Journal of Humanistic Mathematics

Volume 4 | Issue 2 July 2014

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James Marcotte & Matthew Salomone, "Loxodromic Spirals in M. C. Escher's Sphere Surface," *Journal of Humanistic Mathematics*, Volume 4 Issue 2 (July 2014), pages 25-46. DOI: 10.5642/jhummath.201402.04. Available at: https://scholarship.claremont.edu/jhm/vol4/iss2/4

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Loxodromic Spirals in M.C. Escher's *Sphere Surface*[1]

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Abstract

Loxodromic spirals are the analogues in spherical geometry of logarithmic spirals on the plane. M.C. Escher's 1958 woodcut *Sphere Surface* is an image of black and white fish arranged along eight spiral paths on the surface of a sphere. By connecting the plane and spherical models of the complex numbers, we show that *Sphere Surface* is the conformal image on the sphere of a tessellation of fish on the plane, and that the spirals running through the fish are indeed loxodromic spirals to a high degree of accuracy.

1. Introduction

The works of art of M.C. Escher (1898-1972) are renowned for their rich geometric properties. Studies of these properties have variously focused on Escher's use of symmetries and tessellation [5], "impossible" topology [7], and curved, particularly hyperbolic, geometry [2]. Recently, Escher's art has been investigated for its conformal properties: the features of his designs which are preserved when the design is transformed in an angle-preserving fashion. This approach was recently popularized by de Smit and Lenstra's deconstruction of Escher's 1956 lithograph *Prententententoonstelling* (*Print Gallery*) that

[1]This work was partially supported by a grant from the Adrian Tinsley Program at Bridgewater State University.

revealed the "straightened" Maltese landscape conformal to the "twisted" view seen Escher's original image [3]. An analogous technique has also been used to demonstrate that Escher's spiral-shaped 1966 *Path of Life III* is conformally related to a tessellation of bird and fish shapes [1]. The latter employed techniques which uncovered remarkably precise mathematical equations for the red spiral paths in the image.

In this article, we investigate Escher's 1958 woodcut *Sphere Surface*, which may be viewed at http://www.mcescher.com/Gallery/recogn-bmp/LW427.jpg (last accessed June 26, 2014). *Sphere Surface* depicts a progression of black and white shapes ("fish") aligned along eight spiral paths emanating from an off-center point ("pole"). The perspective of the image and the faint curved grid on its surface suggest to the viewer that the the fish are drawn on a sphere.

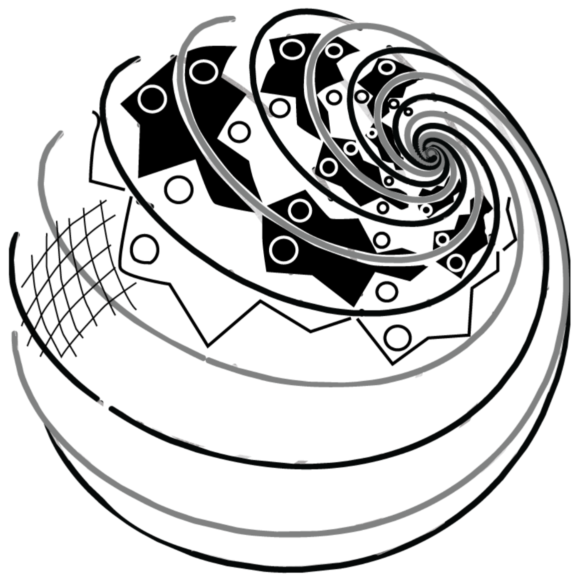


Figure 1: Selected features of M.C. Escher's *Sphere Surface* (1958): eight spirals, with an interlocking fish pattern along each, and a faint grid appearing throughout the image.

Is *Sphere Surface* a perspective drawing of a sphere? Is there a "flat" tessellation to which its fish pattern is conformal? Do the spiral paths followed by the fish on the sphere have the same geometric properties as the red spirals in *Path of Life III*? Can equations be found to match the spiral paths? We will show that all these questions may be answered in the affirmative.

All but the first of these questions may be answered with the very technique used in [1]. We review in Section 2 the role that conformal transformations play in discovering tessellations, and especially how the complex exponential function helps to characterize logarithmic spirals.

However, there are two obstacles to applying this technique directly to *Sphere Surface*. First [1] uses conformal transformations of a flat planar surface; here we must work with a curved sphere. This issue necessitates a two-step remedy. We must first place Escher's two-dimensional image on the surface of a sphere in three dimensions, to negate the effects of perspective in the drawing; we describe this process in Section 3. Next we must map the surface of the sphere conformally onto a flat plane. We do this by imposing a standard system of complex coordinates on the sphere, as described in Section 4.

The second technical obstacle involves the off-center placement of *Sphere Surface*'s pole at which the spirals originate. Since logarithmic spirals by definition converge to the origin (i.e., the pole, or the complex number 0) in a complex coordinate system, it is necessary to transform *Sphere Surface* so as to move its pole—again conformally—to the center of the image. This is accomplished by a rotation of the sphere, detailed in Section 5.

Finally, the application of a conformal transformation, as in Section 6, uncovers *Sphere Surface*'s flat tessellation and exposes its paths as loxodromic spirals on the sphere. A single measurement determines equations for these spirals, which match those drawn by Escher to a high degree of accuracy.

Uncovering the geometric properties of *Sphere Surface*'s pattern allows us to continue this pattern onto the back of the sphere, unseen in Escher's image, and yields a process by which any flat tessellation pattern may be transformed into a *Sphere Surface*-like image. It also supplies further evidence of the remarkable mathematical precision with which Escher created—by hand—this work of art.

2. Conformal Transformations and Spirals

In his younger years, Escher honed his artistic talent with highly realistic sketches of portraits, such as his 1929 *Self-Portrait*; city scenes such as his 1923 woodcut of San Gimignano, Italy; and patterns of interlocking images known as *tessellations* (or *tilings*, if not all figures are congruent). The latter was perhaps facilitated by Escher's frequent use of woodcut stamps, permitting him to repeat identical images side by side, as in the strikingly detailed 1922 tiling pattern *Eight Heads*. As his career progressed, Escher's works retained their realistic qualities but increasingly used distortion to increase the visual interest of the finished work, from distorted portraits (*Hand with Reflecting Sphere*, 1935) to distorted cityscapes (*Print Gallery*) to distorted tilings (from the *Circle Limit* series of the late 1950s to the *Path of Life* series of the 1960s).

