Journal of Humanistic Mathematics

Volume 7 | Issue 2 July 2017

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Cover Page Footnote
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The Biblical Value of Pi in Light of Traditional Judaism[1]

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Abstract

There are numerous attempts at a solution to the puzzle as to why the Biblical value of pi, as demonstrated by the measurements of King Solomon's pool, is exactly 3 and not a better approximation. This article shows that virtually all such published solutions are deficient because volume-based factors are ignored. Issues respecting the volume of this pool are explained, and some possible solutions of the puzzle are presented.

1. Introduction

> "He made the sea (*yam*) of cast [metal] ten cubits from one lip to its [other] lip, circular all around, five cubits its height; a thirty-cubit line (*kav*) could encircle it all around. Knobs under its lip surrounded it ten cubits [in length], girding the sea all around, two rows of the knobs, which were cast with the casting [of the sea]. It stood upon twelve oxen, three facing north, three facing west, three facing south and three facing east; the sea was on top of them, and their hunches were towards the center [of the sea]. Its thickness was one hand-breadth; its lip was like the lip of a cup, with a rose-blossom design; its capacity was two thousand *bath*-measures." [I Kings, 7:23–26][2]

[1]An alternate version of this paper, focused on Torah, was published in the Winter 2016 (volume 22) issue of the Torah journal, *Hakirah*. Common sections are used with permission from *Hakirah*. See http://www.hakirah.org.

[2]Here and elsewhere where Biblical and Talmudic material is quoted, I use the ArtScroll translations.

We learn from I Kings, 7:23–26 that King Solomon commissioned for the Temple a cast bronze (or copper) pool of measured dimensions at 10 cubits diameter, round with a circumference of 30 cubits, 5 cubits high and the wall was 1 handbreadth thick. This yields the ratio of the circumference to diameter, designated as the mathematical constant π, equal to exactly 3.

Nobody would give the value, $\pi = 3$, a second thought if we were dealing with an ordinary document. Who would be surprised if a three-millennium old document were to designate $\pi = 3$? But this is not just some random document; this book is part of the Hebrew Biblical canon. Could it be that the Bible is wrong, or is it possible that the designation $\pi = 3$ has a sophisticated meaning and purpose that is not immediately apparent? We have here a millennia-old cold-case mystery that has generated much literature, including this paper.

Will this paper finally resolve this mystery? Alas, there is no final resolution here, if such a possibility even exists. In fact, this paper makes the situation somewhat worse, as we will show that many of the proposed solutions are flawed when viewed from the perspective of traditional Judaism. The approach here is to consider the implications of $\pi = 3$ within the context of rabbinic, known as Orthodox, Judaism. We will deal not only with the question of why $\pi = 3$ for I Kings, but also why $\pi = 3$ in the Talmud, dated over a thousand years after the time of King Solomon. In the process we will examine some of the many mathematics-based suggestions intended to show that while the direct, or simple, meaning of the text is that π equals 3, there are other meanings embedded in the text to yield a much better approximation for π. We will also introduce some additional suggestions that had not been considered before.

While the primary focus is of a mathematical nature, that is, to find an acceptable approximation for the value of π within the structure of this pool, we take it in this paper that this effort needs to be consistent with the perspective of Torah-based Judaism. Hence we will next provide some context for the reader as to what this "perspective" is about. Also please note that "structure of this pool" refers to the physical shape and volume of the pool, which will be addressed a bit later.

2. What is Torah?

The word "Torah" generally refers to the Hebrew Bible; often meaning the five books of Moses. But this is quite a narrow understanding of what "torah" is about. The word itself means teaching or instruction or doctrine, and how this word is to be understood depends on the context. For example, we have the statement in Proverbs 1:8, "Hear, my child, the discipline of your father, and do not forsake the teaching (*torah*) of your mother." We have a lower case "t" in torah here, as opposed to capital "T" when dealing with the sacred writings of the Biblical Torah. The basic Hebrew Scriptures consisting of the five books of Moses are identified by the word *Chumash*, which is of the same root as the word "five." A direct translation would be the Pentateuch. The full canon of the Hebrew Bible is known by the Hebrew acronym, *Tanach*, consisting of the Torah (the five books) plus the other books, such as the Prophets, which are committed to writing. This constitutes the Written Torah, and the book of I Kings is part of this Written Torah.

There is also an Oral Torah. According to Jewish tradition, this consists of information imparted to Moses at the same time as when the material for the "five books" was given to him. But while the content of the "five books" was to be committed to writing, this related information was to be transmitted orally from teacher to student. We have a list of primary teacher-student pairs starting with Moses and Joshua and up to the time of the Talmudic sages, whereby the Oral Torah was transmitted to us. The Oral Torah is consistent with, but different from the Written Torah. It can be understood as application instructions for the basic rules contained in the Written Torah. If the Written Torah were compared to the characteristics of a mechanism — a computer or airplane — the Oral Torah would be the operating instructions. Both are necessary, and both are equally sacred within traditional Judaism as both stem originally from the same sacred Source via the prophet, Moses.

The Oral Torah, though intended for transmission from teacher to student, is no longer completely oral. It was redacted into written form under the leadership of Rabbi Judah the Prince, in about the year 200 of the Common Era (CE) in response to the dispersion of Torah study across distances and borders and the Roman persecutions that made the teaching of Torah a capital offense. This material is known as *Mishnah* — from the root word meaning "review." The individual *mishnas* that we now have are generally identified with teachers spanning the years 20-200 CE, and this is known as the *Mishnaic* period or alternatively as the *Tannaitic* period based on the title — *Tanna* (teacher, repeater) — for the teachers of the oral *Mishnah*. The individual *mishnas*, loosely grouped by subject matter in six groupings comprising some sixty sections known as tractates, are short and terse as would be expected of material intended for memorization.

Once the existential threat that the Oral Torah might be lost was eliminated, subsequent generations of scholars spent roughly the next three centuries, till about year 500 CE, in study, analysis, and much needed commentary on the content of the *Mishnah*. The result is the *Gemarah* (from the root word to study and learn) and together the *Mishnah* and *Gemarah* comprise the Talmud (study, to learn). Many people - the teachers and primary students from the Torah academies - participated in the development of the Talmud. All together there would have been thousands of scholars over a period of near 500 years in the development of the Talmud.

Discussions in the *Gemarah* scrupulously identify who said what and why. Final conclusions, some of which took several generations of scholars to develop, are traced ultimately to the redacted *Mishnah* or to the Written Torah, and these are considered of equal validity. Sometimes, though, a result is traced to the teachings of the *Tannaim* (Mishnaic teachers) through a path parallel to the redacted and written *Mishnah*. This is because not every possible item in the oral transmission from teacher to student was put into writing. Some items just didn't make it into the written *Mishnah*, but were still remembered and taught in the Torah academies. This material, known by the name of *baraita*, has found its way into the Talmud by way of the *Gemarah* when someone will quote a *baraita* in support of his position. A *baraita* has the same authority and force as a *mishnah* in the traditional Jewish understanding of Torah.

Thus, if we have in a *mishnah* in the Talmud that a circle of circumference equal to 3 has a diameter equal to 1, this cannot be simply dismissed as a mistaken opinion of an ignorant individual. If it is stated in the Talmud that the pool had a certain volume with a reference to a *baraita*, we take it that this was indeed the volume. The position of the Talmud (Oral Torah) and the position of the Torah (Written Torah) are taken as correct. Hence we wonder how to reconcile a statement that the constant we designate as π equals 3 within a context of Torah (in the broad sense), with what we know from mathematics.

3. The cold-case mystery of the Biblical value for π

It would appear from a simple reading of the text that neither the Israelites at the time of King Solomon (ca 950 BCE) nor the Talmudic sages over a thousand years later (20-500 CE) were aware that the ratio of the circumference to diameter of the circle, designated by the symbol π, is greater than 3. As previously noted, the dimensions of what is known as Solomon's pool provided in I Kings (see the quoted text starting Section 1) yield a value for π of exactly 3. Likewise, the Talmudic text of a *mishnah* in Talmud Tractate Eruvin (page 13b) provides, among other matters, information about circles saying in particular, "Whatever has a circumference of three handbreadths has a width of a handbreadth." This text also clearly designates the value $\pi = 3$.

