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What's All the Fuss About?

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The proposal to hold this conference says that, "the teaching of calculus is in a state of disarray and near crisis . . . [with a] failure rate of nearly half at many colleges and universities." An alarm was sounded earlier by the January, 1985 AMS/MAA joint panel, "Calculus instruction, crucial but ailing" [1].

This came as a surprise to me. Why is the teaching only of calculus under scrutiny? Are we doing such a wonderful job with discrete mathematics, linear algebra, differential equations, complex variables, or upper division algebra? Perplexed, I asked some of my colleagues, good mathematicians and fine teachers all, "What's your impression of the teaching of calculus, here and elsewhere?" One professor suggested that we might drop a couple of topics, maybe some integration techniques. Another said, we should meet five times a week instead of four but he doesn't want to. Finding no sense of calamity, I talked to colleagues in the physics and engineering departments. They liked what we do, but urged us to do more of it in the first quarter, especially differentials, vectors, $e^{i\theta}$, Stokes' theorem, and certain differential equations.

Then I went to the placement office, which helps undergraduates obtain summer internships and seniors get jobs. "What have you heard about calculus?" They were not aware that calculus is in disarray and ailing. I asked what employers were looking for. The answer was clear, "Students who can communicate orally and in writing, think, are not afraid of numbers, with a little touch of the computer." Still no complaint about calculus.

I asked my engineer son-in-law what he looks for when he recruits. His answer: "People who can deal with questions on their own." He seeks recommendations from a professor who regularly assigns his class a few open-ended problems. Though not hard in the sense that their solutions requires the insight of a genius, they are not directly related to the day's lesson. One year not one of the professor's three hundred students could solve his problems. My son-in-law did not blame calculus for this tragedy, though it was clear to me that we do little to prevent it.

So I picked up a calculus market research report that McGraw-Hill had done in 1981, based on a questionnaire sent to mathematics professors in over 200 colleges and universities of all sizes. According to the poll, 83 percent of the students in first semester calculus complete the three semester sequence. That was reassuring. Furthermore, if there was a feeling that something was wrong it should show up in the respondents' comments on the texts they were using. But of the 227 replies 170 judged their text's completeness to be "good" or "excellent" and only 47 called it "poor" or "adequate." They seemed quite satisfied with "topic sequence as well" with 173 out of 227 calling it "good" or "excellent."

In spite of these calming numbers, I still felt that there is indeed something in disarray in calculus teaching, something ailing. Whatever it is, we can't blame the publishers. The books they offer us respond to such polls; the manuscripts are read by a panel of independent, conscientious reviewers. We get the texts we ask for. The problem lies with us. Mathematics, the only discipline where all the cards can be laid on the table, and which therefore should be the best taught, is often among the worst taught subjects. One reason is that we haven't decided what we are teaching.

This uncertainty is visible, in the discussion, *The Introductory Mathematics Curriculum*, presented in [1]. There we find such statements as, "We must instead teach how to create mathematics" (R.W. Hamming, p. 388); "Even more essential is the creation of courses that focus on concepts. Ideas and problem solving are the really critical part" (Robert Davis, p. 391); "Our teaching fails to provide students with the Joy of using mathematics to cope with challenging problems" (Wade Ellis, p. 393); "The main fault of the introductory curriculum . . . is an issue of pedagogy as much as of the content" (Patrick Thompson, p. 394); "Curriculum change must be accompanied by severe ques-

tions of current teaching methods" (John Mason, p. 395). Though appearing as asides to the main debate, they call attention to what I feel is the central issue.