One of the distortion techniques Escher used often is what mathematicians term a *conformal transformation*. A conformal transformation turns one image into another in a way that preserves the measure of all its angles (and thus the overall "shape" of its objects), but does not necessarily preserve distances, areas, or curvature. For instance, the slats of the window frame in the eponymous *Print Gallery* meet at the expected right angles, but the slats themselves are profoundly curved and vary in width. Analogously, the fish-and-bird pattern in *Path of Life III* does not itself qualify as a tessellation—since the figures do not all have equal area, and they follow curved paths—but the angles making up each fish and bird's polygonal shape are equal across all fish in the image, and across all birds. These facts suggest that a conformal transformation exists which straightens out the pattern into a true tessellation. Conversely, they also suggest that images such as *Path of Life III* may be *constructed* by applying a conformal transformation to a true tessellation; see Figure 2 below for an illustration.

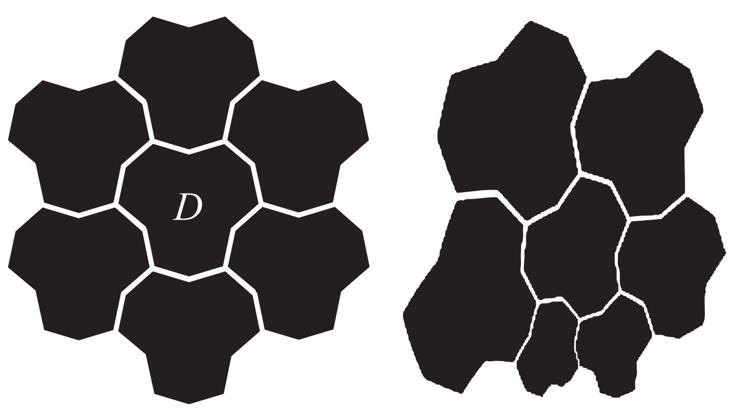


Figure 2: On the left, a tessellation based on the dodecagon D. On the right, a conformally transformed version of the same: the angles and hence the basic shape of the dodecagons is preserved, but their sizes and curvatures now vary.

But, given a particular image, how do we find the appropriate conformal transformation, the one that straightens it out?

A fundamental source of conformal transformations in two-dimensional geometry is differentiable functions of a complex variable [4]. When the xy-plane's usual Cartesian coordinates are treated instead as complex numbers (i.e., the point with coordinates (x, y) envisioned as the complex number $x + iy$, where $i^2 = -1$), any differentiable function $f : \mathbb{C} \to \mathbb{C}$ is a conformal transformation at any point where its derivative f' is nonzero.

The conformal transformation instrumental in analyzing *Print Gallery* and *Path of Life III* was the complex exponential function $f(z) = e^z$. As in real-variable calculus, the complex exponential function is differentiable everywhere, is equal to its own derivative ($f'(z) = e^z$), and since the latter is nonzero at every point, the complex exponential function is a conformal transformation at every point. In coordinates $z = x + iy$ on the complex plane, the exponential function has the expression

$$e^{x+iy} = e^x\, e^{iy} = e^x\big(\cos y + i \sin y\big). \tag{1}$$

In the latter equality, we used Euler's famous formula unifying exponential functions and trigonometry (and implying the famous identity $e^{i\pi} + 1 = 0$).

Print Gallery's distortion was rectified with a combination of the complex exponential function, its inverse the complex logarithm

$$\mathrm{Log}(x + iy) = \ln \sqrt{x^2 + y^2} + i \arctan \frac{y}{x} = \ln r + i\theta \tag{2}$$

(using standard polar coordinates), and a rotation and scaling accomplished by scalar multiplication $f(z) = \omega \cdot z$ for an appropriate choice of complex number ω. Meanwhile, *Path of Life III* is straightened by the complex logarithm itself. Or, viewed another way, it is the distortion of a congruent fish-and-bird tiling by the complex exponential function.

The fact that *Path of Life III*'s spiral paths are straightened by the complex logarithm transformation is what characterizes the paths themselves as *logarithmic spirals*, as follows. In polar coordinates on the plane, a logarithmic spiral is a curve which intersects each radial at an equal angle. Provided this angle is neither 0° (at which the spiral is itself a radial line) nor 90° (at which the spiral is a circle with its center at the origin), a logarithmic spiral approaches the origin only asymptotically, and the point at the origin is not included in these spirals.

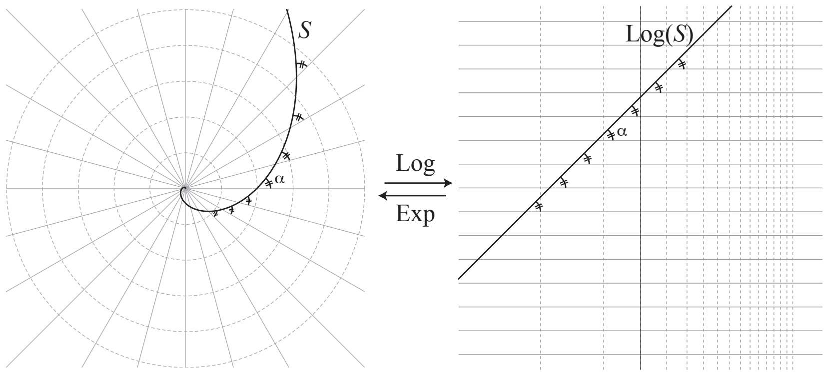


Figure 3: The logarithmic spiral S makes an equal angle α with each radial; its logarithmic image $\mathrm{Log}(S)$ is a straight line making an equal angle with the horizontal. Since Log is conformal, this angle is also equal to α.

