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Representation Theoretical Approach to n-Candidate Voting

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A Decomposition of Voting Profile Spaces

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Abstract

Voting theory as been explored mathematically since the 1780's. Many people have tackled parts of it using various tools, and now we shall look at it through the eyes of a representation theorist. Each vote can be thought of as a permutation of the symmetric group, S_n, and a poll is similar to a linear combination of these elements. Specifically, we will focus on translating and generalizing the works of Donald Saari into more algebraic terms to discover not just one space, but a whole isotypic component essential to positional voting.

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Chapter 1

Introduction

1.1 Background

Voting theory officially began in the 18^{th} century with the Marquis de Condorcet and Jean-Charles de Borda arguing as to the best method of elections for three or more candidates. Borda published *Mémoire sur les élections au scrutin* in 1781 [Borda (1781)], causing the French academy to begin looking at voting. In response, Condorcet put the pairwise voting scheme into a rigorous and publicized essay *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix* in 1785 [Caritat, M. Marquis de Condorcet (1785)]. Neither of these schemes were completely original, but they did spark much debate, as well as a new field of political science.

Although debated furiously for time to come, Borda and Condorcet's methods of tallying votes are still among the most common schemes. In his pamphlets from 1874, not published until 2001 [Dodgson (2001)], Charles Ludwige Dodgson (better known as Lewis Carrol) adjusted Condorcet's method to better ensure the existence of a candidate winning in any given pairwise race by finding who was "closest" to the Condorcet winner.

In the 1950s, Kenneth J. Arrow gave a proof showing that every voting system was prone to a specific sort of paradox. Given criteria, which he proved independent from one another, no method of voting can be guaranteed to escape a paradox which Condorcet described. Arrow's Theorem will be explored in more depth in Chapter 2.

John Kemeny discerned a new method of adjusting the Condorcet vote, which finds the entire ranking "closest" to the voters' election. In recent years, Kemeny's Rule and Dodgson's method have been compared by a few scholars, notably Thomas Ratliff [Ratliff (2002)].

There have been many economists, political scientists, and statisticians who have studied voting theory. Recent years have brought significant progress, and many new people, including Donald G. Saari, professor at the University of California, Irvine. He uses geometric methods of proving impressive theorems, which are popular with the modern voting theory community.

1.2 Overview of Thesis

The language of political science has been very verbose, with few definitions properly nailed down to a formal, rigorous base. It is a primary goal to translate the problems of voting theory into a form suitable to an algebraist, then to verify some of these theorems in as precise a manner as possible. The verification will be done with tools from representation theory to which the whole subject lends itself so naturally.

In particular, this thesis looks at Saari's works using algebra rather than geometry. Some of his ideas are quite simple and elegant when cast in the light of representation theory. However, his decompositions for the space of votes are not precise enough. Representation theory is able to point out the underlying group structures which are invariant in voting so that one might easily separate the portions of the profile space that matter to a final tally.

Using those tools, this thesis will present some original research involving manipulation of modules in applied representation theory. This will be guided by the intuitions afforded from voting theory, and take into account Borda's scheme.

At the end of the thesis, the results will be translated back into the economics of voting theory, hopefully in a mathematically elegant way. In this way, the concepts in voting theory will be brought to a formal level for a broader audience within the mathematical community.

Chapter 2

Theory

2.1 Theorems from Economics

Voting operates on some basic criteria, which all political scientists and economists take into assumption. The first is that every voter will vote honestly with a transitive ranking of all n candidates. That is, if a voter prefers A to B and B to C then the voter prefers A to C. A voter preferring that transitive ranking can then be represented by one vote for $A \succ B \succ C$. Moreover, every voter should have no more weight than any other. These fundamental assumptions lend themselves to the use of a linear function for tallying an election. Any election can be counted in any order, and if there are c voters for a particular ranking of candidates then the voting scheme should allow scalar multiplication of that ranking.

Condorcet's method of pairwise voting takes the list of everyone's votes to a space which assigns points to ordered pairs of candidates. The result of the transformation of the ranking $A \succ C \succ B$ gives one point to the $A \succ C$, one to $A \succ B$, and one to the $C \succ B$ coordinates. Condorcet considered a "winner" to be someone who would beat any other candidate in a two person race. In modern sources this person is called the "Condorcet winner"' since there are other forms of winning an election.

Borda thought that a positional method, involving giving points to each place in a vote, would be more fair than Condorcet's method, after coming up with a paradox similar to this:

votes	ranking
2	$A \succ B \succ C$
1	$A \succ C \succ B$
1	$B \succ A \succ C$
2	$B \succ C \succ A$
2	$C \succ A \succ B$
1	$C \succ B \succ A$

The results of the pairwise test applied to this vote are as follows: $A \succ B$ at $5:4$, $B \succ C$ at $5:4$, and the paradoxical part (as it is not transitive) $C \succ A$ at $5:4$. To more clearly see the paradox, this means that $A \succ B \succ C \succ A$ so there is no single winner.

To combat this intransitivity paradox, Borda assigned points to each candidate in every vote. So, for example, a vote for $A \succ B \succ C$ gives one point to A, $\frac{1}{2}$ a point to B and no points to candidate C.

From Arrow's theorem, even Borda's system has a flaw. Condorcet was quick to come up with a situation in which Borda would have a tough time explaining. If instead of using Borda's $(1, \frac{1}{2}, 0)$ weighting vector, one might use a $(1, \frac{3}{4}, 0)$ vector, although a paradox can be determined for any.

Suppose there are rankings:

votes	ranking
1	$A \succ B \succ C$
1	$A \succ C \succ B$
1	$B \succ A \succ C$
1	$B \succ C \succ A$
0	$C \succ A \succ B$
3	$C \succ B \succ A$

At a weight of $(1, \frac{3}{4}, 0)$, this election comes out as $B \succ C \succ A$ at a ratio of $5:4.5:2.75$. However, if candidate A were to drop out of the race, giving one point to B each time he beats C and vice versa, then the results of the election would be $C \succ B$ at a ratio of $4:3$.

The paradox is that if everyone kept their votes the same, but dropped a candidate from their ballot then the outcome of the election can change.

More recently, Kenneth Arrow won the Nobel Prize in Economic Sciences in 1972 for his acclaimed theorem in economics. The theorem states that a number of criteria desired in voting systems are contradictory. These criteria are as follows: (1) results, (2) monotonicity, (3) irrelevant alternatives, (4) achievability, (5) non-dictatorship.

That is a scheme should always provide some result, even if it is declared as a tie. Monotonicity means that it should never be in a voter's best interests to vote his favorite candidate ranked lower than first. Irrelevant alternatives states that a winner should not change if a non winning candidate drops out of the election. Achievability is when, for any ranking, there should be some profile which when counted affords the ranking as outcome. Non-dictatorship just says that more than one person's vote should count.

New proofs of Arrow's Theorem were given by John Geanakoplos [Geanakoplos (2001)], as opposed to Arrow's original proof. This version bases its proof heavily on set theory, which has the advantage of being relatively easy to follow once the translation from economics into mathematics is understood.

Donald Saari provides the first rigorous attempt at mathematical notation for voting theory, with the formalization of what he terms *voting tallies* as weighting vectors in positional schemes. Moreover he introduces notation to be the linear mechanism of counting votes.

With the big names in the economic aspect of voting theory mentioned, now it is necessary to understand the algebra needed for the thesis.

2.2 Representation Theory

This section contains a review of representation theory needed to understand this paper. Here we shall also examine the notation and terminology used in the paper. All definitions and theorems in this section are found in [Dummit and Foote (1999)] and [Sagan (1991)].

Whenever used, G will designate a finite multiplicative group and 1_G its identity. Where it is unambiguous 1 will suffice to be the notation for the identity. Group action will be written in a multiplicative manner or with a dot. The first three points of the definition below define the group action on V, while the fourth describes the linear component to the definition.

