2005

Connections Between Voting Theory and Graph Theory

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December, 2005

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Acknowledgments

I'd like to thank Professor Francis Su for being my thesis advisor, despite the additional hardship from my thesis being a spring-fall thesis. I'd also like to thank Professor Michael Orrison, both for being my second reader and for suggesting I start my thesis a semester earlier than is usual. Additionally, I'd like to thank Tyler Seacrest for allowing me to discuss my ideas and conjectures with him, which eventually led to our paper on 3-convexity.

Chapter 1

Introduction

Mathematical concepts have aided the progression of many different fields of study. Math is not only helpful in science and engineering, but also in the humanities and social sciences. Therefore, it seemed quite natural to apply my preliminary work with set intersections to voting theory, and that application has helped to focus my thesis. Rather than studying set intersections in general, I am attempting to study set intersections and what they mean in a voting situation. This can lead to better ways to model preferences and to predict which campaign platforms will be most popular. Because I feel that allowing people to only vote for one candidate results in a loss of too much information, I consider approval voting, where people can vote for as many platforms as they like.

1.1 Definitions

If we are concerned with a single issue where opinions range from one extreme to another, we can represent voters' opinions using a linear system. Consider the set $[0, 1] \subset \mathbb{R}$ and assign 0 to one extreme view and 1 to the other. Then every view, or *platform*, in between can also be represented by a real number between 0 and 1. See Figure 1.1 for an example of how we might label the interval in terms of voting preferences. Let the set of platforms for whom person i will vote for be represented by the set A_i. For convenience, we can distribute the sets along the vertical axis, as shown in Figure 1.2.

For the purpose of creating a realistic model, we require that A_i be a closed interval (or the empty set). Thus, if person i will vote for platforms p and q, he will vote for a platform which is intermediate.

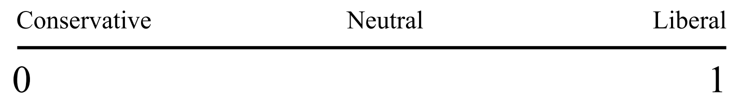

Figure 1.1: One Method of Labeling the Unit Interval

A *society*, S, is a group of voters together with the platforms they can agree with, which we call X. The *order* of a society, $|S|$, is the number of voters in the society. A (k,n)- *society* is a society such that for any subset of n voters, there is at least one platform that at least k of them can agree upon. Notice that the order of a (k,n)-society is at least n. To further describe the society, we say a (k,n,t)-society is a (k,n)-society with order t.

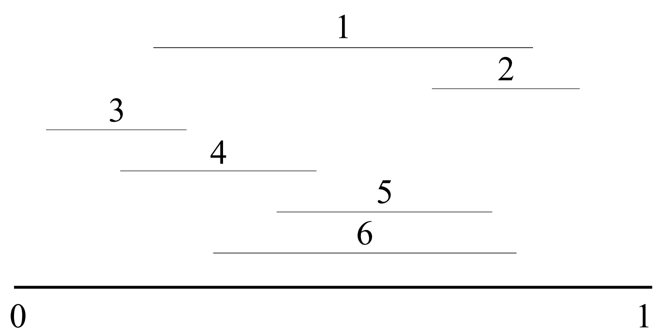

Figure 1.2: Linear Graph of a (2, 3, 6)-Society

The *agreement number* of a platform, $a(p)$, is the number of voters who approve of platform p. The *agreement number* $a(S)$ of a society S is the maximum agreement number over all platforms in X, so $a(S) = \max_{p\in X} a(p)$. The *agreement proportion* of S is simply the agreement number of S divided by the order of S, or $a(S)/|S|$. This concept is useful when we are interested in percentages of the population rather than the number of voters. The *agreement set* of S consists of platforms that receive $a(S)$ votes. Notice that Figure 1.3 shows a society with agreement number 4, and the shaded rectangles cover the agreement set. It is again important to remember that the vertical axis only makes the graph easier for us to interpret and has no mathematical meaning. Finally, we say that a society is *super-agreeable* if $a(S) = |S|$.

The *distance* between voters, $d(a,b)$, is the minimum number of other voters $v_1, v_2, \ldots, v_{d(a,b)}$ required such that $A_a \cap A_{v_1} \neq \{\}$, $A_{v_1} \cap A_{v_2} \neq$

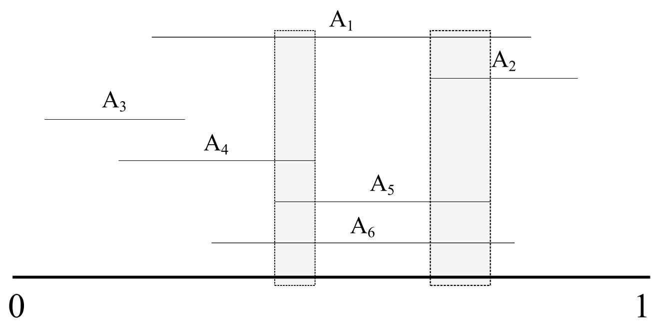

Figure 1.3: Agreement Set of a Society with Agreement Number Four

$\{\}, \ldots, A_{v_d(a,b)} \cap A_b \neq \{\}$. In Figure 1.3, we see that the distance between voters 2 and 3, for example, is 1, because there are no platforms that both 2 and 3 will vote for, but 2 and 3 both have platforms in common with voter 1. If no set of voters will satisfy this condition, we say $d(a,b) = \infty$. If $d(a,b) = 0$, then we say the voters are *adjacent*. In Figure 1.3, for instance, we see that voters 1 and 2 are adjacent. Upon closer inspection, we realize that voter 1 is actually adjacent to every other voter.

1.2 Background

We realize that a society is simply a set of voters, so work concerning set intersections can be applied to this problem. The most well known theorem in this area is Helly's theorem [1], proven in 1923. Helly's theorem states that given t convex sets in $\mathbb{R}^d$, if every $d+1$ of them intersect at a common point, then they all intersect at a common point. The KKM lemma [10], proven in 1929, is similar to Helly's theorem, but is concerned with set intersections on simplices. A more recent theorem, proven by Neidermaier and Su in a paper still in progress, generalizes Helly's theorem to non-convex sets on trees. These theorems gave me a good place to start research, as I could see how they applied to a linear representation of voting theory.

1.3 Motivation

The concept of agreeable societies of voters is useful in voting theory for many reasons. For instance, it can be used to determine the minimum number of platforms that are necessary such that everyone has some platform of which he or she approves. It can also lead to comparisons of voters in those societies in an attempt to determine what kind of people approve of certain platforms. Thus, using these modeling techniques, politicians can attempt to satisfy the greatest number of voters possible.

Another application of this is in consensus theory. Consensus theory is related to voting theory in that it is used for decision making when opinions may not agree. However, consensus theory is more flexible than voting theory, and it attempts to use more information than is generally available in voting theory. Consensus theory tends to be used in a managerial setting. Although it encompasses more than voting theory does, the results of this thesis may be quite useful to consensus theory in general.

Additionally, since this thesis stemmed from work in a variety of other areas, it seems certain that conclusions formed in the context of voting theory will be applicable to many other branches of mathematics.

In Chapter 2, we consider methods of using graphs to explore the properties of different societies, and in Chapter 3, we examine this idea more thoroughly. Chapter 4 introduces concepts related to the piercing number of a graph and explains how the piercing number and agreement number are related. Chapter 5 considers a possible generalization of convexity. In this chapter, we find a necessary and sufficient condition for a set to be 3-convex, but unfortunately, 3-convex sets do not model societies in a realistic way. We draw conclusions, both mathematical and otherwise, in Chapter 6.

Chapter 2

Graphs

2.1 Representations with Graphs

We now examine methods of pictorially representing the agreeableness of a society. Consider a (k, n, t)-society, and let each vertex in an *agreement graph* G represent a voter. Draw an edge between two voters if the voters are adjacent. We notice that a super-agreeable society will produce a complete graph. Additionally, if $d(a, b) = \infty$, then there is no path from a to b, so we say that a and b are in different *components* of the agreement graph.

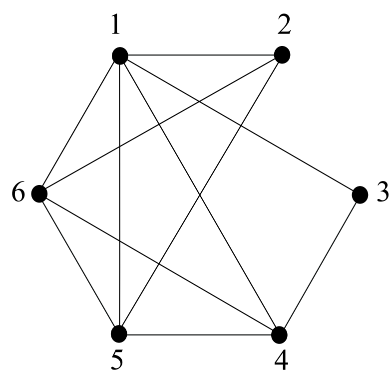

Figure 2.1: Agreement Graph of the (2, 3, 6)-Society from Figure 1.3

Theorem 2.1 (Helly). *A $(2, 2, t)$-society must contain at least one platform that all t voters will vote for.*

This follows as a consequence of Helly's theorem for dimension 1. It also results from a theorem Niedermaier and Su proved for set coverings of trees in a paper still in progress. Below I will give an alternate proof from a different viewpoint.

Proof: Since each voter agrees on at least one platform with every other voter, we see that the sets A_i must be non-empty. Thus, each A_i is a non-empty closed interval in $[0,1]$. We notice that $n = \min_i\{\max\{p \in A_i\}\} \geq \max_i\{\min\{p \in A_i\}\} = m$, because all A_i are adjacent and must share at least one platform p. Therefore, all intervals contain the platforms in the non-empty interval $[m,n]$, so there is at least one platform that all voters will vote for. □

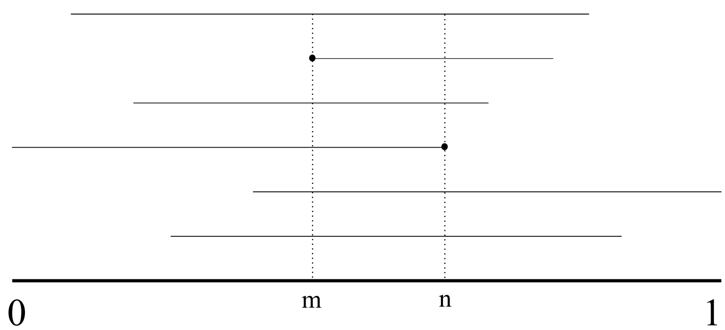

Figure 2.2: Linear Graph of a (2, 2, 6)-Society

2.2 Properties of Agreement Graphs

Theorem 2.2. *The agreement graph of a (k,n)-society has no more than $n-k+1$ disjoint components.*

Proof: Consider a graph G with at least $n-k+2$ disjoint components, and let S be the set of vertices such that we choose one vertex from each component and $k-2$ other arbitrary vertices. Then there are no more than $k-1$ vertices of S in a single component of G. Since voters in distinct components will not agree to vote for any particular platform, there is no set of k voters who will all vote for the same platform, so no choice of n voters is such that k of those voters will agree on a platform. Therefore, the agreement graph of a (k,n)-society has no more than $n-k+1$ disjoint components. □

An example may help make this reasoning even more clear. We see that Figure 2.3 has 2 components. I claim that this is a $(2,3,6)$-society. We can assume the contrary and attempt to find three vertices such that no pair is adjacent. First, choose an arbitrary vertex v, which must be in some triangle. Since triangles are complete, any other vertex in that triangle is adjacent to v, so we must choose a vertex w in the other triangle. However, we have now chosen vertices from each triangle, so we cannot find a third vertex such that it is not adjacent to either of the first two. Thus, this is a $(2,3,6)$-society, and we see that it has $n-k+1=2$ components.

