{U})$.

Now, if we take the branch cuts to be straight down from a and b, the function $f(z) = (1/\pi)\arg(z-b) - (1/\pi)\arg(z-a)$ is harmonic in the upper half plane.

We evaluate the boundary values of f as follows. If $z > 0$, $\arg z = 0$. If $z < 0$, $\arg z = \pi$. This implies that $f(z) = 0$ on (b, ∞), $f(z) = \pi/\pi - \pi/\pi = 0$ on $(-\infty, a)$, and $f(z) = \pi/\pi - 0 = 1$ on (a, b).

Thus,

$$\omega\left(z_0, [a, b], \mathbb{U}\right) = f(z_0) = \frac{1}{\pi}(\arg(z_0 - b) - \arg(z_0 - a)).$$

Since for $r > 1$ the region of $\partial\mathbb{U}$ within a radius r of i is the interval $\left[-\sqrt{r^2-1}, \sqrt{r^2-1}\right]$,

$$h_{\mathbb{U}}(r; i) = \omega\left(i, \left[-\sqrt{r^2-1}, \sqrt{r^2-1}\right], \mathbb{U}\right) = \frac{2}{\pi}\arctan\sqrt{r^2-1}.$$

■

Once we know this fact, by using Theorem 34, we can compute harmonic measure in many other domains.

B.2 Harmonic measure of the disk

B.2.1 Conformal mappings of the disk

Theorem 38 *If $S(z) = \frac{a^*z+b^*}{bz+a}$, and $|a| > |b|$, then S maps the unit disk $\mathbb{D}$ to itself conformally.*

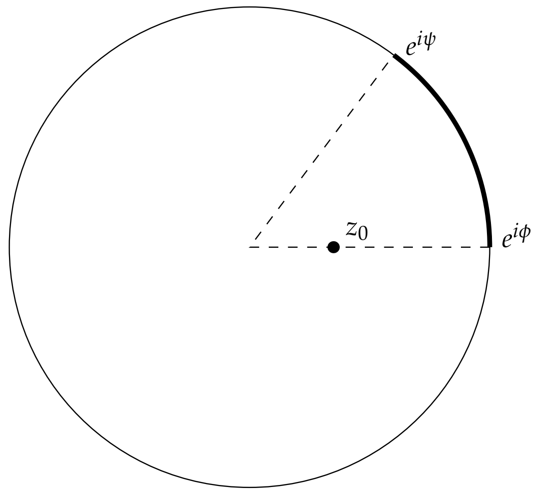

Figure B.2: The unit disk: the domain in Section B.2.

Here a star $(x+iy)^*$ indicates the complex conjugate $x - iy$.
Proof By [**?**, p. 322], S is conformal on the disk, because $bz + a = 0$ only when $|z| = |a/b| > 1$. Also, because S is a Möbius transformation, we know that S will map lines and circles to lines and circles, and will map only boundary points to boundary points. If $z = e^{i\theta}$, (that is, z is on the boundary of the unit disk), then

$$\begin{aligned} S(z) &= \frac{a^* e^{i\theta} + b^*}{b e^{i\theta} + a} \\ &= \frac{a^* e^{i\theta/2} + b^* e^{-i\theta/2}}{b e^{i\theta/2} + a e^{-i\theta/2}} \\ &= \frac{\left(a e^{-i\theta/2} + b e^{i\theta/2}\right)^*}{\left(a e^{-i\theta/2} + b e^{i\theta/2}\right)}. \end{aligned}$$

Since $|a| > |b|$, this is not $\frac{0}{0}$. Hence, since $|z| = |\bar{z}|$, $|S(z)| = 1$ for all $|z| = 1$.

So S maps the boundary of the unit disk into itself. By [**?**, p. 323], S must map the boundary of the unit disk onto itself.

Since $S(z)$ is a Möbius transformation, we know that $\mathbb{D}$ is mapped either to itself or to $\mathbb{C}\backslash\overline{\mathbb{D}}$. $S(0) = \frac{b^*}{a}$. Since $|a| > |b|$, $S(0)$ is inside the unit disk; hence $S(z)$ maps the unit disk to itself. ■

Theorem 39 *The Möbius transformation* $S(z) = \frac{i - iz}{1+z}$ *takes the unit disk to* $\mathbb{U}$.

Proof If $z = e^{i\theta}$, then

$$\begin{aligned} S(z) &= i\frac{1-e^{i\theta}}{1+e^{i\theta}} \\ &= i\frac{e^{-i\theta/2} - e^{i\theta}/2}{e^{-i\theta/2} + e^{i\theta/2}} \\ &= \frac{2}{2i}\frac{e^{i\theta/2} - e^{-i\theta}/2}{e^{-i\theta/2} + e^{i\theta/2}} \\ &= \tan(\theta/2) \end{aligned}$$

and so S maps the boundary of the unit disk to the real line.

Since S is a Möbius transformation, we need only check where a single point inside the disk goes. $S(0) = i$, and so S maps the disk $B(0,1)$ to $\mathbb{U}$. ■

B.2.2 Harmonic measure from 0

Theorem 40 *The harmonic measure in the disk from 0 of an arc of the boundary from $e^{i\phi}$ to $e^{i\theta}$ is $(\theta - \phi)/2\pi$, provided $2\pi \geq \theta - \phi \geq 0$.*

Proof Consider the connected arc I in the boundary of $\mathbb{D}$ from $e^{i\phi}$ to $e^{i\theta}$, with $2\pi \geq \theta - \phi \geq 0$. First, using the conformal mapping $g(z) = ze^{-i(\theta+\phi)/2}$, note that

$$\omega(0, I, \mathbb{D}) = \omega(g(0), g(I), g(D)) = \omega(0, g(I), \mathbb{D}).$$

Now, $g(I)$ is the arc from $e^{i(\phi-\theta)/2}$ to $e^{i(\theta-\phi)/2}$. Define $\psi = (\theta - \phi)/4$, so that $g(I)$ is the arc from $e^{-2i\psi}$ to $e^{2i\psi}$.

We apply the Möbius transform S given in Theorem 39. $S(e^{\pm 2i\psi}) = \pm\tan\psi$, and so

$$\begin{aligned} \omega(0, g(I), \mathbb{D}) &= \omega(S(0), S(g(I)), S(\mathbb{D})) \\ &= \omega(i, [-\tan\psi, \tan\psi], \mathbb{U}) = \frac{2\psi}{\pi} \\ &= \frac{\theta - \phi}{2\pi}. \end{aligned}$$

Thus, again we have that the harmonic measure is equal to the normalized angle of sight. ■

B.2.3 Harmonic measure from a real z_0 inside the disk

Theorem 41 *For an arbitrary real point z_0 inside $\mathbb{D}$ and an arc I from $e^{i\phi}$ to $e^{i\psi}$, if $0 \le \psi - \phi \le \pi$, then*

$$\omega(z_0, I, \mathbb{D}) = \frac{1}{\pi}\left[\arg\left(e^{i\psi/2} - z_0 e^{-i\psi/2}\right) - \arg\left(e^{i\phi/2} - z_0 e^{-i\phi/2}\right)\right].$$

Proof Consider the Möbius transformation $S(z) = \frac{z-z_0}{1-z_0 z}$. By Theorem 38, S maps the unit disk to itself.