But it is virtually impossible that some people were not aware that the result is more than 3. Solomon's Temple, and all connected with it, was built with precision and skill. The king had access to the most skilled and experienced craftsmen. Thus, from I Kings 7:13, "King Solomon sent and took Hiram from Tyre ... He was full of wisdom, insight and knowledge to do all sorts of work with copper, so he came to King Solomon and performed his work." And we can take it for granted that there were no budgetary constraints. These expert builders would have established, by measurement and observation, various approximations to π well before the time of King Solomon. Ordinary people might have been ignorant on this matter, but certainly not those involved in the construction of the Temple. One might say that it does not matter here what the builders of Solomon's Temple knew or did not know, because they did not author the Books of Kings. But we are told that the dimensions given in I Kings 7:23 are based on measurements via a measuring rope (*kav*), see again the beginning of Section 1. Is it possible that the builders left a record of measurements that they made, or was this measurement made at a different time? Could the author of I Kings himself have made the measurements?

All agree that the author of the Books of Kings, whether the prophet Jeremiah according to Jewish tradition (Tractate Bava Basra, 15a), or someone else, lived at the time of the destruction of the Temple some four hundred years after it was constructed. The date of destruction by the Babylonians is given as 587 BCE according to secular historians. Traditional Judaic dating differs,[3] but it does not matter because the sources agree that the Temple stood for near four hundred years. How likely then is it that the author had access to original information from the builders? The author would have personally seen this pool, and we can expect that he would not have described it as round, had it been square. Likewise the rotational symmetry implied by "it stood on twelve oxen" would be verified by the author from personal observation. But while the general appearance would be clear to the author that is not the case for the dimensions. Here we depend on measurements, and that is what the measuring rope (*kav*) is understood to mean. There are two logical possibilities as to the source of the dimensions: The author made the measurements himself or he had access to results of measurements made by someone else. In either case, the error implied by a measurement of 30 for the circumference when it should be nearer to 31.5, or a diameter of 10 when it should be near 9.5 or some combination of both, is difficult to accept given the expectation that the author was a learned person. Did he not know that the ratio of the circumference to diameter of the circle is more than 3? Possibly he did not, but possibly there is more involved here than a simple error in measurement or a matter of ignorance respecting circles. We will examine later what "more involved here" might be.

Even though one might possibly argue that the author of I Kings was ignorant respecting the geometry of circles, it is all but impossible to make this claim for the Talmudic sages about one thousand years later, from 587 BCE to the completion of the Talmud near year 500 CE. We are dealing here with hundreds, possibly thousands, of people over a period of nearly 500 years while the Talmud was finally completed. This is hundreds of years after Archimedes (287-212 BCE) established the relationship

$$\frac{223}{71} < \pi < \frac{22}{7}.$$

Some of these people traveled all over the world and were conversant with the general knowledge of their time. Not only were some of these people up-to-date respecting the mathematical knowledge of their time, but they were

[3] While Judaic dating shows 410 years for the Temple, secular dating yields 370 years. Easily accessible links are a `Wikipedia` article which references the Judaic sources for 410 years (https://en.wikipedia.org/wiki/Solomon%27s_Temple) and a Britannica reference which provides dates which yield 370 years (https://www.britannica.com/topic/Temple-of-Jerusalem).

skilled in mathematical procedures. In particular, Tractate Eruvin (*eruv* means merging), where we find the statement that yields $\pi = 3$, deals with the separation and merging of spaces, areas, and volumes. Hence this tractate includes a great deal of geometric analysis and computation. These people had skill when dealing with mathematical issues.

An example will be useful. Here is a simple problem that calls for very little explanatory background which will illustrate the approach to, and skill in mathematical analysis by the Talmudic sages.

This problem is discussed in Tractate Eruvin pages 23a (*mishnah*) and 23b (*gemarah*). The translation in this example, as elsewhere where the Talmud is quoted, is per the ArtScroll edition.

A *karpaf* is an enclosed area that is significant under some circumstances. We have in the *mishnah* that certain behaviors are permitted in the *karpaf* "provided the *karpaf* is only seventy *amos* (cubits) and a fraction by seventy *amos* (cubits) and a fraction, but not larger." In other words, the square *karpaf* may approach, but not reach, 5000 square cubits. However, an area of a full 5000 square cubits is acceptable for a rectangular area where "its length is twice its width." This is a $50 \times 100 = 5000$ rectangle. Skipping various *mishnah* and *gemarah* considerations we get to the mathematical problem. How do we convert from the 50×100 rectangle with an area of exactly 5000 to a 70+ by 70+ square whose area is a touch less than 5000? The *Gemarah* says: "The Torah instructs: take fifty and surround fifty."

The authoritative commentator Rashi (1040-1105 CE) explains the procedure for "take fifty and surround fifty."

> We cut the 100×50 rectangle into two 50×50 squares. This gives us two squares, one to be surrounded by the other.
>
> We now cut one of the squares into five 10×50 strips with which we will "surround" the remaining 50×50 square. We now lay one strip next to each side of the square which yields 70×70 with empty corners.
>
> The fifth, remaining 10×50 strip is cut into five 10×10 squares, four of which are used to fill in the corners of our newly constructed square which is now a complete 70×70 square. One 10×10 square remains.

> We now cut this square into 30 strips of width $\frac{1}{3}$ and length equal 10. The ratio $\frac{1}{3}$ is easy to achieve because there are 6 handbreadths to each cubit. Hence, the strips are two handbreadths wide. Laid end-to-end we have a strip 300 cubits long and $\frac{1}{3}$ cubit wide. Placing strips along the borders of the square we have a square of width $70\frac{2}{3}$, except we are missing the corners, which are filled in from the remaining 20 cubit long strip $(300-4\times 70 = 20)$.
>
> The remaining strip is $18\frac{2}{3}$ long by $\frac{1}{3}$ wide.

The remaining strip can be further divided for a better approximation, but the result obtained thus far deviates from an area of 5000 by just a bit over 0.1%. And while the procedure is cumbersome and time consuming, it gives us an exact measure of the area by which our square is less than the area of the 50×100 rectangle.

Focusing on rabbinic approximation techniques, modern mathematicians Tsaban and Garber [1] begin their paper with a history of the Talmud and other matters, such as the apparent conclusion by Maimonides that π is irrational, that the mathematics-focused reader might find interesting. But the main topic of interest to us is the analysis of mathematical procedures to be found in the Talmud which yield (according to their formulation) good approximations for π and $\sqrt{2}$. Tsaban and Garber introduce the value $3\frac{1}{7}$ for π, and point to a derivation of $3\frac{15}{106}$ as embedded in the Talmud. This is the mathematical $\pi_2 = \frac{333}{106}$, which we will discuss in the last section of this paper. They also show excellent approximations for the square root of two, including the value 1.413 which is connected to our calculation of the area of the square *karpaf*. We find in their Footnote 33: "It is said that twice the side of a square whose area is 5000 square cubits is equal to $141\frac{1}{3}$ cubits, i.e., $2\sqrt{5000} = 141\frac{1}{3}$, whence... " we arrive at $\sqrt{2} = 1.413$. This is the result, provided that $\sqrt{5000} = 70\frac{2}{3}$ per our approximation for the *karpaf*.

Given the above, it is all but impossible that some of the Talmudic sages did not know that π is not equal to exactly 3. In fact, we have the approximation $\pi = 3\frac{1}{7}$ directly in a book on geometry, *Mishnat ha-Middot*, generally attributed to Rabbi Nehemiah (ca 150 CE). Furthermore, the author connects this approximation to the structure of Solomon's pool where the value 3, and not the more accurate $3\frac{1}{7}$, is obtained by measuring the diameter (10) to the outside brim while the circumference (30) is measured along the inner brim. For a more detailed discussion about this book, and other factors respecting the Biblical and Talmudic choice at $\pi = 3$, see [2].

Yet, the Talmudic sages let stand the statement that a circle with circumference 3 has a diameter 1. What is going on here?