Before we propose the medicine, we had better agree on the diagnosis. The diagnosis depends on what we mean by "health," that is, what we are trying to accomplish in our introductory courses. That may depend to some extent on whether the course serves other majors or our own. (According to the McGraw-Hill poll, enrollment in the basic calculus runs about 60% physical science-engineering, 20 percent life science-biology-economics, 12 percent math, and 8 percent others.) In large schools the second group often has its own calculus sequence; at Davis, with its strong biological emphasis, more students enroll in the short calculus than in the engineering sequence. So the main calculus sequence we are talking about serves simultaneously engineers, physicists, computer scientists, and math majors. That is a boundary condition that any solution must satisfy. But it is not as restrictive as it may appear, since there seems to be a consensus that the students in these varied majors should learn to write, read, and think. The dean of computer scientists, E. W. Dijkstra, has written that the most important requirement for a computer scientist is mastery of his native tongue. And my computer-science colleagues urge us to expect well written answers and proofs in our sophomore course on sets, relations, functions, and induction.

But what about calculus, where the texts have settled into a fairly uniform table of contents? There are always a few sections that the instructor may delete, such as Kepler's laws or Lagrange multipliers. But the instructor could consider deleting some more topics, such as some formal integration techniques or even related rates. Authors have less choice, for if they omit someone's favorite topic, their books will not be adopted and soon will be out of print. After all, calculus committees meet in order to reject books, much in the same way that canneries sort tomatoes. Labelling a section "optional" will surely offend someone who feels his students will then not treat it seriously if he covers it. It seems that a calculus author has the freedom to make only two decisions: Where to put analytic geometry and whether the title should be *Calculus with Analytic Geometry* or *Calculus and Analytic Geometry*. Thus the major revolution in calculus texts in the last decade has been the introduction of a second color. (In high school texts, the number of colors has reached four.) Whatever proposals this conference may make, I predict calculus will begin with functions, limits, derivatives, extrema, integrals, the fundamental theorem, go on to more applications, series, and then reach at least partial derivatives and multiple integrals. Still, there are options, and perhaps this conference will encourage publishers and professors to be more flexible when developing a table of contents or a course syllabus.

The fundamental question is not, "Should discrete mathematics precede calculus, follow it, be woven into it, or be separate and simultaneous." The question should be, "What are we trying to do in calculus and discrete mathematic courses other than cover some definitions, facts, and algorithms?" If the answer is "nothing", then we make no basic changes. If we also want the student to learn to "think" (this is now called 'problem solving' and 'heuristics') and to write, then we should act accordingly. The last thing we should do is ask for texts that mix discrete mathematics and calculus, for invariably, when two subjects are put between the covers of one book either the book grows unacceptably large or one of the two is sacrificed to the other, or both are shortchanged. Witness the fate of analytic geometry in our calculus books or of both algebra and its applications in our applied algebra books.

My own proposals may appear mild. Indeed, the first one is, but the second could encourage a change in emphasis.

The first is specific, and concerns calculus and discrete mathematics. I suggest that a discrete course of a quarter or semester be available to freshman (if that is successful, then iater it could be extended). It could be taken simultaneously with beginning calculus, or alone, or, in the case of non-engineering students, with the calculus delayed. Such a course could help develop maturity and thus prepare students for calculus. It could, incidentally, weed out those who are not ready to go on. (All campuses of the University of California already require passing an exam on high school algebra and trig for entry to calculus.) It would also broaden the student's mathematical perspective earlier.

My second suggestion applies to our curriculum in

general and is a response to what I see as the disarray and the ailment. Implementing this suggestion does not require new courses, nor radically new texts. However, if enough of us act on this suggestion, we may provide the quorum to support certain changes in the texts.

It too is modest, for I find that proposals for abrupt major reform tend to be carried out in form but not in substance, or viewed as something for someone else to implement.

My suggestion is rooted in my definitions of the words "curriculum" and "syllabus." Usually, "curriculum" describes the courses offered and "syllabus" lists the topics in a course. Both "curriculum" and "syllabus" call attention to the material treated. They do not refer to the way it is treated and certainly they do not mention what should be our main goal: to develop the student's ability to read, analyze, write and speak. We easily lose sight of this objective, for facts tend to displace process. We see this bias both in the classroom and in texts. I hope that the reform suggested by this conference gives process at least equal billing with content. And I hope that authors maintain a similar perspective as they try to implement our recommendations.