Given an arc I from $e^{i\phi}$ to $e^{i\psi}$, we can calculate

$$\omega(z_0, I, \mathbb{D}) = \omega(S(z_0), S(I), S(\mathbb{D})) = \omega(0, S(I), \mathbb{D}).$$

Now, $S(I)$ is the closed interval in the real line from $S(e^{i\phi})$ to $S(e^{i\psi})$. Thus,

$$\omega(z_0, I, \mathbb{D}) = \frac{1}{2\pi}\left[\arg\left(S\left(e^{i\psi}\right)\right) - \arg\left(S\left(e^{i\phi}\right)\right)\right].$$

Now,

$$\begin{aligned} S\left(e^{i\psi}\right) &= \frac{e^{i\psi} - z_0}{1 - z_0 e^{i\psi}} \\ &= \frac{e^{i\psi/2} - z_0 e^{-i\psi/2}}{e^{-i\psi/2} - z_0 e^{i\psi/2}} \\ &= \frac{\left(e^{i\psi/2} - z_0 e^{-i\psi/2}\right)^2}{|e^{-i\psi/2} - z_0 e^{i\psi/2}|^2}, \end{aligned}$$

and so $\arg\left(S\left(e^{i\psi}\right)\right) = 2\arg\left(e^{i\psi/2} - z_0 e^{-i\psi/2}\right)$. Thus,

$$\begin{aligned} \omega(z_0, I, \mathbb{D}) &= \frac{1}{\pi}\left[\arg\left(e^{i\psi/2} - z_0 e^{-i\psi/2}\right) - \arg\left(e^{i\phi/2} - z_0 e^{-i\phi/2}\right)\right] \\ &= \frac{1}{\pi}\arctan\left(\tan\left(\frac{\psi-\phi}{2}\right)\frac{1-z_0^2}{1+z_0^2 - 2z_0\frac{\cos(\frac{\psi+\phi}{2})}{\cos(\frac{\psi-\phi}{2})}}\right). \end{aligned}$$

Note that the correct branch of the arctangent is the one from 0 to π, not the ore usual branch from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

We can make this equation a bit clearer by letting $\theta = \frac{\psi-\phi}{2}$, and $\theta_0 = \frac{\psi+\phi}{2}$; thus, θ_0 is the location of the midpoint of the arc we measure, and θ is half the arc length. Thus,

$$\omega(z_0, I, \mathbb{D}) = \frac{1}{\pi}\arctan\left(\tan\theta \frac{1-z_0^2}{1+z_0^2 - 2z_0\frac{\cos\theta_0}{\cos\theta}}\right)$$

In the special case that $\theta_0 = 0$, and so we measure an arc centered on the real axis, this simplifies further to

$$\frac{1}{\pi}\arctan\left(\tan\theta\frac{1-z_0^2}{1+z_0^2-2z_0\sec\theta}\right).$$

In the special case that $\theta = \theta_0$, that is, I runs from 0 to 2θ, it simplifies to

$$\frac{1}{\pi}\arctan\left(\tan\theta\frac{1+z_0}{1-z_0}\right).$$

■

Note that this is *not* the normalized angle of sight, in general. For example, if $z_0 = \frac{1}{2}$, $\phi = -\frac{\pi}{3}$, and $\psi = \frac{\pi}{3}$, then the angle of sight is π, which normalizes to $\frac{1}{2}$, but

$$\omega(z_0, I, \mathbb{D}) = \frac{1}{\pi}\arctan\left(\tan\left(\frac{\pi}{3}\right)\frac{1-z_0^2}{1+z_0^2-2z_0\sec\frac{\pi}{3}}\right) = \frac{1}{\pi}\arctan\left(-\sqrt{3}\right) = \frac{2}{3}$$

which is *not* $\frac{1}{2}$.

B.2.4 An alternative calculation, using the Poisson kernel

Consider a harmonic function u on $\mathbb{D}$ with prescribed boundary values $u\left(e^{i\theta}\right) = f(\theta)$. This is the classic Dirichlet problem, and by [9, p. 135], it has as a solution

$$u\left(re^{i\theta_0}\right) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\frac{1-r^2}{1-2r\cos(\theta-\theta_0)+r^2}d\theta.$$

Define the Poisson kernel

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos(\theta)+r^2},$$

so that

$$u(re^{i\theta_0}) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)P_r(\theta-\theta_0)d\theta.$$

This gives us harmonic measure in the disk: we let the basepoint $z_0 = re^{i\theta_0}$, and let $f(\theta) = \chi_I$, the characteristic function which is 1 if $e^{i\theta} \in I$ and 0 otherwise. Then $u(z_0) = \omega(z_0, I, \mathbb{D})$, for any arc I. Let $\theta_0 = 0$, and let I be the arc from ϕ to ψ, $0 < \psi - \phi \leq 2\pi$.

Let $w(\theta)$ be the expression for $\omega(z_0, [0,\theta], \mathbb{D})$ calculated above, where z_0 is a real positive point in $\mathbb{D}$ and $[0,\theta]$ is the arc of $\partial\mathbb{D}$ from 0 to θ. Thus, $w(\theta) = \frac{1}{\pi}\arctan\left(\tan\left(\frac{\theta}{2}\right)\frac{1+z_0}{1-z_0}\right)$.

Note that

$$\begin{aligned}
\frac{d}{d\theta}w(\theta) &= \frac{1}{2\pi}\frac{1}{1+\left(\tan\left(\frac{\theta}{2}\right)\frac{1+z_0}{1-z_0}\right)^2}\frac{1+z_0}{1-z_0}\sec^2\frac{\theta}{2} \\
&= \frac{1}{2\pi}\frac{1-z_0^2}{(1-z_0)^2\cos^2\frac{\theta}{2}+(1+z_0)^2\sin^2\frac{\theta}{2}} \\
&= \frac{1}{2\pi}\frac{1-z_0^2}{1+z_0^2-2z_0\cos\theta},
\end{aligned}$$

and so the Poisson kernel is just 2π times the derivative of the harmonic measure function found above. Thus, the alternative calculation of harmonic measure, done by solving the Dirichlet problem, must yield

$$\omega(z_0, I, \mathbb{D}) = u(z_0) = \frac{1}{2\pi}\int_0^{2\pi}\chi_I(\theta)P_{z_0}(\theta)d\theta = \frac{1}{2\pi}\int_\phi^\psi P_{z_0}(\theta)d\theta = w(\psi) - w(\phi).$$

B.2.5 Harmonic measure from a real z_0 outside the disk

Consider the Möbius transformation $S(z) = 1/z$.

