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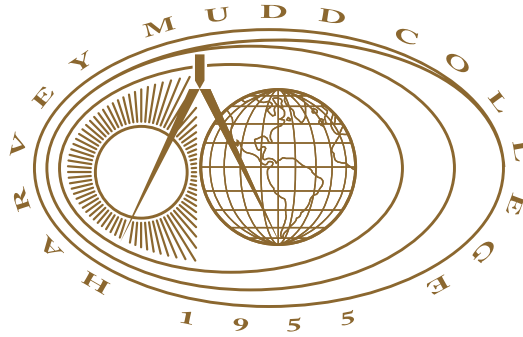
# Combinatorial Consequences of Relatives of the Lusternik-Schnirelmann-Borsuk Theorem

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## HARVEY MUDD
### COLLEGE

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# Abstract

Call a set of $2n + k$ elements *Kneser-colored* when its $n$-subsets are put into classes such that disjoint $n$-subsets are in different classes. Kneser showed that $k + 2$ classes are sufficient to Kneser-color the $n$-subsets of a $2n + k$ element set. There are several proofs that this same number is necessary which rely on fixed-point theorems related to the Lusternik-Schnirelmann-Borsuk (LSB) theorem. By employing generalizations of these theorems we expand the proofs mentioned to obtain proofs of an original result we call the Subcoloring theorem. The Subcoloring theorem asserts the existence of a partition of a Kneser-colored set that halves its classes in a special way. We demonstrate both a topological proof and a combinatorial proof of this main result. We present an original corollary that extends the Subcoloring theorem by providing bounds on the size of the pieces of the asserted partition. Throughout, we formulate our results both in combinatorial and graph theoretic terminology.

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# Acknowledgments

I would like to thank my advisor, Professor Francis Su, for his patience and guidance. I would also like to thank Professor Castro for acting as my second reader.

# Chapter 1

# Introduction

## 1.1 Background: Families of Theorems

Often theorems which are powerful in their own specific fields have extensions or equivalences in other fields of mathematics. Connections between theorems in a particular field can often lend insight to the discovery or exploration of relationships between their ostensibly dissimilar incarnations in other fields.

This thesis is a continuation of the exploration conducted in the summer of 2004 on the following sets of equivalences:

Tucker's Lemma ⇔ Borsuk-Ulam Theorem ⇔ LSB Theorem
Sperner's Lemma ⇔ Brouwer Fixed Point Theorem ⇔ KKM Lemma

We will state some of these theorems later when it is convenient. These families of equivalences each include three fixed-point theorems: one of combinatorial character, one that is topological and one that describes set covering.

In 1997 Su established a direct constructive proof of the Brouwer fixed-point theorem based on the Borsuk-Ulam theorem (14). Since (14) provided the link between the two sets of equivalences, it was clear that the Lusternik-Schnirelmann-Borsuk (LSB) theorem did imply the Knaster-Kuratowski-Mazurkiewicz (KKM) Lemma, and it seemed logical that there should be some direct link. In fact, such a link does exist: we found a direct proof of the KKM Lemma based on a version of the LSB theorem for open sets (see Appendix B for the full text of (13) ).

In this thesis I continue to focus on the LSB theorem and its relatives

but with different goal that I will describe in the following sections.

## 1.2 Motivation: Greene's Proof of Kneser's Conjecture

Kneser's Conjecture is a powerful and well-known result in combinatorics. Kneser made the following conjecture in 1955:

**Kneser's Conjecture.** *Consider the $n$-subsets of a $2n + k$ element set. Divide these into classes such that any two $n$-subsets of the same class are pairwise intersecting. Then $k + 2$ classes are necessary (and this number is sufficient).*

We call the $2n + k$ element set the *ground set* and say that a ground set is *Kneser-colored* when its $n$-subsets have been put into classes such that no two disjoint $n$-subsets are in the same class.

Since a good grasp of this definition is essential to our subsequent discussion we provide an example of a Kneser coloring in Figure 1.1 below.
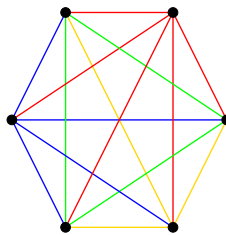


Figure 1.1: A Kneser-colored set

Figure 1.1 demonstrates a Kneser-colored set of $2n + k$ elements where $n = k = 2$. The colors represent classes, and since we are classifying the 2-subsets, this is equivalent to coloring the edges of the complete graph on 6 vertices. Note that the coloring demonstrated uses the minimum possible number of colors (or classes).

The first proof of this conjecture appeared in 1978 when Lovász showed that it was true using algebraic topology in (8). A few weeks later that same year Bárány submitted a paper entitled "A Short Proof of Kneser's Conjecture," (1) that relied on the LSB theorem and a result of Gale. In 2002 Greene published a further simplification under the title, "A New Short Proof of Kneser's Conjecture" (6) that showed that only the LSB theorem itself was actually necessary. Let $S^m = \{x \in \mathbb{R}^{m+1} | \, ||x|| = 1\}$ denote the unit sphere in $\mathbb{R}^{m+1}$, then the LSB theorem states the following (9):

**Lusternik-Schnirelmann-Borsuk (LSB) Theorem.** *If $S^m$ is covered by $m+1$ closed sets then one of the sets contains a pair of antipodes.*

## 1.3    Problem Statement

Greene's recent use of the LSB theorem 1 to provide a direct proof of the combinatorial conjecture of Kneser motivates the exploration of the combinatorial consequences of other topological fixed point theorems related to LSB.

In the we will begin by examining the combinatorial consequences of Fan's generalization of the LSB theorem as stated in 3.2, (4). Using a construction similar to that of Bárány and Greene we prove an original theorem which we will call the Subcoloring theorem. In Chapter 2, we frame this theorem both as a combinatorial result and as a graph theoretic result. In Chapter 3, we demonstrate a proof of this theorem from Fan's generalization of LSB and additionally show a proof that follows closely from a later result of Fan from 1982.

In Chapter 4, we shift our attention to Fan's generalization of Tucker's Lemma (4)(a result that is equivalent to Fan's generalization of LSB). We adapt Matoušek's recent proof of Kneser's conjecture from Tucker's Lemma in order to find an original combinatorial proof of Fan's 1982 result and of the Subcoloring theorem from Fan's generalization of Tucker's Lemma.

In Chapter 5, we prove a corollary of the Subcoloring theorem that bounds the sizes of the pieces of a partition which it asserts. We describe this corollary's effects in both combinatorial and graph theoretic settings.

Finally, in the last 2 chapters, we present a few potential applications of this result and some concluding thoughts on the topic. Two papers written in the last year, "LSB implies KKM" and "Combinatorial consequences of LSB-related topological fixed-point theorems" are included at the end as appendices.

# Chapter 2

# The Subcoloring Theorem

## 2.1  The Main Result: The Subcoloring Theorem

In (4) Fan shows a generalization of the LSB theorem which the LSB theorem as a limiting case. We derive an original Kneser-like conclusion of the non-limiting case using a construction that closely resembles that of Greene and Bárány's proofs of the Kneser result. We will call this original result the Subcoloring theorem.

We restate Kneser's conjecture for convenience:

**Kneser's Conjecture.** *Consider the $n$-subsets of a $2n + k$ element set. Divide these into classes such that any two $n$-subsets of the same class are pairwise intersecting. Then $k + 2$ classes are necessary (and this number is sufficient).*

The related original result which we will prove in Chapters 3 and 4 is the Subcoloring theorem. In the statement of this theorem which follows, we say an $n$-subset of a ground set of points is *complete* when all of its elements occur in the same piece of a partition of the ground set.