What is going on is that the Talmud is not intended to teach mathematics; it is intended to teach Torah (in the broad sense). Furthermore, there is more involved here than the simple text of the specific verses; we need to consider the information in context.

In traditional Judaism, texts are explained using several levels of explanation, of varying degrees of sophistication. Traditionally, these levels are designated by the Hebrew terms: *pshat* (ordinary meaning), *remez* (meaning derived from a hint), and *sod* (meaning derived from a secret). The *pshat* is the meaning that ordinary people can understand, though it may not be simple and may require an explanation from a learned individual. The *remez* requires expert knowledge to follow the trail of a hidden hint. The final level is hidden in a secret (*sod*) that can only be understood by exceptional individuals. Or the secret may have to wait many years to be uncovered as other knowledge becomes available. We will explain how the questions about π are addressed on all three levels.

But first a digression to establish the volume of Solomon's pool which we will need as we analyze various suggested solutions to the puzzle as to why we are informed that $\pi = 3$.

4. A matter of volume

A circumference at 30 cubits and diameter at 10 cubits, per I Kings, introduce our problem whereby π is found to be exactly 3. Suggested solutions take into account various bits of information that we find in I Kings 7:23–26 as quoted earlier. Among these bits of information is the volume which has to be consistent with the given dimensions. Unfortunately the stated volume at "two thousand *bath*-measures" introduces two difficulties; one is a matter of confusion and the other is a matter of substance.

The confusion stems from the volumetric unit which in transliteration from Hebrew to English comes out as "*bath*." This is a unit of volume in the Hebrew language and not connected to the word "bath" in English. To add to the confusion we will also introduce the volume of the ritual bath (the word bath again) known in Hebrew as a *mikvah*. The reader will need to be careful to avoid confusing different "baths."

The substantive difficulty is best explained by quoting Rabbi Adin Steinsaltz [3]: "Units of volume in the Talmud are among the most complicated units of measurement, because of the existence of several independent systems. . . "

A volume consistent with dimensions in "cubits" would be in cubic cubits. Here the reference standard is a unit of length: the cubit. But I Kings states the volume in *baths*, where the reference standard is the volume of an egg. We need to correlate one set of units to the other, and this takes quite a bit of analysis. The interested reader will find this analysis in an Appendix, below. Here we will simply provide the essential results.

- The cubit (*amah* in Hebrew) is the distance from the elbow to the end of the middle finger. This is commonly estimated for ease of use at 18–24 inches; a more accurate range, but more difficult to use without a calculator, is provided in the appendix. There are two cubits: the standard cubit at 6 handbreadths and the short cubit at 5 handbreadths. "Cubit" without a designation usually refers to the standard cubit and the handbreadth is set to be between 3 and 4 inches.
- 2000 *bath* measures comes to 150 ritual bath or ritual pool (*mikvah* in Hebrew) volumes at 3 cubic standard cubits each.
- The result that we have been seeking is that the volume of Solomon's pool was 450 ($= 3 \times 150$) cubic standard cubits at 6 handbreadths to the cubit.

Having established the volume, the Talmud investigates shapes for Solomon's pool that yield a volume of 450 cubic standard cubits. The result is a square shape for the lower 3 cubits with a volume of 300 cubic standard cubits, and a circular shape for the upper 2 cubits with a volume of 150 cubic standard cubits, based on $\pi = 3.0$. The total is the required 450 cubic standard cubits. An alternate suggestion with the bottom 4 cubits square and the upper one cubit circular is rejected as not meeting the volume requirement. This alternative will be of interest to us later.

Now that we have the dimensions and volume of the pool in a consistent set of units based on the cubit, we can proceed to analyze these with the aim of arriving at or deriving a mathematically acceptable approximation for π.

5. *Pshat* — the ordinary meaning

The Talmud not only teaches Torah on a theoretical level, but also on a practical level involved in daily usage. For example, the statement in Eruvin 13b, previously quoted, that: "Whatever has a circumference of three handbreadths has a width of a handbreadth," is preceded by "if it was round we view it as if it were square." Surely these people knew that a circle is not a square, yet they make the ridiculous-seeming claim that one can treat a circle as if it were a square. In fact the treatment of a circle "as if it were a square" is not uncommon in the Talmud, and it appears several times in Tractate Eruvin.

What is the operational meaning of this phrase, and what is the practical reason for this? The operational process is to "square" (*m'rabeah*) the circle. Thus, we find in Eruvin 56b, "The Rabbis taught in a *baraita*: one who squares a circular city…" No, the Talmud did not engage in the impossible construction known as squaring the circle. Rather, the process was to draw a square with sides tangent to the circle. The objective in the case of the circular city of the above quote is to extend the legal (according to Torah) size of the city. A circular city of 2000 *amos* (cubits) in diameter "will be found to have gained four hundred *amos* (cubits) here and four hundred *amos* (cubits) there." That is, the distance to the corners of the square is 1.4 times the diameter, based on the approximation that $\sqrt{2} = 1\frac{2}{5}$. Thus, whatever is *halachically* (Torah-based law) legal to do within the actual city will extend beyond the circular perimeter and into the corners of the square; the city dwellers "will be found to have gained four hundred *amos* (cubits) here and four hundred *amos* there." We see that "squaring" of the circular city has a practical purpose. The result is that whatever one may do within the "actual" city is extended in distance by 1.4. The Talmud, however, does not deal in decimal notation, and the 0.4 is usually stated as $\frac{2}{5}$. The system deals in fractions and irrationals are approximated by a fraction. This applies to the irrational square root of two and it also applies to the irrational value of pi.

We know that π is irrational (even if the rabbis did not) so the rabbis had to deal with an approximation. We will discuss three basic interrelated reasons for choosing $\pi = 3$ for the preferred approximation; more complex reasons are discussed in the literature, see for instance [2].

These reasons include the previously discussed conversion between circles and squares for *halachic* (the adjective of *halachah* = Torah-based law) needs, ease of mental calculation and permitted approximations for that purpose, and a statement of a rule respecting the precision or accuracy to which calculations involving circles are to be carried out. I will now explain these reasons.

The previously quoted statement that we (sometimes) treat a circle as if it were a square, which is followed by the statement that a circle of circumference equal 3 has a diameter equal 1, are the last of a list of various pronouncements in a *mishnah* in Eruvin 13b. The *Gemarah* now proceeds to analyze the implications of this list of statements, which being part of the same *mishnah* are presumed to have some common elements. A page later, towards the end of page 14a, we get to the statement about circles and squares. The *gemarah* asks: "Why do I need this case to be taught" in this *mishnah*, given that this result is obvious from the analyses of previous statements in the same *mishnah*? This question needs to be understood in the context that the *Mishnah* is structured for memorization; hence information already known or alluded to is considered superfluous. The *gemarah* answers that this is not stated in its own right, but rather as an introductory clause to the next statement that a circle of circumference 3 is treated as having a diameter 1.Thus, the ratio 3 : 1 is a continuation of our understanding about the relationship between circles and squares. We see here that the 3 : 1 ratio is not a mathematical statement respecting circles, but rather a guide to the practical application of Torah law (*halachah*).

Halachah (Torah-based law) is not just a theoretical exercise; it is intended as something to be used on an ongoing basis. And not just by an elite group, but by ordinary people who might have some difficulty in the application of the rules. This was especially so two thousand years ago when writing implements were scarce and most calculations were done mentally. Hence we have a number of cases in the Talmud involving approximations chosen to make calculation of results easier.

Here is an example from page 8a of Tractate Succah. The issue involves the determination of the minimum size of the temporary hut called a *succah* which Jews erect on the Feast of Booths, known as Tabernacles. These are usually made with straight walls and angles, but what would be the minimum dimensions if one made the hut circular? Calculations show that the diameter of the circle should be at least 5.6 cubits $(5+\frac{3}{5})$. But the Talmud calls for the diameter to be 6 cubits. This choice introduces a whole number for ease of use by ordinary people, and it also provides a safety factor (metaphorically known as a fence, or safety fence) so that one will not transgress by making the hut smaller than permitted. The Talmud asks: "When can we say that a Talmudic sage was imprecise [that is, not in agreement with a known or calculated result]? Only when the disparity between the precise amount and the approximation is small. [And] where the approximation results in a stringency." The error in this case is $\frac{6-5.6}{5.6}$ or 7%, and the minimum size of the *succah* hut is made larger than it has to be; hence this is "a stringency." This approximation is accepted. Another approximation involving a 40% difference is rejected as not being "small." Thus, approximating π as 3, or a 5% difference, is acceptable.