True to their name, logarithmic spirals are uniquely characterized by the property that, under the complex logarithm transformation, their images are straight lines, as depicted in Figure 3 above. This is a consequence of the following three facts:

1. By definition, logarithmic spirals make a constant angle with radial grid lines in polar coordinates.
2. The complex logarithm transforms central circles into vertical lines, and radials into horizontal lines.
3. The complex logarithm is a conformal transformation — so the same constant angle the logarithmic spiral makes with radials is also the *same, constant angle* its image makes with the horizontal, and any curve making a constant angle with the horizontal must be a straight line.

This observation makes determining an equation for a logarithmic spiral straightforward. Accepting that a straight line may be described by the familiar slope-intercept equation $y = mx + b$, a logarithmic spiral is then the *inverse* image of this line under the complex logarithm, i.e. its image under the complex exponential function, as in Equation (1). Taking y to equal $mx + b$ in this function gives a mathematical description of a logarithmic spiral to be

$$z(x) = e^x\big(\cos(mx + b) + i\sin(mx + b)\big).$$

As demonstrated in [1], this spiral is in polar coordinates (r, θ) characterized by the equation

$$r = Ae^{\theta/m} \qquad \text{for } A = e^{-b/m}, \tag{3}$$

and the angle α at which the spiral intersects radial lines (or, equivalently, at which its image line intersects the horizontal) satisfies $\tan\alpha = m$.

Of particular interest in *Sphere Surface* is the analogue of logarithmic spirals on the surface of the sphere. Where logarithmic spirals make a constant angle to radials, their analogues on the sphere make a constant angle to parallels, the lines of longitude connecting two opposite poles of the sphere, see Figure 4 below. Curves with this property are called *loxodromic spirals* (or *loxodromes*,' or *rhumb lines*). Practically speaking, a loxodrome is the path that an airplane would follow if it were to fly around a spherical Earth at a constant bearing.

A consequence of this definition is that loxodromes, provided they are not circular, limit on *both* the poles of a sphere, approaching these points only asymptotically. In *Sphere Surface*, one pole of the spirals is visible in the upper right of the image; presumably, if its spirals are loxodromes, the spirals also converge toward a pole on the sphere's unseen opposite side. Verification

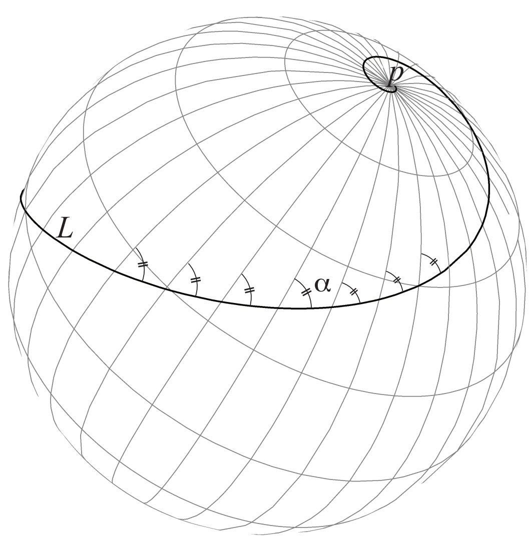


Figure 4: The loxodromic spiral L makes a constant angle α with lines of longitude on the sphere. Both the spiral and the lines of longitude connect the pole p shown to an opposite pole on the unseen reverse of the sphere.

of this fact depends upon a careful consideration first of the perspective in which the sphere is visualized in Escher's image, and then of the particular angle at which the sphere is viewed. In the following sections, we will subject *Sphere Surface* to a series of transformations in order to compensate for these perspective choices and thereby prepare the image for conformal analysis.

3. Two-Dimensional Perspectives of a Sphere

If *Sphere Surface* is indeed a perspective drawing of a sphere, it is necessary to ask: what perspective? In other words, how does Escher's image depict a sphere on a flat surface?

Depicting a spherical object on a flat surface is a central problem in cartography, where the (roughly) spherical surface of the Earth is displayed in the flat form of a paper map. Because of the geometric difficulty inherent in "flattening" a spherical surface, any choice of how to do so comes with a loss of information. The commonly-used Mercator projection, for instance, accurately depicts shapes, directions, and angles at each point — and so is useful for navigation — but does not accurately depict area, a distortion which is minimal near the Equator but increases dramatically toward the Earth's poles [6]. (We will show in Section 6 that the Mercator projection is indeed responsible for the distortion of area in *Sphere Surface*.)

Meanwhile, other projections such as the Lambert cylindrical projection accurately depict area but distort shape and aspect ratio, causing shapes near the equator to look vertically narrowed and shapes near the poles to look horizontally narrowed, (see Figure 5 on the next page for a demonstration). While cartographers have developed dozens of different projections, each faithfully depicts some geometric features of the Earth's surface while distorting others — with implications not only in the geometry but also in the politics of mapmaking.

To surmise which projection Escher has used in *Sphere Surface*, observe that the faint grid Escher has left in the image does not, at many points, meet at right angles, even though it is reasonable to assume that it depicts what would be, on the sphere, a grid of rectangles. Indeed, the grid angles become increasingly deviated from 90° as they approach the edge of the visible image. This suggests that *Sphere Surface* is not an equal-angle visualization of a sphere.