Definition 2.2.1. A representation of G is a finite-dimensional complex vector space V on which G acts linearly.

Thus for all $g,h \in G$, $v,w \in V$, and $a,b \in \mathbb{C}$:
(1) $g \cdot v \in V$
(2) $1 \cdot v = v$
(3) $g \cdot (h \cdot v) = (gh) \cdot v$
(4) $g \cdot (av + bw) = a(g \cdot v) + b(g \cdot w)$

There are several ways of thinking about a representation. At times it may be necessary to think of a representation of G, not as a vector space, but as a homomorphism ϕ from G into $GL(V)$, the general linear group on V. A different way to think of $GL(V)$ is the set of linear bijections from V to itself, where the group operation is composition of functions.

Direct sums of vector spaces are valuable tools in decompositions. Among the direct summands many uses, they pull the space into disjoint pieces which can be used to understand how the vector space behaves when mapped. This provides the motivation for recalling the definition of a direct sum.

Definition 2.2.2. If U and V are subspaces of vector space W which intersect trivially at $\{\mathbf{0}\}$, then their direct sum $U \oplus V$ is defined as

$$U \oplus V = \{u + v \mid u \in U, v \in V\}.$$

To understand this thesis, it is essential to understand the notion of invariance under group action. In terms of voting theory, this would be exemplified by the election being invariant under the names of the candidates, or the order in which they are called. The idea is that if a person supports candidate A that same voter would support the same candidate even if he were called candidate B. Group invariance reflects a structure of fairness among the candidates.

Definition 2.2.3. If V is a nontrivial G-module and a module $U \subseteq V$ has the property that it is closed under the action of G, then U is called *G-invariant*. So for every $g \in G$ and $u \in U$, $g \cdot u \in U$.

Often it is useful when a space is decomposed as much as it can be. An irreducible space is one which has no invariant subspaces aside from itself and the trivial vector space, $\{\mathbf{0}\}$.

Definition 2.2.4. If V is a G-module containing only $\{\mathbf{0}\}$ and V as G-invariant subspaces then V is *irreducible*.

The following theorem is essential in the decomposition of spaces, and fundamental to representation theory itself.

Theorem 2.2.1. ***(Maschke's Theorem)*** *If V is a nontrivial G-module then*

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$$

where each W_i is an irreducible G-module.

This means that every G-module is able to be decomposed into a direct sum of irreducible modules.

Theorem 2.2.2. *If U is a G-invariant subspace of a G-module V then the orthogonal complement of U in V, denoted $U^{\perp_V}$ is also a G-invariant subspace and $V = U \oplus U^{\perp_V}$.*

Using the homomorphism $\phi : G \to GL(V)$ and a fixed basis for V, $\phi(g)$ is an $n \times n$ matrix where $n = dim\, V$. The character χ corresponding to the representation ϕ is defined as the trace of ϕ, sum of the diagonal entries, in this matrix form. In this way, we can consider $\chi : G \to \mathbb{C}$ such that $\chi(g) = \mathrm{tr}(\phi(g))$.

As is known from linear algebra, putting a matrix into a different basis amounts to conjugating it by the proper change of basis matrix. Because $\mathrm{tr}(B^{-1}AB) = \mathrm{tr}(A)$ for any matrices A and B, one can see that the character is independent of a basis chosen for V. Also from that fact, we see that $\chi(h^{-1}gh) = \chi(g)$ for any $g, h \in G$. Such functions from groups to $\mathbb{C}$ which are constant on conjugacy classes are called class functions. Worth remembering is that two representations have the same character if and only if they are isomorphic as submodules of V.

Definition 2.2.5. Let χ and ψ be characters of G. Then their inner product, denoted by $< \chi, \psi >$ is defined by

$$< \chi, \psi > = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}$$

where $\overline{\psi(g)}$ denotes the complex conjugate of $\psi(g)$.

Theorem 2.2.3. *A character χ is irreducible iff $< \chi, \chi > = 1$.*

With this definition and theorem, we can examine the characters of the summands comprising a finite dimensional G-invariant space V.

Definition 2.2.6. An isotypic component of V is the sum of all spaces isomorphic to some submodule W.

$$W + \cdots + W$$

An isotypic component can then be broken into a terms, such that there are exactly a pairwise disjoint submodules of V isomorphic to W. That is the isotypic component is actually written as

$$W \oplus \cdots \oplus W$$

with a copies of W denoted as aW. An interesting note is that if $a \geq 2$ then there are an infinite number of ways of decomposing the isotypic component into a direct sum of exactly a direct summands.

Theorem 2.2.4. *Let V be a representation of G with character χ and*

$$V = a_1 W_1 \oplus \cdots \oplus a_k W_k$$

where the W_i are pairwise non-isomorphic, with a_i summands of each W_i up to isomorphism. If χ_i is the character associated with W_i then $<\chi, \chi_i>= a_i$.

Now $V = M_1 \oplus \cdots \oplus M_k$ where each $M_i = a_i W_i$ for each of the isomorphic copies of W_i. Naturally, if a_i is greater than 1, then infinitely many bases can be chosen for the W_i, but when collected into the isotypic components, the decomposition of V is unique up to ordering. More formally, one can state this as the following theorem.

Theorem 2.2.5. *If V is a nontrivial G-module, then*

$$V = M_1 \oplus \cdots \oplus M_k$$

where the M_i are isotypic G-modules. Moreover this decomposition is unique up to order of the M_i.

In this paper, we will usually be considering the set of formal linear combinations of elements of G over $\mathbb{C}$, called $\mathbb{C}G$, or the group ring of G over $\mathbb{C}$.

We will not use the next theorem directly, but it is important behind the scenes of much of the work of representation theory. It allows many results which will indeed be crucial.

Theorem 2.2.6. *The number of irreducible characters, that is characters of irreducible representations, of G is equal to the number of conjugacy classes of G.*

When $\mathbb{C}G$ is written as $\bigoplus a_i W_i$, the sum of irreducible submodules as just constructed, then each $a_i = dim\ W_i$. From the synthesis of above theorems, every irreducible representation is contained in $\mathbb{C}G$, the following corollary may be stated.

Corollary. $|G| = \sum (dim\, W)^2$ *where the sum is over all non-isomorphic irreducible representations W.*

Schur's Lemma is a central workhorse of representation theory, and there are many ways of stating it. We use the following statement of the theorem, as it will best suit this paper.

Theorem 2.2.7. ***(Schur's Lemma)*** *Let V and W be two irreducible G-modules. If $\theta : V \to W$ is a G-homomorphism, then either*

1. *θ is a G-isomorphism, or*
2. *θ is the zero map.*

The practical use of Schur's Lemma is that if M is a linear map from $\mathbb{C}G$ to $\mathbb{C}^n$ and U is an irreducible G-invariant subspace of $\mathbb{C}G$ then $M(U)$ is $\{\mathbf{0}\}$ or else something isomorphic to U.

Corollary. *Let X and Y be two irreducible representations of G. If A is any matrix such that $AX(g) = Y(g)A$ for all $g \in G$, then either*

1. A is a scalar multiple of the identity matrix, or

2. A is the zero matrix.

2.3 Young Tableaux

At a few times throughout this paper, more complicated machinery such as Specht Modules and Young Tableaux will be required. This thesis requires only a fraction of the theory concerning these structures, so only those theorems which are essential to the paper are presented. More in depth coverage of these topics can be found in [Sagan (1991)]. To maintain consistency, all definitions and theorems come from that same source.

The first definition is that of a partition of a natural number n. This is identical to the number theoretical definition.