**The Subcoloring Theorem.** *(Spencer, Su) Consider the $n$-subsets of a $2n + k$ element set. Divide these into classes $C = \{1, 2, ...m\}$ such that any two $n$-subsets of the same class are pairwise intersecting. Fix these classes.*

- *Suppose exactly $k + 2$ classes are used. Then, for any division of the classes into two sets $C_1$ and $C_2$ whose sizes differ by at most 1, there exists a partition of the original set of elements into two sets $P_1$ and $P_2$ such that all complete $n$-subsets of classes in $C_i$ are in $P_i$ (with all classes in $C_i$ observed in $P_i$).*

- *Suppose more than $k+2$ classes are used. Then there exists some sequence of $k+2$ increasing classes, which has the property that when the classes in odd positions in the sequence are put into $C_1$ and the classes in even positions of the sequence are put into $C_2$, there exists a partition of the original set of elements into two sets $P_1$ and $P_2$ such that all complete $n$-subsets of classes in $C_i$ are in $P_i$ (with all classes in $C_i$ observed in $P_i$).*

Note that in both cases, one of the $C_i$ described has size $\lfloor \frac{k+2}{2} \rfloor$, and the other has size $\lceil \frac{k+2}{2} \rceil$.

In the case where more than $k+2$ classes are used we note that the result holds for any numbering of the classes/colors. That is, for any numbering of the classes we can find a partition of the type described where some set of $k+2$ classes appearing in $C$ has the property that when ordered by number, the classes alternately appear in either $P_1$ or $P_2$. We will refer to this as the *increasing alternating sequence property* of the partition.

Also, in the case where more than $k+2$ classes are used, complete $n$-subsets of classes that are not in the sequence of classes that the result asserts may appear in either none or exactly one of the $P_i$.

In order to clarify the result we provide a simple example for the case when the number of classes is $k+2$ in Figure 2.1 below:
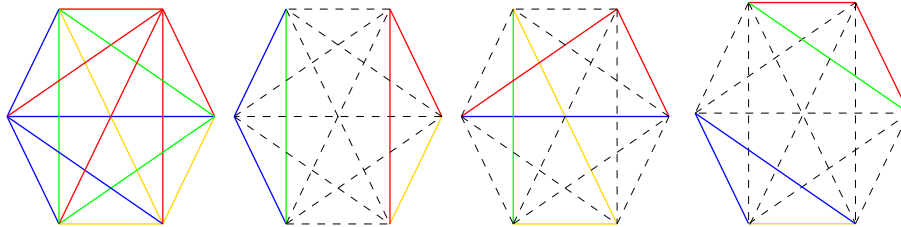


Figure 2.1: Subcoloring Theorem Results

Figure 2.1 demonstrates a Kneser-coloring of a set that has $2n+k$ points where $n = k = 2$. The leftmost graph shows a Kneser-coloring of the set with the minimal number of colors as in Figure 1.1. The other graphs show partitions (which obey the properties specified by the Subcoloring theorem) of the ground set of elements for each possible division of the color classes: for instance, the rightmost graph is a partition that results from partitioning the colors into the groups blue/yellow and red/green. The 2-subsets which are not entirely contained in a single part of the partition are shown as dashed lines; only the *complete* 2-subsets are colored.

## 2.2   A Graph Theoretic Formulation

Though the version of the Subcoloring theorem stated in section  2.1 focuses on the combinatorial character of the result, the result may also be formulated in terms of more graph theoretic vocabulary. This will require a formal introduction to a class of graphs known as *Kneser Graphs*. As this name suggests, Kneser graphs are closely related to the Kneser colorings that we discussed in section 1.2.

Letting $s \leq r$ be natural numbers, the Kneser graph $K_r^{(s)}$ is constructed as follows. The vertex set of $K_r^{(s)}$ is the set of $s$-subsets of $\{1, 2, ...r\}$. Two vertices are joined by an edge if and only if they are disjoint subsets. Figure 2.2 shows a famous example of a Kneser graph popularly known as the Petersen graph.
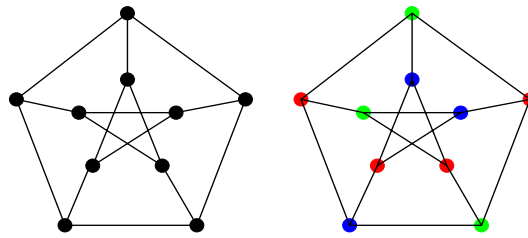


Figure 2.2: The Peterson Graph.

The Petersen graph is the Kneser graph $K_5^{(2)}$.  Each vertex in the Petersen graph is a 2-element subset of the set $\{1,2,3,4,5\}$. A pair of vertices is adjacent in the graph if the sets which correspond to them are disjoint. On the right of Figure 2.2 is a proper coloring of the Petersen graph by the minimal number of colors: $k + 2 = 3$.

What we previously described as a Kneser coloring of the $n$-subsets of a $2n + k$ element set corresponds to a proper coloring of the vertices of $K_{2n+k}^{(n)}$ (that is, a coloring which has no two adjacent vertices colored the same). Translating Kneser's conjecture into this setting we get an equivalent statement:

**Kneser's Conjecture.** *The chromatic number of $K_{2n+k}^{(n)}$ is $k + 2$.*

We are now ready to state a graph theoretic formulation of the Subcoloring theorem. When considering subgraphs of a colored graph we will refer to the coloring of the subgraph under the original coloring as an *inherited coloring*. We will say that a union of Kneser subgraphs of a Kneser graph is

*maximal* when the addition of any vertex not in the union of subgraphs to any of the subgraphs in the union would make that subgraph not a Kneser graph.

**The Subcoloring Theorem.** *(Spencer, Su) Let $n, k$ be natural numbers. For any proper coloring of the vertices of $K_{2n+k}^{(n)}$ by the colors $C = \{1, 2, ..., m\}$ :*

- *Suppose $m = k + 2$. Then given any division of the colors into two sets $C_1$ and $C_2$ that differ in size by at most one, there exist two disjoint induced subgraphs of $K_{2n+k}^{(n)}$, $G_1$ and $G_2$, such that all the colors in $C_i$ appear in the inherited coloring of $G_i$ (with no other colors appearing in $G_i$), and the $G_i$ are Kneser graphs. Further, $G_1 \cup G_2$ is maximal.*

- *Suppose $m \geq k + 2$. Then there exists a sequence of $k + 2$ increasing colors with the property that when the colors in odd position in the sequence are put into set $C_1$ and the colors in even position in the sequence are put into set $C_2$ that there then exist two disjoint induced subgraphs of $K_{2n+k}^{(n)}$, $G_1$ and $G_2$, which are both Kneser graphs and whose union is maximal, such that every color in $C_i$ appears in the inherited coloring of $G_i$.*

As in the original statement of the theorem, in both cases one of the $C_i$ described has size $\lfloor \frac{k+2}{2} \rfloor$, and the other has size $\lceil \frac{k+2}{2} \rceil$.

The reader should note that the asserted Kneser subgraphs are not necessarily colored by the minimum possible number of colors. That is, $\chi(G_i)$ may be less than the number of colors in the inherited coloring of $G_i$.

We can also describe the result in alternate graph theoretic terms used by Simonyi and Tardos in (12). For Kneser graphs, the Subcoloring theorem asserts the existence of special complete bipartite subgraphs which have the complete bipartite subgraphs asserted by the Zig-Zag theorem in (12) as subgraphs. The complete bipartite subgraphs graphs asserted by the Subcoloring theorem are special in that they are colored in a particular way, they are in some sense *maximal*, and they have the property that the graphs induced by each piece of their bipartitions are also Kneser graphs.

Saying that a bipartite subgraph $K$ of a Kneser graph is complete is equivalent to saying that the ground set of points corresponding to any vertex in the first piece of the bipartition of $K$ is disjoint from the ground sets of points corresponding to any vertex in the second piece of the bipartition in $K$ and vice-versa.