We can then consider this $(2,3,6)$-society with an additional vertex in a new component, as shown in Figure 2.4. Is this a $(2,3,7)$-society? We see that it is not, because the three circled vertices form a set of three vertices such that no two are adjacent. The addition of a third component is what made the difference, since all sets of three independent vertices involve the new component. Thus, we see that the agreement graph cannot have $n-k+2=3$ components. This reasoning generalizes to any choice of k, n, and t such that $k \leq n \leq t$.

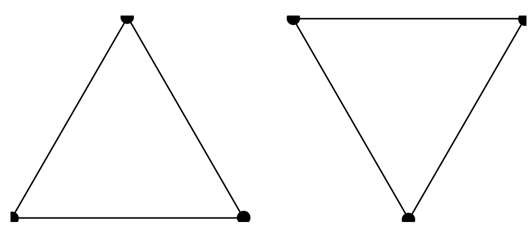

Figure 2.3: Agreement Graph of a $(2,3,6)$-Society with Two Components

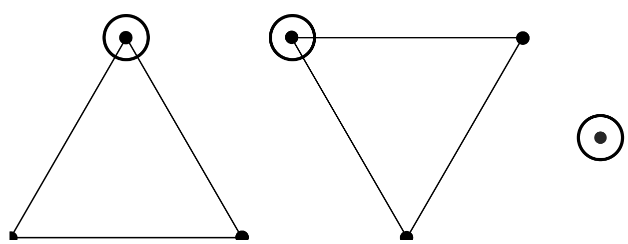

Figure 2.4: Agreement Graph of a Similar Society with Three Components

Corollary 2.1. *The agreement number of a (k,n,t)-society is at least $\lceil t/(n-k+1) \rceil$. In particular, the agreement number of a $(2,3,t)$-society is at least $\lceil t/2 \rceil$.*

Proof: By Theorem 2.2, we know the agreement graph of a (k, n, t)-society has no more than $n - k + 1$ disjoint components. Therefore, since every voter must be in some component, by the pigeonhole principle, there must be some component with at least $\lceil t/(n - k + 1) \rceil$ voters. In particular, the agreement proportion of a $(2, 3, t)$-society is at least $\frac{1}{2}$, so there must be at least one platform that at least half of the population will vote for. $\square$

These results are extremely reminiscent of the piercing number of a set of sets [7]. Although they were developed independently, the piercing number and the agreement number are related. The piercing number of a society of sets is the minimum number of points required such that every set contains at least one point. We notice that the agreement number of a society is equivalent to the maximum number of sets that can be taken care of with a single point. Thus, if there are n sets in a society and the agreement number of the society is k, we know that the piercing number is less than or equal to $n - k + 1$.

Chapter 3

Set Intersections, Perfect Graphs, and Voting in Agreeable Societies

I spent the summer of 2005 at Harvey Mudd College, doing research with Professor Su. We co-authored a paper with Robin Thomas, Serguei Norine, and Paul Wollan that pertains to this thesis. The paper is currently being prepared for submission for publication, and it is included here in its complete form.

3.1 Introduction

When is agreement possible? An important aspect of group decision-making is the question of how a group makes a choice when individual preferences may differ. Clearly, people cannot all have their "ideal" preferences, i.e, the options that they most desire, if those ideal preferences are different. However, for the sake of agreement, people may be willing to accept as a group choice an option that is merely "close" to their ideal preferences.

A good example of such a situation is voting for candidates along a political spectrum. We normally think of this spectrum as one-dimensional, with conservative positions on the right and liberal positions on the left, as in Figure 3.1. While we may represent our ideal preference at some point x on this interval, we might be willing to vote for a candidate that positions himself at some point close to x.

In this article, we ask the following: given such preference sets on a

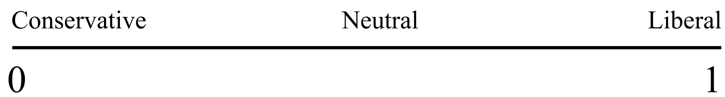

Figure 3.1: Labeling the Unit Interval for Bipartisan Politics

political spectrum, when can we guarantee the existence of some fraction (say, a majority) of the population who would agree on some candidate? By "agree", we mean in the sense of *approval voting*, in which voters declare which candidates they find acceptable.

Approval voting has not yet been adopted for political elections in the United States. However, many scientific and mathematical societies, such as the Mathematical Association of America and the American Mathematical Society, use approval voting for their elections. Additionally, countries other than the United States have used approval voting or an equivalent system. For details, see Brams and Fishburn [4], who give many reasons why they believe approval voting is advantageous. In what follows, our study of *agreeability* will help us understand when we can guarantee a majority under approval voting.

Suppose that approval voting were used in the 2003 California gubernatorial recall election, with 135 candidates in the mix [5]. We might imagine these candidates positioned at 135 points on the line. If each California voter approved of candidates within some range of positions (call this the voter's *approval set*), we might wonder if and when there might be a point on the interval covered by a majority of the voter approval sets, i.e., a platform on which a majority of the voters agree.

An extremely strong condition that guarantees this (and quite a bit more) is that every pair of voters agrees on some platform, i.e., each pair of approval sets has a non-empty intersection. If this condition is met, call the society of voters *super-agreeable*. This local condition guarantees a strong global property, namely, that there is a platform of which *every* voter approves! As we shall see in Theorem 3.3, this is a consequence of Helly's theorem about intersections of convex sets.

In this article, we consider a variety of similar theorems. For instance, we relax the condition above and call a society *agreeable* if among every 3 voters, there is some pair of voters who agree on some platform. Then we prove the following:

Theorem 3.1 (The Agreeable Society Theorem)**.** *In an agreeable society, there is a platform which has the approval of a majority of voters, i.e., a winning platform.*

For example, Figure 3.2 shows approval sets for an agreeable society of six voters, and indeed there are platforms of which a majority of voters approve. As another application of our theorem, consider a situation in which each voter's approval set spans 1/3 of the total interval. Then the theorem above guarantees a winning platform. We also consider other degrees of agreeability and prove more general results in Theorems 3.5 and 3.9.

As we shall see, these questions motivate the study of theorems about set intersections and perfect graphs, since they have natural interpretations in this voting context.

3.2 Definitions

In this section, we fix some terminology and explain some of the basic concepts upon which our results rely. Let us suppose that the set of possible preferences is modeled by a set X, called the *spectrum*. Each element of the spectrum is a *platform*. Assume that there is a finite set of *voters*, and each voter v has an *approval set* A_v of platforms. The set X together with all the approval sets of all voters will be called a *society*. Of particular interest to us will be the case where $X = [0, 1]$. The political spectrum is often modeled this way, but this is by no means limited to politics. Another situation that is well-described by this model is preferences over a single issue where opinions range from one extreme to another— with 0 as one extreme view and 1 as the other. See Figure 3.1 for an example of how we might label the interval in terms of voting preferences.

In Figure 3.2, for ease of display, we have separated the approval sets vertically so that they can be distinguished.

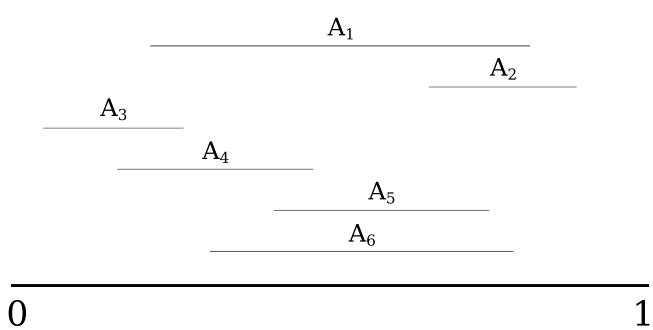

Figure 3.2: Linear Graph of a (2, 3, 6)-Society

A society is (k, m)-*agreeable*, where $1 \leq k \leq m$ are integers, if it has at least m voters, and for any subset of m voters, there is at least one platform that at least k of them can agree upon. For a society S, the *agreement number* of a platform, $a(p)$, is the number of voters in S who approve of platform p. The *agreement number* $a(S)$ of a society S is the maximum agreement number over all platforms in the spectrum, so

$$a(S) = \max_{p \in X} a(p).$$

Note that a society is super-agreeable if $a(S) = |S|$. The *agreement proportion* of S is simply the agreement number of S divided by the order of S, or $a(S)/|S|$. This concept is useful when we are interested in percentages of the population rather than the number of voters. The *agreement set* of S consists of platforms that receive $a(S)$ votes, and this is a subset of X, the set of all possible platforms of which voters can approve. Figure 3.3 shows a society with agreement number 4, and the shaded rectangles cover the agreement set. The *agreement number* of a platform, $a(p)$, is the number of voters who approve of platform p.

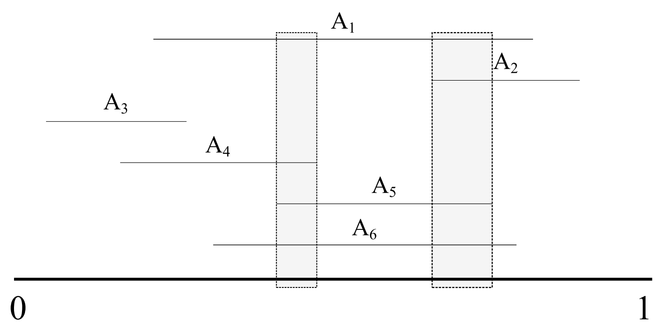

Figure 3.3: Agreement Set of a Society with Agreement Number Four

3.3 Helly's Theorem and Super-Agreeable Societies

Let us say that a society is $\mathbb{R}^d$-*convex* if the spectrum is $\mathbb{R}^d$ and each approval set is a closed convex subset of $\mathbb{R}^d$. Note that a linear society is an $\mathbb{R}^1$-convex society. In case of $\mathbb{R}^d$-convex societies, work concerning set intersections can be applied to the agreement number problem. The most well known theorem in this area is Helly's theorem. This theorem was proven by Helly in 1913, but the result was not published until 1921, by Radon [14].

Theorem 3.2 (Helly)**.** *Given t convex sets in $\mathbb{R}^d$ where $d < t$, if every $d+1$ of them intersect at a common point, then they all intersect at a common point.*

Note that in dimension 1, Helly's condition for approval sets is equivalent to the condition for a super-agreeable society. The conclusion of Helly's theorem therefore has a nice interpretation:

Theorem 3.3 (The Super-Agreeable Society Theorem)**.** *A $(2,2)$-agreeable society must contain at least one platform that is acceptable to all voters.*

A proof of Helly's theorem for general d may be found in [11], but below we give an alternate proof for the case $d = 1$, e.g., Theorem 3.3.