Definition 2.3.1. A partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of n, written $\lambda \vdash n$, has $\sum \lambda_i = n$, $\lambda_i \geq \lambda_{i+1} > 0$, and $\lambda_i \in \mathbb{Z}$.

A more generalized definition of S_n, the group of permutations on n letters, is S_A, the group of permutations on the elements of the set A. The Young subgroup (of S_n) corresponding to λ a partition of n is now defined as:

$$S_\lambda = S_{\{1,2,\ldots,\lambda_1\}} \times S_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \cdots \times S_{\{n-\lambda_l+1,\ldots,n\}}.$$

A Ferrers diagram or shape of λ is an array of n dots into l left justified rows with row i containing λ_i dots for $1 \leq i \leq l$. For the convenience of this section, when an example is chosen, $\lambda = (3,3,2,1)$ unless otherwise specified. A Ferrers diagram for the partition λ is then:

$$\begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \end{array}$$

Convention has it that the final dot in this particular Ferrers diagram is at position $(4,1)$, so this example allows for the generalization of coordinates in Ferrers diagrams.

Using the same partition, the Young subgroup is

$$S_{(3,3,2,1)} = S_{\{1,2,3\}} \times S_{\{4,5,6\}} \times S_{\{7,8\}} \times S_{\{9\}}$$

which in turn is isomorphic to

$$S_3 \times S_3 \times S_2 \times S_1.$$

Definition 2.3.2. Given $\lambda \vdash n$, a Young tableau of shape λ is an array replacing the dots of the Ferrers diagram of λ with the numbers $1,\ 2,\ldots,\ n$ bijectively.

With $\lambda = (3,3,2,1)$, one Young tableau is:

$$\begin{array}{lll} 4 & 6 & 1 \\ 2 & 3 & 7 \\ 9 & 8 & \\ 5 & & \end{array}$$

A Young tableau of shape λ is often called a λ-tableau. Two λ-tableaux, t_1 and t_2 are row equivalent, $t_1 \sim t_2$, if corresponding rows of the two tableaux contain the same elements.

Definition 2.3.3. A λ-tabloid, $[t]$, is the row equivalence class of t:

$$[t] = \{t_1 | t_1 \sim t\}.$$

The permutation module corresponding to λ, M^λ is defined next. As for group rings, $\mathbb{F}A$ denotes the set of formal linear combinations of the elements of a set A with coefficients from the field $\mathbb{F}$.

Definition 2.3.4. With $\lambda \vdash n$,

$$M^\lambda = \mathbb{C}\{[t_1],\ldots,[t_k]\}$$

where $\{[t_1],\ldots,[t_k]\}$ is a complete set of λ-tabloids.

This is equivalent to saying

$$M^\lambda = \left\langle \{[t^\lambda]\} \right\rangle.$$

As is typical when examining vector spaces, the dimensionality usually reveals something important. The following proposition is proved in [Sagan (1991)], and is thus omitted from here.

Proposition 2.3.1. *M^λ is generated by any one λ-tabloid and dim $M^\lambda = \frac{n!}{\lambda_1!\dots\lambda_l!}$*

Later it will be necessary to see dominance in partitions. This occurs in an application of Schur's Lemma and in the general decomposition of $\mathbb{C}S_n$ into Specht Modules, which will be seen soon enough.

Definition 2.3.5. Let $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\mu = (\mu_1, \dots, \mu_m)$ be partitions of n. Then λ dominates μ, written $\lambda \triangleright \mu$ if

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$$

for every $i \geq 1$. If $i > l$ or $i > m$ then $\lambda_i = 0$ or $\mu_i = 0$ respectively.

The last definition involving purely Young Tableaux, without Specht Modules, involves group actions. To talk about group action, let $\sigma \in S_n$ and t a λ-tableau, then $\sigma t = (\sigma t_{i,j})$ where $t_{i,j}$ is the (i,j) coordinate of tableau t.

2.4 Specht Modules

All irreducible submodules of $\mathbb{C}S_n$ eventually turn out to be what are called Specht Modules. These correspond to tableaux, and decompose each permutation module in a natural way.

First, however, there will be more notation leading up to the definition of Specht Modules.

Definition 2.4.1. Suppose tableau t has columns $C_1, C_2, \dots, C_k$. Then define

$$C_t = S_{C_1} \times S_{C_2} \times \cdots \times S_{C_k}.$$

Definition 2.4.2. Define

$$\kappa_t = \sum_{\sigma \in C_t} sgn(\sigma)\sigma.$$

Definition 2.4.3. Given t a tableau, the associated polytabloid is

$$e_t = \kappa_t[t].$$

Definition 2.4.4. For any partition λ, the corresponding Specht Module, S^λ is the submodule of M^λ spanned by the polytabloids e_t where t is the shape of λ.

To show that the Specht Modules are irreducible, the following lemma is needed.

Lemma 2.4.1. *If t is a tableau and σ a permutation, then $\kappa_{\sigma t} = \sigma \kappa_t \sigma^{-1}$, and $e_{\sigma t} = \sigma e_t$.*

Proof: The following are equivalent:

(1) $\pi \in \kappa_{\sigma t}$
(2) $\pi[\sigma t] = [\sigma t]$
(3) $\sigma^{-1}\pi\sigma[t] = [t]$
(4) $\sigma^{-1}\pi\sigma \in \kappa_t$
(5) $\pi \in \sigma\kappa_t\sigma^{-1}$

Now $e_{\sigma t} = \kappa_{\sigma t}[\sigma t] = \sigma\kappa_t\sigma^{-1}[\sigma t] = \sigma\kappa_t[t] = \sigma e_t$. □

Because of the previous lemma, the next proposition is proved.

Proposition 2.4.2. *S^λ is a cyclic module (one with a single generator), generated by any given polytabloid e_t where t has shape λ.*

With symbolic and technical effort, one can prove the following theorem and corollary.

Theorem 2.4.3. *The S^λ for $\lambda \vdash n$ form a complete list of irreducible S_n-modules over $\mathbb{C}$.*

Corollary. *The permutation modules decompose as*

$$M^\mu = \bigoplus_{\lambda \trianglerighteq \mu} m_{\lambda\mu} S^\lambda$$

for choices of $m_{\lambda\mu}$ specified in [Sagan (1991)].

Armed with the understanding afforded by Specht Modules, tackling parts of voting theory will prove to be shorter and perhaps more elegant.

Chapter 3

Translations from Economics

Translating the world of voting theory into algebra was the first goal of the thesis. This chapter begins with the previously established vocabulary of economics, specifically dealing with positional voting. It then continues with some new terms and draws on the combinatorial structure of Latin squares.

3.1 Basic Vocabulary

Looking at voting from a mathematical standpoint relies on several algebraic structures, the most important of which is the symmetric group. Invariance under action from a permutation corresponds to changing the names of the candidates according to the very permutation picked. Given that a voter should not care about the name of his favorite candidate, the invariance induced by this group is considered to be of primary importance.

Definition 3.1.1. The group of permutations on n letters, called the symmetric group, is designated by S_n.

Both the two line and disjoint cycle notations for elements in S_n are used. The following is an example of a permutation in S_5.

Permutation	Two-Line	Cycle
$1 \to 2$ $2 \to 1$ $3 \to 5$ $4 \to 3$ $5 \to 4$	$\binom{12345}{21534}$	$(1\ 2)\ (3\ 5\ 4)$

After the next definition, the reason for using two-line notation will reveal itself. However, there will be a substitution of letters for numbers. Here is how the candidates can be encoded into the symmetric group by the process of ranking.

Definition 3.1.2. A ranking of candidates $X_1, X_2, \ldots, X_n$ is a transitive sequence $X_{\sigma(1)} \succ X_{\sigma(2)} \succ \cdots \succ X_{\sigma(n)}$ for some $\sigma \in S_n$. Thus a ranking can be given by σ when no confusion will arise.