Given an arc I from $e^{i\phi}$ to $e^{i\psi}$, $0 \le \psi - \phi \le 2\pi$, note that

$$\omega(z_0, I, \mathbb{C}\backslash\mathbb{D}) = \omega(S(z_0), S(I), S(\mathbb{C}\backslash\mathbb{D})) = \omega\left(\frac{1}{z_0}, S(I), \mathbb{D}\right).$$

Now, $S(I)$ is the arc from $e^{-i\psi}$ to $e^{-i\phi}$, and so

$$\omega(z_0, I, \mathbb{U}\backslash\mathbb{D}) = \frac{1}{\pi}\arctan\left(\tan\left(\frac{\psi-\phi}{2}\right)\frac{z_0^2-1}{1+z_0^2-2z_0\frac{\cos\left(\frac{\psi+\phi}{2}\right)}{\cos\left(\frac{\psi-\phi}{2}\right)}}\right).$$

B.3 Harmonic measure of the straight edge of a semicircle

Let Ω be $\mathbb{U} \cap B(0,r)$, the half-disk in the upper half-plane. Consider the map $f(z) = -\left(2r/(r+z)-1\right)^2$. This maps Ω conformally onto $\mathbb{U}$.

So

$$
\begin{aligned}
\omega(z,[0,r],\Omega) &= \omega\left(-\left(\frac{2r}{r+z}-1\right)^2,[-1,0],\mathbb{U}\right)\\
&= \frac{1}{\pi}\left(\arg\left[-\left(\frac{2r}{r+z}-1\right)^2\right]-\arg\left[-\left(\frac{2r}{r+z}-1\right)^2+1\right]\right)\\
&= \frac{1}{\pi}\left(\arg\left[-\left(\frac{r-z}{r+z}\right)^2\right]-\arg\left[\frac{4rz}{(r+z)^2}\right]\right)\\
&= \frac{1}{\pi}(\arg(-1)+2\arg(r-z)-2\arg(r+z)\\
&\quad -\arg(4rz)+2\arg(r+z))\\
&= \frac{1}{\pi}\left(\pi+2\arg(r-z)-\arg(z)\right).
\end{aligned}
$$

If $z=a+ie$, then

$$
\begin{aligned}
\omega(z,[0,r],\Omega) &= 1+\frac{1}{\pi}\left(2\arg[r-a-ie]-\arg[a+ie]\right)\\
&= 1-\frac{1}{\pi}\left(2\arctan\left[\frac{e}{r-a}\right]+\arctan\left[\frac{e}{a}\right]\right).
\end{aligned}
$$

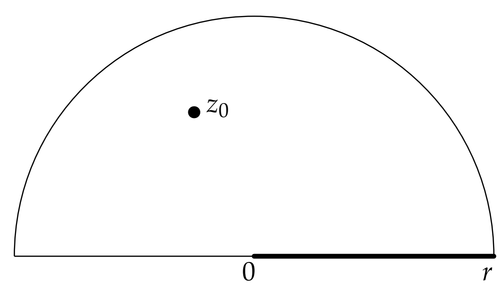

Figure B.3: A semicircle: the domain in Section B.3.

Figure B.4: The plane minus a slit: the domain in Section B.4.

B.4 The plane minus a long slit

Let $\Omega = \mathbb{C}\backslash[0,\infty)$. Then, using the conformal map $f(z) = \sqrt{z}$, with branch cut along the positive reals, we compute that

$$h_\Omega(r;-1) = \omega(-1,[0,(r-1)],\Omega) = \omega(i,[-\sqrt{r-1},\sqrt{r-1}],\mathbb{U}) = \frac{2}{\pi}\arctan\sqrt{r-1}$$

for $r \geq 1$. For $r < 1$, $h_\Omega(r;-1) = 0$.

B.5 The half-plane minus a short slit

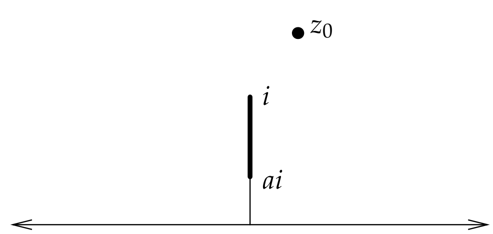

Figure B.5: The half-plane minus a short slit: the domain in Section B.5.

Let $\Omega = \mathbb{U}\backslash[0,i]$. Let $z_0 \in \Omega$, and let $E = [ai,i]$.
First, note that $f(z) = \sqrt{1+z^2}$ maps Ω to $\mathbb{U}$ conformally.
Then:

$$\begin{aligned}
\omega(z_0,E,\Omega) &= \omega(f(z_0),f(E),\mathbb{U}) \\
&= \omega\left(\sqrt{1+z_0^2},\left[-\sqrt{1-a^2},\sqrt{1-a^2}\right],\mathbb{U}\right) \\
&= \frac{1}{\pi}\left(\arg\left(\sqrt{1+z_0^2}-\sqrt{1-a^2}\right)-\arg\left(\sqrt{1+z_0^2}+\sqrt{1-a^2}\right)\right).
\end{aligned}$$

If $z_0 = bi$ for some real $b > 1$, then

$$\begin{aligned}
\omega(z_0, E, \Omega) &= \frac{1}{\pi}\left(\arg\left(\sqrt{1-b^2}-\sqrt{1-a^2}\right)-\arg\left(\sqrt{1-b^2}+\sqrt{1-a^2}\right)\right)\\
&= \frac{1}{\pi}\left(\arg\left(i\sqrt{b^2-1}-\sqrt{1-a^2}\right)-\arg\left(i\sqrt{b^2-1}+\sqrt{1-a^2}\right)\right)\\
&= \frac{1}{\pi}\left(-\arctan\sqrt{\frac{b^2-1}{1-a^2}}-\arctan\sqrt{\frac{b^2-1}{1-a^2}}\right)\\
&= -\frac{2}{\pi}\arctan\sqrt{\frac{b^2-1}{1-a^2}}.
\end{aligned}$$

B.6 The disk minus a slit

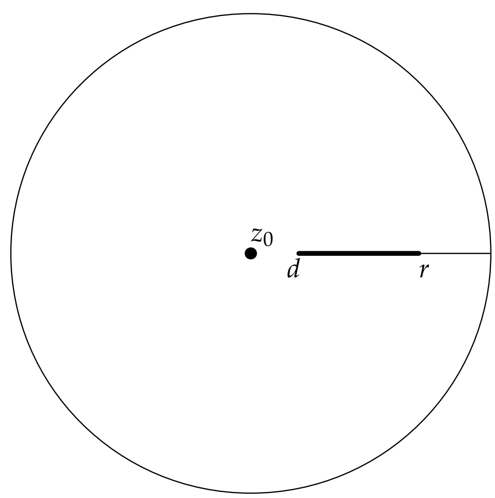

Figure B.6: The disk minus a slit: the domain in Section B.6.