My suggestion is only a modest step toward rescuing process from subservience to content.

I propose that in whatever course we teach we include a significant number of what might be called "open-ended" or "exploratory problems." Though not routine, they should not be difficult in the sense of a Putnam problem. I mean that when a student sees the solution, he will say "I should have gotten it." These problems should encourage experimentation and independent work. The answer should require the student to write coherent sentences. That means that the instructor or some other qualified person should read and evaluate what is turned in. He should demand suitable revision. The solution should not be in the solutions manual; it should not be closely tied to the particular section in the book that is being covered in class. The assignment should not be due the next day, so that the student will have time to mull it over.

Some examples will bring this proposal down to earth. To demonstrate my neutrality on the relative merits of calculus and discrete mathematics, I will choose some examples from both disciplines. I begin with examples that parallel the standard calculus.

Example 1: Let $f(x) = ax^2 + b$ be a polynomial of degree 2. Is there a polynomial g of degree 3 such that the two compositions, fog and gof, are equal?

Remarks: If the students have trouble, then you might suggest that they look at a specific f(x). Little in their earlier education has suggested such a bold step. The computations involve nothing more than cubing a quadratic or squaring a cubic. The algebra is not mysterious and the final result is both elegant and surprising. Moreover, the student should be urged to write the solution with more than a string of equations. We have a right to expect an introduction and a conclusion. We should demand that a sentence begins with capital letter and ends with a period. The left margin should be straighter than the right margin. The student may complain that such request are inappropriate in a math course. But that same student may one day be writing software manuals and internal memoranda. For us to demand less is to short-change our students.

Example 2: Are there continuous functions f such that $f(x + y) = f(x) + f(y)$ for all real numbers x and y?

Remarks: The student may or may not come up with some examples. You may have to steer him out of a rut. If he finds $f(x) = kx$, you might then ask, "Are there more?" (In a discrete course, the domain could be Z instead of R.) Of course one could also ask for solutions of $f(xy) = f(x) f(y)$.

Such exercises are usually delayed until the Junior year, but they are appropriate during the lower division courses as well. Perhaps we could delete a few topics from the standard curriculum, whether calculus or discrete mathematics, lowering the pressure so students would have more time for this type of problem.

Example 3: Let R be a bounded plane convex set. Is there a chord that bisects its area?

Remarks: For us this is a trivial exercise in the intermediate value theorem, but most students will need

help. They cannot turn back a couple of pages for the example that's just like this exercise. After this problem is solved one might ask whether there is a chord that bisects the area and the perimeter at the same time.

Example 4: What happens to x^y when x and y are near 0 but positive?

Example 5: Which polynomials of degree at most 3 have inflection points?

Remarks: Much is lost in a more conventional wording, such as, "Show that every polynomial of degree 3 has an inflection point." One might then ask about polynomials of degree 5.

Example 6: Let f be an increasing positive function on the interval [0,1]. What, if anything, can we say about the centroid of the region R under the graph of f and above [0,1]?

Remarks: A variant is to demand that f also be differentiable and concave down and ask about the centroid of its graph. Or we could ask whether there is any relation between the centroid of R and the centroid of the solid of revolution obtained by revolving R around the x axis.

Example 7: Let R be a bounded plane convex set and P_0 a point in R. Assume that each chord of R through P_0 has length at most a. What can be said about the area of R?

Remarks: This question ultimately takes the student back to the formula for area in polar coordinates and extrema problems. For a discussion of this example see [2].

Now for some illustrations in discrete mathematics.

Example 8: You could compute x with five multiplications by writing $x^6 = x(x(x(x(xx))))$. But you could also write $x^6 = (x^2x^2)x^2$, which requires only three distinct multiplications. (Assume that once a multiplication is done, the result remains available.) Investigate the smallest number of multiplications needed to compute x^n.