An approximation that yields a strict result is still recognized as an approximation which is made for the sake of convenience or legal (Torah-based law) safety. But there are cases in *halachah* where distances are ignored as if they did not exist. This is more than just an approximation, as the missing distance has no "legal" status. For example, a wall that has a gap of under 3 handbreadths can in some circumstances be considered as a continuous wall as if the gap simply did not exist. The Rosh (late 13^{th}–early 14^{th} century commentator) indicates that this is the case for a choice of 3, rather than a more accurate value, for π. He bases his position on the fact that the Talmud cites the description of King Solomon's pool in choosing a ratio of $3:1$. He asks why it was necessary to cite a scriptural verse for something that is easily determined by measurement, and especially when the verse does not give an accurate value. He concludes that this is meant to teach us that we are to ignore the difference and treat the circumference as only 3 for *halachic* (Torah law) purposes.

Thus there are good reasons to accept that the ratio $3:1$ was chosen for practical and *halachic* (Torah law) purposes and is not in any way indicative of a statement respecting the mathematical properties of circles. The constant, π, after all, is irrational; no matter what value we choose, it is all an approximation. In what way then is 3 not an acceptable value? Elishakoff and Pines [2] put the matter this way:

> "*How good is good enough?* Even the 1.2 trillion digit approximation of π made by Professor Yasumasa Kanada of Tokyo University in 2002 is still only an approximation. It is humbling to realize that there is something that we can never really know, and π provides us with this experience."

It is difficult to exaggerate the accomplishments of Moses Maimonides (1135-1204 CE) in Torah, astronomy, mathematics, philosophy, medicine… So it should not come as a surprise that many hold that, though he did not provide a mathematical proof, Maimonides argued that the mathematical constant we now call π has the characteristics that we associate with the designation "irrational".[4] Tsaban and Garber [1] quote from Maimonides' commentary on our *mishnah*: "You need to know that the ratio of the circle's diameter to its circumference is not known and it is never possible to express it precisely. This is not due to lack in our knowledge…, but it is in its nature that it is unknown… but it is known approximately…" They conclude their paper with several approaches to the matter that we are investigating, the first of which is: "The rational approach of Maimonides holds that, since we cannot know the exact value, the Bible tells us that we do not have to worry about this and that it suffices to use the value 3."

This should be sufficient to close the matter. But there are people who insist that there is more involved here; that there are hidden hints and secrets within the description of Solomon's pool that yield a credible approximation to the value of π.

6. *Remez* — a hint to a more accurate value

We begin with a slight digression to analyze the calculation of volume in the Talmud. Tsaban and Garber [1], citing various space-altering miracles claimed for the Temple, state that "… Munk [see *sod*, below] suggests a mystical explanation...; In the temple, the ratio of the circumference of a circle to its diameter was exactly π_0 [that is 3.0]." In other words, the space around the pool was miraculously non-Euclidean and the volume of the pool was exactly 450 cubic standard cubits, as calculated. This would conclude

[4] The time of Maimonides was some 600 years after the completion of the Talmud, and mathematical knowledge was much advanced. But let us not forget that this was also some 600 years before Lambert proved in 1767 that π is indeed irrational.

the matter and there would be no need to proceed further. Usually, however, we take it that the space around the pool was normal, that is, Euclidean, and the ratio of the circumference to diameter of the circle is π and not 3.0. Hence, while the calculated volume is 450 cubic standard cubits, the true volume is more. The lower 3 cubits in the shape of a square are not affected, and this is 300 cubic standard cubits. But the upper two circular shaped cubits have an actual volume not at 150 cubic standard cubits, but $150\left(\frac{\pi}{3}\right) \approx 157$ cubic standard cubits. The total volume is 457 cubic standard cubits, which yields a near 1.6% increase to 450 cubic standard cubits. Using the approximation that $\pi = 3$, introduces a "small" difference in volume, and because the actual volume is larger than calculated, we have a "stringency." That is, we have a 7 cubic standard cubits safety factor (fence) within the calculation using $\pi = 3$.

There is no problem with the slightly larger volume, and all is also well if we accept a "mystical" explanation. But in Euclidean space an inner diameter of 10 means that the inner circumference must be more than 30. So we are back to the original question — why is it that I Kings and the Talmud use 10 and 30? The simplest explanation is to argue that the 10 cubit diameter is the outside diameter of the cylindrical pool, while the 30 cubit circumference is for the inside of the pool. This yields a ratio that is greater than 3.

The earliest such suggestion, previously noted, is from Rabbi Nehemiah (ca 150 CE), where π is set at $3\frac{1}{7}$; a value known from Archimedes. So if this *Mishnah*-era scholar knew a more accurate value for π, why was this not reported in the Talmud? One can conjecture various reasons, including that Rabbi Nehemiah was not able to influence the choice for the Talmud. But the simplest and most obvious reason is that this value was reported in a book about mathematics where a mathematically correct (or as correct as was known at the time) value was used. The Talmud, however, is not a book about mathematics and the value reported in the Talmud has its own, non-mathematics-based reasons, as was previously discussed. See also references [1] and [2] for additional information about this question.

A more recent and more significant suggestion along the same lines comes from Rabbi Levi ben Gershon, known as the Ralbag (an acronym for his name) (1288-1344). The Ralbag was a noted scholar of Torah, philosopher, mathematician, and scientist. In addition to his publications on Torah, he also authored several important books on mathematics and science, and he is known for his introduction of the astronomical instrument known as Jacob's staff. No one can doubt his commitment to Torah and no one can doubt his knowledge of mathematics. Hence his attempt at a solution to our puzzle is important.

The Ralbag proceeds with the shape proposed by the Talmud, with the 3 bottom cubits in a square and the top 2 in a cylinder. The result, noted by Simonson [4] shows a volume of 446.8 (this would be $446 + \frac{4}{5}$) cubic standard cubits under the best assumptions, and Ralbag states that his result is approximate; that is, he does not ignore the need for 450 cubic standard cubits. This volume is easy to calculate. Thus, the lower 3 cubits are in the shape of a 10×10 square with a volume of 300 cubic standard cubits. The upper two cubits are cylindrical with an outside diameter of 10 cubits, equal to 60 handbreadths at 6 per cubit. The walls are 1 handbreadth thick; hence the inner diameter is 58 handbreadths. Equivalently, the diameter is $\frac{58}{60} \times 10$ cubits, and we have that $2 \times \pi R^2 = 146.78$ cubic standard cubits.[5] It appears that the mathematically correct value for π was used in the reported result. The volume would be a bit less had the computed value for π, based on the proposed solution, been used.

Whatever the mode of calculating the volume, the discrepancy versus 450 cubic standard cubits, at near 1%, is a "small" amount. But the result is not a "stringency" since the volume is less than the called for 450 cubic standard cubits. One can only speculate why the Ralbag, who was thoroughly conversant with and totally devoted to the cause of Torah, would violate the dictum that calls for a stringency. Here are two possibilities.

The primary purpose of "stringency" is to build a protective fence (*seyag* in Hebrew) against a transgression. This is in the same category as the addition of a "safety factor" in common usage. No one would specify the use of a material in construction at the limit of its load bearing capacity; the specification includes a safety factor or margin. Likewise in Torah-law (*halachah*). Thus, no one would build a *mikvah* (ritual bath) to hold just the minimum required volume. But this pool, which was used for ablutions of the priests as they officiated in the Temple, had a capacity to hold 150 minimum *mikvah* volumes. The safety factor is enormous; hence, the concept of "stringency" does not apply here.

[5] This is my reconstruction of how the 446.8 result might have been obtained, but Ralbag would not have used a decimal notation.