With this definition, we can write $A \succ C \succ B$ to mean A beats C who in turn beats B. Again, when it is not ambiguous will arise, $A \succ C \succ B$ can be written as ACB. Replacing A with 1, B with 2, and C with 3, ACB is turned into $1 \succ 3 \succ 2$, which is interpretable in two line notation as $\binom{123}{132}$ or in cycle notation as $(2\ 3)$. Thus this ranking corresponds to switching the second and third candidates. In this paper, the phrase "two-line notation" will be used through this bijection frequently.

For the purposes of standardization, the use of lexicographic order provides a convenient convention in any list of candidates. Thus $A \succ B \succ \cdots \succ N$ will be considered as the first permutation, and $N \succ M \succ \cdots \succ A$ last. When necessary, σ_i will denote the i^{th} permutation under this ordering.

For the sake of example, here is the lexicographic ordering on three candidates.

Number	Two-Line	Cycle
1	ABC	1
2	ACB	$(2\ 3)$
3	BAC	$(1\ 2)$
4	BCA	$(1\ 2\ 3)$
5	CAB	$(1\ 3\ 2)$
6	CBA	$(1\ 3)$

The space of possible votes, or profiles, is now just a linear combination of the rankings. The profile space for n candidates is an $n!$-dimensional S_n-module over $\mathbb{C}$, that is a vector space with multiplication naturally defined like that in the group ring. A profile is an element of the profile space, which we can identify with an $n!$-dimensional vector where the i^{th} component is the number of votes attributed to σ_i. Formally, the profile space has the following simple definition.

Definition 3.1.3. The profile space, Ω_n is the group ring $\mathbb{C}S_n$. Elements of Ω_n are called profiles.

Using the lexicographical ranking is arbitrary, but makes computer coding a bit easier, and it gives a standard basis for Ω_n.

A good way to think of an element of Ω_n, isomorphic to $\mathbb{C}S_n$ is as an entire poll, where each entry is the number of votes for the corresponding ranking. Note that complex numbers of votes are allowed. This is to allow the structure of representation theory to operate smoothly. Never, however, will this paper use anything but real values for the number of votes. In fact, although in this paper it is written as the complex field $\mathbb{C}$, the rational field $\mathbb{Q}$ is sufficient in all theorems here when the group considered is the symmetric group. [James and Kerber (1981)] The notation $\mathbb{C}$ is kept despite its unnecessary extra elements for the purposes of appearing in the form of the statements of theorems.

With the votes in place, now it is necessary to examine how to form a procedure to count the votes. In the same way as Saari and Borda, we define weighting vectors.

Definition 3.1.4. A weight vector $\mathbf{w} = [w_1 \; w_2 \; \ldots \; w_n]$ is a vector in $\mathbb{R}^n$ such that $1 = w_1 \geq w_2 \geq \cdots \geq w_n = 0$.

The fact that the vector $\mathbf{w}$ begins at 1 and ends at 0 is a convention. Given a vector beginning at 7 and ending at 5 would work just as well, since it could be converted linearly to a proper weighting vector by subtracting five then dividing by two.

Let $\mathbf{w}_\sigma$ denote the vector $[w_{\sigma(1)} \; \ldots \; w_{\sigma(n)}]$, not necessarily a weight vector. Also let $M_\mathbf{w}$ be the matrix with permutations of the weight vector $\mathbf{w}$ as its columns. That is $M_\mathbf{w} = [w_{\sigma_1} \; w_{\sigma_2} \; \ldots \; w_{\sigma_{n!}}]$, and we can write $M_\mathbf{w} : \Omega_n \to \mathbb{C}^n$. $M_\mathbf{w}$ is called the positional map with weighting vector $\mathbf{w}$.

The arbitrary form of the three candidate positional map is given by the following matrix where $0 \leq s \leq 1$.

$$\begin{bmatrix} 1 & 1 & s & 0 & s & 0 \\ s & 0 & 1 & 1 & 0 & s \\ 0 & s & 0 & s & 1 & 1 \end{bmatrix}$$

Throughout the thesis, whenever a ρ or τ is used, it will always indicate a particular element of the profile space. Define ρ to be the cycle $(1 \; 2 \; \ldots \; n)$ and τ to be the product of transpositions $(1 \; n)(2 \; n-1)\ldots(\frac{n}{2} \; \frac{n}{2}+1)$ if n is even or $\tau = (1 \; n)(2 \; n-1)\ldots(\frac{n-1}{2} \; \frac{n+3}{2})$ if n is odd. To denote the cyclic group generated by ρ, P is used.

3.2 New Definitions

Within Ω_n there exists an S_n-invariant subspace T, determined by a weighting vector such that for every vote p in T, $M_\mathbf{w} p$ is a complete tie among the n candidates. Thus T can be decomposed into $NS(M_\mathbf{w}) \oplus \mathbf{1}_{n!}$, where $\mathbf{1}_{n!}$ is the space spanned by the profile containing exactly one vote per candidate and $NS(A)$ is the null space of the matrix A.

Definition 3.2.1. A *Latin square*, denoted by λ_i is a profile in T with n votes distributed such that any candidate in any position in a ranking can be found in that profile exactly once. The coefficients on the group elements must be either 0 or 1.

Thus $ABCDE + BADEC + CDEBA + DEACB + ECBAD$ is a Latin square. Because this method of transcription is relatively tedious and gives little insight, this example is recopied below into the form of a typical looking Latin square:

Latin Square
ABCDE
BADEC
CDEBA
DEACB
ECBAD

A space, Λ, can be formed by taking the span of the collection of the Latin squares. An interesting combinatorial question at this point is to ask the dimension of Λ.

Saari indirectly uses Latin squares of a rather restricted type to form what he calls the Condorcet space. Here, these will be called picky Latin squares and will be generable from a single vote. The definition is for the sake of formality; the following example may prove more helpful. Recall $P = \langle \rho \rangle$.

Definition 3.2.2. A picky Latin square is a profile in T generated by a given ranking. The picky Latin square generated by $X_{\sigma(1)} \succ X_{\sigma(2)} \succ \ldots \succ X_{\sigma(n)}$ is $\ell_\sigma = [\ell_1\ \ell_2\ \ldots\ \ell_{n!}]$ where $\ell_i = 1$ if $\sigma_i \in \sigma P$ or else 0.

The example of the previous Latin square was not picky, but this example is. Pick σ to be the cycle $(2\ 4\ 5)$ in S_5, making

$$\ell_\sigma = (2\ 4\ 5) + (1\ 4\ 2\ 3\ 5) + (1\ 3\ 2\ 5\ 4) + (1\ 5\ 3) + (1\ 2)(3\ 4).$$

In the more convenient box notation:

$\ell_{(2\ 4\ 5)}$
ADCEB
DCEBA
CEBAD
EBADC
BADCE

In talking about invariant spaces, and generating sets, $\langle S \rangle$ denotes the smallest S_n-invariant space containing the set S.

We now define the Condorcet Space as:

$$C = \langle \{\ell_\sigma - \ell_{\sigma\tau}\} \rangle .$$

The algebraic proof of $dim\, C = \frac{1}{2}(n-1)!$ is found in Proposition 4.1.3, in contrast to the more geometric proof of Saari. In [Saari (1999)], Saari proved that within the Condorcet Space is a subspace responsible for all intransitivity paradoxes. This sort of paradox occurs in pairwise voting, which is not covered in this paper. The proof of the dimensionality of the Condorcet Space is included, however, because it represents a translation of an economic theory into algebra.