We say that a complete bipartite subgraph $K$ of a graph $G$ is *maximal* when no vertices of $G$ that are not in $K$ can be added to $K$ in order to obtain a larger complete bipartite subgraph. A complete bipartite subgraph

of a Kneser graph is thus maximal when all vertices not in $K$ correspond to subsets of the ground set of elements which already appear in the union of the ground set elements that correspond to vertices in $K$. That is, when the union of the ground set elements corresponding to the vertices of $K$ is the entire original ground set. To relate this to our earlier terminology: saying that a complete bipartite subgraph with each piece of the bipartition a Kneser graph is maximal is the same as saying that the pieces of the bipartition of the graph are disjoint Kneser graphs whose union is maximal.

Suppose that $c$ is a coloring of a Kneser graph $G$ by $\chi(G)$ colors. In the limiting case the Subcoloring theorem states that for any set $C_1$ of $\lfloor \frac{\chi(G)}{2} \rfloor$ of the colors in $c$, there exists a maximal complete bipartite subgraph $K$ of $G$ which has for the pieces of the bipartition of $K$ two sets of vertices: one set induces a Kneser subgraph colored by the colors of $C_1$ (with each color in $C_1$ appearing), the other set induces a Kneser subgraph colored by all the colors not in $C_1$ (with all of these colors appearing).

In the next two chapters we will prove the Subcoloring theorem in a variety of ways. In Chapter 3 we will demonstrate a topological proof that follows from Fan's generalization of LSB. In Chapter 4 we will demonstrate a combinatorial proof that follows from Fan's generalization of Tucker's Lemma.

# Chapter 3

# A Topological Proof of the Subcoloring Theorem

## 3.1   Greene's Topological Proof of Kneser's Conjecture

Since Greene's proof plays a major role in an original proof which we will establish in this Chapter, I will reproduce his proof nearly unaltered here. As Kneser showed, it is obvious that $k + 2$ classes is sufficient to Kneser-color the $n$-subsets of a $2n+k$ element set (there is a simple argument of this fact in the next section). To show that this number is necessary, Greene's proof assumes that we have a classification of the $n$ element subsets of a $2n + k$ element set into $k + 1$ classes (or fewer), which has the property that disjoint $n$-subsets are in different classes. First we will review some necessary notation.

Recall that $S^m = \{x \in \mathbb{R}^{m+1} | \, ||x|| = 1\}$ denotes the unit sphere in $\mathbb{R}^{m+1}$. Let $H(a) = \{x \in S^m | a \cdot x > 0\}$, the open hemisphere centered at $a$.

*Proof.* First, we show that $k+2$ classes is sufficient. Let the numbers $1, 2, ..., 2n+k$ denote the elements of the $2n + k$ element set. For each $i$ in this set let $K_i$ be the collection of all $n$-subsets that have smallest element $i$. Then, by construction the $k + 2$ sets $K_1, K_2, ..., K_{k+1}$, and $K_{k+2} \cup ... \cup K_{n+k+1}$ are classes which satisfy the desired property (that any two $n$-subsets within a class have a non-trivial intersection).

Greene proves the following short lemma which extends the LSB theorem as stated in section 1.2:

**Lemma 1.** *(LSB for open and closed sets) If $S^m$ is covered with $m + 1$ sets, each of which is either open or closed, then one of the sets contains a pair of antipodes.*

I will not include the proof of this lemma since we use a different technique in the our later construction, and we will not need this lemma again. Now we are ready to show that less than $k + 2$ classes is insufficient to Kneser-color the set.

Distribute $2n + k$ points on $S^{k+1}$ in general position; thus strictly less than $k + 2$ points lie on any great $k$-sphere. Classify the $n$-subsets of these points into $k + 1$ classes denoted $A_1, A_2, ..., A_{k+1}$. For $i = 1, 2, ..., k+1$ let $U_i$ denote the set of all points a of $S^{k+1}$ for which $H(a)$ contains an $n$-subset in the class $A_i$. The $U_i$ are open sets, hence $F = S^{k+1} \backslash (U_1 \cup ... \cup U_{k+1})$ is closed. Together $F$ and the $U_i$ are $k + 2$ sets that cover $S^{k+1}$. From the lemma then, some set must contain a pair of antipodes. The set $F$ cannot contain a pair of antipodes, since this would mean that some $H(a)$ and $H(-a)$ would each contain less than $n$ points, such that at least $k + 2$ points would occur on the great $k$-sphere that is the boundary of $H(a)$. Thus, some $U_i$ must contain a set of antipodes such that there exists an $a$ for which $H(a)$ and $H(-a)$ each contain an n-subset of class $i$. Clearly these $n$-subsets are disjoint, such that the $A_i$ could not have been a Kneser coloring of the original $2n + k$ points. This argument easily generalizes for any number of classes less than or equal to $k + 1$.

Therefore, $k + 2$ classes are both necessary and sufficient.

$\square$

## 3.2   Adapting Greene's Proof to Prove the Subcoloring Theorem

In order to prove the Subcoloring theorem we will establish a version of Fan's generalization of the LSB theorem for open sets, and rely on a condition of Gale. Fan's LSB generalization as generally stated is the same as the following except that the $F_i$ are closed sets(4).

**Fan's LSB Generalization.** *(For Open Sets) Let $k, m$ be two arbitrary positive integers. If $m$ open subsets $F_1, F_2, \ldots, F_m$ of the k-sphere $S^k$ cover $S^k$ and if no one of them contains a pair of antipodal points, then there exist $k + 2$ indices $a_1, a_2, \cdots, a_{k+2}$ such that $1 \le a_1 < a_2 < \ldots < a_{k+2} \le m$ and*

$$F_{a_1} \cap -F_{a_2} \cap F_{a_3} \cap \cdots \cap (-1)^{k+1} F_{a_{k+2}} \ne \emptyset$$

*where $-F_i$ denotes the set antipodal to $F_i$.*

*Proof.* Begin with open sets $F_1, F_2, \ldots, F_m$ which cover $S^k$ and do not contain antipodes. Since $S^k$ is compact, any open cover of it has a finite subcover. Construct an open cover as follows: since $F_i$ is open, every point, $x \in F_i$ is an interior point such that $F_i$ contains an open neighborhood of $x$ of some positive radius. For every point, $x$, on $S^k$, include in the open cover an open neighborhood of radius $0.9$ times the radius of an open neighborhood of $x$ that is contained within all $F_i$ that contain $x$ (clearly the latter exists since $x$ may be contained only in finitely many $F_i$). A finite number of such neighborhoods cover $S^k$. Denote these neighborhoods $N_j$. Now let

$$C_i = \{\cup \overline{N_j} | N_j \subseteq F_i\}.$$

By this construction, the $C_i$ are closed subsets of the corresponding $F_i$ that cover. Since the $F_i$ contained no antipodes, the $C_i$ cannot. Also, note that $-C_i \subseteq -F_i$. Now apply the standard version of Fan's LSB generalization (4) to the $C_i$ to find that:

$$\emptyset \neq C_{a_1} \cap -C_{a_2} \cap C_{a_3} \cap \cdots \cap (-1)^{k+1} C_{a_{k+2}} \subseteq F_{a_1} \cap -F_{a_2} \cap F_{a_3} \cap \cdots \cap (-1)^{k+1} F_{a_{k+2}},$$

As desired. $\qquad\square$

It is easy to check that the limiting case of Fan's generalization is the LSB theorem: if we let $m = k + 1$, then since the assertion of the theorem cannot be true (there simply are not enough sets to find such a collection of $a_i$), it must be that at least one of the $F_i$ conatains a pair of antipodes. This is the LSB theorem.

**Lemma 1.** *(Gale) There is a distribution of $2n + k$ points on $S^k$ such that every open hemisphere of $S^k$ contains at least $n$ points.*

A proof of this result can be found in Gale's original paper. We are now ready to proceed with the proof of the main theorem. We will provide a general construction and then attend to the two types of cases.