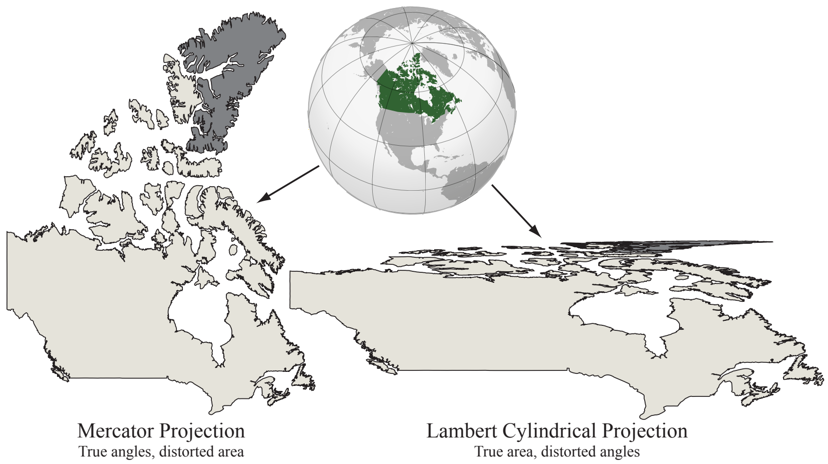


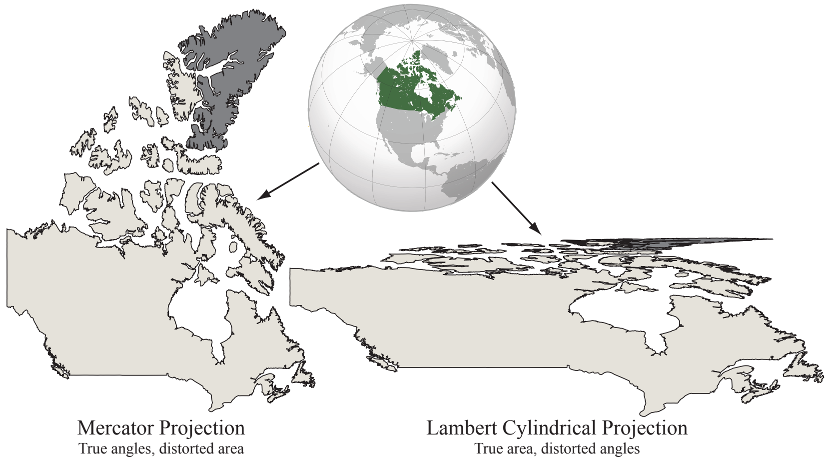
Figure 5: Two map projections of the outline of Canada, with Ellesmere Island shaded. The island's shape is accurately depicted in the conformal Mercator projection, but its area relative to the rest of Canada is not: it comprises roughly 2% of Canada's total land area but more than 13% of area on the Mercator map. The Lambert map depicts its relative area accurately, but distorts its shape. These distortions are especially pronounced due to the island's proximity to the North Pole.

Instead, we hypothesize that Escher has drawn this sphere from the perspective in which an observer would see it from some distance away, as though projecting parallel rays of light directly onto a viewing screen to which they are perpendicular. (See Figure 8(a)-(b).) We call this an *orthographic projection* of the sphere onto a flat plane, and it has the obvious limitation that a viewer may only see one hemisphere at a time.

Situating the sphere with its center at the origin of the usual three-dimensional xyz-space $\mathbb{R}^3$, the orthographic projection may be seen as "flattening" the sphere's northern hemisphere onto the xy-plane by simply forgetting the height (z-) coordinate of each point:

$$R(x, y, z) = (x, y). \tag{4}$$

The orthographic projection may then be inverted by restoring the missing z-coordinate to each point on the xy-plane, elevating it to the point directly above it on the upper hemisphere. If, for simplicity, we imagine the sphere's radius to be 1, then its equation in xyz-coordinates is $x^2+y^2+z^2 = 1$ and, solving this equation for z and choosing its positive root, we have the



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$$r(x, y) = (x, y, \sqrt{1 - x^2 - y^2}). \tag{5}$$

Applying r to Escher's flat image turns *Sphere Surface* into a hemisphere situated in xyz-space. (See Figure 8b, below.)

Within this context we ask: are the spiral paths formed by the fish loxodromic spirals on the sphere? Additionally, has the orthographic projection indeed turned *Sphere Surface*'s grid into rectangles? The answers to these questions lie in the conformal geometry of the sphere on which the image now resides. This is a point of departure from previous studies of Escher's conformal geometry: where *Print Gallery*'s and *Path of Life III*'s patterns were the conformal images on the complex plane of a regular tiling, we will show that *Sphere Surface* is the conformal image on a unit sphere of a regular tiling. In order to make use of analogous techniques, we need only to connect the unit sphere and the complex plane—conformally—in a way that permits us full use of the latter's arithmetic and geometric properties.

4. Complex Numbers on a Sphere

The conformal properties of Escher's prior works, such as *Print Gallery* and *Path of Life III*, for all their geometric intricacy, were ultimately discovered by treating the images as flat. The transformations used were, therefore, transformations of a flat, two-dimensional plane. Giving that plane complex coordinates—in other words, selecting an origin point corresponding to the number 0 and associating to each point with Cartesian coordinates (x, y) the complex number $x + iy$—is not only a convenient choice, it is one that unlocks a wealth of conformal transformations to choose from, as described in Section 2.

For the present work, however, since we hypothesize that *Sphere Surface* does indeed depict a sphere, the flat complex plane is not sufficient. In principle, any question regarding the geometry of Escher's pattern in this image must be answered in the context of spherical geometry. Like all curved geometries, spheres bend the familiar rules of Euclidean geometry: for instance, distances on a sphere are measured not along straight lines but along great circles, and the sum of the interior angles of a spherical triangle strictly *exceeds* 180°.

Fortunately, many difficulties inherent in spherical geometry are overcome when the sphere is given complex coordinates. In the same way as with the plane, an origin point is selected to correspond to the number 0, and each point (X, Y, Z) on the sphere is assigned a unique complex number P to serve as its coordinates. While there would appear to be many different ways to accomplish this task, in fact, once an origin point is selected, there is only one choice of "standard" complex coordinates to which all others are conformal. This choice of coordinate system defines what is known as the *Riemann sphere*.

Introduced by Bernhard Riemann in the 19th century, the Riemann sphere associates to each of its points $w = (X, Y, Z)$ a complex number $z = x + iy$ (i.e., a point in the complex plane) via *stereographic projection*. This projection situates both the complex plane and unit sphere within xyz-space, and associates together the two intersection points of each ray emanating from the sphere's origin, or *pole*, with the sphere and the plane. (See Figure 6 below.) Every such ray will intersect the sphere and the plane exactly once each, and so the stereographic projection associates to every point z on the complex plane exactly one point w on the sphere.