Proof. Since each voter agrees on at least one platform with every other voter, we see that the sets A_i must be non-empty. Thus, each A_i is a non-empty closed interval in $[0,1]$. Let $x = \max_i\{\min\{p \in A_i\}\}$ and $y = \min_j\{\max\{p \in A_j\}\}$.

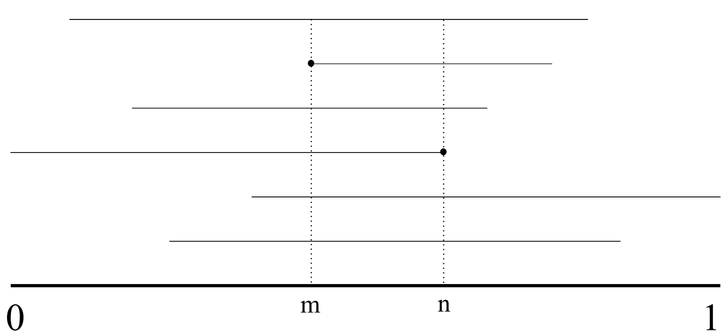

Figure 3.4: Linear Graph of a (2, 2, 6)-Society

We claim that $x \leq y$. Why? Let i be the voter whose approval set minimum is maximal, and let j be the voter whose approval set maximum is minimal. Note that i and j are adjacent, and the only way this could be the case is if $x \leq y$.

Therefore, all approval sets contain the platforms in the non-empty interval $[x, y]$, so there is at least one platform that all voters will vote for. $\square$

Besides Helly's theorem, another famous theorem about set intersections is the KKM lemma [10], which is concerned with set intersections on simplices. More recently, Niedermaier and Su [12] proved a set intersection theorem on trees that generalizes both Helly's theorem and the KKM lemma to this context. Since a line is a tree, Theorem 3.3 for $d = 1$ can also be proved from their results.

Here is an example demonstrating that the convexity assumption is essential. Let $n \geq 2$ be an integer and let the spectrum of a society S consist of all 2-element subsets of $\{1, 2, \ldots, n\}$. Let S have n voters numbered $1, 2, \ldots, n$, and let the approval set of voter i consist precisely of those 2-element subsets of $\{1, 2, \ldots, n\}$ that include i. Then S is a $(2, 2)$-agreeable society with agreement number 2, which is in sharp contrast with Theorem 3.3.

3.4 Graph Representations

If we are to understand other kinds of agreeability beyond super-agreeability, it will be helpful to examine methods of pictorially representing the agreeableness of a society. A *graph* G consists of a finite set $V(G)$ of *vertices* and a set $E(G)$ of 2-element subsets of $V(G)$, called edges. If $e = \{u, v\}$ is an edge, then we say that u, v are the *ends* of e, and that u and v are *adjacent* in G. We use uv as another notation for the edge e.

Given a society S, we construct an *agreement graph* G of S by letting each vertex represent a voter and drawing an edge between two voters if the voters are adjacent, meaning that there is some candidate of which they both approve. Note that a super-agreeable society will produce a complete graph. Also, the distance between two voters (as defined earlier) is just the graph distance in the agreement graph. Moreover, if $d(a, b) = \infty$, then there is no path from a to b, so we say that a and b are in different *components* of the agreement graph.

We now have some tools with which to study (k, m)-agreeability.

Theorem 3.4. *The agreement graph of a (k, m, n)-society has no more than $m - k + 1$ disjoint components.*

Proof. Consider a graph G with at least $m - k + 2$ disjoint components, and let S be the set of vertices such that we choose one vertex from each component and $k - 2$ other arbitrary vertices. Then there are no more than $k - 1$ vertices of S in a single component of G. Since voters in distinct components will not agree to vote for any particular platform, there is no set of k voters who will all vote for the same platform, so no choice of m voters is such that k of those voters will agree on a platform. Therefore, the agreement graph has no more than $m - k + 1$ disjoint components. $\square$

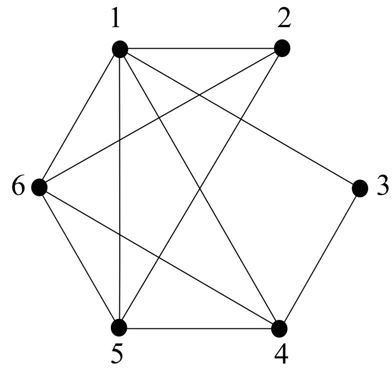

Figure 3.5: Agreement Graph of a (2, 3, 6)-Society

An example may help make this reasoning even more clear. We see that Figure 3.6 has 2 components. We claim that this is a $(2,3,6)$-society. We can assume the contrary and attempt to find three vertices such that no pair is adjacent. First, choose an arbitrary vertex v, which must be in some triangle. Since triangles are complete, any other vertex in that triangle is adjacent to v, so we must choose a vertex w in the other triangle. However, we have now chosen vertices from each triangle, so we cannot find a third vertex such that it is not adjacent to either of the first two. Thus, this is a $(2,3,6)$-society, and we see that it has $m-k+1=2$ components.

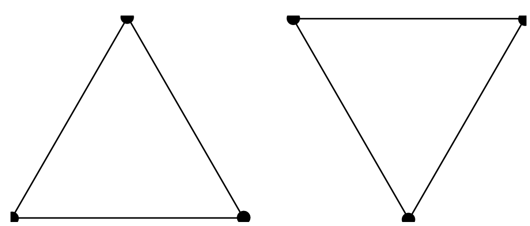

Figure 3.6: Agreement Graph of a $(2,3,6)$-Society with Two Components

We can then consider this $(2,3,6)$-society with an additional vertex in a new component, as shown in Figure 3.7. Is this a $(2,3,7)$-society? We see that it is not, because the three circled vertices form a set of three vertices such that no two are adjacent. The addition of a third component is what made the difference, since all sets of three independent vertices involve the new component. Thus, we see that the agreement graph cannot have

$m - k + 2 = 3$ components. This reasoning generalizes to any choice of m, k, and n such that $k \leq m \leq n$.

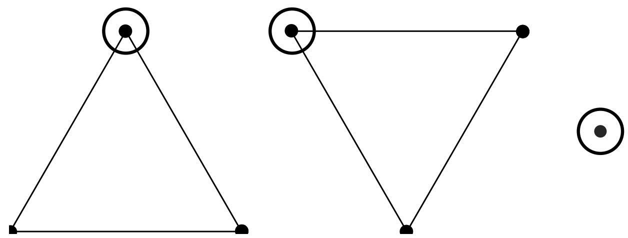

Figure 3.7: Agreement Graph of a Similar Society with Three Components

Theorem 3.5. *The agreement number of a (k, m, n)-society is at least $\lceil n/(m - k + 1) \rceil$. In particular, the agreement number of a $(2, 3, n)$-society is at least $\lceil n/2 \rceil$.*

Proof. By Theorem 3.4, we know the agreement graph of a (k, m, n)-society has no more than $m - k + 1$ disjoint components. Therefore, since every voter must be in some component, by the pigeonhole principle, there must be some component with at least $\lceil n/(m - k + 1) \rceil$ voters. In particular, the agreement proportion of a $(2, 3, n)$-society is at least $\frac{1}{2}$, so there must be at least one platform that at least half of the population will vote for. $\square$

These results are reminiscent of the *piercing number* Π of a collection of sets [7]. Although they were developed independently, the piercing number and the agreement number are related. The piercing number of a collection of sets is the minimum number of points required such that every set contains at least one of those points. Note that the agreement number of a society is equivalent to the maximum number of sets that can be pierced by a single point. Thus, if there are n sets in a society S, then $\Pi \leq n - a(S) + 1$. Additionally, since the piercing number is the smallest number of points such that each set contains at least one point and the agreement number is the largest number of sets that can be pierced by one point, the total number of sets must be at least the agreement number times the piercing number. In other words, $\Pi \geq n/a(S)$.

3.5 Interval Graphs and Perfect Graphs

The *chromatic number* of a graph G, written as $\chi(G)$, is the minimum number of colors necessary to color vertices such that no adjacent vertices have the same color. This can tell us information about relationships between voters' preference sets. In general, the chromatic number of an agreement graph rises as the agreeability of the society rises.

The *clique number* of G, written $\omega(G)$, is the greatest integer n such that $K_n \subset G$. In an agreement graph, the clique number is the agreement number of the society. Notice that in all cases, $\chi(G) \geq \omega(G)$.

A graph G is called an *interval graph* if every vertex x represents a real interval I_x and $xy \in E(G)$ if and only if $I_x \cap I_y \neq \emptyset$. We notice that because we restrict voters to approving of contiguous closed intervals (or the empty set), agreement graphs are interval graphs.

Given a cycle on n vertices, an edge e is a *chord* if it is adjacent to two vertices of the cycle but is not in the cycle itself. If a graph G is such that any cycle of length greater than three has a chord, it is a *chordal graph*, sometimes called a *triangulated graph*.

Theorem 3.6. *Interval graphs are chordal.*

Proof. We prove the contrapositive: if a graph is not chordal, it cannot be an interval graph. Let G be a non-chordal graph, so there is some cycle C of order $n > 3$ such that C contains no chords. Label the vertices of the cycle $v_1, v_2, \ldots, v_n$ such that v_1 is adjacent to v_n, which we write as $v_1 \sim v_n$, and $v_i \sim v_{i+1}$ for $i \in \{1, 2, \ldots, n-1\}$. Assume by way of contradiction that G is an interval graph, and let $I_i = [a_i, b_i]$ be the interval corresponding to vertex v_i. Without loss of generality, assume $a_1 \leq a_i$ for all $i \in \{2, 3, \ldots, n\}$. Because $[a_1, b_1] \cap [a_2, b_2] \neq \emptyset$, $[a_2, b_2] \cap [a_3, b_3] \neq \emptyset$, and $[a_1, b_1] \cap [a_3, b_3] = \emptyset$, we see that $a_1 \leq a_2 \leq b_1 < b_2$. By symmetry, $a_i \leq a_{i+1} \leq b_i < b_{i+1}$ for all $i \in \{1, 2, \ldots, n-2\}$.

We know $a_{n-1} \leq a_n$, so $a_1 < a_n$. Thus, since $I_{n-1} \cap I_n \neq \emptyset$, $a_n \leq b_{n-1} \leq b_n$. Also, since $I_1 \cap I_n \neq \emptyset$ but $I_1 \cap I_{n-1} = \emptyset$, we know $a_n \leq b_{n-1} \leq a_1$, which is a contradiction. Therefore, G is not an interval graph, so all interval graphs must be chordal graphs. □

Theorems 3.7 and 3.8 and their proofs are adapted from Diestel's *Graph Theory* [6]. We first introduce some terminology. Note that *pasting* graphs G_1 and G_2 along a subgraph $S \subset (G_1 \cap G_2)$ means that we essentially overlap some of the identical parts of G_1 and G_2. In Figure 3.8, S is composed of edges in blue and their associated vertices, and the two graphs on the left are pasted along S to produce the graph on the right. Additionally, an *induced cycle* is simply a cycle with no chords. Note that for $S \subset V(G)$, we write $G[S]$ to mean the graph induced by S, so $G[S]$ contains all vertices in S and all edges in G that have both endpoints in S. Finally, a *minimal separating set* between vertices x and y consists of the fewest number of vertices that we must remove such that x and y are in different components of the resulting graph.