Consider the domain $D = B(0,R)\backslash[d,R]$. Let

$$k(z) = \left(\frac{i}{z+R}-\frac{i}{2R}\right)\frac{2R(R+d)}{R-d}.$$

Then $k(R) = 0$, $k(d) = i$, and $k(D) = \mathbb{U}\backslash[0, i]$. Let $d < r < R$. Then

$$\begin{aligned}
\omega(0, [d, r], D) &= \omega(k(0), k([d, r]), \mathbb{U}\backslash[0, i]) \\
&= \omega\left(i\frac{R+d}{R-d}, \left[\left(\frac{i}{r+R} - \frac{i}{2R}\right)\frac{2R(R+d)}{R-d}, i\right], \mathbb{U}\backslash[0, i]\right) \\
&= \frac{2}{\pi}\arctan\sqrt{\frac{1 - \left(\left(\frac{1}{r+R} - \frac{1}{2R}\right)\frac{2R(R+d)}{R-d}\right)^2}{\left(\frac{R+d}{R-d}\right)^2 - 1}} \\
&= \frac{2}{\pi}\arctan\sqrt{\frac{(R-d)^2 - \left(\left(\frac{2R}{r+R} - 1\right)(R+d)\right)^2}{4Rd}} \\
&= \frac{2}{\pi}\arctan\sqrt{\frac{(R-d)^2 - \left(\frac{4R^2}{(r+R)^2} - 2\frac{2R}{r+R} + 1\right)(R+d)^2}{4Rd}} \\
&= \frac{2}{\pi}\arctan\sqrt{\frac{-4Rd - \left(\frac{4R^2}{(r+R)^2} - \frac{4R}{r+R}\right)(R+d)^2}{4Rd}} \\
&= \frac{2}{\pi}\arctan\sqrt{-1 - \frac{1}{d}\left(\frac{R}{(r+R)^2} - \frac{r+R}{(r+R)^2}\right)(R+d)^2} \\
&= \frac{2}{\pi}\arctan\left(\frac{R+d}{R+r}\sqrt{\frac{r}{d} - 1}\right).
\end{aligned}$$

Appendix C

Further investigation of circle domains

Circle domains are very interesting. In Chapter 4, we used them to construct some h-functions, and in Chapter 2, we used them to show that there is something wrong with Conjecture 2, the partial converse to Theorem 1 for multiply connected domains.

Therefore, we would like to investigate them further. We would like to be able to show, for example, that any sequence of circle domains with identical h-functions must satisfy Conjecture 2.

Throughout this section, let D and D' be circle domains with k closed boundary arcs a_n and a'_n centered at $r_n e^{i\theta_n}$ and $r_n e^{i\theta'_n}$. Further suppose that D and D' have the same h-function.

Let the angles subtended by the arcs be $2\psi_n$ and $2\psi'_n$; thus,

$$a_n = \left\{ e^{i\phi} r_n e^{i\theta_n} \mid |\phi| < \psi_n \right\}.$$

C.1 Carathéodory convergence

If we have a sequence of circle domains, under what conditions do they converge in the sense of Carathéodory?

Theorem 42 *Suppose we have a sequence of circle domains $\{\Omega_n\}$ and some circle domain Ω. Suppose further that Ω has $k+1$ boundary arcs a_m, (including the outermost circle a_{k+1} at $r_{k+1} = 1$), each centered at $r_m e^{i\theta_m}$ with arc length $2\psi_m$, and that each Ω_n similarly has $k+1$ arcs $a_{n,m}$, each centered at $r_{n,m} e^{i\theta_{n,m}}$ with arc length $2\psi_{n,m}$. If $r_{n,m} \to r_m$, $\theta_{n,m} \to \theta_m$, and $\psi_{n,m} \to \psi_m$, then $\Omega_n \to \Omega$ in the sense of Carathéodory.*

See Fig. C.1.

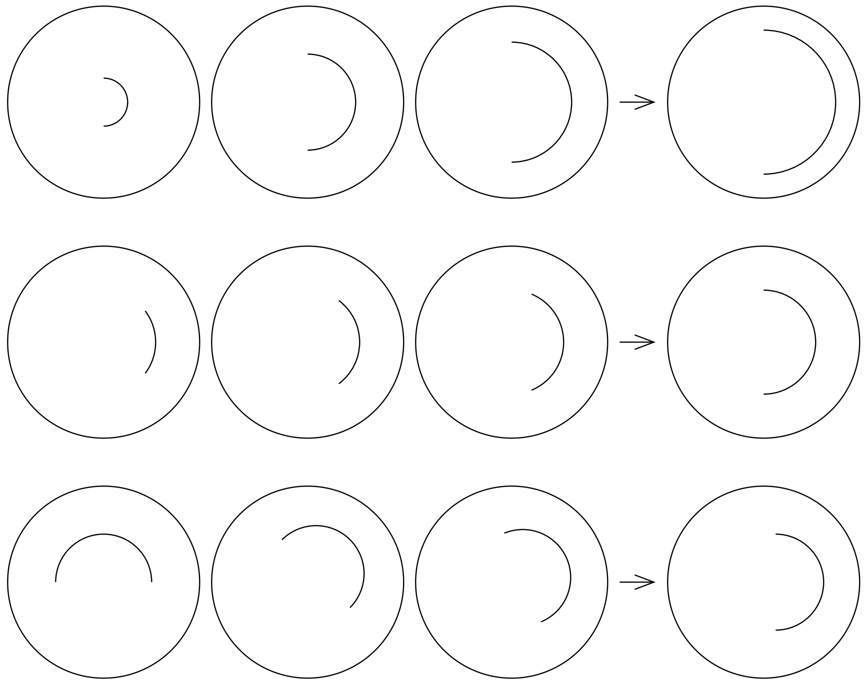

Figure C.1: Circle domains converging in the sense of Carathéodory.

Proof Recall Definition 7 in Section 2.3, the definition of Carathéodory convergence. Let $B(w, \rho)$ be the neighborhood of w with radius ρ.

We address the two conditions in Definition 7 one at a time.

(1) Fix $w \in \Omega$. Let

$$\rho = \frac{1}{2} \inf_{z \in \partial\Omega} |z - w|.$$

Since Ω is open, $\rho > 0$. There is some M such that, for all $n \geq M$ and $1 \leq m \leq k+1$, $|r_{n,m} - r_m| < \rho/\sqrt{2}$, $\max(r_m, r_{n,m})|\psi_{n,m} - \psi_m| < \rho/2\sqrt{2}$, and $\max(r_m, r_{n,m})|\theta_{n,m} - \theta_m| < \rho/(2\sqrt{2})$.