Containing C is a larger space related to Latin squares.

$$D = \left\langle \{\lambda_i - \lambda_j\} \right\rangle .$$

Another space of great use toward decomposition of Ω_n is the space of profiles which are annihilated by any positional map. A more precise way to write that is

$$E = \bigcap_{\mathbf{w}} NS(M_{\mathbf{w}}).$$

Again, this means that for any profile $p \in E$ and any weighting vector $\mathbf{w}$, then $M_{\mathbf{w}}p = \mathbf{0}$. This has applications to economists and others, since the projection of a profile onto this space may be neglected when tallying positional elections.

Now that the definitions, new and old, have been established, one can look at some containment, generalizations, and decompositions of spaces.

Chapter 4

Application of Representation Theory

4.1 Early Work and Theorems

Ultimately it is a goal to show that the space of linear combinations of Latin squares is the entire space of profiles which will be ties under any positional map. The first step is to show that the combinations of Latin squares is the same as the direct sum of the trivial representation and differences of Latin squares. Recall it is already known that $D \oplus \mathbf{1}_{n!} \leq \Lambda$ by virtue of the definitions of all the terms involved.

Examining the interchangeability of labels of candidates, the following remark needs no proof.

Remark. $\sum \lambda_i$ is a scalar multiple of $\mathbf{1}_{n!}$.

With that said, one is naturally led to the question of double containment of Λ and $D \oplus \mathbf{1}_{n!}$.

Lemma 4.1.1. $\Lambda \leq D \oplus \mathbf{1}_{n!}$.

Proof: It suffices to show that any Latin square is an element of $D \oplus \mathbf{1}_{n!}$, as their span is all of Λ. Let λ_j be a Latin square and set α to be intrinsically defined as $\sum_i \lambda_i = \alpha \mathbf{1}_{n!}$. It should be fairly easy to notice that there are $\frac{\alpha}{n}$ distinct Latin squares. Call this number m. Then

$$m\lambda_j = m\lambda_j - \alpha\mathbf{1}_{n!} + \alpha\mathbf{1}_{n!} = m\lambda_j - \sum_i \lambda_i + \alpha\mathbf{1}_{n!} = \sum_i \lambda_j - \lambda_i + \alpha\mathbf{1}_{n!}.$$

Now it is seen that λ_j is in $D \oplus \mathbf{1}_{n!}$, so $\Lambda \leq D \oplus \mathbf{1}_{n!}$.□

Corollary. $\Lambda = D \oplus \mathbf{1}_{n!}$.

Now the tools are in place to show that the Latin squares are in the complete tie space.

Proposition 4.1.2. $D \leq E$.

Proof: Let $\sum \alpha_{ij}(\lambda_i - \lambda_j)$ be an arbitrary element of D. Now map this element by $M_{\mathbf{w}} \sum \alpha_{ij}(\lambda_i - \lambda_j)$. This equals $M_{\mathbf{w}} \sum \alpha_{ij}\lambda_i - M_{\mathbf{w}} \sum \alpha_{ij}\lambda_j$ which in turn equals $\sum \alpha_{ij} M_{\mathbf{w}} \lambda_i - \sum \alpha_{ij} M_{\mathbf{w}} \lambda_j$. Knowing that the multiplication of the map by a Latin square gives us

$$\sum \alpha_{ij}(\mathbf{w} \cdot \mathbf{1}_n)\mathbf{1}_n - \sum \alpha_{ij}(\mathbf{w} \cdot \mathbf{1}_n)\mathbf{1}_n = \mathbf{0}$$

for any $\mathbf{w}$. Hence for all $d \in D$, $d \in E$, so $D \leq E$.□

The next paragraph can potentially begin the proof of the following corollary. In this way, it motivates further research into Latin squares and their connections with representation theory.

Let $\mathbf{w}_i = [1\ 1 \ldots 1\ 0 \ldots 0]$ where there are i 1's and e is a non-zero element in E. Then since $e \in N_{\mathbf{w}}$ for any weight vector $\mathbf{w}$, $M_{\mathbf{w}_i} e = \mathbf{0}$ for every i. Rewriting as

$$e = \sum \alpha_\sigma (X_{\sigma(1)} \succ \ldots \succ X_{\sigma(n)}),$$

it should be noticed that $M_{\mathbf{w}_1} e = \mathbf{0}$ implies every candidate gets first place the same number of times, which is zero total. Continuing this process yields the fact that every candidate gets i^{th} place the same number of times, zero total. This property is shared by everything in D, so this might lead one to conjecture the following statement.

Conjecture 4.1.1. $E \oplus \mathbf{1}_{n!} = \Lambda$.

As a side note, both Ω_3 and Ω_4 are evidence for the conjecture. The calculations to determine the number of linearly independent Latin squares on five candidates require finding the rank of a matrix with 120 columns.

To get even more of a handle on the complete tie space, the element ρ is used to provide information about the Condorcet space and the alternating representation, $S^{(1,1,\ldots,1)}$, denoted A for simplicity.

Recall once again that $\langle \rho \rangle = P$. With $\sigma_i \in S_n$, let $\Pi_i = \sigma_i P \cup \sigma_i \tau P$ such that

$$\biguplus \sigma_i P \cup \sigma_i \tau P = S_n$$

where $\biguplus$ is the disjoint union. Let $\mathscr{B}_C = \{(\sigma_i - \sigma_i \tau) \sum \rho^j\}$ This means that $|\mathscr{B}_C|$ is the number of $\Pi_i = \frac{|S_n|}{2|P|} = \frac{1}{2}(n-1)!$. Since $\mathscr{B}_C$ is a basis for C, the Condorcet space has dimension $\frac{1}{2}(n-1)!$.

The next two propositions concern the location of the alternating representation in relation to the Condorcet space. First, note that when the number of candidates is divisible by 4 then A is not a subspace of the Condorcet space.

Proposition 4.1.3. *If* $n \equiv 0 \ (mod\ 4)$ *then* $A \not\leq C$.

Proof: Recall $\tau = (1\ n)(2\ n-1)\ldots(\frac{n}{2}\ \frac{n}{2}+1)$ so $sgn\ \tau = 1$ when $n \equiv 0\ (mod\ 4)$. Similarly $\rho = (1\ 2 \ldots n)$ gives $sgn\ \rho = 1$. Let $\mathbf{a} \in A$. Now suppose that $\mathbf{a} \in C$. Then (Without loss of generality, each α_σ can be positive, so let it be as such.)

$$\mathbf{a} = \sum \alpha_\sigma(\ell_\sigma - \ell_{\sigma\tau}) = \sum \alpha_\sigma \ell_\sigma - \sum \alpha_\sigma \ell_{\sigma\tau}$$
$$= \alpha_\sigma \left(\sum(\sigma - \sigma\tau)\right)\left(\sum \rho^i\right) = \alpha_\sigma \left(\sum \sigma\right)(1-\tau)\left(\sum \rho^i\right) = \mathbf{a}_e - \mathbf{a}_o$$

Here, $\mathbf{a}_e$ is comprised of only the even component of $\mathbf{a}$ and similarly, $\mathbf{a}_o$ is the odd component.

This means $\alpha_\sigma(\sum\sigma)(\sum\rho^i) = \mathbf{a}_e$. This in turn implies that each σ in the indexing set must be even since $sgn\ \rho^i = 1$ for every i. But then $\alpha_\sigma(\sum\sigma)\tau(\sum\rho^i) = \mathbf{a}_o$ implies that each σ must have odd sign. This is a contradiction, thus completing the proof. □

This proposition guarantees that the alternating representation is a subspace of C whenever the number of candidates is equivalent to 2 modulo 4.

Proposition 4.1.4. *If* $n \equiv 2\ (mod\ 4)$ *then* $A \leq C$.