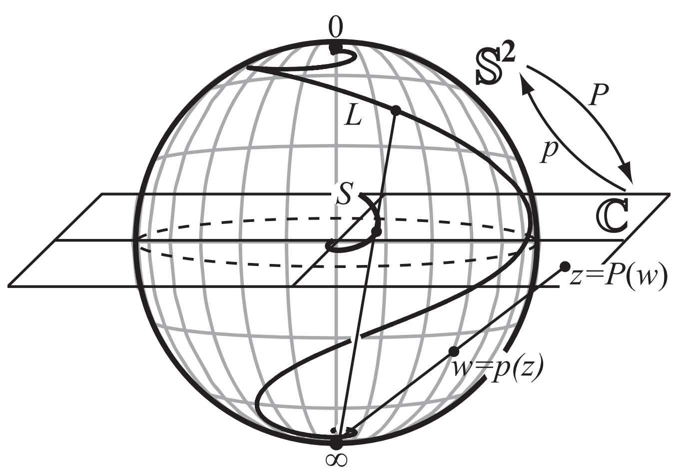

Figure 6: The stereographic projection conformally relates the complex plane $\mathbb{C}$ and the Riemann sphere $\mathbb{S}^2$. It also relates loxodromes L on the sphere to logarithmic spirals S on the complex plane. In this projection, the origin of $\mathbb{C}$ is associated (0) with the north pole of the sphere, and the "horizon" of the plane (∞) with the south pole.

Assuming for simplicity that this sphere has radius 1, the formulas for the stereographic projection may be found using analytic geometry. Situating the complex plane as the horizontal xy-plane, the unit sphere with its center at the origin, and choosing the "south pole" of the sphere $(0, 0, -1)$ to be the source of our rays, the map from plane onto sphere is given by

$$p(x + yi) = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{1 - x^2 - y^2}{1 + x^2 + y^2}\right) \tag{6}$$

and from sphere onto plane by

$$P(X, Y, Z) = \frac{X + Yi}{1 + Z}. \tag{7}$$

However, while the stereographic projection pairs up each point of the complex plane with a point on the sphere, one point on the sphere is left without a partner on the plane: the pole itself, from which we draw the rays. Note in the formulas above that, while the plane-to-sphere map p is well defined for all values of x and y, the sphere-to-plane projection P is undefined exactly at the south pole, where $Z = -1$ makes its denominator zero.

To remedy this difficulty, and to make P a continuous function, we are forced to place the image of the pole at a point which is nearby to the images of the points that surround it, but the shallow rays through these southernmost points on the unit sphere carry them far off toward the horizon of the complex plane, in all directions. Since there is no point on the complex plane which is capable of being on the horizon in every direction, we associate to the sphere's pole a new point, which we suggestively call *infinity* (∞). So it is that this unit sphere, the Riemann sphere, is a model of the *extended complex plane* $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Owing to the association of the pole with infinity, the abusive colloquialism "one over zero is infinity" becomes true in the arithmetic of the Riemann sphere.[2]

The stereographic projection is not only continuous; it is a differentiable function of the complex variable $x + yi$. This makes it a conformal transformation from the complex plane to the Riemann sphere, and hence makes

[2] Geometrically, the reciprocal function $f(z) = 1/z$ reflects points on the Riemann sphere through the x-axis in three dimensions. The zero-infinity pair makes the reciprocal function continuous—indeed, differentiable—on the entire sphere!

it exactly the right tool to investigate the conformal geometry of *Sphere Surface*, since all the angles of the original image will be preserved in its *flattening out* onto the complex plane.

Better still, the stereographic projection provides the crucial link between logarithmic spirals on the complex plane and loxodromes on the Riemann sphere. As indicated in Figure 6 earlier, a loxodrome on the sphere becomes, when transformed by the stereographic projection, a logarithmic spiral on the complex plane. This is an immediate consequence of conformality, analogously to the argument in Section 2:

1. By definition, logarithmic spirals make a constant angle with radial grid lines in polar coordinates.
2. The stereographic projection transforms radial lines into lines of longitude on the sphere, and central circles into lines of latitude.
3. The stereographic projection is a conformal transformation — so the same constant angle the logarithmic spiral makes with radials is also the *same, constant angle* its image makes with lines of longitude, and by definition curves making a constant angle with lines of longitude on the sphere are loxodromes.

This link also permits equations to be found for loxodromes. In polar coordinates, the stereographic projection in Equation (6) takes the form

$$p(r,\theta) = \left(\frac{2r\cos\theta}{1+r^2}, \frac{2r\sin\theta}{1+r^2}, \frac{1-r^2}{1+r^2}\right),$$

and a logarithmic spiral having the form $r = Ae^{-\theta/m}$ (as in Equation (3)) therefore becomes, under stereographic projection, a curve on the sphere whose parameterization is

$$p(Ae^{-\theta/m},\theta) = \left(\frac{2Ae^{-\theta/m}\cos\theta}{1+A^2e^{-2\theta/m}}, \frac{2Ae^{-\theta/m}\sin\theta}{1+A^2e^{-2\theta/m}}, \frac{1-A^2e^{-2\theta/m}}{1+A^2e^{-2\theta/m}}\right).$$

A simpler expression for this curve on the sphere uses the spherical coordinates θ (longitude) and ϕ (co-latitude, so named since $\phi = 0$ corresponds to the north pole rather than the equator). In these coordinates, the loxodrome has the equation

$$\sin\phi = \frac{2Ae^{-\theta/m}}{1+A^2e^{-2\theta/m}}. \tag{8}$$

If the spirals in Escher's *Sphere Surface* are indeed loxodromes, we expect to be able to determine equations of this type to model them.

When the stereographic projection is applied to *Sphere Surface*, we immediately notice that Escher's grid intersects in right angles — and conclude, thanks to the conformality of the stereographic projection, that the grid depicted in *Sphere Surface* is, if Escher has viewed the sphere in orthographic projection, a grid of rectangles on the surface of the sphere. (See Figure 8c below.)

This lends further credibility to the hypothesis that *Sphere Surface* is an orthographic view of an image on the Riemann sphere. Indeed, it is likely that Escher used this grid of rectangles in the creation of *Sphere Surface*, in the same way he used a grid to create *Print Gallery* (though the latter grid was not a visual element of that image in the end) [3].