Then for each $r_{n,m}e^{i\phi} \in \partial\Omega_n$ with $n \geq M$, because of the bound on $|\theta_{n,m} - \theta_m|$ and $|\psi_{n,m} - \psi_m|$, there is some ε such that $\max(r_m, r_{n,m})|\varepsilon| < \rho/\sqrt{2}$ and $r_m e^{i(\phi+\varepsilon)} \in \partial\Omega$. Then

$$\left| r_{n,m}e^{i\phi} - r_m e^{i(\phi+\varepsilon)} \right| < \sqrt{\left(\frac{\rho}{\sqrt{2}}\right)^2 + \left(\frac{\rho}{\sqrt{2}}\right)^2} = \rho.$$

Thus by the triangle inequality,

$$|r_{n,m}e^{i\phi} - w| \geq \left| r_m e^{i(\phi+\varepsilon)} - w \right| - \left| r_{n,m}e^{i\phi} - r_m e^{i(\phi+\varepsilon)} \right| \geq \rho,$$

and so $r_{n,m}e^{i\phi} \notin N(w,\rho)$. So $N(w,\rho) \subset \Omega_n$ for all $n > M$.

(2) Let $w = r_m e^{i\phi} \in \partial\Omega$. Define w_n as follows: Let $\phi_n = \min_{z \in a_{n,m}} |\arg z - \phi|$. (Note that ϕ_n may be 0!) Then $\phi_n \leq |\theta_n - \theta_{n,m}| + |\psi_n - \psi_{n,m}|$. Let $w_n = r_{n,m}e^{i\phi \pm i\phi_n}$ such that $w_n \in \partial\Omega_n$. Then, since $r_{n,m} \to r_m$, $\psi_{n,m} \to \psi_m$, and $\theta_{n,m} \to \theta_m$, $w_n \to w$.

Thus, $\Omega_n \to \Omega$ in the sense of Carathéodory. ∎

C.2 Containment results and the two-arc case

So, if we have a sequence of circle domains with identical h-functions, showing Carathéodory convergence is equivalent to showing that the lengths and midpoints of the arcs converge.

The condition that the sequence of domains has identical h-functions implies that each domain has arcs at the same radii.

Now, we would like to show that sequences of circle domains with identical h-functions satisfy Conjecture 2. Any sequence of circle domains with a fixed number of arcs must have a subsequence with convergent midpoints, because the locations of the midpoints form a compact space $[0, 2\pi] \times [0, 2\pi] \times \ldots \times [0, 2\pi]$. We wish to show that this subsequence must have convergent arc lengths.

So far, we have been able to prove this only in the special case where the domains have only two arcs. We do this by constructing a bound on the change in arc lengths that is dependent on the difference in midpoints. That is, given two two-arc domains with the arcs at the same radii, we show that the difference in arc lengths is no more than the difference in midpoints. We can do this simply by showing that neither domain can be a subset of the other.

Theorem 43 *Suppose we have a nonempty set S of the boundary arcs of D such that $\theta'_n = \theta_n$ if $a_n \notin S$, and $\theta'_n = \theta_n + \delta$ if $a_n \in S$. Let $S' = \{a'_n \mid a_n \in S\}$. Then for some $a_j \in S$, $a_j \not\subset a'_j$ and $a'_j \not\subset a_j$.*

Proof Suppose not. Then for each n, either $a_n \subset a'_n$ or $a'_n \subset a_n$. (See Fig. C.2.)

Let

$$E = \bigcup_{a_n \subset a'_n} a_n, \quad E' = \bigcup_{a_n \subset a'_n} a'_n, \quad F = \partial D - E, \quad F' = \partial D' - E'.$$

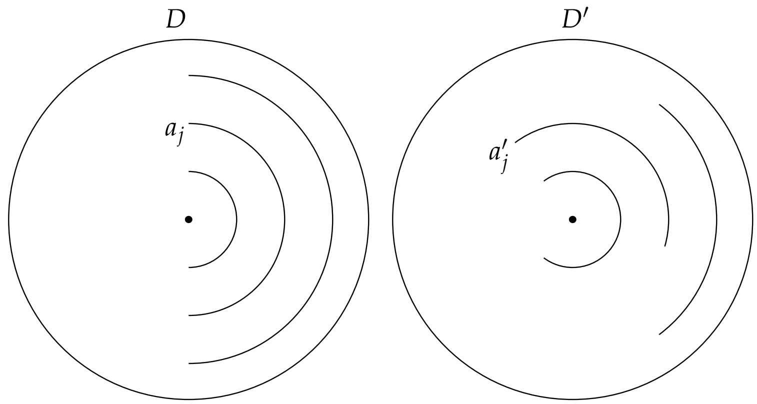

Figure C.2: The two domains in Theorem 43. If all rotated arcs in D' are either subsets of or contain the original arcs in D, then all arcs in D' are subsets of or contain the arcs in D.

Then since the h-functions of D and D' are identical, $\omega(0, E, D) = \omega(0, E', D')$.

Let D_0 have boundary E and F'. Then $D \subset D_0$, and so

$$\omega(0, E, D) \leq \omega(0, E, D_0).$$

But by Lemma 36,

$$\omega(0, E, D_0) < \omega(0, E', D').$$

This is a contradiction; hence $a_j \not\subset a'_j$. Similarly, $a'_j \not\subset a_j$. ∎

If $k = 2$, so there are two boundary arcs, rotate D' by $-\delta$. Clearly, rotating a_1 by δ produces exactly the same transformation of the lengths of the arcs as rotating a_2 by $-\delta$. Thus, by Theorem 43, $|\psi_1 - \psi'_1| < 2|\delta|$ and $|\psi_2 - \psi'_2| < 2|\delta|$. So, this means that for a sequence of two-arc circle domains D_n, with identical h-functions, convergence of arc midpoints implies convergence of arc lengths.

C.3 A two-arc circle domain with closely spaced arcs

As a digression, we investigate the following question.

If you rotate an arc of the circle domain, while keeping the h-function fixed, do the boundary arcs get shorter or longer?

To investigate this, we examine two-arc circle domains in the limit where the arcs are very close together.

Suppose D is a circle domain with an arc a at r and an arc b at $r+\varepsilon$, where we take ε to be very small..

Suppose the arc a is centered along the line $\arg z = 0$ and b is centered along $\arg z = \theta$ with $0 \leq \theta \leq \pi$, and that they have arc lengths $2\psi_a$ and $2\psi_b$, respectively.

If we allow $\varepsilon < 0$, then we may assume $\psi_a \geq \psi_b$.