Proof: If $n \equiv 2\ (mod\ 4)$ then $sgn\ \tau = -1$. Consider

$$\sum_{\sigma \in A_n} (\ell_\sigma - \ell_{\sigma\tau}) = \left(\sum_{\sigma \in A_n} (\sigma - \sigma\tau)\right)\left(\sum \rho^i\right)$$

$= \mathbf{a}\sum\rho^i = \mathbf{a}\rho + \cdots + \mathbf{a}\rho^n = n\mathbf{a} \in A$. Thus $A \leq C$.□

With such a large chunk of Ω_n occupied by the complete tie space, Saari proves that there is an $(n-1)$-dimensional space determining the first place winner in the outcome of an election [Saari (2000b)]. This space, which Saari termed, "Basic," can be generalized to say that a Basic space B is any (except $\mathbf{1}_{n!}$) for which $M_{\mathbf{w}}b \neq 0$ for all $b \in B$.

Any Basic space corresponds to the tableau with $\lambda = (n-1,1)$, which dominates all but the trivial representation, so by Schur's Lemma all the remaining spaces are sent to $\mathbf{0}$.

Saari's Basic space is given by:

$$B = span\left\{\sum_{\sigma(1)=i} \sigma - \sum_{\sigma(n)=i} \sigma\right\}_{i=1}^{n-1}.$$

All Basic Spaces, naturally including Saari's, are isomorphic. This fact will be shown in Section 3 of this chapter. A Basic Space can be found for any weighting vector by finding the orthogonal complement of a null space corresponding to a weighting vector. If $\{\,^{\circ}{}_i\}$ is a basis for the null space of $M_{\mathbf{w}}$ then the corresponding Basic, $B_{\mathbf{w}}$ is $NS([\mathbf{1}_{n!}\ v_1 \ldots v_{n!-n}]^T)$. This works because of the Specht Module reasoning below.

Using Specht Modules it is known that $M_{\mathbf{w}} : M^{(1,1,\ldots,1)} \to M^{(n-1,1)}$ which in conjunction with Schur's Lemma means that only $S^{(n)}$ and $S^{(n-1,1)}$ can be mapped to be orthogonal to the null space. Of course, $M_{\mathbf{w}}\mathbf{1}_{n!} = (\mathbf{1}_n \cdot \mathbf{w})(n-1)!\mathbf{1}_n$ which is not $\mathbf{0}$. Since *rank* $M_{\mathbf{w}} = n$, there are $n-1$ more dimensions to fill in the domain such that $M_{\mathbf{w}}$ is onto. Thus there is a unique module $S^{(n-1,1)}$ such that $M_{\mathbf{w}}b = \mathbf{0}$ implies $b = \mathbf{0}$ for all $b \in S^{(n-1,1)}$.

4.2 Profile Decomposition of Ω_3

To analyze a profile, the simplest method is to project onto intuitive orthogonal spaces. This allows one to see each piece that affects the election then sum them up to find the overall profile again. Each of these orthogonal spaces should be S_n-invariant, again because the "naming" of candidates is irrelevant in economics. The orthogonal spaces being invariant leads to them being representations of the group. For the case of S_3, there should be four irreducible spaces, two 1-dimensional, and two 2-dimensional.

Recall from Chapter 3 that the arbitrary positional map $M_{\mathbf{w}}$ looks like this, with respect to the lexicographic order, where $0 \le s \le 1$:

$$\begin{bmatrix} 1 & 1 & s & 0 & s & 0 \\ s & 0 & 1 & 1 & 0 & s \\ 0 & s & 0 & s & 1 & 1 \end{bmatrix}$$

Since the matrix, $M_{\mathbf{w}}$, has nullity 3 and $M_{\mathbf{w}}\mathbf{1}_6 = 2(\mathbf{1}_3 \cdot \mathbf{w})\mathbf{1}_3$, there must be a one dimensional space and a two dimensional space comprising the $NS(M_{\mathbf{w}})$.

In the 3-candidate race, the space of complete ties, E is the one dimensional alternating representation. That is

$$E = A = span\left\{ \sum_{\sigma \in S_n} sgn(\sigma)\sigma \right\}$$

or alternatively

$$E = A = span\left\{ \sum_{\sigma \in A_n} \sigma - \sum_{\sigma \in (1\ 2)A_n} \sigma \right\},$$

where A_n is the alternating group. For any n-dimensional case, the alternating representation will always be a subspace of E.

The remaining two 2-dimensional spaces are given in the form of Saari's Basic space, and the orthogonal complement of the direct sum of the other spaces.

$$B = span\left\{ \sum_{\sigma(3)=1} \sigma - \sum_{\sigma(3)=2} \sigma, \sum_{\sigma(3)\neq 3} \sigma - 2 \sum_{\sigma(3)=3} \sigma \right\}$$

$$F = span\{2(1) - 2(1\,2) + (2\,3) - (1\,3\,2) + (1\,2\,3) - (1\,3), (1\,2\,3) + (1\,3\,2) - (1\,3) - (2\,3)\}.$$

Saari calls F the Reversal Space, but that terminology is avoided here, as it stems from geometrical reasoning, and from an algebraic standpoint, F is also a Basic Space.

4.3 New Generalizations

We will now define a more generalized notion of Saari's basic space. The generalization will carry the same intuition with it, so we may decompose the isotypic component of Basic spaces with great ease.

Definition 4.3.1. The basic vector for candidate X_i and position j is given by

$$b_{i,j} = \sum_{\sigma(j)=i} \sigma - \sum_{\sigma(n)=i} \sigma$$

where i and j are restricted by $1 \leq i, j \leq n-1$.

The basic space corresponding to position j is given as $B_j = \left\langle \{b_{i,j}\}_{i=1}^{n-1} \right\rangle$. Setting $j = 1$ gives Saari's basic space, with the $b_{i,1}$ vectors as Saari's "Borda Profiles" [Saari (2000b)].

Proposition 4.3.1. *B_j is a S_n-invariant space.*

To see this, we examine the action of an arbitrary element of S_n on a basis vector, $b_{i,j}$, of the basic space B_j. Before showing this fact, note that every candidate, except X_n has a basic vector corresponding to it. It is convenient to the proof that $b_{n,j}$ is defined to be linearly dependent on the others in the following way:

$$b_{n,j} = -\sum_{i=1}^{n-1} b_{i,j}.$$

Since any element of S_n can be written as a product of transpositions, it is enough to show that B_j is closed under the action of any transposition. So long as $i \neq l$ and $i \neq k$,

$$(l\ k) \cdot b_{i,j} = b_{i,j}.$$

If, however $l = i$ then we have

$$(i\ k) \cdot b_{i,j} = b_{k,j}.$$

This leads naturally to the identity

$$\sigma \cdot b_{i,j} = b_{\sigma(i),j}$$

which is in B_j. It is now necessary to show that each B_j is actually an irreducible invariant module. Using the standard technique of creating a matrix representation with the $b_{i,j}$ as basis vectors, the character of B_j becomes obvious.

Theorem 4.3.2. *The character of any basic space is given by $\chi_{B_j}(\sigma) = k-1$ where σ fixes k letters. This character is irreducible.*

There are two proofs to this theorem provided in Appendix A. Both are left there because they are technical and provide little insight. The first proof uses the notions of Specht Modules, while the second is combinatorial in nature, which may be more pleasing to an audience not as familiar with Specht Modules and Young Tableaux.

Remark. With $\chi_{B_j}(1) = n-1$, the dimension of the space is $n-1$.

Because the character of the space is irreducible, the space itself is irreducible.

Given the $n-1$ isomorphic irreducible invariant spaces, it remains to be shown that they are in fact distinct. In the proof of this, we will need to create the weighting vector $\mathbf{q} = [1\ 1\ \dots\ 1\ 0]$ (all 1's with a final 0), which leads to the next statement.