*Proof.* Embed a ground set of $2n+k$ points on $S^k$ such that the Gale property is met. Classify the $n$-subsets into $m$ classes (indexed $1, 2, ..., m$) such that disjoint $n$-subsets are in different classes. Let $H(x)$ denote the open hemisphere centered at $x$. Let $F_i$ be the set of all coordinate points, $x$, of $S^k$ for which $H(x)$ contains an $n$-subset of class $i$. By construction, the $F_i$ are all open, and from Lemma 1, the $F_i$ cover $S^k$. Clearly, $F_i$ cannot contain a pair

of antipodes since this would imply the existence of two disjoint $n$-subsets (each contained in an open hemisphere, the two of which are disjoint) of the same class.

**Case 1: (Subcoloring theorem with exactly $k + 2$ classes)** If there are exactly $k + 2$ classes, the set of indices given by Fan's LSB generalization corresponds to the set of all classes such that the expression for the non-empty intersection simplifies to the following:

$$F_1 \cap -F_2 \cap F_3 \cap \cdots \cap (-1)^{k+1} F_{k+2} \neq \emptyset.$$

That is, there is a nontrivial open intersection in which every point has the property that its open hemisphere contains $n$-subsets of classes 1, 3, 5, etc, and its antipode's open hemisphere contains $n$-subsets of classes 2, 4, 6, etc. Since the set of points with this property is open and there are only finitely many ground set points on the sphere, there is at least one coordinate point in the open intersection which has the property that the boundary of its open hemisphere does not contain any ground set points. Call this coordinate point $b$.

Let $P_1$ consist of all points of the ground set in $H(b)$. Let $P_2$ consist of all points of the ground set in $H(-b)$. It is obvious from our construction that $P_1$ and $P_2$ are disjoint and that their union contains all of the $2n + k$ points.

As described above, $P_1$ contains n-subsets of classes 1, 3, 5, etc, where every class is realized. Similarly $P_2$ contains n-subsets of classes 2, 4, 6, etc, where every class is realized. Since $P_1 \subseteq H(b)$ and $P_2 \subseteq H(-b)$ any class that is observed on a complete set in $P_1$ cannot be observed in a complete $n$-subset in $P_2$ (if this happened it would have necessarily followed from a violation of our original assumption that disjoint $n$-subsets are in different classes). Thus, all complete $n$-subsets of odd class are observed in $P_1$ and every odd class is observed in $P_1$. The analog for $P_2$ and even classes follows by the same argument.

Since the labelling of the classes was arbitrary, any relabelling is equally valid. Thus, any $\lceil \frac{k+2}{2} \rceil$-subset of classes may be chosen to be those which are indexed for membership in $P_1$ (this is simple: give any class that is desired to be represented in $P_1$ an odd index).

**Case 2: (Subcoloring theorem with more than $k + 2$ classes)** The main difference from the case where the number of classes is $k + 2$ is that now the expression in Fan's LSB generalization involves only a subset of the set of $F_i$. The conclusions are thus somewhat different.

Fan's LSB generalization stipulates that there exist $k+2$ indices $a_1, a_2, \cdots, a_{k+2}$ such that $1 \le a_1 < a_2 < \ldots < a_{k+2} \le m$ and

$$F_{a_1} \cap -F_{a_2} \cap F_{a_3} \cap \cdots \cap (-1)^{k+1} F_{a_{k+2}} \ne \emptyset$$

The nonempty open intersection described contains coordinate points whose open hemispheres contain $n$-subsets of classes $a_1$, $a_3$, $a_5$, etc, and whose antipodes' open hemispheres contains $n$-subsets of classes $a_2$, $a_4$, $a_6$, etc. As before, there exists some coordinate point with this property that has no ground set points which lie on the boundary of its open hemisphere. Call this coordinate point $b$.

Let all ground set points in $H(b)$ be in the first set $P_1$ of the partition. Let all ground set points in $H(-b)$ be in the second set $P_2$ of the partition. It is obvious from our construction that $P_1$ and $P_2$ are disjoint and their union contains all the $2n + k$ points. As described above, $P_1$ contains n-subsets of classes $a_1$, $a_3$, $a_5$, etc, where every class is realized. Similarly $P_2$ contains n-subsets of classes $a_2$, $a_4$, $a_6$, etc, where every class is realized. From our original assumption that disjoint $n$-subsets are in different classes, $P_1$ can not contain any complete $n$-subsets of class $a_i$ for $i$ even. The analog says that $P_2$ can not contain any complete $n$-subsets of class $a_i$ for $i$ odd. Note that we are guaranteed by Fan's LSB generalization that the $a_i$ are associated with an increasing alternating sequence of classes. This confirms the increasing alternating sequence property for the non-limiting case. Note also that since only a subset of the $F_i$ corresponding to the original classes are described in the intersection expression it may be the case that other classes which were not described (that is, sets not indexed by $a_i$) are observed in $P_1$ or $P_2$.    □

A Note: In Case 2 we are not given a choice of which classes will be observed in $P_1$ and $P_2$. Fan's LSB generalization essentially chooses for us. In the first case we were given the choice because when exactly $k+2$ classes were present all of the $F_i$ were in some sense interchangable in the expression given by Fan's LSB generalization. In the second case, because the application of Fan's LSB generalization selects a subset of the $F_i$ in a way that does not consider the indices to be equivalent (namely, the assertion of the theorem is that it is possible to identify a particular combination of actual sets, regardless of their labels, which has the intersection described non-empty), we are not able to generalize through relabelling.

## 3.3   A Proof Using Fan's 1982 Result

In 1982, Fan proved the following result in (5) as a consequence of his generalization of the LSB theorem:

**Theorem 1.** *Let $E$ be a* ground set *of $2n + k$ points, and suppose each $n$-subset of $E$ is assigned one of $m$ colors $\{1, ..., m\}$ such that no two disjoint $n$-subsets have the same color. Then there exist colors $i_1 < i_2 < ... < i_{k+2}$ and corresponding $n$-subsets $A_1, ..., A_{k+2}$ colored $i_1, ..., i_{k+2}$ respectively such that $\cup_{j \ odd} A_j$ is disjoint from $\cup_{j \ even} A_j$.*

This result is a direct consequence of the Subcoloring theorem. To see this, consider the $C_1$ and $C_2$ asserted by the Subcoloring theorem. Observe that to exhibit a class $C_i$ must contain an $n$-subset of that class. From the alternating sequence property of the Subcoloring theorem and this observation, we can find $A_i$ such that $\cup_{j \ even} A_j \subseteq C_1$ and $\cup_{j \ odd} A_j \subseteq C_2$, colored as specified by Theorem .

Also, from this result we can find an another proof of the Subcoloring theorem.

*Claim:* The Subcoloring theorem can be proved from Fan's 1982 result.

*Proof.* Consider the sets $\cup_{j \ odd} A_j$ and $\cup_{j \ even} A_j$ asserted by Fan's 1982 result. These sets each exhibit as many classes as the Subcoloring theorem asserts will appear as a lower bound in each piece of the partition and have the increasing alternating sequence property. However, $(\cup_{j \ odd} A_j) \cup (\cup_{j \ even} A_j)$ is not necessarily the whole ground set $E$. Let $C_1 = \cup_{j \ odd} A_j$ and let $C_2 = E \setminus (\cup_{j \ odd} A_j)$. Since $C_1$ and $C_2$ are still disjoint, no classes will appear in both of them and the other properties are preserved. In the case when $m = k + 2$ no classes that are not in $\cup_{j \ even} A_j$ can appear in $C_1$: the disjointness of $C_1$ and $C_2$ forbids the presence in $C_1$ of any classes in $C_2$, and all classes already appear in either $C_1$ or $C_2$, thus the complete $n$-subsets in $C_1$ must all be of classes exhibited by $\cup_{j \ even} A_j$. Thus, this $C_1$ and $C_2$ satisfy the conclusion of the Subcoloring theorem.