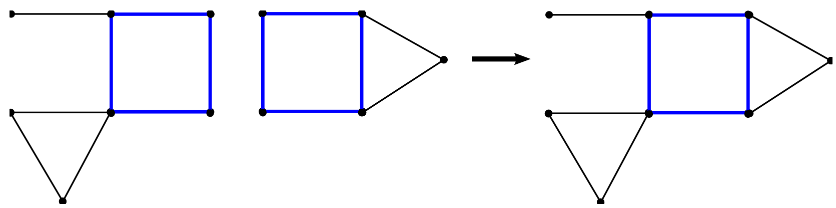

Figure 3.8: Two Graphs Pasted Along a Common Subgraph

Theorem 3.7. *A graph G is chordal if and only if it can be constructed recursively by pasting two chordal graphs along a complete graph.*

Proof. Let G be a graph constructed of two chordal graphs, G_1 and G_2, pasted along a complete subgraph $S \subset (G_1 \cap G_2)$ and let C be a cycle in G. If $C \cap (G - G_1)$ and $C \cap (G - G_2)$ are both non-empty, then the cycle must contain at least two disjoint edges of S. However, S is a complete graph, so the vertices in those two disjoint edges are all adjacent. Thus, C contains a chord. Therefore, any induced cycle of G must be a subset of either G_1 or G_2. By definition, this means any induced cycle of G is an induced cycle of G_1 or G_2, which are chordal graphs. Thus, all induced cycles of G are triangles, so G is chordal.

We prove the other direction by induction. Let G be a chordal graph. If $|G| = 1$, the construction is trivial. In fact, if G is complete, the construction is trivial, since all induced cycles of a complete graph are triangles. Thus, we assume G is not complete, $|G| > 1$, and all chordal graphs smaller than G can be constructed in the manner specified. Because G is not complete, we can find two vertices, x and y, that are not adjacent. Let $M \subset V(G)$ be a minimal separating set between x and y, and consider $G - M$. Let X be the component of $G - M$ containing x. Let $G_1 = G[M \cup V(X)]$ and $G_2 = G - X$. Then G can be constructed by pasting G_1 and G_2 along $G[M]$. Since G_1 and G_2 are both induced subgraphs of G, they must be chordal, by the inductive hypothesis. Thus, we simply need to show that $G[M]$ is complete.

Suppose the contrary, that $G[M]$ is not complete. Then we can find two non-adjacent vertices in $G[M]$, which we'll label a and b. Because $G[M]$ is a *minimal* separating set, each must be adjacent to some vertex in X. Since X is a single component, there must be some minimal path P between a and b contained entirely in X, which is in G_1. Similarly, there must be some minimal path P' between a and b contained entirely in the component containing y, which is in G_2. Thus, $P \cup P'$ is a cycle of length at least 4. However, since this is constructed from minimal paths, this cycle has no chords, which contradicts the fact that G is chordal. Thus, we cannot find non-adjacent vertices in $G[M]$, so $G[M]$ is complete. Therefore, we can construct an arbitrary chordal graph G by taking two chordal graphs, G_1 and G_2, and pasting them along a complete graph, $G[M]$. □

If every induced subgraph H of a graph G is such that $\chi(H) = \omega(H)$, then G is a *perfect graph*. Perfect graphs have applications in many branches of mathematics and computer science, since they have a more precise structure than graphs in general. This structure can be used in proofs, as in Theorem 3.9.

Theorem 3.8. *Chordal graphs are perfect.*

Proof. Complete graphs are perfect, and by Theorem 3.7, we can construct any chordal graph by pasting two chordal graphs along a complete graph. Thus, we simply show that two perfect graphs pasted along a complete graph yields a perfect graph.

Let G_1 and G_2 be perfect graphs and S be a complete graph contained in both G_1 and G_2. We show that G, which results from pasting G_1 and G_2 along S, is perfect. Let H be some induced subgraph of G. If we can show that $\chi(H) = \omega(H)$, then G is perfect.

By definition, $\chi(H) \geq \omega(H)$, so we simply show $\chi(H) \leq \omega(H)$. Let $H_1 = G_1 \cap H$ and $H_2 = G_2 \cap H$, and let $S' = S \cap H$. Because H is an induced subgraph, all chords are included, so since S is complete, S' is complete. Thus, H is made by pasting H_1 and H_2 along S'. Because the H_i are induced subgraphs of the G_i, they are perfect, so they can be colored with $\omega(H_i)$ colors. Since S' is complete, we relabel colors if necessary and can color H with $\max\{\omega(H_1), \omega(H_2)\}$ colors. However, we see that $m \leq \omega(H)$, so $\chi(H) \leq \omega(H)$, as desired. Thus, $\chi(H) = \omega(H)$, so G is perfect. □

3.6 (k, m)-Agreeable Societies

We now use the concept of a perfect graph to allow us to prove that the agreement number of a (k, m, n)-society is at least $(k-1)n/(m-1)$.

Theorem 3.9 (The (k, m)-Agreeable Society Theorem)**.** *If G is the agreement graph of a (k, m, n)-society, then*

$$\omega(G) \geq \left(\frac{k-1}{m-1}\right) n.$$

In other words, in a (k, m)-agreeable society, there is some platform whose agreement proportion is at least $(k-1)/(m-1)$.

Note that this extends the Agreeable Society Theorem (in which $k = 2, m = 3$ and the guaranteed agreement proportion is 1/2) and the Super-Agreeable Society Theorem (in which $k = m$ and the guaranteed agreeement proportion is 1).

Proof. Because agreement graphs are perfect, $\chi(G) = \omega(G)$, so we can color G using ω colors. Additionally, since G contains a K_k, we know that $\omega(G) \geq k$. Thus, we can consider the $k-1$ largest color classes, which we'll call $\{C_1, C_2, \ldots, C_{k-1}\}$. Let $A = \bigcup_{i=1}^{k-1} C_i$. The average order of a color class is n/ω, so the order of the union of the $k-1$ largest color classes is at least $n(k-1)/\omega$. Thus, $|A| \geq n(k-1)/\omega$. Because A can be colored with only $k-1$ colors, no k members of A agree. Since this is a (k, m, n)-society, we know that $|A| < m$. Thus, $m - 1 \geq n(k-1)/\omega$. Therefore, $\omega \geq (k-1)n/(m-1)$. □

If we compare this bound to $\lceil n/(m-k+1) \rceil$, the bound found in Theorem 3.5, we see that if $k = 2$ or $k = m$, then the bounds are the same. Otherwise, $2 < k < m$, so $(k-1)n/(m-1)$ is generally a better lower bound on the agreement number of a (k, m, n)-society.

3.7 Speculation and Open Questions

As we have seen, set intersection theorems can provide a useful framework to model and understand the relationships between preference sets in many social contexts.

Additionally, recent results in discrete geometry have social interpretations. The piercing number [7] of approval sets can be interpreted as the minimum number of platforms that are necessary such that everyone has some platform of which he or she approves.

We suggest several directions which the reader may wish to explore.

Thus far, we have considered approval voting on a single issue with infinitely many platforms. It is interesting to consider how to extend these conclusions to different situations. For example, what if we only had a finite number of platforms, rather than a continuous spectrum? Alternatively, we might wish to allow platforms to lie on a plane or a d-dimensional cube rather than a line, to more accurately represent multiple issues. Which of these theorems still apply, and which could be extended?

A quick examination shows us that Theorem 3.3 works in d dimensions if we have a (d, d)-agreeable society (with convex approval sets) instead of a $(2, 2)$-agreeable society. This is simply due to Helly's theorem [14]. However, extending the other theorems to the d-dimensional context or the discrete setting may be more difficult.

Additionally, we must examine our initial assumptions. While convexity seems to be a rational assumption in the linear case, in multiple dimensions, there are additional considerations that may need to be made. We have also assumed that voters place candidates along the spectrum in exactly the same manner, but people will not necessarily agree that one platform is more conservative than another.

The original concept of an agreement graph could be applied to d-dimensional preferences, but we would like to be able to indicate that, for example, two people's preferences are the same in $d - 1$ dimensions. To do so, one may wish to consider an agreement graph with weighted edges.

Finally, we might wonder about the agreement parameters k and m for various issues which affect us personally. For instance, a society considering outlawing murder would probably be much more agreeable than that same society considering tax reform. Not only do the issues matter, however, but also the societies. Groups of similar people seem likely to be more agreeable than groups consisting of a more diverse population. Currently, we can empirically measure these parameters only by surveying large numbers of people about their preferences. It is interesting to speculate about methods for estimating k and m from limited data.

Chapter 4

Convex Sets with k of Every m Meeting

During the summer of 2005, Professor Su and I wrote another paper pertaining to this thesis, which looks at the same subject from a different angle. The paper is included here in its complete form.

4.1 Introduction

In this article, we consider d-dimensional convex sets on a d-plane such that k out of every m intersect. We ask when we can guarantee that some fraction (say, a majority) of the sets will intersect and how many points $\{p_i\}$ are needed such that every set contains at least one point p_i. Much of this work was inspired by the study of approval voting [3].

In a previous paper [3], we considered the $d = 1$ case. By Helly's theorem [14], for instance, if $d = 1$ and $k = m = 2$, then all sets have a non-empty intersection. By the (k, m)-Agreeable Society Theorem [3], if there are n sets, then some $\left(\frac{k-1}{m-1}\right) n$ sets will have a non-empty intersection.

In this paper, rather than considering the maximum number of sets that have a non-empty intersection, which is the *agreement number* $a(S)$, we focus on the minimum number of points needed such that every set contains some point, called the *piercing number* $\Pi(S)$. This concept was considered in depth in a series of papers by Alon and Kleitman, beginning with a paper from 1992 [2]. We develop relationships between these two numbers, the most notable being that $a(S) + \Pi(S) \leq n + 1$ (Theorem 4.2) and $a(S)\Pi(S) \geq n$ (Theorem 4.3).

4.2 Definitions

In this section, we fix some terminology and explain some of the basic concepts upon which our results rely.

We begin by considering intersections of closed convex sets in dimension 1. In Figure 4.1, for ease of display, we have separated the sets $\{A_i\}$ vertically so that they can be distinguished.