We should also note that the Ralbag was careful to not mix his mathematics and Torah. Thus "[h]is mathematics rarely contains spiritual discussions, and his Biblical commentary does not often contain mathematics, but there is at least one notable exception" [4]. This "exception" is the computation discussed here. This is a computation from a mathematical perspective respecting basic information taken from Torah. And while Ralbag was careful to show that the mathematical relationship for the circle is a ratio greater than 3, he would, no doubt continue to use a ratio of 3 : 1 in matters of Torah, as this is not intended to be a mathematical statement. The only deficiency, if the word "deficiency" may be used, is that we get a rather poor approximation for π. Thus, the outer diameter is 10 cubits which is 60 handbreadths at 6 handbreadths per cubit. The inner diameter is 58 handbreadths with a wall thickness at 1 handbreadth. The inner circumference is 30 cubits which is 180 handbreadths. Hence the ratio is $\frac{180}{58} = 3 + \frac{3}{29} = 3.10\ldots$; a rather poor approximation for the value of π.

Other attempts, using variations of the procedure proposed by Ralbag, yield more accurate values for π. But these are disallowed either because of problems with the volume or the ratio of the cubit to handbreadth. Let us look at an example.

A very accurate value for π attributed to one Bob Graff is reported on the web site: http://www-gap.dcs.st-and.ac.uk/~history/Miscellaneous/other_links/Graf_theory.html, accessed on June 1, 2017. Here the cubit is set at a very small 17.75 inches and the handbreadth is set at its maximum value of 4 modern inches. This makes the outer diameter of the cylindrical pool 177.5 inches, and the inner diameter is 169.5 inches. The inner circumference is $30 \times 17.75 = 532.5$ inches, and $\pi = \frac{532.5}{169.5} = 3.141593$; precise to five decimal places. The web site shows that the choice of 17.75 inches and 4 inches is not arbitrary, as $\frac{532.5}{169.5} = \frac{355}{113}$. This is the third convergent of the simple continued fraction expansion of π, known as π_3. We will discuss this value, π_3, later. At this time it will suffice to note that this suggestion is not acceptable because the ratio of cubit to handbreadth is assumed to be $\frac{17.75}{4} = 4.4375$, while the actual relationship is 6 for the standard cubit and 5 for the small cubit.

There are a number of other ingenious suggestions to solve the matter differently, but these also violate conditions imposed by the Talmudic Sages. The reader will find an extensive collection of shapes and suggestions out of this dilemma compiled by Andrew Simoson [5]. The interested reader will find it worth the time to look at this paper as we will not replicate its content. Some of the suggestions are obviously not to be taken seriously; Simoson describes these as "whimsical." But many of the suggestions are of a serious nature; unfortunately they usually fail the volume or cubit to handbreadth ratio test. We leave it to the interested reader to determine the volumes and ratios for these shapes.

Here are two examples not discussed by Simoson, to add to Simoson's extensive list.

The hexagonal pool solution. The description of the pool in I Kings includes the phrase "rose-blossom design." The Hebrew word, *shoshan*, is translated here as "rose." But this word also means lily. This flower has six petals, sort of in the shape of a hexagon. A regular hexagon whose side s is 5 will have a circumference of 30 and a maximal diameter of 10. This fits perfectly the ratio of $3:1$. Hence some people suggest that the pool was not perfectly round, but hexagonal in shape.

The area of this shape is $\frac{3\sqrt{3}}{2}s^2 \approx 64.95$ square cubits for $s = 5$, so the volume is not quite 325 cubic standard cubits; significantly less than the called for 450 cubic standard cubits, so this suggestion does not work.

The flared lip solution. There are a number of analyses based on the idea that the pool had a flared upper shape wherein the diameter on top was larger than just below, which was of cylindrical shape. This, flared lip shape, is suggested by the language in I Kings which Peter Aleff translates as "made like a cup, shaped like the calyx of a lily." Aleff argues for a flared upper lip on his weblog [6], and as explained in an adaptation of his work [7].

Here the 10 cubit diameter is measured across the flared top, while the circumference is measured on the outside lower cylindrical body. Referencing various works on archeology and ancient science/mathematics (e.g., van der Waerden: *Science Awakening, Egyptian, Babylonian and Greek Mathematics* and Leen Ritmeyer: *The Temple and the Rock*), Aleff provides a well annotated argument for a fairly accurate value of π. He uses a rim thickness computed from 7 handbreadths to the cubit based on archeological analysis and Egyptian units. The bottom of the pool is taken as one handbreadth thick, and this is deducted from the 5 cubit height. Unlike other calculations that ignore this feature, the required volume of 2000 liquid measure *bath* is accounted for in the calculations. And Aleff shows that it all works out perfectly. His calculation of the volume after accounting for the thickness of the bottom and the flared lip is 304.04 cubic cubits, which he equates to 2000 *bath*. This appears to be contrary to the position of Talmud Tractate Eruvin where 2000 *bath* is equated to 450 cubic standard cubits. But it is not that simple, because Aleff uses a super large cubit of 7 handbreadths. The volumetric ratio to the Talmudic cubit of 6 handbreadths is $\left(\frac{7}{6}\right)^3$ and the two volumes can be made to agree by a judicious choice of the volume lost to the flared rim. The only issue is the choice of 7 rather than 6 handbreadths to the cubit.

A new suggestion. It is highly likely that the skilled craftsmen employed by Hiram on behalf of King Solomon understood the properties of circles in a practical way. They likely had rules of thumb based on accumulated experience, and/or possibly also mathematical approximations which we have from ancient documents. We know of two sources predating the time of King Solomon that might have been available to these craftsmen. The Egyptian Rhind papyrus shows how to calculate the area of a circle from which one can derive that $\pi = \frac{256}{81} \approx 3.16$. We will ignore this one because it does not provide a value for π directly, and because the derived value is not in the form of a simple fraction. The Babylonian approximation, based on the results from the analysis of the sexagesimal notation of a clay tablet from Susa (1900-1680 BCE), which yields the approximation $3 + \frac{1}{8}$, is more in line with our interests. The matter of this approximation, 3.125 ($= 3 + \frac{1}{8}$), is quite complicated and controversial. For one thing, this tablet is the only place from which one can obtain this approximation while other Babylonian tablets show a value of 3.0. A computer search on this matter will generate many responses. Quoting from one such item, by Jason Dyer [8]: "Because this is given as an actual fixed ratio (rather than being extrapolated from a circle area procedure) it's arguably the first discovered value for π. [...] Because this is given as an actual fixed ratio," and this ratio fits within the procedures we have in the Talmud, is why this approximation is "more in line with our interests."

We now ask: is there a shape for the pool that approximately fits the position of the Talmud that will yield a volume of 450 cubic standard cubits with π set at 3.125 ($= 3 + \frac{1}{8}$)? The answer is yes, as described below.

The bottom 4 cubits is rectangular with a volume of 400 cubic standard cubits. The upper cubit (6 handbreadths high) is circular with an outside diameter of 10 small cubits (at 5 handbreadths each) and an inner circumference of 30 small cubits. The wall thickness is one handbreadth; hence the inner diameter is 48 handbreadths. The circumference is $30 \times 5 = 150$, and so $\pi = 150/48 = 25/8 = 3.125$. We calculate the volume of this short cylinder as follows. The inside diameter is 48 handbreadths, or 8 regular cubits. The square of the radius is 16, and with π at 3.125 we get a volume of 50 cubic standard cubits. The total volume is the required 450 cubic standard cubits.

The mathematics works and the volume is right, but there are some concerns associated with this shape. Here are the concerns, and some possible responses.

- The Talmud clearly states that the proportion of the pool was 3 cubits in a square and 2 in a circle. Are we permitted to change this to 4 and 1? Perhaps we are because the Talmud considers both possibilities — 3 square and 2 circular, and also 4 square and 1 circular. The choice of 3 and 2 was made on the basis of volume calculations to yield 450 cubic standard cubits. But now we show that a shape of 4 and 1 also yields the correct volume. Hence this might be a permitted possibility.