Suppose we rotate b by an angle $\delta < \pi - \theta$ while keeping the h-function constant, producing a circle domain D' with arcs a' and b', with arc lengths $2\psi_a'$ and $2\psi_b'$.

Now, if $\theta \leq \psi_a + \psi_b$ and $\theta + \delta \leq \psi_a' + \psi_b'$, then D and D' are each very similar to a circle domain with only one arc. (This arc runs from $-\psi_a$ to ψ_a or $\theta + \psi_b$ in D, and from $-\psi_a'$ to ψ_a' or $\theta + \delta + \psi_b'$ in D'.) Then the arc length of the approximating single arc must be nearly unchanged, since we must keep the harmonic measure of the outer circle constant. Note that, from Theorem 43, we know that $|\psi_b - \psi_b'| < \delta$ and $|\psi_a - \psi_a'| < \delta$, and so

$$\theta + 3\delta \leq \psi_a + \psi_b \Rightarrow \theta + \delta \leq (\psi_a - \delta) + (\psi_b - \delta) < \psi_a' + \psi_b'.$$

So we can, in fact, force $\theta + \delta \leq \psi_a' + \psi_b'$ merely by choosing δ small enough with respect to D, without knowing anything about D'.

We can break down this problem into several cases, based on the initial arrangement of the arcs and their final distribution; that is, whether they start out overlapping, whether they start out with the arc b entirely behind the arc a, and so forth. We have analyzed only two of the cases, as shown in Fig. C.3.

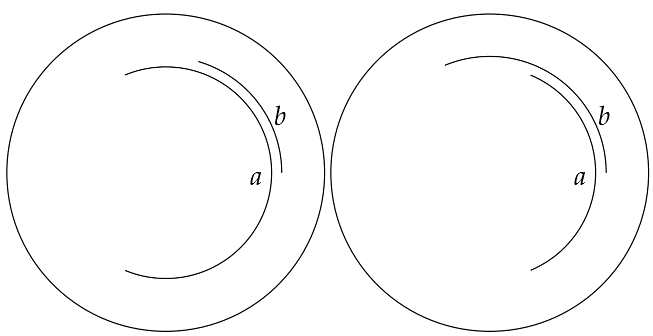

Figure C.3: Two possible arrangements of nearby close arcs.

C.3.1 The arcs overlap, but neither covers the other

Suppose $\theta + \psi_b \geq \psi_a$ and $\psi_a + \theta \geq \psi_b$. This is the arrangement of arcs shown on the right of Fig. C.3.

Generate D' using a $\delta < \frac{1}{3}(\psi_a + \psi_b - \theta)$.

Since $b \not\subset b'$, we have that the arc $a \cup b$ must gain on the positive-θ side. Hence it must shrink on the other side. So

$$\psi_a - \psi_a' \approx \delta + \psi_b' - \psi_b > 0.$$

But $\delta > \psi_a - \psi_a'$ and so $\psi_b > \psi_b'$.

Thus, as we rotate the arcs farther apart, they both shrink.

C.3.2 a covers b

Suppose that $\psi_a > \theta + \psi_b$. This is the arrangement of arcs shown on the left of Fig. C.3.

Behavior of a

If $\psi_b' + \theta + \delta > \psi_a'$, and so the arcs in D' are arranged as shown on the right of Fig. C.3, then a must shrink. (This is true even if $\psi_a' + \psi_b' < \theta + \delta$; consider $\omega(0, a' \cup b', D')$.)

If $\psi_b' + \theta + \delta \leq \psi_a'$, and so the arrangement of arcs does not change, then $\psi_a' \approx \psi_a$. We therefore cannot determine whether a grows or shrinks.

Behavior of b

Suppose $\theta + \psi_b = \psi_a$. Then let $\gamma = \psi_a - \psi_a'$. (From above, we know that γ is positive or, at least, not very negative.)

From earlier results, we know $\gamma < \delta$.

If $\theta + \delta \leq \psi_a' + \psi_b'$, then note that

$$\psi_a' + \theta + \delta + \psi_b' = \psi_a + \theta + \psi_b \Rightarrow \psi_b - \psi_b' = \delta - \gamma > 0,$$

and so the arc b must also shrink.

C.3.3 How closely does this approximate the behavior of the real arcs?

We wish to show that we can force two closely spaced arcs to behave like a single arc by making them be sufficiently close together. That is, we would

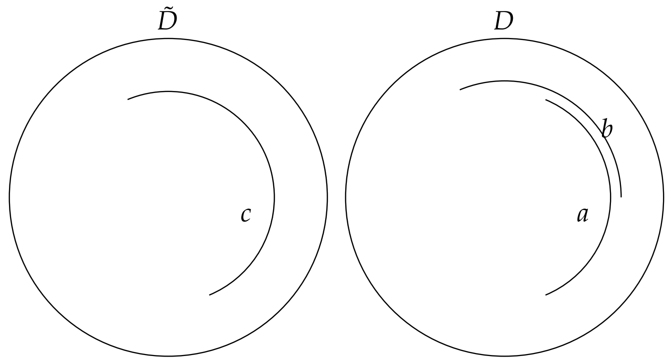

Figure C.4: A circle domain with two closely spaced arcs and a similar domain with a single arc.

like to show that the h-functions of the two domains in Fig. C.4 are very similar.

Theorem 44 *Let D be a circle domain with an arc a of arc length $2\psi_a$, centered on the positive real axis at radius r, and an arc b of arc length $2\psi_b$, centered at $(r+\varepsilon)e^{i\theta}$. Assume $\theta \leq \psi_a + \psi_b$.*

Let $\tilde{D}$ be a circle domain with an arc c, of arc length

$$2\psi_c = \psi_a + \max(\theta + \psi_b, \psi_a),$$

centered at $re^{i(\psi_c - \max(\psi_a, \psi_b - \theta))}$. D and $\tilde{D}$ are shown in Fig. C.4.

Then

$$|\omega(0, c, \tilde{D}) - \omega(0, a \cup b, D)| < \frac{1}{\pi} \frac{2\sqrt{\delta'/\rho'}}{1 - \delta'/\rho'},$$

where $\rho' = (1 - (r+\varepsilon))/(1 + (r+\varepsilon))$ and

$$\delta' = \varepsilon \frac{2r + \varepsilon}{r^2 + (r+\varepsilon)^2 + 2r(r+\varepsilon)\cos(\theta + \psi_b)}.$$

This means that unless b reaches all the way to the negative-real axis, we can make $\omega(0, c, \widetilde{D})$ arbitrarily close to $\omega(0, a \cup b, D)$ by forcing ε small enough.

Proof Let $D' = D \cap \widetilde{D}$. Let u be harmonic on D with boundary values of 1 on $a \cup b$ and 0 else, and let $\widetilde{u}$ be harmonic on $\widetilde{D}$ with boundary values of 1 on c and 0 elsewhere.