$\square$

# Chapter 4

# A Combinatorial Proof of the Subcoloring Theorem

Matoušek recently published the first combinatorial proof of Kneser's conjecture(10). Based on the fact that Fan's generalizations of the LSB theorem and Tucker's lemma are equivalent, our work in the previous section motivates the question of whether an extension of Kneser's conjecture can be found by extending Matoušek's proof (or using a similar construction) so that Fan's generalized Tucker's lemma can be applied.

First we will review Tucker's lemma. Let $B^n = \{x \in \mathbb{R}^n| \; ||x|| \leq 1\}$ denote the unit ball in $\mathbb{R}^n$. Recall that the *octahedral subdivision* of the $n$-ball is the division of the ball induced by its intersection with the coordinate hyperplanes in $\mathbb{R}^n$. Also, a *barycentric derived subdivision* is a subdivision derived by successive application of a finite number of barycentric subdivisions. We will use the version of Tucker's lemma stated in (11):

**Tucker's Lemma.** *Let $K$ be a barycentric subdivision of the octahedral subdivision of the $n$-ball $B^n$. Suppose that each vertex of $K$ is assigned a label from $\{\pm 1, \pm 2, ..., \pm n\}$ in such a way that labels of antipodal vertices sum to zero. Then some pair of adjacent vertices of $K$ have labels that sum to zero.*

Fan's generalization of Tucker's lemma states the following(4):

**Fan's Tucker Generalization.** *Let $K$ be a barycentric derived subdivision of the octahedral subdivision of the $n$-ball $B^n$. Let $m$ be a fixed positive integer independent of $n$. To each vertex of $K$, let one of the $2m$ numbers $\{\pm 1, \pm 2, ..., \pm m\}$ be assigned in such a way that the following two conditions hold:*

- *labels at antipodal vertices sum to zero and*

- *labels at adjacent vertices do not sum to zero.*

*Then there are an odd number of $n$-simplices whose labels are of the form $\{k_0, -k_1, ..., (-1)^n k_n\}$, where $1 \leq k_0 < k_1 < ... < k_n \leq m$. In particular $m \geq n + 1$.*

As with LSB, it is easy to check that the limiting case of Fan's Tucker generalization is Tucker's lemma itself: if we let $m = n$, and have a labeling that has antipodal sums of $0$, then since the assertion of the theorem cannot be true (there simply are not enough labels to find such a collection of $k_i$), it must be that there is at least one pair of adjacent vertices whose labels sum to $0$. This is Tucker's Lemma.

Matoušek's combinatorial proof of Kneser's conjecture begins by constructing a barycentric subdivision of the octahedral subdivision of $S^{n-1}$. By preserving information about the inclusion of simplices in the subdivision, and assigning labels according to two cases (one of which is based on an attempted Kneser coloring by $k + 1$ classes) such that the conditions of Tucker's lemma are met, Matoušek is able to apply Tucker's lemma to obtain a contradiction.

We will slightly expand Matoušek's construction so that Fan's Tucker generalization can be used to obtain a combinatorial proof of the Subcoloring Theorem discussed in the previous chapter. The first part of the proof that follows is taken directly from (10).

We restate the Subcoloring theorem for convenience:

**The Subcoloring Theorem.** *(Spencer, Su) Consider the $n$-subsets of a $2n + k$ element set. Divide these into classes $C = \{1, 2, ...m\}$ such that any two $n$-subsets of the same class are pairwise intersecting. Fix these classes.*

- *Suppose exactly $k + 2$ classes are used. Then, for any division of the classes into two sets $C_1$ and $C_2$ whose sizes differ by at most 1, there exists a partition of the original set of elements into two sets $P_1$ and $P_2$ such that all complete $n$-subsets of classes in $C_i$ are in $P_i$ (with all classes in $C_i$ observed in $P_i$).*

- *Suppose more than $k+2$ classes are used. Then there exists some sequence of $k + 2$ increasing classes, which has the property that when the classes in odd positions in the sequence are put into $C_1$ and the classes in even positions of the sequence are put into $C_2$, there exists a partition of the original set of elements into two sets $P_1$ and $P_2$ such that all complete $n$-subsets of classes in $C_i$ are in $P_i$ (with all classes in $C_i$ observed in $P_i$).*

We will prove this Theorem with the aid of the following notation. Let $B^{2n+k}$ denote the unit ball in $\mathbb{R}^{2n+k}$ under the $l_1$-norm. Let $S^{2n+k-1}$ denote its boundary, and let $K_0$ be the natural triangulation of $B^{2n+k}$ induced by the coordinate hyperplanes (where each $n$-dimensional simplex corresponds uniquely to orthant in $\mathbb{R}^{2n+k}$). Call a triangulation $K$ of $B^{2n+k}$ a *special triangulation* if it refines $K_0$ and is antipodally symmetric about the origin.

*Proof.* We will construct a *special triangulation $K$*, label it in a way that incorporates a proper Kneser coloring and meets the conditions for applying Fan's Tucker generalization, and finally apply this result and interpret its assertions in our construction.

We begin by defining the triangulation $K$. Let $L_0$ be the subcomplex of $K_0$ consisting of the simplices lying on $S^{2n+k-1}$. Note that the nonempty simplices of $L_0$ are in one-to-one correspondence with nonzero vectors from $V = \{-1, 0, 1\}^{2n+k}$. The left diagram in Figure 4 shows $K_0$ and $L_0$ for $B^2$. The inclusion relation on the simplices of $L_0$ corresponds to the relation $\preceq$ on $V$, where $u \preceq v$ if $u_i \preceq v_i$ for all $i = 1, 2, ..., n$ and where $0 \preceq 1$ and $0 \preceq -1$.

Let $L_0'$ be the first barycentric subdivision of $L_0$. Thus, the vertices of $L_0'$ are the centers of gravity of the simplices of $L_0$ and the simplices of $L_0'$ correspond to chains of simplices of $L_0$ under inclusion. A simplex of $L_0'$ can be uniquely identified with a chain in the set $V \setminus \{(0, ..., 0)\}$ under $\preceq$. Now we define $K$: it consists of the simplices of $L_0'$ and the cones with the origin for an apex over such simplices. We have constructed a *special triangulation* of $B^{2n+k}$ as in Fan's Tucker's generalization. The right diagram in Figure 4 demonstrates $K$ of $B^2$.
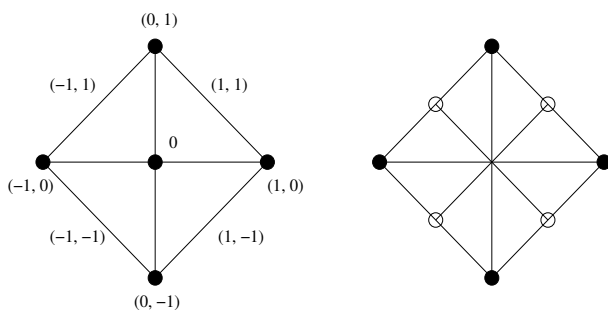


Figure 4.1: The Triangualtions $K_0$ and $L_0$ (left), and K (right).

Let $E = \{1, 2, ..., 2n + k\}$ denote a set of $2n + k$ elements. Suppose that $c$ is a proper Kneser-coloring of the $n$-subsets of $E$ by $m$ colors. In particular, $m$ must be at least $k + 2$. For tactical convenience we will label the colors $2n, 2n + 1, ..., 2n + m - 1$. We will now define a labeling of the vertices of $K$ as in Fan's Tucker generalization. These vertices include $0$, so they can be identified with the vectors of $V$, and we want to define a labeling $\lambda : V \rightarrow \{\pm 1, \pm 2, ..., \pm (2n + m - 1)\}$.