Proposition 4.3.3. *For any candidate X_i and position j, the following identity holds:*

$$M_{\mathbf{q}} b_{i,j} = (n-2)!(ne_i - \mathbf{1}_n)$$

where e_i is the i^{th} standard basis vector.

Proof: Note that $M_{\mathbf{q}} b_{i,j}$ gives the same number of votes to all candidates not X_i. This should be evident by the symmetry of $b_{i,j}$ on all candidates not X_i. By elementary counting arguments, one can see that $M_{\mathbf{q}} b_{i,j}$ awards X_i exactly $(n-1)!$ points in the tally. So far, $M_{\mathbf{q}} b_{i,j}$ can be written as $(n-1)!e_i + \sum_{k \neq i} \alpha e_k$ where α is some number depending only on n. Because there were zero total votes cast in $b_{i,j}$, there must be zero total points awarded, giving the equation:

$$(n-1)\alpha + (n-1)! = 0.$$

Solving for α, this gives $\alpha = -(n-2)!$. Now this gives

$$M_{\mathbf{q}} b_{i,j} = (n-1)! e_i - (n-2)! \sum_{k \neq i} e_k$$

which can be simplified (by adding zero in the form of $(n-2)e_i - (n-2)e_i)$ to arrive at

$$(n-2)!(ne_i - \mathbf{1}_n)$$

thus completing the identity. □

The most recent proposition is sufficient to show that no $b_{i,j}$ is in E.

Theorem 4.3.4. *For any $j_1 \neq j_2$, the basic spaces B_{j_1} and B_{j_2} coincide only at $\mathbf{0}$.*

Proof: Suppose $B_{j_1} \cap B_{j_2} \neq \mathbf{0}$ for $j_1 \neq j_2$. Then because the spaces are irreducible, $B_{j_1} = B_{j_2}$. A basis vector for B_{j_2} can thus be written as a linear combination of basis vectors for B_{j_1}. Secondly, $M_{\mathbf{q}} b_{i,j_2} = M_{\mathbf{q}} b_{i,j_1}$ follows from the proposition above. Naturally it follows that $M_{\mathbf{q}}(b_{i,j_2} - b_{i,j_1}) = \mathbf{0}$. Thus $\langle \{b_{i,j_2} - b_{i,j_1}\} \rangle \leq NS(M_{\mathbf{q}})$. But $\langle \{b_{i,j_2} - b_{i,j_1}\} \rangle \leq B_{j_1}$ also. Thus we find that anything in this space must be $\mathbf{0}$ or that $B_j \leq NS(M_{\mathbf{q}})$. From the remark, $M_{\mathbf{q}} b_{i,j} \neq \mathbf{0}$, so $b_{i,j_2} - b_{i,j_1} = \mathbf{0}$. This would imply that the following equation be true:

$$\sum_{\sigma(j_1)=i} \sigma = \sum_{\sigma(j_2)=i} \sigma.$$

But that is clearly a contradiction so long as $j_1 \neq j_2$. Because j_1, j_2, and i were arbitrary, all the B_j are distinct. □

With the B_j distinct, we can now say that the isotypic component, known to be of dimension $(n-1)^2$ by the theorems of Chapter 2, does not intersect E, except trivially, the space of annihilation under any $M_{\mathbf{w}}$. Another way of saying this is that every profile can be projected onto the Basic component in a way which does not lose any information vital to positional voting.

The last theorem of this section merely states the dimensionality of E.

Theorem 4.3.5. *The dimension of $E = \cap NS(M_{\mathbf{w}})$ is exactly $n! - (n-1)^2 - 1$.*

Proof: As already seen, $dim\, E = n! - 1 - \alpha(n-1)$ where α is the number of Basic spaces which are not subspaces of E. Given now that there are $n-1$ distinct Basic spaces, none of which are subspaces of E, and the fact that there are exactly $n-1$ isomorphic copies of the Basic space, α must be $n-1$. □

4.4 Profile Decomposition of Ω_n

The profile space Ω_n decomposes much the same way as Ω_3, but the pieces being accounted for are larger. According to positional voting, the space will decompose as

$$\Omega_n = \mathbf{1}_{n!} \oplus E \oplus \mathscr{B}.$$

Again $\mathbf{1}_{n!}$ is the trivial representation in this $n!$-dimensional case, E is the intersection of null spaces for all $M_{\mathbf{w}}$ and $\mathscr{B}$ is the isotypic component of Basic spaces.

When $n = 4$, the dimensions of each space are 1 for the trivial representation, 14 for E, and 9 for $\mathscr{B}$. Although there are infinitely many bases for $\mathscr{B}$, the most appropriate basis, considering positional voting, uses the general positional Basic space basis given above.

Chapter 5

Conclusion

5.1 Results

$$\sum_{k=1}^{p}\left(\frac{k}{p}\right)\varepsilon^{k}=\sqrt{p}$$

This thesis provided a decomposition of the profile space of n voters according to the positional voting method. Closed forms for the dimensions of all spaces involved are deduced, and the bases for the Basic spaces in particular are concise and intuitive. The Basic space corresponding to first place matches with Saari's, and the others are isomorphic with similar descriptions. Each Basic space carries with it the information to declare who won the position corresponding to the space. Thus, to find out who received second place in an election, one needs only to to examine B_1 and B_2.

In terms of combinatorics, a connection is made between Latin squares and voting theory. Moreover, bringing the symmetric group into the mix raises questions about partitions of natural numbers. Nowhere before has anyone recorded this particular equivalence class of Latin squares, which means that the vector spaces spanned by them is novel.

5.2 Impact

Clearly voting theory has impact in political science and economics, but also in and around any social organization. Voting has become a staple of daily life in modern society, and it is important to realize the possibilities for paradoxes when deciding who will stay at what hotel or who the next math chair will be. Computation of large candidate voting can be tedious if done using a positional scheme, but with the knowledge that the result is the same having projected onto the Basic space, smaller and fewer numbers may be used to tally these votes.

Representation theory itself lends itself the study of groups, as well as areas of physical chemistry. Working with Latin squares may provide additional contributions to these fields in addition to combinatorics.

The decomposition of pairwise schemes is a start to the general decomposition of $\mathbb{C}S_n$ and aids in understanding the breakdown of the space under different voting methods. Condorcet's method carries with it a subspace of E which survives the pairwise map, to influence the vote and potentially cause paradoxes or other interesting outcomes.

With more decompositions, it is foreseeable that there is a space in which paradoxes are fewest in some sense. Using algebraic methods can quickly sift through much of the valueless data and cut straight to the underlying problematic areas.

5.3 Open Questions

Naturally there are many questions left about voting theory. Some of these include investigating properties of the pairwise map, specifically the portion of Saari's Condorcet space that actually survives the mapping. Other questions involve the probability of paradox given a number of voters, or examining other voting mechanisms.

A conjecture of this paper was the dimensionality of Λ, the space of Latin square combinations. If $dim\ \Lambda = n! - (n-1)^2$, then it is the case that there is an interesting categorization for any vote which tallies to a complete tie, no matter the weight. Such a vote is comprised of exactly Latin squares.

Still in question is an extension of Saari's bold claim that his Basic space is the only to matter to positional schemes. This is true when considering only first place, but to know who received second, third, or n^{th} is left to the other Basic spaces, defined here. It is suspected that the projection of a vector onto B_j carries all the information necessary to declare who received j^{th} place.

Regardless or any new breakthroughs or further decompositions, as long as Arrow's axioms are accepted, voting will never be paradox free, so there will remain questions so long as people vote.

Appendix A

This appendix contains two proofs of the irreducibility of an arbitrary Basic Space. The first is due to Sagan. The second is an original work by the author and E. Segarra.