Then
$$\omega(0,c,\widetilde{D}) - \omega(0,a\cup b,D) = \widetilde{u}(i) - u(i).$$

Consider $f(z) = (2ir)/(z+r) - i$. Let $\Omega = f(D)$, $\widetilde{\Omega} = f(\widetilde{D})$. Then a and c map to straight lines, and b maps to a curved arc.

Now, consider $\Omega' = \Omega \cap \widetilde{\Omega}$. Then $(\widetilde{u} - u) \circ f^{-1}$ is harmonic on Ω' with boundary values of 0 on $f(a)$ and $f(\partial D)$, and boundary values of $1 - u \circ f^{-1}$ on $f(c \setminus a)$, and $\widetilde{u} \circ f^{-1} - 1$ on $f(b)$.

Now, let $\delta = \max_{z\in f(b)} |\operatorname{Im} f(z)|$. Note that

$$\begin{aligned}
\delta &= \left|\operatorname{Im} f((r+\varepsilon)e^{i(\theta+\psi_b)})\right| \\
&= \left|\operatorname{Im} \frac{2ir}{(r+\varepsilon)e^{i(\theta+\psi_b)} + r} - i\right| \\
&= \left|\operatorname{Im} \frac{2ir((r+\varepsilon)e^{-i(\theta+\psi_b)} + r)}{((r+\varepsilon)e^{-i(\theta+\psi_b)} + r)((r+\varepsilon)e^{i(\theta+\psi_b)} + r)} - i\right| \\
&= \left|\operatorname{Im} \frac{2ir(r+\varepsilon)e^{-i(\theta+\psi_b)} + 2ir^2}{(r+\varepsilon)e^{-i(\theta+\psi_b)}((r+\varepsilon)e^{i(\theta+\psi_b)} + r) + r((r+\varepsilon)e^{i(\theta+\psi_b)} + r)} - i\right| \\
&= \left|\operatorname{Im} \frac{2ir(r+\varepsilon)e^{-i(\theta+\psi_b)} + 2ir^2}{(r+\varepsilon)^2 + r(r+\varepsilon)e^{-i(\theta+\psi_b)} + r(r+\varepsilon)e^{i(\theta+\psi_b)} + r^2} - i\right| \\
&= \left|\operatorname{Im} \frac{2ir(r+\varepsilon)(\cos(\theta+\psi_b) - i\sin(\theta+\psi_b)) + 2ir^2}{(r+\varepsilon)^2 + r(r+\varepsilon)e^{-i(\theta+\psi_b)} + r(r+\varepsilon)e^{i(\theta+\psi_b)} + r^2} - i\right| \\
&= \left|\operatorname{Im} \frac{2ir(r+\varepsilon)(\cos(\theta+\psi_b) - i\sin(\theta+\psi_b)) + 2ir^2}{(r+\varepsilon)^2 + 2r(r+\varepsilon)\cos(\theta+\psi_b) + r^2} - i\right| \\
&= \left|\frac{2r(r+\varepsilon)\cos(\theta+\psi_b) + 2r^2}{(r+\varepsilon)^2 + 2r(r+\varepsilon)\cos(\theta+\psi_b) + r^2} - 1\right| \\
&= \left|\frac{2r(r+\varepsilon)\cos(\theta+\psi_b) + 2r^2}{(r+\varepsilon)^2 + 2r(r+\varepsilon)\cos(\theta+\psi_b) + r^2} - \frac{(r+\varepsilon)^2 + 2r(r+\varepsilon)\cos(\theta+\psi_b) + r^2}{(r+\varepsilon)^2 + 2r(r+\varepsilon)\cos(\theta+\psi_b) + r^2}\right| \\
&= \left|\frac{r^2 - (r+\varepsilon)^2}{(r+\varepsilon)^2 + 2r(r+\varepsilon)\cos(\theta+\psi_b) + r^2}\right|.
\end{aligned}$$

Fix some $z = x - iy \in f(b)$. Then $0 < y \le \delta$, and

$$-\frac{2\sin(\theta+\psi_b)}{2+2\cos(\theta+\psi_b)} \le x \le \frac{2\sin\psi_a}{2+2\cos\psi_a}.$$

Suppose we let $\rho = |f(1) - f(r)| = (1-r)/(1+r)$. Then for all $z \in c$ and $w \in \partial\mathbb{D}$, $|f(z) - f(w)| \ge \rho$.

Let D' be the disk of radius ρ with the interval from 0 to ρ deleted. Let v be harmonic in D' with boundary values of 1 on $[0,\rho]$ and 0 on $\partial\mathbb{D}$. Then $\widetilde{u}\circ f^{-1}(z) \geq v(-iy)$.

Let $k(z) = i(1-z/\rho)/(1+z/\rho)$. Then $k : D' \to \mathbb{U}\setminus[0,i]$. Let $g(z) = \sqrt{1+z^2}$. Then $g(\mathbb{U}\setminus[0,i]) = \mathbb{U}$ and $g([0,i]) = [-1,1]$.

Now,

$$\begin{aligned} v(-iy) &= \omega(-iy,[0,\rho],D') = \omega(k(-iy),[0,i],\mathbb{U}\setminus[0,i]) \\ &= \omega(g\circ k(-iy),[-1,1],\mathbb{U}) \\ &= \frac{1}{\pi}\arg\left[g\circ k(-iy)-1\right] - \frac{1}{\pi}\arg\left[g\circ k(-iy)+1\right]. \end{aligned}$$

But we have that, for $A=\pm 1$,

$$\begin{aligned} \arg\left[g\circ k(-iy)-A\right] &= \arg\left[\frac{1}{1-i\frac{y}{\rho}}\sqrt{\left(1-i\frac{y}{\rho}\right)^2-\left(1+i\frac{y}{\rho}\right)^2}-A\right] \\ &= \arg\left[\frac{\rho^2+iy\rho}{\rho^2+y^2}\sqrt{\frac{-2iy}{\rho}}-A\right] \\ &= \arg\left[\sqrt{-i}\left(1+i\frac{y}{\rho}\right)\sqrt{\frac{2y}{\rho}}-A\sqrt{-i}\sqrt{i}\left(1+\frac{y^2}{\rho^2}\right)\right] \\ &= \arg\left[2\sqrt{\frac{y}{\rho}}+A\left(1+\frac{y^2}{\rho^2}\right)+2i\frac{y}{\rho}\sqrt{\frac{y}{\rho}}+Ai\left(1+\frac{y^2}{\rho^2}\right)\right]-\frac{1}{4\pi} \\ &= \arctan\left[\frac{2(y/\rho)\sqrt{y/\rho}+A(1+y^2/\rho^2)}{2\sqrt{y/\rho}+A(1+y^2/\rho^2)}\right]-\frac{1}{4\pi} \end{aligned}$$