We fix some arbitrary linear ordering $\leq$ on $2^{[2n+k]}$ that refines the partial ordering according to size (which has that $|A| < |B|$ implies $A < B$). Let $v \in V$ and define $\lambda(v)$ as follows. Consider the ordered pair $(A, B)$ of disjoint subsets of $E$ defined by

$$A = \{i \in [2n + k] : v_i = 1\}, B = \{i \in [2n + k] : v_i = -1\}.$$

We distinguish two cases. If $|A| + |B| \leq 2n - 2$ (Case 1) then

$$\lambda(v) = \begin{cases} |A| + |B| + 1 & \text{if } A > B \\ -(|A| + |B| + 1) & \text{if } A < B \end{cases} \tag{4.1}$$

If $|A| + |B| \geq 2n - 1$ (Case 2) then at least one of $A$ and $B$ has size at least $n$. If, say, $|A| \geq n$ we define $c(A)$ as $c(A')$, where $A'$ consists of the first $n$ elements of $A$, and for $|B| \geq n$, $c(B)$ is defined similarly. We set

$$\lambda(v) = \begin{cases} c(A) & \text{if } A > B \\ -c(B) & \text{if } A < B \end{cases} \tag{4.2}$$

Thus, in Case 1 we assign labels in $\{\pm 1, \pm 2, ..., \pm (2n - 1)\}$ while labels assigned in Case 2 are in $\{\pm 2n, \pm 2n + 1, ..., \pm (2n + m - 1)\}$.

We now will verify that $\lambda$ meets the conditions necessary to apply Fan's Tucker generalization. First, we note that $\lambda$ is a well-defined mapping from $V$ to $\{\pm 1, \pm 2, ..., \pm (2n + m - 1)\}$. To see that $\lambda$ labels antipodes so that their sum is $0$, i.e. that $-\lambda(v) = \lambda(-v)$, we observe that from our definitions of $A$ and $B$, $A_i = B_{-i}$ where $A_i$ denotes the set $A$ that corresponds to the $i$th vector $v$. Thus labels assigned by both cases will label antipodes with additive inverses.

Next, we need to check that there are no 1-simplices whose vertices labels sum to $0$ (that is, there are no complementary edges). Because of the way we defined $\lambda$, any complementary edge would have to have had both of its vertices labelled by either Case 1 or Case 2. Suppose that there is a complementary edge between vertex $i$ and vertex $j$. If both labels were

assigned by Case 1, then because of our observation about simplices corresponding to chains in $V$ under $\preceq$, we would get (after a possible relabelling) that $A_i \subseteq A_j$ and $B_i \subseteq B_j$ with at least one of these inclusions being proper. But this gives that $|A_i| + |B_i| \neq |A_j| + |B_j|$ so that there is no way that Case 1 could have assigned complementary labels to the $i$th and $j$th vertices. Suppose both labels were assigned by Case 2, and that, without loss of generality, $A_i \subseteq A_j$ and $B_i \subseteq B_j$. This would mean that the label of the $i$th vector (which corresponds to the color of a $k$-subset of $A_i$ after a possible relabelling) was the negative of the label of the $j$th vector (which corresponds to a $k$-subset of the same color in $B_j$). But since $A_i \subseteq A_j$ and $A_j \cap B_j = \emptyset$ this would imply that we had found two disjoint $k$-subsets of the same color. Since $c$ is proper Kneser-coloring, this cannot be the case. Thus, $\lambda$ has no complementray edges.

Since $\lambda$ has that $-\lambda(v) = \lambda(-v)$ and contains no complementary edges, we can apply Fan's Tucker generalization. Fan's Tucker generalization gives that there are an odd number of $2n+k$-simplices in $K$ whose labels are of the form $S = \{l_0, -l_1, ..., (-1)^n l_{2n+k}\}$, where $1 \leq l_0 < l_1 < ... < l_{2n+k} \leq 2n + m - 1$. In particular there is at least one $2n + k$-simplex with this property. Referring to our construction of $\lambda$, at least the $2n+k+1-(2n-1) = k+2$ highest of these labels were assigned by Case 2. Index these $k + 2$ vertices that were labelled by Case 2 with the indices $\{1, 2, ..., k + 2\}$. Recalling that the vertices of our $2n + k$-simplex correspond to entries of a chain in $V$ under $\preceq$ we find that (after a possible reindexing):

$$A_1 \subseteq A_2 \subseteq ... \subseteq A_{k+2}$$

$$B_1 \subseteq B_2 \subseteq ... \subseteq B_{k+2},$$

where $A_i \cap B_i = \emptyset$. Now let $P_s = A_{k+2}$ and $P_l = \{E \setminus A_{k+2}\}$. We make several observations:

- Note: a positive label was assigned when $A > B$ and a negative labels was assigned when $B > A$. Thus, the label $j$ occurring on the asserted $2n + k$ simplex in $\lambda$ would follow from $n$-subset of color $i$ being contained in sets $A_i \subseteq ... \subseteq A_{k+2} \subseteq P_s$ for some $i$. Similarly, the label $-r$ occurring on the asserted $2n + k$ simplex in $\lambda$ would follow from an $n$-subset of color $r$ being contained in the sets $B_j \subseteq ... \subseteq B_{k+2} \subseteq P_l$ for some $j$.

- One of $A_{k+2}$ or $B_{k+2}$ contains $n$-subsets of $\lfloor \frac{k+2}{2} \rfloor$ colors. The other contains $n$-subsets of the other $\lceil \frac{k+2}{2} \rceil$ colors. Thus, one of $P_l$ or $P_s$ con-

tains $k$-subsets of at least $\lfloor \frac{k+2}{2} \rfloor$ colors. The other contains $n$-subsets of at least $\lceil \frac{k+2}{2} \rceil$ other colors.

- Since $c$ was a proper Kneser-coloring and $P_s$ and $P_l$ are disjoint by construction, there are no colors that are exhibited by $n$-subsets in both $P_s$ and $P_l$.

From our construction, Fan's generalization of Tucker's Lemma asserts the existence of a $2n + k$ simplex with a sequence $S$ of $k + 2$ labels (where each label represents a color) whose absolute values increase monotonically and whose signs alternate. From our first observation, all positive colors in $S$ must have been exhibited in some $A_i$ and all negative colors in $S$ must have been exhibited in some $B_j$. Thus $A_{k+2}$ contains $n$-subsets of each of the positive colors and $B_{k+2}$ contains $n$-subsets of each of the negative colors, and since $A_{k+2}$ and $B_{k+2}$ are disjoint by construction, we have Fan's 1982 result that we introduced in section 3.3.

Suppose $m = k + 2$. All of the $n$-subsets of $A_{k+2}$ are indexed by either exclusively the even-indexed colors or exclusively the odd-indexed colors. Similarly, all of the $n$-subsets of $B_{k+2}$ are indexed by the other parity colors. Since there are no colors exhibited by $P_s$ and $P_l$ which are not exhibited by $A_{k+2}$ and $B_{k+2}$ respectively, the same property holds for the $P_i$. Thus, $P_s$ and $P_l$ are the two pieces of the partition described in the Subcoloring theorem. Since the indices assigned to the colors were arbitrary, this result holds for any relabelling: for any set of $\lfloor \frac{k+2}{2} \rfloor$ colors, there exists a partition which meets the conditions described in the Subcoloring theorem in which they appear in the same piece of the partition.

Thus, we conclude the proof. □

# Chapter 5

# Bounds and Applications

Subsequent results and observations Now that we have established the Subcoloring theorem, there are a number of questions that we can ask about its assertions. This section includes an exploration of some of those questions.