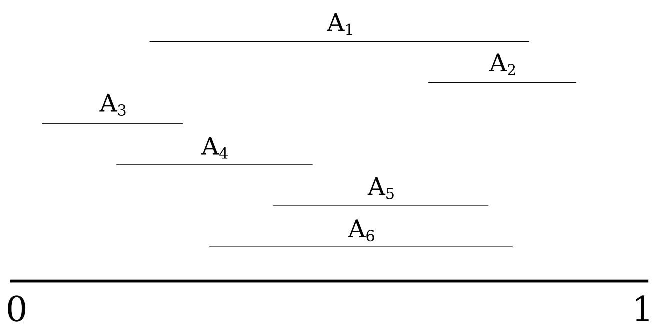

Figure 4.1: Linear Graph of a (2, 3, 6)-Society

We define a (k, m, n)-*society* as a collection of n sets such that for any subset of m sets, there are at least k sets with a non-empty intersection. Notice that $k \leq m \leq n$. Unless otherwise specified, we assume $k \geq 2$. If the order of the society does not matter, we simply call it a (k, m)-*society*.

Due to the nature of (k, m)-societies, the following three statements are always true. These facts are easily verifiable and allow us to use the knowledge we have to generate bounds on societies for which we do not yet have information.

1. A (k, m)-society is a $(k-1, m)$-society
2. A (k, m)-society is a $(k, m+1)$-society
3. A (k, m)-society is a $(k-1, m-1)$-society

The *agreement number* of a point p in X, our set of platforms, is $a(p)$, which is the number of sets which contain p. The *agreement number* $a(S)$ of a society S is the maximum agreement number over all points in X, so

$$a(S) = \max_{p \in X} a(p).$$

Finally, the *agreement set* of S consists of points that are contained in $a(S)$ sets. Figure 4.2 shows a society with agreement number 4, and the shaded rectangles cover the agreement set.

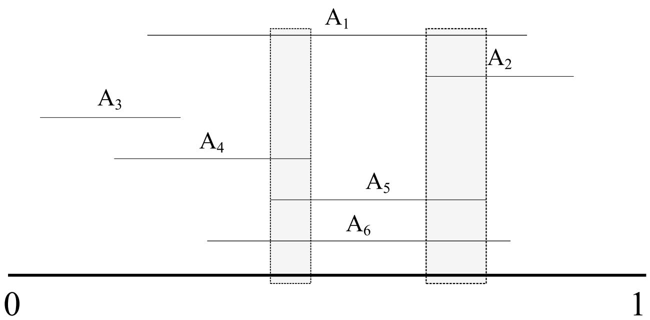

Figure 4.2: Agreement Set of a Society with Agreement Number Four

The *piercing number* of a collection C of sets, $\Pi(C)$, is the minimum number of points necessary such that every set contains at least one point. A *piercing set* of C is a collection of points such that every set in C contains at least one point in the collection. If we wish to emphasize that C is d-dimensional, we write the piercing number as $\Pi^d(C)$. We sometimes need to know the largest possible piercing number given certain conditions, such as for a (k,m)-society. In this case, we write $\Pi(k,m)$ or, to be more explicit, $\Pi^d(k,m)$. In certain cases, $\Pi^d(k,m)$ depends on n, the total number of sets; the remainder of the time, it depends solely on d, k, and m.

Figure 4.3 depicts a collection of three sets such that each two intersect, but the intersection of all three sets is empty. This means that the piercing number must be at least two, and indeed, two points that pierce all sets are shown in the figure.

The *distance* between sets, $d(a,b)$, is the minimum number c such that there are sets $v_1, v_2, \ldots, v_c$ for which the following intersections are all non-empty: $A_a \cap A_{v_1}, A_{v_1} \cap A_{v_2}, \ldots, A_{v_c} \cap A_b$.

In Figure 4.2, we see that the distance between sets 2 and 3, for example, is 1, because their intersection is empty, but sets 2 and 3 both have points in common with set 1. If no collection of sets will satisfy this condition, we say $d(a,b) = \infty$. If $d(a,b) = 0$, then we say the sets are *adjacent*. In Figure 4.2, for instance, we see that sets 1 and 2 are adjacent. Upon closer inspection, we realize that set 1 is actually adjacent to every other set.

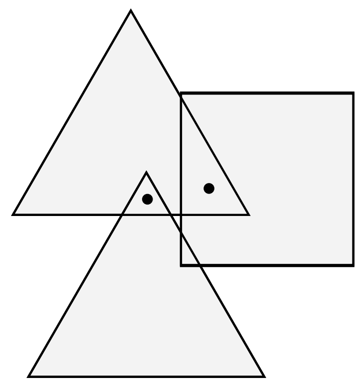

Figure 4.3: Collection of Sets with Piercing Number Two

4.3 Helly's Theorem

A society is simply a set of sets, so work concerning set intersections can be applied to this problem. The most well known theorem in this area is Helly's theorem. This theorem was proven by Helly in 1913, but the result was not published until 1921, by Radon [14]. A proof of Helly's theorem may be found in [11].

Theorem 4.1 (Helly)**.** *Given t convex sets in $\mathbb{R}^d$ where $d < t$, if every $d+1$ of them intersect at a common point, then they all intersect at a common point.*

Besides Helly's theorem, another famous theorem about set intersections is the Knaster-Kuratowski-Mazurkiewicz (KKM) lemma [10], which is concerned with set intersections on simplices. More recently, Niedermaier and Su [12] proved a set intersection theorem on trees that generalizes both Helly's theorem and the KKM lemma to this context. Since a line is a tree, their results are applicable to the $d = 1$ case.

An excellent source of information on Helly's theorem and related theorems is Wenger's "Helly-Type Theorems and Geometric Transversals" [17]. This paper includes theorems, explanations, and open problems in the field.

4.4 Relationships Between Agreement and Piercing Numbers

In this section, we introduce some theorems that relate the agreement numbers and piercing numbers of societies.

Theorem 4.2. *The sum of a society's agreement number and its piercing number is no more than the order of the society plus one, so* $a(S) + \Pi(S) \leq n + 1$.

Proof. Note that the agreement number of a society is equivalent to the maximum number of sets that can be pierced by a single point. Thus, if there are n sets in a society S, we can choose a point p in the intersection of $a(S)$ sets and then one point for each of the $n - a(S)$ sets not containing p. Together, these are $n - a(S) + 1$ points such that each set contains at least one of those points, so $n - a(S) + 1 \geq \Pi(S)$. □

Theorem 4.3. *The product of a society's agreement number and its piercing number is at least the order of the society, so* $a(S)\Pi(S) \geq n$.

Proof. We know $a(S)$ represents the largest number of sets that can be pierced by a single point and $\Pi(S)$ is the smallest number of points needed to ensure that each set contains at least one point. Therefore, there can be no more sets than $a(S)\Pi(S)$, which represents the largest number of sets in this configuration that could ever be pierced by $\Pi(S)$ points. □

One nice property about the piercing number is that there are many inequalities which follow directly from the definition. For instance,

Theorem 4.4. $\Pi^d(k, m) \geq \Pi^{d-1}(k, m)$.

Proof. Given a (k, m)-society in $d - 1$ dimensions, we consider the (k, m)-society in d dimensions formed by taking the product of the sets of the original society and $[0, 1]$. This new society has the same piercing number as the original society. Since $\Pi^d(k, m)$ is the maximum possible piercing number of a (k, m)-society in d dimensions, $\Pi^d(k, m) \geq \Pi^{d-1}(k, m)$. □

4.5 Piercing Numbers and Agreement Graphs

To further illustrate methods of finding the piercing number of a (k, m, n)-society, we consider the collection of sets shown in Figure 4.4. Figure 4.4 is a collection of 10 sets such that in any 3 sets, there is some point that is contained in some 2 of them. In other words, it is a $(2, 3, 10)$-society, which we call S.

Theorem 4.5. *The society in Figure 4.4 has piercing number* 5.

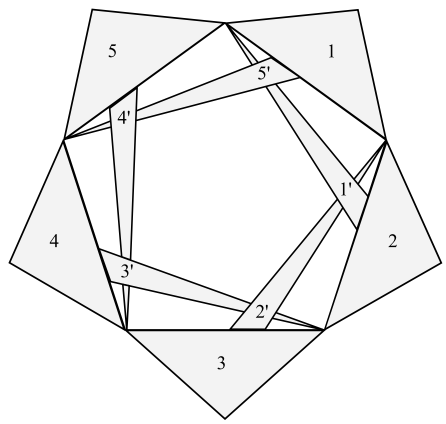

Figure 4.4: A $(2, 3, 10)$-Society of Convex Sets

Proof. Note that no point in S is contained in more than three convex sets. Thus, $a(S) = 3$. By Theorem 4.3, this means that $\Pi^2(S) \geq \lceil \frac{n}{a(S)} \rceil = \lceil \frac{10}{3} \rceil = 4$.

Suppose $\Pi^2(S) = 4$. Since $a(S) = 3$ and there are 10 sets total, we see that at least two of the points must pierce three sets each, and that these sets must be unique. Therefore, these two points must be in the agreement set and must not be adjacent. Without loss of generality, we can choose the points shown in Figure 4.5. This leaves four sets which are not pierced and two points left to place. However, this configuration cannot be pierced with only two points, so $\Pi^2(S) > 4$.

We see that the five points where the large triangles meet pierce all of the sets, so $\Pi^2(S) = 5$. $\square$

If we are to understand other kinds of agreeability, it is helpful to examine methods of pictorially representing the agreeableness of a society.

We construct an *agreement graph* G by letting each vertex represent a set and drawing an edge between two sets if the sets are adjacent. Notice that the distance between two sets (as defined earlier) is just the graph distance in the agreement graph. Moreover, if $d(a, b) = \infty$, then there is no path from a to b, so we say that a and b are in different *components* of the agreement graph.

Agreement graphs do not yield perfect information, as we see later, but we can determine whether the original graph is a $(2, m)$-society for any given m by examining the agreement graph. This is because agreement

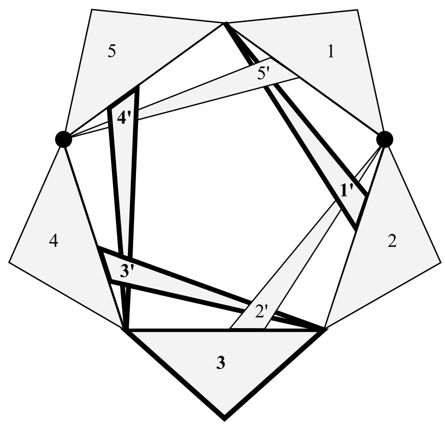

Figure 4.5: A Collection of Ten Sets with Piercing Number Five

graphs show 2-intersections, but not 3-intersections.

Theorem 4.6. *The society in Figure 4.4, whose agreement graph is shown in Figure 4.6, is a* $(2,3)$*-society.*

Proof. Suppose the contrary, that we can find three sets that do not share any points. We first notice that we cannot choose both sets a and 1, because they have a non-empty intersection. Additionally, sets 1 and $1'$ are adjacent to the same sets (namely $2, 2', 5$, and $5'$). Therefore, choosing set 1 is equivalent to choosing set $1'$, so we can effectively eliminate the vertex $1'$ from our agreement graph. By symmetry, we can also eliminate vertices $2', 3', 4'$, and $5'$, so we are left with a 5-cycle. It is trivial to verify that a 5-cycle meets the $(2,3)$ condition, and this implies that the society in Figure 4.4 is a $(2,3)$-society. $\square$

Thus, we see that $\Pi^2(2,3) \geq 5$. Notice that we can similarly construct a society S of size $2r$ for $r \in \mathbb{Z}$ such that $\Pi(S) = r$. All societies constructed in this manner are $(2, \lceil r/2 \rceil)$-societies and have agreement number 3.