Having situated *Sphere Surface* within the complex plane, we are almost in a position to apply the techniques of [1] to assess the nature of its spiral paths and uncover its regular tiling.. The remaining difficulty is that, unlike *Path of Life III*, the origin of these paths is not, in fact, located in the center of the image (i.e., the origin of the complex plane). Since the Riemann sphere associates its north pole with the origin of the complex plane, repairing this difficulty requires a rotation of the Riemann sphere that places the center of the spirals at its north pole.

5. Centering Escher's Spirals

A consequence of the conformal equivalence of the Riemann sphere with the (extended) complex plane is that transformations of the sphere are simultaneously transformations of the plane, and vice versa. We wish to rotate the Riemann sphere to accomplish the goal of moving the center of its spirals to the north pole; we may, therefore, instead apply a transformation to the complex plane which accomplishes the same task. What transformation occurs to the complex plane when the Riemann sphere is rotated?

Rotations of the Riemann sphere are one among several types of its *automorphisms*, conformal transformations not only preserving its angles but also its arithmetic structure. In other words, when two points on the sphere are rotated, their sum, difference, product, and quotient are rotated in identical fashion.

It can be shown that all automorphisms of the Riemann sphere are represented on the complex plane by *Möbius transformations*. A Möbius transformation, also known as a linear fractional transformation, is a function having the form

$$M(z) = \frac{az+b}{cz+d} \tag{9}$$

where a, b, c, d are complex constants [4]. In order to prevent this transformation from being a constant function (and collapsing every point of its domain to a single point in its range), we assume that $ad - bc \neq 0$. Each Möbius transformation is also a transformation of the Riemann sphere once we allow for the no-longer abusive maxim "one over zero is infinity."

Möbius transformations come in a variety of types: some are rotations of the Riemann sphere, while others are reflections or "dilations" pulling points away from one of the sphere's poles and toward the other. The most general Möbius transformation combines the effects of all three types, and the values of the parameters a, b, c, d uniquely determine which effects are present and in what proportion.

Möbius transformations which correspond to rotations of the Riemann sphere were found by Gauss in 1819 to have the form

$$M(z) = \frac{az+b}{-\overline{b}z+\overline{a}}. \tag{10}$$

In this special case of Equation (9), a and b are complex constants and $\overline{a}, \overline{b}$ their complex conjugates. This form makes it straightforward to find the specific Möbius transformation that will rotate the sphere in such a way that the center of *Sphere Surface*'s spirals will now be at the origin. (See Figure 7 on the next page.)

To do this, we need only to require that the function must send the spirals' pole u to 0, i.e. $M(u) = 0$. This can be done by setting the numerator of $M(u)$ to zero, so that

$$au + b = 0, \text{ or } b = -au.$$

Substituting for b then gives Equation (10) the form

$$M(z) = \frac{az - au}{\overline{a}\overline{u}z + a} = \frac{z-u}{\overline{u}z+1}. \tag{11}$$

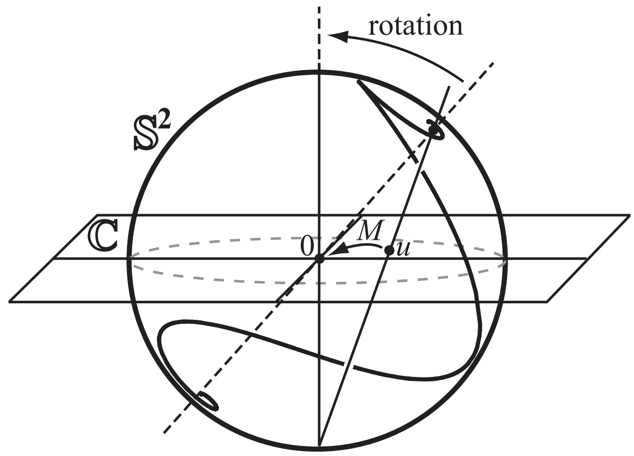

Figure 7: Rotating the sphere to bring the center of the spirals to its north pole will, on the complex plane, bring the center of the spirals to the origin. On the complex plane, this rotation takes the form of a Möbius transformation M which sends the off-center pole point u to the origin 0 (Equation (11)).

Approximate coordinates for u can be found by measuring the stereographic view of *Sphere Surface* plotted on the complex plane, see Figure 8c. Measured in image-editing software, this gives the approximation

$$u \approx 0.3 + 0.3i$$

so that Equation (11), customized to rotate *Sphere Surface* to bring its spirals' center to the north pole, has the form

$$M(z) = \frac{z - (0.3 + 0.3i)}{(0.3 - 0.3i)z + 1}. \tag{12}$$

The resulting image, of black and white fish following spiral paths out of a central origin, is now directly analogous to *Path of Life III.* (See Figure 8d.) It may therefore be subjected to the techniques of [1] to determine whether it is a conformal image of a regular tiling, and whether its spirals are indeed logarithmic spirals on the complex plane, and hence loxodromes on Escher's sphere.

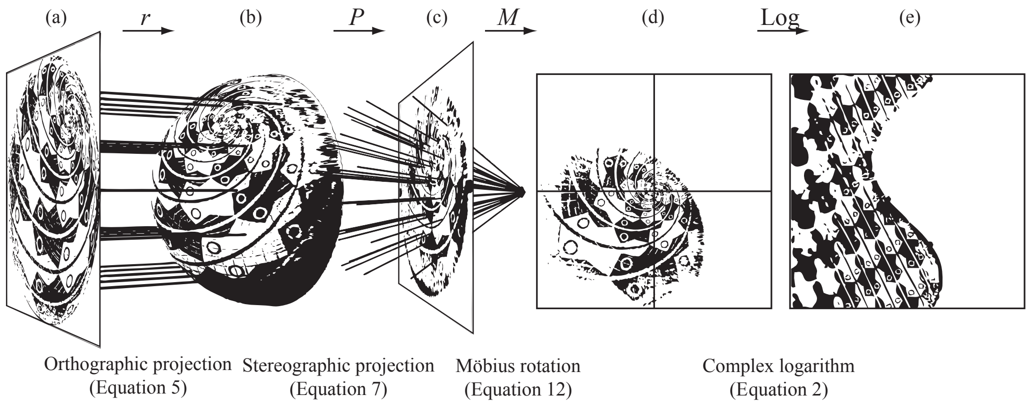

Figure 8: The inverse orthographic projection r transforms Escher's image (a) onto the unit sphere (b), nullifying the effect of perspective. The stereographic projection P then conformally transforms the sphere onto the complex plane (c), where once its pole is conformally moved to the origin (d), the logarithm uncovers its pattern (e).