## 5.1 Bounding the Pieces of the Partition

Though the Subcoloring theorem asserts that we can find a partition of the set of $2n + k$ points with the properties described, it tells us nothing about the sizes of the pieces of the asserted partition. Thus, we aim to construct bounds on the sizes of these pieces. Denote the larger piece of the partition by $P_l$ and the smaller piece by $P_s$ where $|P_s| \leq |P_l|$. We will say that a class is *exhibited* by $P_i$ if some $n$- subset of $P_i$ is in that class. For the present we consider the limiting case where the number of classes is equal to $k + 2$.

### An obvious bound

First we consider an obvious bound. Since $P_s$ must have at least $\lfloor \frac{k+2}{2} \rfloor$ classes exhibited, clearly it must have at least that number of $n$-subsets contained within it:

$$\lfloor \frac{k + 2}{2} \rfloor \leq \binom{|P_s|}{n}. \tag{5.1}$$

When $k = 2$ and for any $n$, this bound produces the conclusion that $|P_s| = |P_l| = n + 1$. This result is quite obvious because if $|P_s|$ were any smaller, it could contain at most one $n$-subset, and thus have at most one class exhibited. Though this example is not particularly surprising, the

equal sizes of $P_s$ and $P_l$ would be useful if it could be generalized. Unfortunately, with $n$ held constant, this concept for bounding creates a lower bound for $|P_s|$ that grows much slower than $k$.

### A more clever bound

Still focusing on the limiting case, we shift our attention to creating an upper bound for $|P_l|$. First, we observe that since the full set of $n$-subsets is colored in a Kneser-way (that is, disjoint sets are different classes/colors) the subcolorings induced on the partition pieces must also be Kneser colorings. That is, within $P_i$ disjoint $n$-subsets are colored differently.

From the Subcoloring Theorem, the $n$-subsets of $P_l$ are colored with at most $\lceil \frac{k+2}{2} \rceil$ colors/classes. Since we already observed that $P_l$ is Kneser-colored, we can now apply Kneser's conjecture in a reverse style: $\lceil \frac{k+2}{2} \rceil$ colors are sufficient to Kneser-color the $n$-subsets of a set that has at most $2n + (\lceil \frac{k+2}{2} \rceil - 2)$ points. That is, if a set has more than this many points, then $\lceil \frac{k+2}{2} \rceil$ colors will not be enough to color its $n$-subsets in a Kneser way. Since $P_l$ is colored in a Kneser way with at most $\lceil \frac{k+2}{2} \rceil$ colors, it must be that $|P_l| \leq 2n + (\lceil \frac{k+2}{2} \rceil - 2)$.

The resulting bound on $|P_s|$ is $|P_s| \geq (2n + k) - (2n + (\lceil \frac{k+2}{2} \rceil - 2)) = k - \lceil \frac{k+2}{2} \rceil + 2 = k - \lceil \frac{k}{2} \rceil + 1 = \lfloor \frac{k}{2} \rfloor + 1$, that is:

$$|P_s| \geq \lfloor \frac{k}{2} \rfloor + 1 \tag{5.2}$$

In many cases this is a much higher lower bound than our previous method produced. In particular, for constant $n$, as $k$ increases, the relative sizes of $P_l$ and $P_s$ approach equality. Unlike the other bounding method, this method's lower bound grows at a rate linearly related to $k$.

For $n$ large compared to $k$, this bound is not very helpful. In fact, in some cases it is actually lower than the "obvious" bound. In cases where $n$ is much larger than $k$ however, regardless of what $k$ is, there must be at bare minimum $n$ points in $P_s$ (since at least one class must be represented in $P_s$) so we obtain a similar result of nearly equally sized partition pieces.

This motivates the question of a general minimum bound for the size of $P_s$ in cases where the relative magnitude of $n$ and $k$ are not known. Computing some bounds, we observe that for small values of $n$ with arbitrary $k$ the minimum bound (of the obvious bound and the clever bound) is very close to one quarter. It turns out that it is simple to show that at least one of our bounds will always be greater than one quarter.

**Corollary 1.** *Let $K$ denote a set of $2n + k$ points. Suppose the $n$ element subsets of $K$ are classified in a Kneser way using exactly $k + 2$ classes. The Subcoloring Theorem guarantees the existence of a division of the original point set into two new sets, $P_l$ and $P_s$, such that specified properties hold. It is also the case that for all such divisions, $|P_s|/|K| > \frac{1}{4}$.*

*Proof.* We will suppose the contrary and use our obvious and clever bounds to arrive at a contradiction.

We begin by manipulating inequality 5.2. Suppose that this clever lower bound for $|P_s|$ is less than or equal to $\frac{1}{4}$ of the total number of points:

$$\frac{\lfloor \frac{k}{2} \rfloor + 1}{2n + k} \leq \frac{1}{4}.$$

Multiplying through by four and $2n + k$ we obtain:

$$4\left(\lfloor \frac{k}{2} \rfloor + 1\right) \leq 2n + k.$$

If $k$ is even then the equation becomes $2k + 4 \leq 2n + k$ such that $k \leq 2n - 4$. If $k$ is odd then the equation becomes $2k + 2 \leq 2n + k$ such that $k \leq 2n - 2$. In either case, it is certainly true that $k \leq 2n - 2$.

Leaving the clever bound for now, suppose that the obvious bound in inequality 5.1 gives a value of $|P_s|$ such that $\frac{|P_s|}{2n+k} \leq \frac{1}{4}$. Manipulating this expression we get $|P_s| \leq \frac{1}{4}(2n + k) = \frac{n}{2} + \frac{k}{4}$. Since $|P_s|$ must be an integer, we know we can write $|P_s| \leq \lfloor \frac{n}{2} + \frac{k}{4} \rfloor$. We will only make the right-hand side of inequality 5.1 greater by substituting $\lfloor \frac{n}{2} + \frac{k}{4} \rfloor$ for $|P_s|$ as follows:

$$\lfloor \frac{k}{2} \rfloor + 1 \leq \binom{\lfloor \frac{n}{2} + \frac{k}{4} \rfloor}{n}.$$

Now we will substitute in our expression for $k$ from the clever lower bound computations we did earlier with equation 5.2. Since $k \leq 2n - 2$, we get:

$$\lfloor \frac{k}{2} \rfloor + 1 \leq \binom{\lfloor \frac{n}{2} + \frac{2n-2}{4} \rfloor}{n} = \binom{\lfloor \frac{n}{2} + \frac{n}{2} - \frac{1}{2} \rfloor}{n} = \binom{\lfloor n - \frac{1}{2} \rfloor}{n} = \binom{n-1}{n}.$$

But $\binom{n-1}{n}$ equals 0 and the left side of the equation is clearly positive. Thus we arrive at a contradiction.

Thus we must reject our assumption and conclude that at least one of the bounds is greater than $\frac{1}{4}$.

<div align="right">□</div>

### 5.1.1   Bounds for various quantities in the non-limiting case

In the non-limiting case, the assertion of the Subcoloring theorem does not give any information about what happened with the classes that are not indexed by the $a_i$. For this reason is is difficult to imagine a bound tighter than the obvious one we discussed in the previous section: the upper limits on the number of colors exhibited by a piece of the partition are equal to the total number of $n$-subsets within those pieces respectively (note that this limit is reached when all $n$-subsets within $P_i$ are differently colored). Also, there is no reasonable way to bound the number of colors exhibited by each piece other than to say that it must excede $\lfloor \frac{k+2}{2} \rfloor$ and be less than the total number of colors less $\lfloor \frac{k+2}{2} \rfloor$. These bounds induce bounds on the sizes of the $P_i$ that are less restrictive than the obvious bound discussed in the previous section in nearly every case.