This gives lower bounds, but these bounds can be improved significantly by considering the example shown in Figure 4.7. We can view each line segment as a convex set, and we see that every line segment intersects every other line segment. Additionally, the intersection of three or more of these sets is always empty, so the agreement number of this society is 2.

Notice that by Theorem 4.3, if we have n such sets and the agreement number is 2, the piercing number is $\lceil \frac{n}{2} \rceil$. In other words, the maximum

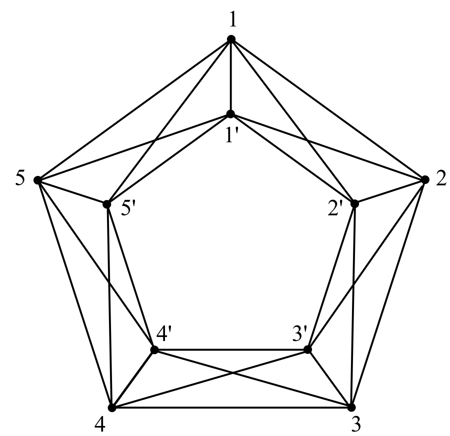

Figure 4.6: Agreement Graph of Figure 4.4

possible piercing number for a $(2,2)$-society in the plane is unbounded. Because $(2,m)$-societies are $(2,2)$-societies for all $m \geq 2$, this result extends to $(2,m)$-societies, as well.

We can easily extend the example in Figure 4.7 to $d > 2$ dimensions by taking the product of each set with $[0,1]^{d-2}$. Because we extend each set in the same manner and directions, the agreement number remains the same. Thus, we see that $\Pi^d(d,m) = \lceil \frac{n}{2} \rceil$ for $d \geq 2, m \geq d$.

4.6 Families of Societies

Certain societies of sets yield inequalities for piercing numbers of (k,m)-societies that can be generalized. For instance, in Figure 4.8, we have a $(3,7)$-society with piercing number 5. However, we can easily generate a $(3,6)$-society with piercing number 4 by using four inner triangles and four outer triangles arranged in the same manner as those in Figure 4.8, and we can similarly create a $(3,8)$-society with piercing number 6 by using six inner and six outer triangles. Let societies of this type as described be denoted by C_m, where C_m is a $(3,m)$-society. In fact, in this family, n is uniquely determined by m, namely, $n = \lfloor \frac{3m}{4} \rfloor$. Therefore, these are $(3,m,2\lfloor \frac{3m}{4} \rfloor)$ -societies.

In general, C_m has n triple intersections, and these intersections are one of two types. A Type I intersection is the intersection of two large triangles and one small triangle, such as the intersection of triangles $2,3$, and $3'$. A

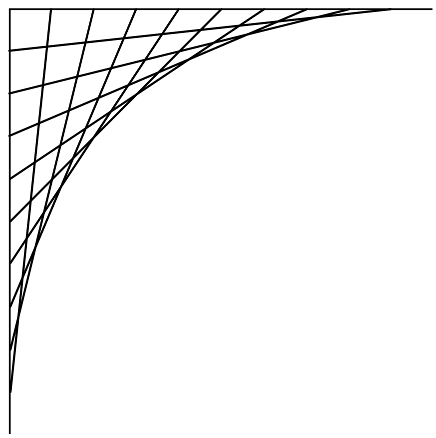

Figure 4.7: $(2, 2)$-Society with Agreement Number Two

Type II intersection is the intersection of two small triangles and one large triangle, such as the intersection of triangles $2', 3$, and $3'$. Due to symmetry, we see that there are $\frac{n}{2}$ intersections of each type.

Theorem 4.7. $\Pi^2(C_m) = \lfloor \frac{3m}{4} \rfloor$ *for integer* $m > 3$. *Thus,* $\Pi^2(3, m) \geq \lfloor \frac{3m}{4} \rfloor$ *for integer* $m > 3$. *In particular,* $\Pi^2(C_7) = 5$.

Proof. A key insight in this proof is that for any set i pierced by a point, there is some set j such that a single point pierces both sets i and j; similarly, for any sets i and j pierced by a point, there is some set k such that a single points pierces i, j, and k. Therefore, we may as well place each point in the piercing set in some triple intersection.

For convenience, we label the large triangles of C_m by $1, 2, \ldots, \frac{n}{2}$ in a clockwise manner, and we similarly label the small triangles $1', 2', \ldots, \frac{n}{2}'$, in such a way that triangle i' has an edge contained in an edge of triangle i'. In Figure 4.8, we have done this for the case $n = 10$.

Without loss of generality, we choose the intersection of sets $2, 3$, and $3'$ as a point in our piercing set. We can now ignore sets $2, 3$, and $3'$, since they already contain a point in the piercing set. Now consider splitting Figure 4.8 into a chain of triangles, as shown in Figure 4.9. We shall bear in mind that the solid dot and the dotted dot represent the same point.

We need some point in our piercing set to be contained in $2'$, and the only triple intersections involving $2'$ are $1 \cap 2 \cap 2'$, $1' \cap 2 \cap 2'$, and $2' \cap 3 \cap 3'$. We no longer need to consider 3 and $3'$, since they have already been pierced, so we choose either $1 \cap 2 \cap 2'$ or $1' \cap 2 \cap 2'$ to be in our piercing set.

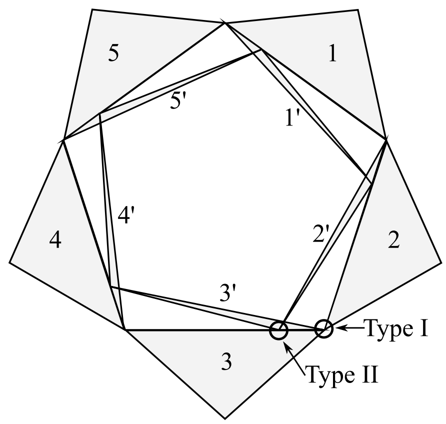

Figure 4.8: A $(3,7)$-Society with Agreement Number Three and Piercing Number Five

Figure 4.10 shows the agreement graph of C_7, where the dark triangle represents the point chosen to be in the piercing set and the light triangles represent the points that could be chosen to be in the piercing set such that set $2'$ is pierced. We see that for each node j in the agreement graph, j and j' are interchangeable, in the sense that they are adjacent to the same nodes. Thus, without loss of generality, we choose $1 \cap 2 \cap 2'$ to be in our piercing set. In the same manner, we continue counter-clockwise along the chain until we have chosen $3 \cap 4 \cap 4'$. This is a total of $\lfloor \frac{3m}{4} \rfloor$ points, and it is minimal because we chose an optimal point at each juncture. Thus, $\Pi^2(C_m) = \lfloor \frac{3m}{4} \rfloor$, and consequently, $\Pi^2(3,m) \geq \lfloor \frac{3m}{4} \rfloor$. $\square$

Notice that if we ignore the shading, Figure 4.10 is the same as the agreement graph of Figure 4.4, though the original graphs are different. Thus, we see that agreement graphs are not unique.

4.7 Summary

In this section, we summarize some results from the literature and some that we have proved in this paper.

1. $3 \leq \Pi^2(3,4) \leq 13$ [9]
2. $\Pi^2(3,m) \geq \lfloor \frac{3m}{4} \rfloor$ (Theorem 4.7)

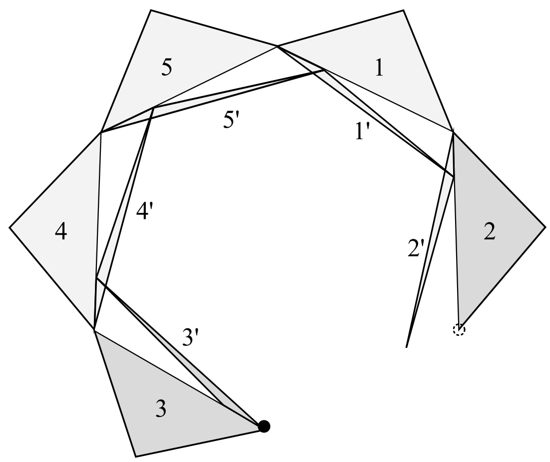

Figure 4.9: A $(3,7)$-Society with Three Pierced Sets

3. $\Pi^d(d+1, d+1) = 1$ [14]
4. $\Pi^1(2,3) \leq \lfloor \frac{n}{2} \rfloor + 1$ (Theorem 4.1)
5. $\Pi^d(k,m) \geq \Pi^{d-1}(k,m)$ (Theorem 4.4)
6. $\Pi^d(k,m) \leq \Pi^d(k,m+1)$
7. $\Pi^d(k,m) \leq \Pi^d(k-1,m)$
8. $\Pi^d(k,m) \leq \Pi^d(k-1,m-1)$

Item (1) is the result of a paper by Kleitman, Gyárfás, and Tóth [9], which focuses on finding $\Pi^2(3,4)$. The lower bound is found by constructing C_4, and the upper bound is found by identifying "special configurations" that can be pierced with at most 5 points. They then show that any two-dimensional $(3,4)$-society contains at most two of these configurations, and that the sets in the $(3,4)$-society but *not* in the configurations can be pierced with at most 3 points.

Notice that (6), (7), and (8) are direct consequences of the three statements in section 4.2.

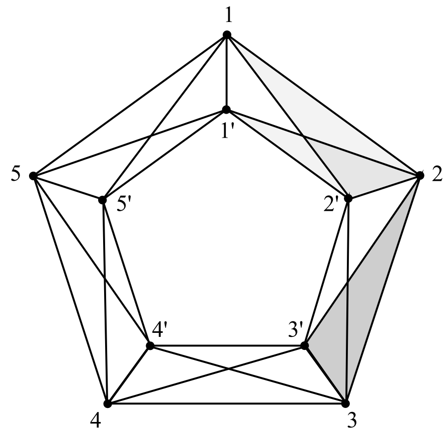

Figure 4.10: Agreement Graph of Figure 4.8 with 3 Intersections Shaded

4.8 Cubes

Cubes have many symmetries, so we can study them in order to better understand polytopes in general. We first examine some basic properties of cubes. A d-dimensional cube, or d-cube, can be represented as $[0,1]^d$. Additionally, d-cubes have 2^d vertices and $2^{d-1}d$ edges. The number of edges results from choosing one of d dimensions that the edges can be in, and then choosing one of 2^{d-1} vertices from which the edge can originate. Notice that the vertex that the edge must end at is uniquely determined by the dimension and the originating vertex, so there are exactly $2^{d-1}d$ edges.

When we let the edges of a d-cube C be sets, the piercing number is 2^{d-1}. Because there are $2^{d-1}d$ edges and each vertex is incident with d edges, we see that the inequality in Theorem 4.3 is a strict equality, since $a(S)\Pi(S) = n$. Figure 4.11 shows a 4-cube pierced with 8 colored points. Each edge is pierced by exactly one point, so the edges are colored according to which point pierces them.