6. Discovering the Loxodromic Pattern

With *Sphere Surface*'s image planted firmly in the complex plane, and its spirals limiting on the origin, we may at last employ the technique of [1]. If indeed the spirals on the sphere are loxodromic, then the spirals in this plane will be logarithmic and their image under the complex logarithm will be straight lines.

Using `Mathematica` software to apply this series of transformations to *Sphere Surface* does result in a strikingly regular image; see Figure 8e. In this image, the black and white spiral paths have transformed into straight lines, and the pattern of fish along the paths has transformed into a tessellation pattern in which all fish are equal shape, size, and orientation. The latter observation confirms that *Sphere Surface* does indeed depict the conformal image of a tessellation of fish on the sphere; the former verifies that the spirals seen in the previous image (Figure 8d) are indeed logarithmic spirals, and hence the spirals on *Sphere Surface* are loxodromes to a very high degree of accuracy.

Remarkably, the faint curved grid in *Sphere Surface* which enhances its spherical perspective is, in this logarithmic image, a nearly-perfect square grid of vertical and horizontal lines, see Figure 9. Not only does this speak highly of Escher's precision in having depicted, in orthographic perspective,

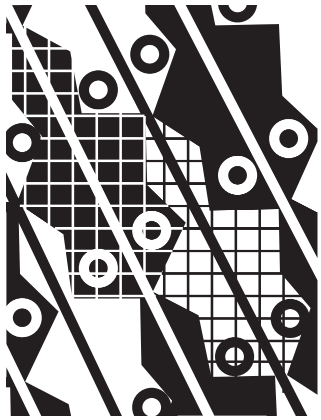

Figure 9: A tracing of the tessellation of Figure 8e shows that the logarithmic images of Escher's spirals are straight lines of slope $m \approx -2$, and the black and white fish tessellate the plane. Escher's faint curved grid through the original image now appears square.

a square spherical grid, we believe this suggests the method Escher employed in order to create *Sphere Surface*. He would have sketched the tessellation pattern of Figure 9 atop a square grid, developed a separate curved grid having the geometric properties he desired in the finished work, and then carefully transferred the tessellation onto that curved grid, square by square.

Indeed, such grids appear across Escher's oeuvre in many of his conformally distorted works. A grid is visible in the conformal tiling *Path of Life III*, as well as the 1958 woodcut *Sphere Spirals* whose spherical spirals-on-grid pattern bears strong resemblance to that of *Sphere Surface*. While there is no visible grid in *Print Gallery* to indicate its conformal distortion, a hand-sketched grid was found in Escher's personal effects that he used to produce the image [3]. The technique of using grids to transform images is not unique to Escher. That modern computer software performs image transformations nearly identically to Escher's hand sketches, however, speaks volumes to the incomparable precision with which he worked.

Thanks to this precision, the slope of these lines as estimated by the "rise over run" along the square grid of Figure 9 agrees with the slope as measured by image editing software. The slope is approximately $m \approx -2$. According to Equation (3), these logarithmic spirals therefore have equations of the form

$$r = Ae^{-\theta/2}.$$

The values of A are chosen so as to equally space all eight spirals. Since the variable θ is periodic, repeating every 2π radians, we have that

$$r = e^{-\theta/2} \quad \text{and} \quad r = e^{-(\theta+2\pi)/2} = e^{-\pi} e^{-\theta/2}$$

describe one and the same spiral. An equal spacing of eight spirals can therefore be accomplished by shifting θ by multiples of $\frac{2\pi}{8} = \frac{\pi}{4}$, giving eight equations for eight distinct spirals:

$$r_k = e^{-(\theta+k\pi/4)/2} = e^{-k\pi/8} e^{-\theta/2} \qquad \text{for } k = 0, 1, 2, \ldots, 7.$$

Finally, since these spirals are logarithmic spirals on the plane, the stereographic projection carries them into loxodromic spirals on the surface of the Riemann sphere. According to Equation (8), these spirals have the spherical coordinate equations

$$\sin\phi = \frac{2e^{-k\pi/8} e^{-\theta/2}}{1 + e^{-k\pi/4} e^{-\theta}} \qquad \text{for } k = 0, 1, 2, \ldots, 7. \tag{13}$$

A plot of these eight spirals, generated by `Mathematica`, is shown in Figure 10 below. The agreement between these spirals and Escher's is quite strong.

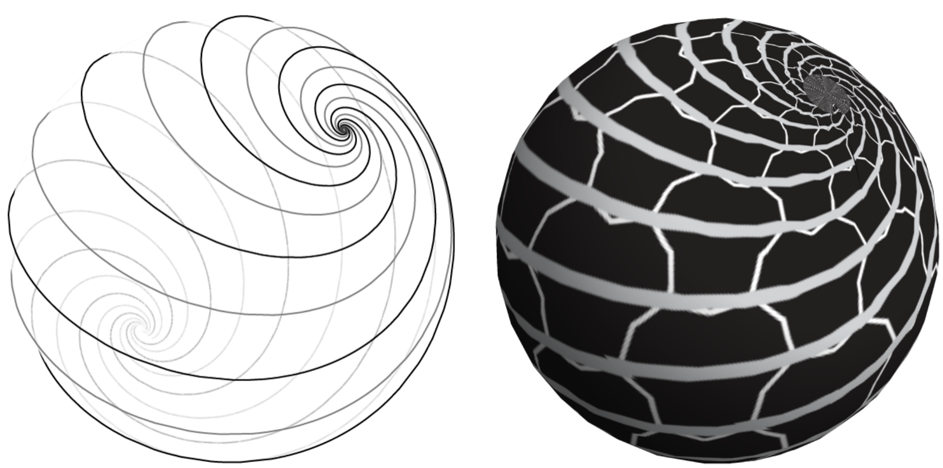

Figure 10: On the left, the eight spirals of Equation (13), visualized in `Mathematica`, show strong agreement with those in *Sphere Surface* (and suggest its unseen reverse side). On the right, a *Sphere Surface*-like image based on the tessellation of Figure 2.