Remarks: The exact formula is not known, though eventually students can show, with the aid of an induction, that the number is at least $\log_2 n$ and equals $\log_2 n$ when n is a power of 2.

Example 9: In which linear graphs can we find a path that passes through each edge exactly once?

Remarks: This is usually given in the "theorem and proof" form, but I think it far more instructive for the students to discover the result themselves. When I have raised the question in a liberal arts class, it isn't long before students observe that the vertices of odd degree give trouble and find the necessary condition quickly. Of course, sufficiency is harder to demonstrate.

Example 10: Let f be a permutation or a finite set. Is there necessarily a positive integer k such that f^k is the identity function of that set?

Remarks: The approach may depend on whether this is given before or after the cycle decomposition of a permutation. In the first case the student will be more likely to experiment. That means choosing some specific sets and functions, again a traumatic experience for students not used to such freedom and responsibility.

Example 11: In a finite graph is there anything that one can say about the number of vertices of even degree or about the number of vertices of odd degree?

Remarks: This exercise usually appears as a theorem. Too often we ask a question and then answer it before the student has had a chance to live with the question. By answering our own questions we turn the students into spectators, putting a barrier between them and the material. The temptation to do this is usually irresistable and is often justified by the "need to cover the syllabus." But what if the syllabus includes "teach students how to explore, to make conjectures, to write clearly?"

Example 12: Is there any relation between the number of vertices and the number of edges in a finite tree?

Remarks: The comments on Example 11 apply to this example as well. In both cases we can ask the students to prove their conjectures. There are several ways to justify both, including induction. These there-

fore serve as legitimate induction problems. The sooner we reduce the number of traditional induction problems like, "Show by induction that $1 + 2 + \ldots + n = n(n+1)(2n+1)/6$", the better. In a realistic induction problem, the student should propose the statement to be proved. (Recall Example 8.)

The next exercise gives students far more trouble than might be expected, both in carrying out their experiments and in explaining their conclusions.

Example 13: The function of $f: A \to B$ induces functions $F: P(A) \to P(B)$ and $G: P(B) \to P(A)$. For which f is
(a) F one-to-one?
(b) F onto?
(c) G one-to-one?
(d) G onto?

More examples discussed from a slightly different perspective are to be found in [2], but it is not hard to make up your own. Some can be derived from the statements of theorems. In some only an exploration and a conjecture are to be expected. In some a complete argument would be in order.

It may be easier to offer individual guidance and feedback in a smaller class than in a large one, but the organizational challenge in a large class should be negotiable. Though we might prefer to think our task done when we give a clear lecture, we may have to acknowledge that giving good feedback is equally important. Grading homework and examinations, which usually just offers the student the guidance of a number, is hardly adequate feedback. I suspect we, charmed by the clarity of our lectures, could go through an entire semester and never see a single page of a student's work. (I confess that this has happened with me.) It therefore may be necessary to give some time to see what the students write. It may be advisable to sacrifice content to achieve other goals.

My proposal is simply an attempt to respond to the concerns expressed by Hamming, Davis, Ellis, Thompson, and Mason that I cited. I want us to consider the goals of our teaching. Do they go beyond transmitting content? If not, we should say so in our catalogs and encourage others to introduce "problem-solving" courses to compensate for the narrowness of our mission.

If we want our students to be able to think on their own and to express their thoughts, we should give them a chance, even in the introductory curriculum, whether calculus or discrete mathematics, even in service courses even if we propose only two or three open-ended problems in a semester. If enough of us urge publishers to include an ample supply of such problems, with variations and solutions discussed only in the instructor's manual, they will comply. But we don't need to wait for them.

REFERENCES

1. The introductory mathematics curriculum: misleading, outdated, and unfair, College Mathematics Journal, Vol. 15, November 1984, 383 - 399.

2. S. K. Stein, Routine Problems, ibid, Vol. 16, November 1985, 383 - 385.