- The more troubling item is the introduction of the small cubit of 5 handbreadths, and especially the mixing of the regular and small cubit in the same structure. The use of the small cubit, while unusual, is not rare and there are a number of items in Solomon's Temple that are measured in units of the small cubit. Mixing of different cubits in the same item is more unusual and is rare, but not without precedent.

- The surface area of this version, at a radius of 4 regular cubits, is smaller by the ratio $(5/4)^2 = 1.5625$ compared to the pool of the Talmud with a radius of 5 regular cubits. This means that fewer people could enter this pool together.

- On the whole, there is nothing that I know of that prohibits Solomon's pool to have consisted of a square shape on the bottom four fifths of its height and a circular section on top with the short cylinder consisting of both the regular and small cubits. Nevertheless, we should note that there is no commentary that suggests this idea.

- Possibly the most serious negative is not based on the requirements of the Talmud but on the grounds of esthetics. Having built a miniature model of this design, I find it difficult to see how this ungainly and peculiar looking structure could be considered an embellishment to the artistically elegant Temple.

A second new suggestion. The previous suggestion assumes the builders knew the approximation 3.125 for π and built the pool accordingly. Well, maybe they knew this value and maybe they did not. We don't really know. A claim that 3.125 was used by the builders is conjecture. But it is not conjecture that the ancients were excellent builders. We have physical evidence to support this position. Hence it is reasonable to expect that if the builders intended to build a circular pool, the rim would be highly circular. An accurate measurement of the ratio of the circumference to diameter would show a good approximation for π, say 3.14. Is there a way to get such a result using the information that we have in I Kings? Yes there is. I propose three related possibilities.

(a) The language in I Kings initially provides the large dimensions at 30, 10, and 5 cubits, and also introduces the measuring rope (*kav*) for determining these values. We then have a descriptive section involving knobs, oxen, a decorative floral design, and within this descriptive section we are informed that the wall was 1 handbreadth wide. We are not told how or even whether the 1 handbreadth was measured. Is it possible that the wall thickness was not measured at all? Could 1 handbreadth be a visual approximation that is "described" along with knobs and flowers, rather than stated as a measured result?

We find that a 1.34 handbreadth wall width will yield $\pi = 3.14\ldots$. A 34% discrepancy in a visual approximation is a bit much, but not impossible. The volume of this pool, however, will be only near 1970 *bath*; 1.5% less than the 2000 *bath* given in I Kings. Might the stated volume also be approximate? Ralbag appears to accept this possibility. A further refinement is to increase the height of the lower, square, section by 0.238 cubits and to reduce the height of the upper, circular, section by the same amount to achieve a volume of 2000 *bath*.

(b) We don't know how or whether the wall width was measured. But we do know that the large dimensions were determined by means of a measuring rope (*kav*). Taking the circumference at 30.2 cubits (0.67% deviation), and taking the diameter at 9.95 cubits (0.5% deviation) yields $\pi = 3.14\ldots$ Today we have tape measures with high-density markings to provide fairly accurate measurements. Here we expect marks at no closer than one handbreadth. Still, a 0.2 cubit deviation cannot be ascribed to estimation to the nearest mark error, as this is more than one handbreadth at $1/6 = 0.167$ cubits. But the distance involved is about 50 feet, so the way the rope is manipulated could introduce a one handbreadth deficiency.

(c) Finally, we can combine the two previous suggestions in multiple ways. For example. Taking the wall to be 1.1 handbreadths with a circumference at 30.1 cubits and diameter at 9.95 cubits, yields $\pi = 3.14\ldots$

7. *Sod* — a solution based on secret knowledge

For years this solution was attributed to Elijah, the Gaon of Vilna (18$^{\text{th}}$ century), known as the Gra. The Gra was a major Torah sage and also an accomplished mathematician. Hence it was assumed that he must be the source of this conjecture, even though no one could point to a specific reference from his writings. However, it is now well-established that this solution was first proposed by 20$^{\text{th}}$ century Torah scholar, Rabbi Matityahu Hakohen Munk (Max Munk). Belaga provides details of this discovery [9], and I have specifics on its publication from Rabbi Dr. Shnayer Leiman, contributor to the Biblical Encyclopedia.[6] The result uses an (alleged) secret hidden in the spelling of the Hebrew word *kav*, meaning measuring rope, with which the dimensions of Solomon's pool were determined.

The reader will need some background respecting numerical equivalents for the letters of the Hebrew alphabet, and the matter of written (*ketiv*) and spoken (*qeri* or *keri*) versions of the Hebrew scriptures for a full understanding of the implications of this "secret" hidden in the spelling of the word *kav*.

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[6] Per private communication from Professor Rabbi S. Leiman of Brooklyn College. "The Gra did talk about pi, but never suggested the secret interpretation ascribed to him. That interpretation was first suggested by Max Munk in 1939. He published his suggestion in *Shalosh Ba'ayot Handasiyot be-Tanakh uve-Talmud*, Sinai 51 (1962) 218-227."

Biblical-based writing does not have separate symbolic representation for numbers. These are presented in an ordinal manner based on the sequence of the letters of the alphabet. This is also how the ancient Greeks did it. The fact that letters and words can be understood as a numerical value gives rise to a system, or several systems, of numerology, known as *gematria* (pl. *gematriot* or *gematrios*), whereby one could discover deeper or hidden meanings in the text. This is obviously subject to abuse as one can cleverly introduce the wording of one phrase to fit the number value of another phrase. This is not permitted and only *gematriot* of proven provenance may be used, and only by a thoroughly learned individual. Torah commentator Nachmanides (Ramban) cautions in *Sefer Ha'Geula*, "One is not allowed to delve into the calculation of *gematriot* and deduce there-from some subject which has occurred to him." But there are *gematriot* which are embedded in the text itself, and these are not only permitted to be used, but point to a need for further explanation. We find the following in the introduction to the Ramban's monumental commentary on the Torah (the five books) [10]: "... all was written in the Torah, explicitly, or by allusion through certain words, either through *gematrios* or the forms of the letters... " Rabbi Munk uses the *gematria* values in the varied spellings for the word, *kav*, to arrive at a proposed solution to a better approximation for π. But first we need to learn the equivalence between letters and numbers in the Hebrew alphabet, and the implications of the varied spellings that one finds in the Torah (Torah in the broad sense).

The equivalence between the letters and numbers is simply a matter of the ordinal count of the letters. We start at one (1) for the first letter and end at 400 for the last of the twenty-two letters in the basic Hebrew alphabet. The first letter, *aleph* (A) = 1, the second letter *beth* (B) = 2... the fifth letter *hei* (H) =5, the sixth letter *vav* (V) = 6... Once we reach ten we start counting in tens till we get to the nineteenth letter, *qof* or *kof* (Q or K) = 100, etc.

The word *kav* is a two letter word, spelled *kof-vav* with a *gematria* value of $100 + 6 = 106$. This is the usual spelling of this word and this is how it is given in II Chronicles where the pool is described. But the spelling in I Kings adds a silent *hei* to the end of the word, for a *gematria* value of $100+6+5 = 111$. Manipulation of the values, 106 and 111, will yield a good approximation for π, as we will soon see; but first a word about variations in spelling in Biblical texts.

Occasionally a word is to be pronounced or read (*qeri* or *keri*), differently than how it is written (*ketiv*). This distinction may not be violated. A reader who happens to read a word as written in a formal setting when it has a *keri* designation will be stopped and the reading has to be repeated. A scroll that is written as per a *keri* rather than the *ketiv* is invalid and may not be used.

The modern printed version of I Kings will clearly show both spellings for the word *kav*. I will spell the *keri* as KV (to be pronounced *kav*) and the written version as KVH (also pronounced *kav*). Hence the printed version looks like this: KVH [$\mathrm{kV_k}$]. The subscript "k" in the brackets tells us that there is a *keri* involved here. People of even moderate knowledge might arrive at a lesson or purpose when the *keri* and *ketiv* differ in meaning or at least in pronunciation. But the matter is more difficult when there are no such differences. Torah-based hermeneutics introduce possible reasons for extra, silent, letters; a numerical interpretation via *gematria* is one of these. The fact that we have a *keri* and a *ketiv* tells us that there is a message embedded in the variant spellings. But that does not mean that we must deal with a *gematria* here. There might be a different purpose. Rabbi Munk has suggested a *gematria*, and what that consists of. But he is not a prophet; he has no direct access to hidden knowledge and his suggestion could be wrong. Nonetheless it is important to understand that it is incontrovertible within this tradition that there is a secret code embedded within the variant spellings of *kav*. There is a message here. Is it the one suggested by Rabbi Munk? An analysis of the mathematical implications can help us decide.