A further consequence of Escher's spirals being loxodromes is that, were a Mercator projection (a la Figure 5) used to construct a map of *Sphere Surface*, the spirals on that map would appear as straight lines, and the tiling of fish would appear as an equal-area tessellation; see Figure 11 on the next page for an illustration. This projection is a conformal transformation sending loxodromic spirals on the unit sphere to helices of uniform pitch on a circumscribed cylinder. If the sphere and the cylinder both have unit radius, this transformation may be expressed using the spherical coordinates (θ, ϕ) and the cylindrical coordinates (θ, z) by

$$\theta = \theta \text{ and } z = \tan\phi.$$

The process by which the spirals were uncovered may also be employed in reverse to generate new *Sphere Surface*-like images from any regular tiling of the plane. Beginning with any tessellation such as in Figure 8e, the inverse transformations may be applied: from right to left, the complex exponential function exp (Equation (1)) is followed by the inverse Möbius transformation M^{-1}, the inverse stereographic projection p (Equation (6)), and finally the orthographic projection R (Equation (4)). One such result was shown on the right in Figure 10.

7. Conclusion

M.C. Escher's *Sphere Surface*, in much the same way as several of his other works, owes its geometric properties to a clever use of angle-preserving transformations. It is likely that Escher produced this image by constructing a regular tiling of black and white fish atop a rectangular grid, then transforming it square by square onto a polar grid atop the surface of a sphere, viewed in a "straight-on" orthographic perspective. Just as with *Path of Life III*, this process gives rise to visible spirals in the completed image which limit on the sphere's north (and unseen south) poles.

The geometric precision of *Sphere Surface* permits Escher's original fish tessellation, and a faint square grid beneath it, to be recovered from the original image. The process by which this is accomplished proves that *Sphere Surface* accurately depicts, in orthographic perspective, a loxodromic spiral pattern of fish on the Riemann sphere.

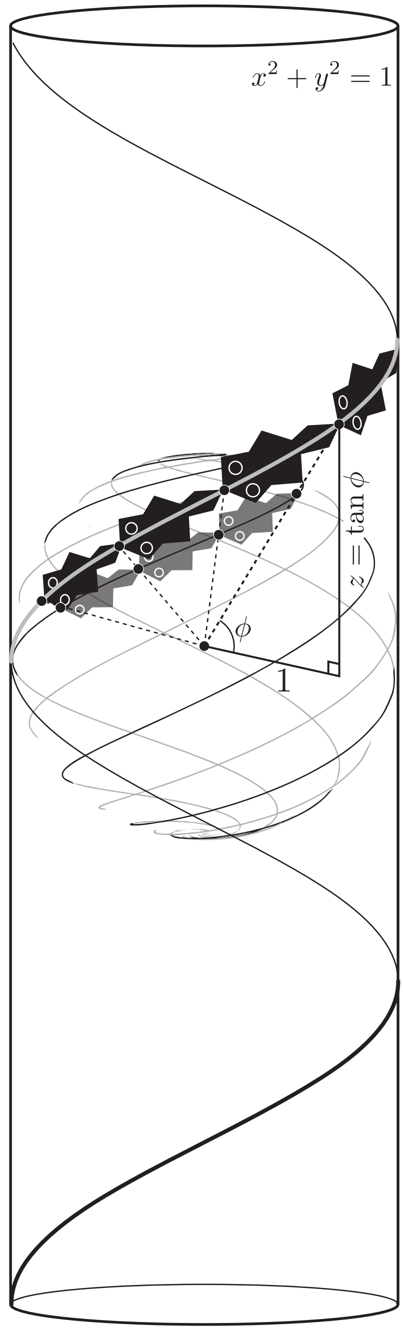

Figure 11: A Mercator projection conformally relates *Sphere Surface*'s tiling to an equal-area tessellation on a cylinder, and each loxodromic spiral on the sphere to a helix on the cylinder.

This not only situates *Sphere Surface* alongside Escher's many other works of conformal geometry, it also provides a recipe by which any tessellation may be realized as a loxodromic spiral pattern on a sphere.

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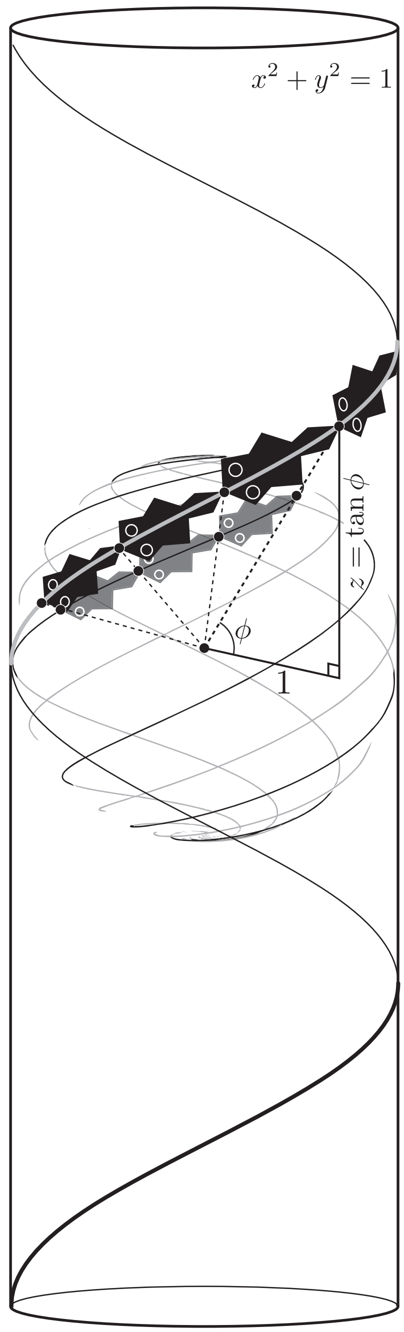

Figure 11: A Mercator projection conformally relates *Sphere Surface*'s tiling to an equal-area tessellation on a cylinder, and each loxodromic spiral on the sphere to a helix on the cylinder.