As in the case of a Basic Space, the tableau corresponding to the surviving module is that of $\lambda = (n-1,1)$. Any one tabloid of this can be written as:
$[t] = \begin{matrix}\mathbf{i}\cdots\mathbf{j}\\ \mathbf{k}\end{matrix}$ thus $[t]$ can be identified by only $\mathbf{k}$. This tabloid has $e_t = \mathbf{k} - \mathbf{i}$, and the span of such vectors is given by

$$S^{(n-1,1)} = \{c_1\mathbf{1} + \cdots + c_n\mathbf{n} | c_1 + \cdots + c_n = 0\}.$$

A basis of $S^{(n-1,1)}$ is chosen as

$$\{\mathbf{2}-\mathbf{1}, \mathbf{3}-\mathbf{1}, \ldots, \mathbf{n}-\mathbf{1}\}.$$

Computing the action of a permutation on the basis, gives the character to be χ_{B_j}. Knowing that $S^{(n-1,1)}$ is irreducible and has the appropriate character means that the Basic Space, which shares the same character, is also irreducible. $\square$

Here is the combinatorial approach to the problem.

Let χ be the character in question, namely

$$\chi(\sigma) = k - 1$$

where σ fixes k elements of the symmetric group.

This proof requires four lemmas to directly compute the inner product of this character with itself.

Lemma .0.1. *For any α greater than n, the following identity holds:*

$$\sum_{k=0}^{n} k^{\alpha} \binom{n}{k} (-1)^{n-k} = 0.$$

Proof: Base Case of $\alpha = 0$ is proved by the binomial theorem. Suppose the identity is true for some α. Then

$$\sum_{k=0}^{n} k^{\alpha+1} \binom{n}{k} (-1)^{n-k} = \sum_{k=0}^{n} k^{\alpha} n \binom{n-1}{k-1} (-1)^{n-k}$$

by the common identity: $k\binom{n}{k} = n\binom{n-1}{k-1}$. This in turn equals

$$n\left[\sum_{k=0}^{n} k^{\alpha} \binom{n}{k} (-1)^{n-k} - \sum_{k=0}^{n} k^{\alpha} \binom{n-1}{k} (-1)^{n-k}\right].$$

By the induction hypothesis this is

$$-n \sum_{k=0}^{n} k^{\alpha} \binom{n-1}{k} (-1)^{n-k}.$$

Splitting the sum where $k = n$ and removing a -1, this is

$$n\left[\sum_{k=0}^{n-1} k^{\alpha} \binom{n-1}{k} (-1)^{n-k-1} + 0\right].$$

This equals zero by the induction hypothesis again. □

Lemma .0.2. *For any n the following identity holds:*

$$n^2 = \sum_{k=0}^{n} (k-1)^2 \binom{n}{k-1} (-1)^{n-k}.$$

Proof: Consider an n by n grid. How many ways are there to pick a square? Clearly there are n^2. Now consider counting the squares one can choose after restricting choices in certain columns and rows. For example, how many ways can you choose a spot that is not in the first row or first column? There are $n-1$ choices for both row and column, so there are $(n-1)^2$. There are $\binom{n}{1} = \binom{n}{n-1}$ rows/columns to restrict. But this leads to over counting. One must subtract out the cases where both row/column 2 and row/column 3 are restricted. Now there are $(n-2)^2$ places left to choose and $\binom{n}{2} = \binom{n}{n-2}$ pairs of rows/columns to restrict. Continue this for the triples, and so on. Eventually the following equation is obtained.

$$(n-1)^2\binom{n}{n-1} - (n-2)^2\binom{n}{n-2} + \cdots \pm 2^2\binom{n}{2} \mp 1^2\binom{n}{1}$$
$$= \sum_{k=2}^{n} (k-1)^2 \binom{n}{k-1} (-1)^{n-k}.$$

When $k = 0$ or $k = 1$ then the product comes out to zero, so this can be extended to $\sum_{k=0}^{n} (k-1)^2 \binom{n}{k-1} (-1)^{n-k}$. Since these two formulas count the same thing, they must be equal. □

Lemma .0.3. *The following identity holds:*

$$n^2 = \sum_{k=0}^{n} (k-1)^2 \binom{n+1}{k} (-1)^{n-k}.$$

Proof: From the previous lemma,

$$n^2 = \sum_{k=0}^{n} (k-1)^2 \binom{n}{k-1} (-1)^{n-k}.$$

By Pascal's identity, this sum is the same as:

$$\sum_{k=0}^{n} (k-1)^2 [\binom{n+1}{k} - \binom{n}{k}] (-1)^{n-k}.$$

Splitting up the sum yields:

$$\sum_{k=0}^{n} (k-1)^2 \binom{n+1}{k} (-1)^{n-k} + \sum_{k=0}^{n} (k-1)^2 \binom{n}{k} (-1)^{n-k}.$$

By an obvious extension of the first lemma, the second sum is zero. Arriving at

$$n^2 = \sum_{k=0}^{n} (k-1)^2 \binom{n+1}{k} (-1)^{n-k}$$

concludes the proof. □

Lemma .0.4. *The next identity holds for any n and D_m denotes the number of derangements on m items:*

$$n! = \sum_{k=0}^{n} (k-1)^2 \binom{n}{k} D_{n-k}.$$

Proof: Base case of $n = 2$ is done by simple arithmetic. Now suppose the identity is true for some n. Consider

$$\sum_{k=0}^{n+1} (k-1)^2 \binom{n+1}{k} D_{n+1-k}$$

which by separating out the final term is

$$\sum_{k=0}^{n} (k-1)^2 \binom{n+1}{k} D_{n+1-k} + n^2 \binom{n+1}{n+1} D_0.$$

Replacing by the single step recursive identity of D_m yields:

$$\sum_{k=0}^{n}(k-1)^2\frac{(n+1)!}{k!(n+1-k)!}[(n-k+1)D_{n-k}+(-1)^{n-k+1}]+n^2.$$

Further simplification gives the expression:

$$(n+1)\sum_{k=0}^{n}(k-1)^2\binom{n}{k}D_{n-k}-\sum_{k=0}^{n}(k-1)^2\binom{n+1}{k}(-1)^{n-k}+n^2.$$

Then using the induction hypothesis and the previous lemma:

$$(n+1)n!-n^2+n^2=(n+1)!.$$

Thus by induction, the claim of the lemma is proved. □

Theorem .0.5. *Let $\chi(\sigma)=k-1$ where $\sigma\in S_n$ fixes exactly k letters. Then χ is irreducible.*

Proof: Recall character inner products and examine that of χ with itself.

$$<\chi,\chi>=\frac{1}{|S_n|}\sum_{\sigma\in S_n}\chi(\sigma)\overline{\chi(\sigma)}.$$

Because χ is real valued, this simplifies to

$$\frac{1}{n!}\sum_{\sigma\in S_n}\chi(\sigma)^2.$$

The second sum in the next line is over the σ which fix exactly k letters. So it should be clear that the inner product equals

$$\frac{1}{n!}\sum_{k=0}^{n}\left(\sum\chi(\sigma)^2\right)=\frac{1}{n!}\sum_{k=0}^{n}\left(\sum(k-1)^2\right).$$

Because χ is constant over those sigma, let α_k be the number of $\sigma\in S_n$ such that σ fixes exactly k letters. Now it is the case that

$$\frac{1}{n!}\sum_{k=0}^{n}(k-1)^2\alpha_k.$$

Now it is easy to see that $\alpha_k=\binom{n}{k}D_{n-k}$. This is seen by picking k of the n elements to fix. Three are $\binom{n}{k}$ ways of doing this. Then derange the rest. There are D_{n-k} ways of doing that. Finally, one has that

$$<\chi,\chi>=\frac{1}{n!}\sum_{k=0}^{n}(k-1)^2\binom{n}{k}D_{n-k}=1$$

by the previous lemma. The inner product being 1 means that χ is irreducible. □

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