The proposed explanation is that we are to apply the correction factor 111/106 to the ratio 3 given in the text, and $\pi = 3(111/106) = 3.1415\ldots$. The purpose of the extraneous *hei* would be unknown even to the author who is identified in Jewish tradition as the prophet Jeremiah (Talmud Tractate Bava Basra 15a), who would have used this variant spelling on the basis of prophetic knowledge. But progress in mathematics eventually lead to recognition of the significance of the extra *hei* and the secret was revealed.

This factor, $111/106 \times 3 = 333/106$, has some interesting mathematical properties as discussed in the literature, see for instance [11]. We note the following: "[T]he simple continued fraction expansion of π is $[3, 7, 15, 1, 292, 1, \ldots]$ and its convergents are: $\pi_0 = 3$, $\pi_1 = 22/7$, $\pi_2 = 333/106$, $\pi_3 = 355/113\ldots$ The surface meaning of the text gives the value π_0, but this is deceptive, those in the know (so the story goes) see hidden in the text the much more accurate value π_2. Now either the Rabbinical tradition is responsible for π_2, and the author of I Kings surreptitiously coded into his text an extremely accurate value of π, or else we have a most remarkable numerical coincidence."

The "most remarkable numerical coincidence" notwithstanding, Simoson [5] quoting Deakin and Lausch [11] argues that what he calls "the hidden key solution" is most unlikely. Thus, from Simoson: "Deakin points out that if the deity truly is at work in this phenomenon of scripture revealing an accurate approximation of π, a much better fraction not far removed from 333/106 would most definitely have been selected instead." The proposed choice is 355/113 and we are given reasons why it should be so. This is certainly an excellent choice from a mathematical point of view. But it is virtually impossible from a practical or operational point of view. Remember that the objective is not only to provide a more accurate value for π but also to have the information hidden. Otherwise the author could have simply provided us with a more accurate value to start with. The key point is that we must start with $\pi = 3$ and the hidden knowledge is a correction to that value. But no correction is possible by multiplication as for 333, since 3 is not a factor of 355. A correction could possibly be contrived mathematically by judicious addition, but what Hebrew words or phrases could be found to surreptitiously introduce this value which needs to be in the form of a fraction? One can think of a number of reasons why the "hidden key" is not the answer. But there is no better result than 333/106 that works on a practical level.

Simoson also brings up possibly the most obvious objection: "A natural question with respect to this method is, why add, divide, and multiply the letters of the words? Perhaps an even more basic question is, why all the mystery in the first place?" "The mystery in the first place" is within the inherent nature of π, which cannot be directly expressed by any written number whatsoever. How does one provide a "true" value without teaching the advanced mathematical concepts inherent in the meanings of irrational numbers? Furthermore, how does one provide a simple-to-use approximation while also indicating that there is much more involved? Might one provide a simple answer for the general public and in parallel also a sophisticated, hidden answer, for those who are searching for something deeper?

There are a number of questions about this procedure.

- It is not permitted to use numerology-based calculations as a proof except for high level Torah scholars or when we have a tradition respecting the matter in question. This prohibition is intended to prevent just anybody who wishes from constructing a frivolous "proof" by an ingenious choice of words and numbers. Our proof, however, is presented in the name of a major Torah scholar, and the use of a *gematria*-based proof is sanctioned when we have the variant spellings embedded in the text itself.

- The word *kav* appears only twenty-eight times in the Hebrew Scriptures in eighteen sections, some of which have the word multiple times. This is not sufficient to show any statistical importance to the extra *hei*. Furthermore, the extra *hei* appears three times — in I Kings, and also Jeremiah 31:39 and Zechariah 1:16. Presumably there is a reason for the extra *hei* in all three places, but no one knows what that reason might be except with respect to Solomon's pool. Of course, the matter is a secret, and just because the secret has not yet been revealed does not mean that it does not exist. Still the proof would be more compelling if we knew why the *hei* appears in Jeremiah and Zechariah as well. A reviewer who saw an advance copy of this paper made an interesting suggestion. We have three (3) instances of the variant spelling to call our attention to the base number, 3, to which we apply the correction factor.

- One more issue is that King Solomon's pool is also described in II Chronicles 4:2, except that *kav* is spelled conventionally, without the *hei*. Why is I Kings worthy of the secret and II Chronicles not? I have a conjecture on that. I Kings provides the volume of the pool as 2000 *bath*. But II Chronicles gives the volume as 3000 *bath*. Why? The Talmud discusses this matter (Eruvin 14b) and determines that the measure in I Kings is in liquid measure *bath*, while II Chronicles refers to dry measure *bath* which are at a ratio of 1.5 to the liquid measure. My conjecture is that II Chronicles has volume-based complications that could cause confusion. Hence this is not the proper place to introduce additional "secret" calculations. The text in I Kings is clear of confusion; hence this is the place to embed the "secret."

- In spite of the above issues, there is an interesting positive point to consider. Taking 106 as fixed, given the normal spelling of *kav*, what should the second number be to give us a good approximation of π? It turns out that 111 is it. In fact, we need to go three more decimals to 111.003 before we get a better result. So while a variant spelling among only twenty-eight *kav*s is not significant, the choice of 111 among the thousands of other possibilities gives a different impression.

8. Conclusion

We know much more than people knew a few hundred years ago, let alone thousands of years ago. And what we don't directly know is so easily accessible to us that we tend to fall into the trap of equating basic knowledge and access to knowledge with ability or intelligence. I know much more in some areas of mathematics than Archimedes did, but that does not detract from his genius or elevate my modest abilities. There were people of genius and ability thousands of years ago when Solomon's pool was built, when the books of Kings were composed and when the Talmud was developed. If there was basic knowledge that what we call π is more than 3, then surely some of these people would have known this. "They were ignorant and did not understand" is not an acceptable answer to explain the puzzle why many of these people appear to claim that $\pi = 3$. No one claims that the Egyptian pyramids were built by incompetent people, or that the heavy stones at Stonehenge assembled on their own. There are many theories and suggestions even if we don't have definitive answers about these ancient artifacts. The same applies here. This puzzle too, calls for investigation and a search for answers. This paper, along with the references, provides a number of possibilities to consider. But as noted in the introduction: "Alas, there is no final resolution here, if such a possibility even exists." Nevertheless, we move a few steps forward.

Separating what we know, or what we conjecture, into categories of complexity and sophistication (*pshat*, *remez*, *sod*) permits progress in one area of inquiry even when other areas are at an impasse. While there is no "proof" in the mathematical sense, it should be clear from the evidence presented or alluded to throughout this paper that the expression $\pi = 3$ is not intended as a statement about mathematics. True, those who made the statements about π were motivated by a religious perspective. But the evidence stands on its own logic, and the reader need not accept any religious conviction thereby. All we need is to understand the reasoning and logic of those who made these statements. This is why this paper devotes significant space to an explanation of the logic and basis that leads to the claim that π equals 3. I believe that we have an acceptable explanation, or explanations, why the text says that $\pi = 3$.

We are not in as good a position with respect to *remez*, where we try to find a mathematically acceptable value for π within the physical structure of the pool. Here is where the statement "there is no final resolution here," applies. Still, some progress was made. We have added two choices, possibly of some merit, and we have invalidated a number of existing choices. In particular it should be clear that we need to take into consideration all available information about this pool, including the cubit to handbreadth ratio and also the volume. One piece of information that we did not consider is that the metal for this pool was cast. It would be a significant accomplishment to cast such a large item; about 20 feet across and near 10 feet high, even today (see the appendix for cubits to feet conversion). But this was done at the time of King Solomon (ca 950 BCE), at the end of the Bronze Age. It might be useful to examine which proposed shapes are more and which less likely to have been cast at that time.