One question we may wish to ask is whether cubes are the only configurations of sets such that $a(S)\Pi(S) = n$. We see that we need a certain amount of symmetry, since this equality requires that each point pierce the same number of sets.

If we consider line segments as sets, we see that an even number of sets in a path or in a cycle will yield $a(S) = 2, \Pi(S) = n/2$, so cubes are not the only configurations so that $a(S)\Pi(S) = n$. Additionally, if $a(S)$ or $\Pi(S) = 1$, it must be the case that $a(S)\Pi(S) = n$. We see that different types of

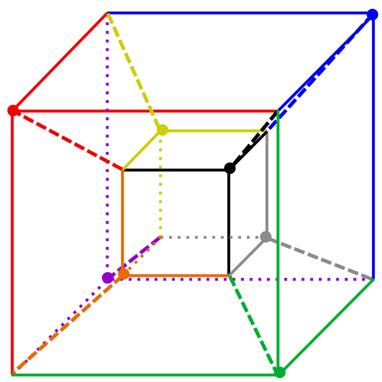

Figure 4.11: Edges of a 4-Dimensional Cube Pierced with Eight Points

configurations have this equality, so we may wish to find a characterization of all such collections of sets.

We now work through a simple example by using the d-cube. In Table 4.1, we view the edges of a d-cube as sets in a collection. In general, there are $2^{d-1}d$ such edges, and the piercing number of the d-cube, as shown in the third column, is 2^{d-1}. The agreement number is simply the dimension of the cube, since every vertex is incident with one edge in each of the d directions.

Dimension	(k,m)	$\Pi(S)$	$a(S)$	n
0	—	0	0	0
1	—	1	1	1
2	$(2,3)$	2	2	4
3	$(2,5)$	4	3	12
4	$(2,9)$	8	4	32
d	$(2,2^{d-1}+1)$	2^{d-1}	d	$2^{d-1}d$

Table 4.1: Edges of d-cubes as Sets in a Collection

By examining some of the properties of cubes, we have laid the groundwork for future explorations of d-dimensional voting spaces.

4.9 Speculation and Open Questions

One problem with the agreement graph is that it currently only indicates whether two sets have anything in common. We may wish to see if some three sets have at least one point in common, and we cannot do this with the current method. Therefore, one area of future exploration is in the direction of making an agreement graph that is more helpful with regards to more than two sets.

Additionally, we might wish to relax requirements and allow voters' preferences to be any convex d-dimensional figure, rather than restricting the sets to d-cubes.

Chapter 5

The Prison Warden's Dilemma: A Characterization of 3-Convexity

Thus far, we have dealt only with convex sets, as generalizing to any kind of set yields no significant results. In July of 2005, Tyler Seacrest and I began discussing generalizations of convexity, to attempt to find non-convex sets about which conclusions could be drawn. Star-shaped sets, which are sets S such that for some point $p \in S$, $\overline{px} \subset S$ for all $x \in S$, seem not to have enough convexity to produce good results, and so we considered sets S such that for every three points in S, at least one of the line segments between two of those points is included in S. Following is our paper discussing these sets, which are commonly known as 3-convex.

Abstract

In this paper, we study properties of *m-convex* sets in $\mathbb{R}^n$, which are sets such that for any m points, some line segment between two of them is entirely contained in the set. We give a necessary and sufficient condition for a set to be 3-convex, namely, that for every point $p \in S$, S is the union of a star-shaped set centered at p and a convex set.

Suppose a prison has m prisoners, and the warden knows that if any 2 of them can collaborate, they will be to able work out a plan to escape. The warden can overhear the prisoners, so they cannot communicate verbally, but he worries that the prisoners might use hand signals to communicate plans of escape. In what kind of rooms could the warden place prisoners such that the prisoners cannot see each other?

To answer a question such as the above, we first consider the case where $m = 3$.

Definition 5.1. *A set S is m-*convex *if for any m points $p_1, p_2, \ldots, p_m \in S$, at least one of the line segments created by pairs of these points is completely contained inside S.*

Thus, if a set is 3-convex, three prisoners cannot be placed such that they cannot see each other. These 3-convex sets are also known as having the *three-point convexity property* P_3 [16]. For this paper, note that we work within the space of closed sets $\mathcal{S}$ and in $\mathbb{R}^n$.

We say that two points p_1 and p_2 *see each other* with respect to some set S if the line segment $\overline{p_1 p_2}$ lies entirely within S. A set of points $\{p_\alpha\}$ is *visually independent* [8] if no pair of points in the set can see each other. Additionally, given a set S, we denote the *star* around a point $p \in S$ by S_p; this is the set of all points in S that can see p.

A set S is *locally convex* at a point if there exists some neighborhood N around that point such that $N \cap S$ is convex. Otherwise, the set is *locally non-convex* at that point.

In this paper, we prove the following theorem.

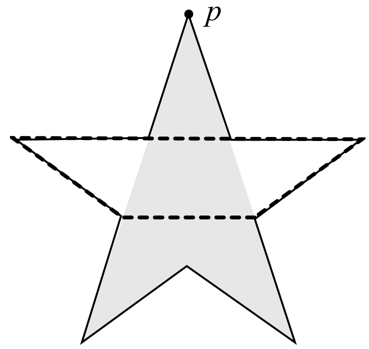

Figure 5.1: Union of a Star around p and a Convex Set

Theorem 5.1. *A closed set S is 3-convex if and only if for every point $p \in S$, S is the union of S_p and a convex set.*

For example, in Figure 5.1, we see that the star is the union of the shaded star around the point p and the dotted convex set. It is fairly simple to prove that for a five-pointed star S, any point p is such that S is made of a star-shaped set centered at p and some convex set. Notice that this figure shows that a closed, 3-convex set S need not be the union of two convex sets.

Note also that the statement of Theorem 5.1 would be false if the closed condition were removed. Figure 5.2 is an example of a set that is 3-convex but not closed, due to a point hole labeled a. Given the point p as shown in the figure, we see that the figure cannot be written as the union of S_p (the shaded region in the figure) and a convex set.

Valentine [16] showed that if a closed, connect set in $\mathbb{R}^2$ is 3-convex, then the set is the union of at most 3 convex sets that have a non-empty intersection, which he proved considering the set of points of local non-convexity of S. He also claimed that this number could not be improved, as we showed in Figure 5.1. Other authors have used points of local non-convexity as a powerful tool for investigating 3-convex sets. Kay and Guay characterized when 3-convex sets are the union of two convex sets [8]. They also found that an m-convex set is the union of at most $m-1$ star-shaped sets, a result we improve upon in the case of $m = 3$. Perles and Shelah determined that an m-convex set in $\mathbb{R}^2$ is the union of at most $(m-1)^6$ convex sets [13].

To prove Theorem 5.1, we first establish two lemmas.

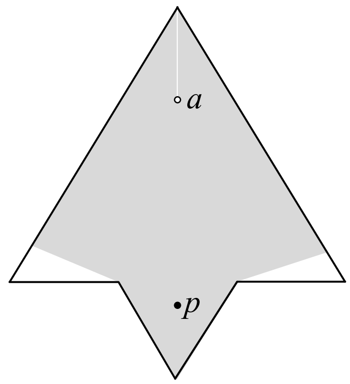

Figure 5.2: A Non-Closed Counter Example to Theorem 5.1

Lemma 5.1. *Let S be a closed 3-convex set. If P is a convex polytope of dimension at least two whose boundary lies within S, then the interior of P lies completely within S.*

We provide two proofs for this lemma, the first of which is self-contained and the second of which relies on previous work by Valentine [16] and Tietze [15].

Proof. For the sake of contradiction, suppose there is some point x in the interior of P such that $x \notin S$.

Consider a 2-dimensional plane containing x, and set x as the origin. Let r_1, r_2, r_3, and r_4 be the four rays extending from x along the coordinate axis, as shown by the thick dotted lines in Figure 5.3. We note that some point of r_i is in S, since r_i intersects the boundary of P, which is contained in S. Furthermore, we know that there is some first point of r_i that is in S, since S is closed. We therefore define p_i to be the first point along r_i that is contained in S.

By the definition of the p_i, the interiors of the line segments $\overline{p_1 p_3}$ and $\overline{p_2 p_4}$ are disjoint from S. In order to satisfy the 3-convex condition, we see that either both $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$ must lie entirely within S, or both $\overline{p_1 p_4}$ and $\overline{p_2 p_3}$ must lie entirely within S. Without loss of generality, suppose S contains both $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$, as in Figure 5.3.

By elementary geometry, we know there must be some ray r_5 extending from x that does not intersect either of the lines containing $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$. One such ray is shown in Figure 5.3 as a thin dotted line. Let p_5 be $\liminf(r_5 \cap S)$, where points are again ordered with respect to distance

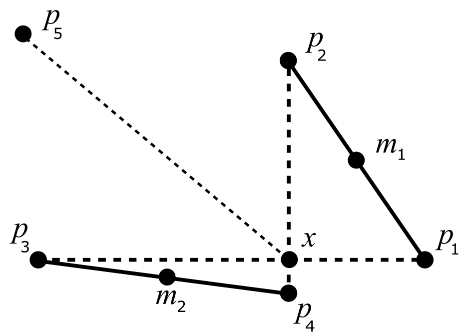

Figure 5.3: Construction Used in the First Proof of Lemma 5.1

from x. Notice that the interiors of the dotted line segments $\overline{xp_i}$ are disjoint from S.

Let m_1 be the midpoint of $\overline{p_1p_2}$ and m_2 be the midpoint of $\overline{p_3p_4}$. We claim that p_5, m_1, and m_2 cannot satisfy the 3-convex condition. We see $\overline{p_5m_1}$ is not contained within S because it would intersect either $\overline{xp_1}$ or $\overline{xp_2}$, (depending on where p_5 is located), which were disjoint from S. A similar argument shows that $\overline{p_5m_2}$ is not contained in S, so by the 3-convexity property, $\overline{m_1m_2}$ must be in S. However, we easily see that $\overline{m_1m_2}$ intersects both the interiors of $\overline{p_2p_4}$ and $\overline{p_1p_3}$ (the dotted lines in Figure 5.3), which are not in S. This contradicts the 3-convexity of S. □

Alternate proof. Let S be a closed, 3-convex set. From Valentine [16], we know that such a set is star-shaped with respect to each of its points of local non-convexity. If S has no points of local non-convexity, then by Tietze [15], S is convex. Therefore, suppose there is some point p of local non-convexity, so S is star-shaped with respect to p. Suppose there exists some point x in the interior of P that is not contained in S.

As shown in Figure 5.4, consider the line containing p and x. We see that since x is surrounded on all sides by the boundary of P, there must be some point q on this line such that q is in the boundary of P and $x \in \overline{pq}$.