Because $\arctan(a)-\arctan(b) = \arctan\left((b-a)/(1+ba)\right)$, this implies that

$$\begin{aligned} v(-iy) &= \frac{1}{\pi}\arctan\left[\frac{2\frac{y}{\rho}\sqrt{\frac{y}{\rho}}+1+\frac{y^2}{\rho^2}}{2\sqrt{\frac{y}{\rho}}+1+\frac{y^2}{\rho^2}}\right]-\frac{1}{\pi}\arctan\left[\frac{2\frac{y}{\rho}\sqrt{\frac{y}{\rho}}-1-\frac{y^2}{\rho^2}}{2\sqrt{\frac{y}{\rho}}-1-\frac{y^2}{\rho^2}}\right] \\ &= -\frac{1}{\pi}\arctan\left[\frac{\frac{-2(y/\rho)\sqrt{y/\rho}+1+y^2/\rho^2}{-2\sqrt{y/\rho}+1+y^2/\rho^2}-\frac{2(y/\rho)\sqrt{y/\rho}+1+y^2/\rho^2}{2\sqrt{y/\rho}+1+y^2/\rho^2}}{1+\frac{-2(y/\rho)\sqrt{y/\rho}+1+y^2/\rho^2}{-2\sqrt{y/\rho}+1+y^2/\rho^2}\,\frac{2(y/\rho)\sqrt{y/\rho}+1+y^2/\rho^2}{2\sqrt{y/\rho}+1+y^2/\rho^2}}\right] \\ &= -\frac{1}{\pi}\arctan\left[\frac{4\sqrt{y/\rho}(1+y^2/\rho^2)(1-y/\rho)}{(1+y^2/\rho^2)^2-(2\sqrt{y/\rho})^2+(1+y^2/\rho^2)^2-(2(y/\rho)\sqrt{y/\rho})^2}\right] \\ &= -\frac{1}{\pi}\arctan\left[\frac{2\sqrt{y/\rho}}{1-y/\rho}\right]. \end{aligned}$$

The correct branch of the arctangent here is that between $-\pi$ and 0. This implies that $v(-iy) \geq 1 - \frac{1}{\pi}\frac{2\sqrt{y/\rho}}{1-y/\rho}$.

Since $\frac{2\sqrt{y/\rho}}{1-y/\rho}$ is monotonically increasing in y, this means that $v(-iy) \geq 1 - \frac{1}{\pi}\frac{2\sqrt{\delta/\rho}}{1-\delta/\rho}$.

Hence $1 - \widetilde{u} \leq 1 - \min v(-iy) \leq \frac{1}{\pi}(2\sqrt{\delta/\rho})/(1-\delta/\rho)$.

Similarly, if

$$\delta' = (2r\varepsilon + \varepsilon^2)/(r^2 + (r+\varepsilon)^2 + 2r(r+\varepsilon)\cos(\theta + \psi_b))$$

and $\rho' = (1-(r+\varepsilon))/(1+(r+\varepsilon))$, then $u - 1 \geq -\frac{1}{\pi}(2\sqrt{\delta'/\rho'})/(1-\delta'/\rho')$. So

$$|u - \widetilde{u}| < \frac{1}{\pi}\max\left(\frac{2\sqrt{\delta'/\rho'}}{1-\delta'/\rho'}, \frac{2\sqrt{\delta/\rho}}{1-\delta/\rho}\right) = \frac{1}{\pi}\frac{2\sqrt{\delta'/\rho'}}{1-\delta/\rho'}.$$

This can be seen by applying $f' = (2i(r+\varepsilon))/(z+r+\varepsilon) - i$.

This proves the theorem. ■

Bibliography

[1] Lars Valerian Ahlfors. *Conformal Invariants*. McGraw-Hill, New York, 1973.

[2] Lars Valerian Ahlfors. *Complex Analysis*. McGraw-Hill, New York, 1979.

[3] D. A. Brannan and W. K. Hayman. Research problems in complex analysis. *Bull. London Math. Soc.*, 21(1):1–35, 1989.

[4] Kai Lai Chung. *A course in probability theory*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, second edition, 1974. Probability and Mathematical Statistics, Vol. 21.

[5] Otto Cortez. Brownian motion and planar regions: Constructing boundaries from h-functions. Senior thesis, Harvey Mudd College, 2000.

[6] J.B. Garnett and D.E. Marshall. Harmonic measure. preliminary version.

[7] John B. Garnett. *Applications of Harmonic Measure*. University of Arkansas Lecture Notes in the Mathematical Sciences, 8. John Wiley & Sons Inc., New York, 1986. A Wiley-Interscience Publication.

[8] Shizuo Kakutani. Two-dimensional Brownian motion and the type problem of Riemann surfaces. *Proc.Imp. Acad. Japan*, 21:138–140 (1949), 1945.

[9] J. David Logan. *Applied partial differential equations*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1998.

[10] Stanisław Łojasiewicz. *An introduction to the theory of real functions*. A Wiley-Interscience Publication. John Wiley & Sons Ltd., Chichester,

third edition, 1988. With contributions by M. Kosiek, W. Mlak and Z. Opial, Translated from the Polish by G. H. Lawden, Translation edited by A. V. Ferreira.

[11] Christian Pommerenke. *Boundary Behavior of Conformal Maps*. Springer-Verlag, Berlin, 1992.

[12] Thomas Ransford. *Potential theory in the complex plane*, volume 28 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1995.

[13] William O. Ray. *Real analysis*. Prentice Hall Inc., Englewood Cliffs, NJ, 1988.

[14] H. L. Royden. *Real analysis*. Macmillan Publishing Company, New York, second edition, 1968.

[15] Walter Rudin. *Principles of mathematical analysis*. McGraw-Hill Book Co., New York, third edition, 1976. International Series in Pure and Applied Mathematics.

[16] Marie A. Snipes. Brownian motion and the shape of a region's boundary: Generating step functions from circle domains. Senior thesis, Harvey Mudd College, Claremont, CA, 1999.

[17] Marie A. Snipes and Lesley Ward. Step functions and convergence properties of harmonic measure distributions for planar domains. in preparation.

[18] Masatsugu Tsuji. *Potential Theory in Modern Function Theory*. Maruzen Co., Tokyo, 1959.

[19] Byron L. Walden and Lesley A. Ward. Distributions of harmonic measure for planar domains. In Laine and Martio, editors, *XVIth Rolf Nevanlinna Colloquium (Joensuu, 1995)*, pages 289–299. Walter de Gruyter & Co., Berlin, 1996. UNSW Pure Mathematics Report PM95/21.

[20] Byron L. Walden and Lesley A. Ward. Asymptotic behaviour of distributions of harmonic measure for planar domains. *Complex Variables Theory Appl.*, 46(2):157–177, 2001.