### 5.1.2   The Implications of Bounds in the Graph Theoretic Formulation

Recall the statement of the subcoloring theorem in terms of the Kneser graph:

**The Subcoloring Theorem.** *Let $n, k$ be natural numbers. For any proper coloring of the vertices of $K_{2n+k}^{(n)}$ by the colors $C = \{1, 2, ..., m\}$ :*

- *Suppose $m = k + 2$. Then given any division of the colors into two sets $C_1$ and $C_2$ that differ in size by at most one, there exist two disjoint induced subgraphs of $K_{2n+k}^{(n)}$, $G_1$ and $G_2$ such that all the colors in $C_i$ appear in the inherited coloring of $G_i$ (with no other colors appearing), and the $G_i$ are Kneser graphs. Further, $G_1 \cup G_2$ is maximal.*

- *Suppose $m \geq k + 2$. Then there exists a sequence of $k + 2$ increasing colors with the property that when the colors in odd position in the sequence are put into set $C_1$ and the colors in even position in the sequence are put into set $C_2$ that there then exist two disjoint induced subgraphs of $K_{2n+k}^{(n)}$, $G_1$ and $G_2$, which are both Kneser graphs and whose union is maximal, such that every color in $C_i$ appears in the inherited coloring of $G_i$.*

An interesting feature of this formulation of the Subcoloring theorem is that although between $G$ and $H$ there are at least $k + 2$ colors, which is a sufficient number to color the original Kneser graph, $G_1$ and $G_2$ may be quite small (especially when $k$ is not big compared to $n$). For instance, in

the case of the Petersen graph $K_5^{(2)}$ in Figure 2.2, if it is colored by $k+2=3$ colors then one of the asserted subgraphs is a single vertex and the other is three vertices with no edges. In fact, reflecting on the definition of a Kneser graph, it is obvious that the asserted subgraphs will have no edges as long as they correspond to partitions of the $2n + k$ points in which fewer than $2n$ points are in either piece of the partition: if $|P_i| < 2n$ then there can be no pairs of disjoint $n$-subsets in $|P_i|$.

We can construct some simple bounds about when there will be edges in the asserted Kneser subgraphs. Since one of the $P_i$ must conatain at least $\lceil \frac{2n+k}{2} \rceil$ points we are guaranteed edges in at least one of the asserted subgraphs when

$$\lceil \frac{2n + k}{2} \rceil \geq 2n.$$

Similarly, we can set a bound above which both Kneser subgraphs must have edges. From Corollary 1 in this chapter we have that the smaller $P_i$ must contain at least $\frac{1}{4}$ of the total number of points. Thus, both subgraphs are guaranteed to contain edges when

$$\lceil \frac{2n + k}{4} \rceil \geq 2n.$$

## 5.2   Applications

The kinds of situations that the Subcoloring Theorem describes naturally are those in which the Kneser coloring condition makes sense. There should be some reasonable rationale for having to color disjoint $n$-subsets differently. A situation in which this condition naturally arises is a prospective competition between $n$-subsets of the population in which no ties are allowed. In this setting, the limiting case of the theorem allows us to identify an "elite" collection of competitors that has size strictly between $\frac{1}{4}$ and $\frac{3}{4}$ of the total number of competitors. Some examples have logical reasons for adopting the limiting case of allowed classes. Examples of relevant situations include:

### 5.2.1   Division of Labor in high-cost-training situations

Suppose you are building a factory and training workers. In particular, suppose that every product that you are considering making takes $n$ workers to complete (but the time to complete a product varies) and your workers will be allowed to take unpaid breaks when they want to so long as they

are not in the middle of working on a product. For the factory to be optimal you would like to initally train the workers so that any time $n$ of them are ready to work they can go to a station that they are trained in using and start on a product. Also, it is prohibitively expensive to retrain workers. If you have $2n + k$ workers, how many different products must you choose to make for these conditions to be met? Kneser's conjecture says you must make at least $k + 2$ products.

Suppose you built the factory to manufacture $k + 2$ products but now you wish to break the factory into two smaller factories with the same property that anytime $n$ workers are ready they can start on some product. Also, you don't want to retrain any workers. The Subcoloring theorem says that for any division of the products into two halves (approximately equally sized) you can find a division of workers into two groups that will each have the desired property. Further, the groups of workers will not be too lopsided since at least a quarter will be in each group.

### 5.2.2   Tennis Club

There is a tennis club of size $(4 + k)$ which assigns every possible doubles pair a color of jersey to play under. They want to insure that any two doubles teams which could possibly compete play under differently colored jersies. From Kneser's conjecture, the minimum number of jersey colors which must be assigned is $k + 2$.

Now suppose that the club wanted to split into two clubs in which each doubles pair contained within a new small club plays under the same color jersey that they did originally. The Subcoloring theorem says that we can divide the colors of jersies into two approximately equally sized groups, Group 1 and Group 2 (with sizes within 1 of each other) in any way and there will exist some division of players such that all doubles pairs within the first new club will play under a jersey color in Group 1, and all colors of jersies in Group 1 will be worn by some pair in the first club. An analogous statement about Group 2 and the second club also holds.

We can obtain similar results for larger teams of players.

### 5.2.3   Combination Therapy

Assessing the promise of medical therapies individually can be misleading when in true practice therapies are often used in combinations or cocktails whose effectiveness has a substantial dependence on the interactions between the therapies applied.

Suppose that we are interested in evaluating combination therapies of $2n+k$ experimental treatments. We would like to identify a class of promising therapies for increased research funding. We survey experts on the treatments by allowing them to rank the combinations of size $n$ into $k + 2$ categories of promise with the condition that between any two totally separate treatment combinations, one must be preferred over the other. From the Subcoloring theorem, we know that we will be able to identify an "elite" class of treatments in which all combinations have rank in the upper half of the promise classifications, and no promising combinations occur within the "non-elite" class. By concentrating study on this reduced field of treatments (each valuable to several combinations) we mobilize resourses in a more effective way.

### 5.2.4   Party Politics and Picking a Running Mate

Suppose that there are $2n + k$ candidates from the same party who intend to run for an office that is elected as a team of $n$ members. If the $n$-subsets are ranked by popularity such that any two totally disjoint teams have one which is more popular, and no more than $k + 2$ classes of popularity are used, then there exists a partition of the potential candidates into two classes such that all $n$-teams within one class dominate all $n$-teams that are within the other class. If slush funds are being distributed, the party might prefer to concentrate their publicity on the candidates that are in the "elite" popularity class, since these candidates have a uniformily high potential value in the election with respect to all potential $n$-teams they might run with.

# Chapter 6

# Conclusions

By extending Greene's proof using Fan's generalization of the LSB theorem we were able to prove the result we have called the Subcoloring theorem. Though the Subcoloring theorem is equivalent to an earlier result of Fan in (5), the insight gained in this process made it a natural step to adapt a combinatorial proof of Kneser's conjecture. By expanding Matoušek's combinatorial proof we were able to find an original proof of Fan's result and the Subcoloring theorem.

This project has probably been most interesting because of its interdisciplinary nature. The first part of the project used topological methods to reach a combinatorial conclusion. We were able to find a parallel proof within combinatorics. Throughout, we have been able to reformulate our results in a graph theoretic framework that relates to areas of current research, for example, the work of Simonyi and Tardos in (12).

This thesis motivates the question of what other topological theorems can be used in combinatorial settings. Even if other applications of topological theorems do not result in original combinatorial theorems, the bridge that they could provide would greatly enhance understanding of the interplay between these areas of mathematics. Even among theorems closely related to the LSB theorem there are many promising directions. For instance, given the work in this thesis, it seems logical that there should be some direct proof of Kneser's Conjecture from the Borsuk-Ulam theorem. If this link could be established, there are many well-known generalizations of the Borsuk-Ulam theorem that might have interesting combinatorial consequences.

# Appendix A

# Combinatorial Consequences of LSB-Related Theorems Paper (In Progress)

# Appendix B

# LSB implies KKM Paper

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