Therefore, $q \in S$, but the line segment $\overline{pq}$ cannot be completely contained in S, since it contains x. This contradicts the fact that S is star-shaped with respect to p. Therefore, our supposition was incorrect, and no such x exists. Thus, the interior of P lies inside S. □

Lemma 5.2. *Let S be a closed 3-convex set. If P is a convex polytope of dimension at least two whose edges and vertices are completely contained in S, then all of P is contained in S.*

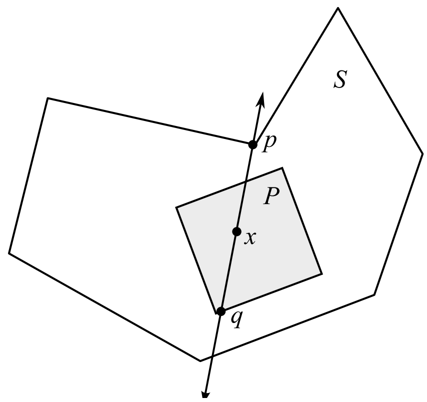

Figure 5.4: Construction Used in the Alternate Proof of Lemma 5.1

Note that this differs from Lemma 5.1 because we require only the edges and vertices to be in S, rather than requiring that the entire boundary be contained in S.

Proof. We shall show, in fact, that the statement is true for the k-faces of P. Note that if P is of dimension n, then P is an n-face of itself. We prove the lemma by inducting on k. Suppose that every k-face of P lies inside S. Notice this is true by supposition for $k = 1$, the base case. By definition, this means that the boundary of every $k+1$-face lies inside S, as this boundary consists of k-faces of P. Using Lemma 5.1, we see that the interior of every $k+1$-face lies in S. Then an inductive argument shows that P is contained in S. $\square$

Having established the previous lemmas, we now prove Theorem 5.1.

Proof. Suppose S is closed and 3-convex, and choose an arbitrary point $p \in S$. Let C be $S - S_p$, so C consists solely of points that cannot be seen by p. Choose $c_1, c_2 \in C$, and notice that $\overline{c_1c_2} \subset S$, because $\overline{c_1p}$ and $\overline{c_2p}$ are not in S. This is the case for all points in C. Let C' be the union of all polytopes with vertices in C. Clearly, $C \subset C'$ since every point in C is a polytope and therefore contained within C'. We now show C' is convex and completely contained within S; this will show that S is the union of S_p and a convex set C'.

Figure 5.5 depicts an example of the results of such a process. The set S is a five-pointed star, and S_p is the shaded portion of the figure. The set

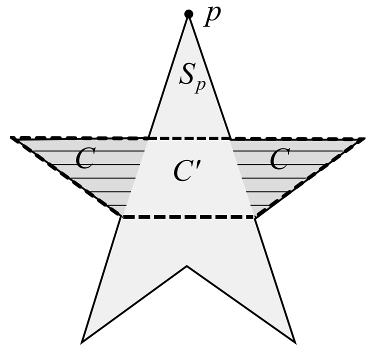

Figure 5.5: Star-Shaped and Convex Sets in a Five-Pointed Star

C is disconnected, and its two components are striped in our figure. The set C' is shown surrounded by a dotted line. While the results of such a procedure can vary, this example helps provide an idea of the workings of our proof.

Let a and b be points in C'. By definition, a is in some polytope A with vertices in C, and b is in some polytope B with vertices in C. Therefore, both a and b are in the polytope defined by the union of the vertices of A and B. Since a and b are in a polytope contained in C', we know $\overline{ab} \subset C'$. Therefore C' is convex.

Consider an arbitrary point $x \in C'$, which is by definition in some polytope P with vertices in C. Notice that every edge of P is contained in S, because line segments between two points of C must be in S. Therefore, P is a polytope whose edge set is in a 3-convex set S. By Lemma 5.2, P is completely contained in S, so $x \in S$. Therefore, $C' \subset S$.

We now prove the contrapositive: suppose that for a closed set S and any point $p \in S$, S is the union of a star-shaped set centered at p and a convex set C. We shall show that S is 3-convex. Choose three arbitrary points, p_1, p_2, and p_3, and consider S_{p_1}. If p_i is in S_{p_1} for $i = 2$ or 3, then $\overline{p_1 p_i} \in S$. Otherwise, both p_2 and p_3 are in the convex set C, so $\overline{p_2 p_3} \in S$. Thus, for three arbitrary points, at least one of the lines connecting some two of those points must lie entirely in S. □

Notice that this proof is valid for all finite dimensions, though our examples consider sets in the plane.

We have thus far considered 3-convex sets, but we may wish to consider m-convex sets. Consider the following example.

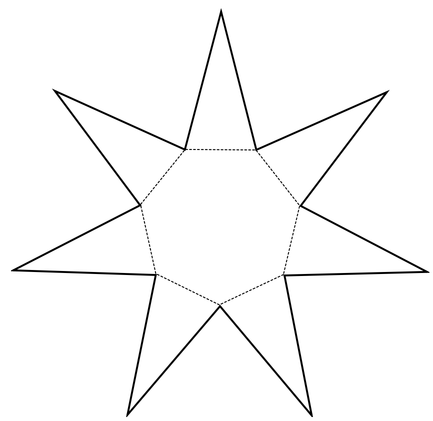

Figure 5.6: A Regular Seven-Pointed Star

Theorem 5.2. *The regular seven-pointed star S shown in Figure 5.6 is* 4*-convex.*

Proof. Suppose by contradiction that there are 4 visually independent points in S. Let G_i be the dark gray region associated with vertex i, and let H_i be the entire shaded region associated with vertex i, as illustrated in Figure 5.7. Notice that the star of any point in G_i contains all of the shaded region H_i. Additionally, note that the union of the G_i covers S, so any point must be in at least one G_i. Without loss of generality, suppose we choose a point p_1 in G_1. We can see that $H_1 \subset S_{p_1}$, so since we choose our next point from $S - S_{p_1}$, p_2 cannot be in H_1. In fact, we see from the picture that the next point must be in at least one of G_2, G_3, G_6, and G_7, and without loss of generality, we can assume that the next point is in at least one of G_2 and G_3. If $p_2 \in G_3$, then since $S - (H_1 \cup H_3)$ is a convex set, we know p_3 and p_4, which must be in $S - (H_1 \cup H_3)$, can see each other, which is a contradiction. However, if $p_2 \in G_2$, $S - (H_1 \cup H_2)$ is still a convex set, so by the same principle, p_3 and p_4 are not visually independent. Thus, we cannot choose four points that are visually independent in this figure, so it is 4-convex.

□

Is there an analog to Theorem 5.1 for 4-convex sets, or even for m-convex sets in general?

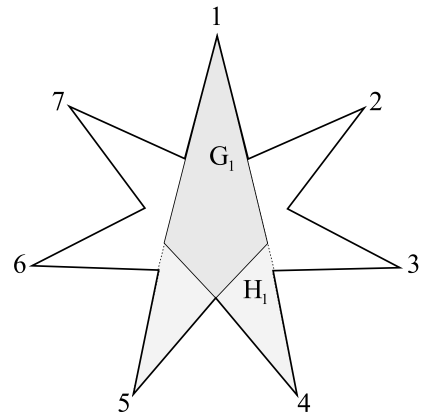

Figure 5.7: Regions in a Regular Seven-Pointed Star

Acknowledgements

The authors gratefully acknowledge the advice and encouragement of Francis Su and the support of his research grant NSF DMS-0301129.

5.1 Discussion

While we were successful in proving an interesting theorem concerning the nature of 3-convex sets, further study revealed no useful applications of these sets to voting theory. This certainly does not mean that there are no applications, but there do not appear to be applications that extend previous results in a natural manner.

Chapter 6

Conclusions

During this year of research, I have found that graph theory is an excellent tool with which to explore the concepts of voting theory. I have formed a structure in which voting can be studied, using one-dimensional representations of voters' views, and I have invented and adapted terminology to facilitate my work. I have applied prior theorems, such as Helly's theorem [1], to this area, and I have also developed some theorems of my own.

By working with Professor Robin Thomas, Professor Su, and Tyler Seacrest, I have deepened my understanding of certain areas of graph theory and convexity. I have considered various generalizations of the convexity argument and concluded that non-convex sets are not good ways of modeling people's preference sets, either in terms of realism or in terms of drawing conclusions.

While working, I have been amazed at the connections between different areas of mathematics. I have found material from my Social Choice and Decision Making class and my Graph Theory class very helpful, and to my surprise, I found that knowledge gained from my Topology class was also useful for voting theory. Additionally, the importance of viewing problems from different perspectives has been impressed upon me several times, both by academic papers and by other people with whom I share my ideas.

Another important lesson I have learned during my thesis is that questions can be either deceptively simple or deceptively complex. For example, I was surprised that Kleitman et al. [9] wrote an entire paper on what I call $(4,3,t)$-societies, as this is an extremely specialized case. However, after further study, I realized that even that single case was quite hard. In fact, their investigations have not yet unearthed proof of a tight bound of the maximum possible piercing number of a $(4,3,t)$-society. They currently have an example of a $(4,3,6)$-society with a piercing number of three, which is shown in Figure 6.1, and a proof that the piercing number cannot be greater than 13.

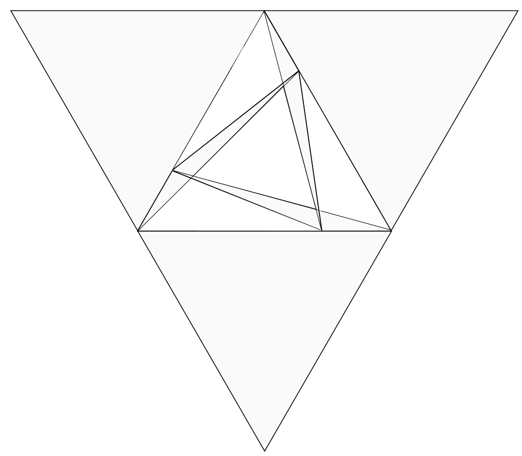

Figure 6.1: A $(3,4,6)$-Society with Piercing Number Three

Notice that the example that has been found is not trivial, and it is expected that if an example of a $(4,3,t)$-society with a piercing number of four can be found, it will be even more complicated. This society consists of the three large shaded triangles and the three shaded triangles inside the center large triangle. We see that at each vertex of the center large triangle, three sets meet, but since the remaining three do not all intersect, the piercing number cannot be two. However, we see that the three vertices of the center triangle form a set of three points such that each triangle contains at least one point, so the piercing number must be three. Note that this problem deals with convex sets in general, so they do not have to be triangles or even polygons; this is merely an example.

I have also learned a lot about submitting papers to research journals. My paper with Tyler Seacrest has already been reviewed, and some of the reviewers' comments have proven to be quite helpful. On the other hand, some of the comments have been cryptic or have shown that the reviewer did not fully understand parts of the paper, so we have attempted to clarify those parts.

Perhaps the most important thing that I learned while writing this thesis is that ideas in one field can lead to advances in other fields, even if those fields may not seem to be related. I have also learned the importance of getting a fresh perspective on ideas by explaining them to others.

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