The hidden or secret (*sod*) analysis is simultaneously fully developed and also unacceptable. I believe that the conventional arguments against this choice, such as the ones proposed by Professor Simoson, can be successfully refuted. But it requires a religious or spiritual outlook to accept the reality of prophetic knowledge. To borrow from Einstein's complaint about "spooky action" in quantum mechanics, prophetic knowledge is spooky action of another sort and difficult for most people to accept. Spooky action in quantum mechanics was eventually proven to be real as we now have experimental verification for the phenomenon known as "entanglement." But we do not have the same possibility respecting the "spooky" encryption about π. Does this mean that only those with a spiritual inclination can or will accept this secret path to a good approximation for π? Perhaps not!

Assume that the author of I Kings, whoever this was and not necessarily a prophet, knew of a good approximation for π at 3.1+. This might be 3.125 or 3.16 or something else used as a rule of thumb by expert builders. He would not know this number in decimals, but we will use decimals to fit current practice. The author knows the dimensions of the pool as about 30 and 10 cubits. The author knows that ordinary people approximate π as 3. What to do to stay within common convention and also not neglect his knowledge of a more accurate value for π? He provides the dimensions of the pool as we have these and also introduces a *keri/ketiv* to point to the better value that he knows. The fact that this "better value" happens to have modern mathematical implications, and is known as π_2, is incidental to the attempt at achieving a result at 3.1+. Is this a reasonable scenario? Perhaps not! But we don't have to have an exact scenario to accept the statement from Deakin and Lausch [11] that "the author of I Kings surreptitiously coded into his text an extremely accurate value of π." Prophetic knowledge provides an excellent reason for those who will accept the idea of "prophetic knowledge," but we don't have to know the reason to accept that it could happen.

So which of the three levels of explanation is correct? There is no way to absolutely prove any of these. But this is not an "either / or" matter. All three levels could be correct, as the *pshat*, *remez*, and *sod* analyses are structured for parallel levels of understanding. The value 3 is there to help us make practical decisions in our normal lives, and especially the lives of people several thousand years ago when calculation of areas and volumes was a major task. The value 3.125 is in line with the scientific and mathematical knowledge of the time, and we have the approximation $3 + 1/8$ on a Babylonian clay tablet. This is not a value that would be used by ordinary people (who would use 3) but may have been used by the skilled builders of Solomon's Temple. Or perhaps the builders focused on a different value. In any event, it is clear that a reasonable approximation for π can be found within the structure of this pool, though we do not have a "final resolution" as to how it was done. Finally, we have a secret result based on prophetic knowledge (or even without prophecy) which is in line with our time and our mathematical sophistication.

This is a long paper to write, and even longer to read. And even with all this effort there is no "final resolution" here. Why do it then? We do it because it's fun to contemplate a mystery. That is why "who done it" literature has a popular following. I hope that you, the reader, had as much fun exploring this cold-case mystery as I had in explaining it.

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A. Appendix — Volume calculation details

The Talmud determines that the volume of the pool was 450 cubic standard cubits (at 6 handbreadths to the cubit). This information invalidates a number of suggested shapes that yield a good approximation for π, but the shapes do not provide the called-for volume. Before we proceed to a computation of the volume, however, we need to learn something about the Talmudic units of measurement. We will introduce only those units which are needed for our purposes, and only to a minimal extent. This is because, quoting Rabbi Steinsaltz [3], "These units do not derive from a single historical period. Indeed their development extends from Biblical times through the entire Talmudic period." This, and other difficulties, makes this a significant topic in its own right, which we will not pursue.

We need to introduce only two units of length.

- The *amah* (cubit) "is the distance from the elbow to the end of the middle finger (this finger is also called *amah* in Hebrew). The common *amah*, known as the standard *amah*' is six *tefach* (handbreadths) long, 18.9 in. according to the Na'eh scale, and 22.7 in. according to the Hazon Ish scale. However, the Talmud also mentions a short' *amah*, which is five *tefachs* (handbreadths) long..."

We should note that Na'eh and Hazon Ish are 20$^{\text{th}}$ century Torah scholars whose conclusions as to the maximum and minimum length of the cubit in modern units are generally accepted as definitive. In common usage it is easier to stick to a range of $18 - -24$ inches ($1.5 - -2$ feet) and 20 inches as an average compromise. The max/min values are important for religious usage because of the principle of "stringency" which is discussed elsewhere in this paper. Thus, in a case a few years ago where I was obliged for religious-law purposes to not exceed 4 cubits, I made sure to stay under 6 feet. Had my obligation been to exceed 4 cubits I would have used 8 feet.

- The *tefach* (handbreadth) is "the width of a clenched fist..."

We should note that while we do not know the exact length of the cubit or handbreadth in modern units of measure, we do know the ratio of these, and that is really all we need for our purposes.

Units of volume pose more of a problem than units of length. To quote Rabbi Steinsaltz: "Units of volume in the Talmud are among the most complicated units of measurement, because of the existence of several independent systems ..." In our case we will deal with volume in cubic cubits, which derive from a basis of length, and also the *bath*, which we find in I Kings, which derives from the volume of an egg (*beitza*) as the reference standard. Also, "even within the same system, there were different measures used for dry and liquid capacity. This was not simply a matter of nomenclature; different types of containers were used for measuring liquids and solids... As a result, measures that were to hold dry materials were designed to permit a 50% overflow... " We have something similar today when we distinguish between a level spoonful and a heaping spoonful.

- *Beitza* is "The bulk of an average egg. In a sense, this unit is one of the most important; as it is used as the basis for calculating all the other measures, both dry and liquid..."
- A "*seah* has a volume of 144 eggs."
- "Forty *seah* ... the equivalent of 5,760 eggs. The minimum quantity of water necessary for a *mikvah* (ritual bath)... The Talmud tells us that the dimensions of a *mikvah* (ritual bath) must be 3 *amah* by 1 *amah*, and that its volume must be 40 *seah*. Thus, according to the Na'eh scale, a *mikvah* must contain 332 liters of water (87 U.S. gal.), and according to the Hazon Ish, 573 liters (151 U.S. gal.)."
- *bath*. "This is equivalent in volume to one *eifah*." The *eifah* has a volume of 3 *seah*. This is per the definition in [3]. We also have a derivation in the Talmud, Eruvin 14b, that shows that the *bath* is 3 *seah*. Thus: "How much is a *bath*? Three *seah*, as it is written: the *bath* is the tithe (10%) that shall be taken from a *kor*," per Ezekiel 45:14. And since a *kor* (unit of volume) is 30 *seah*, the *bath*, at 10%, is 3 *seah*. This gives us a Biblical basis for the volume of the *bath*.

We conclude that the 2000 *bath* volume of the pool had a volume of 6,000 *seah*. Since 40 *seah* makes a *mikvah*, 6,000 *seah* make 150 *mikvahs*. And, because a *mikvah* has a volume of 3 cubic standard cubits, it follows that the pool had a volume of 450 cubic standard cubits.

In a sense we already have the final result that Solomon's pool had a volume of 450 cubic standard cubits at 6 handbreadths to the cubit. But it will be instructive to see how this result is arrived at by the Talmud.

We have in Tractate Eruvin 14a: "Rav Chiya taught the following *baraita*; the pool that Solomon made could hold one hundred fifty *mikvahs* (ritual baths) of purity." "Of purity" because the purpose of the *mikvah* is for someone to immerse in its water in order to be purified.

Quoting Rabbi Steinsaltz [3]: "The Talmud accepts the contents of the *Mishnah* as incontrovertible facts... This special authority and importance is not accorded solely to the *Mishnah*, but also to other collections of statements of the *Tannaim* - ... *Baraitot...*" If we have in a *baraita* that the volume of Solomon's pool was equivalent to 150 ritual bath volumes, then the *Gemarah* accepts this as fact. But the *Gemarah* is not satisfied with factual statements without examining the details. Hence the question, "Now let us see: A *mikvah* is how large? Forty *seah...*" But that is not all. We eventually get to the statement that: "... he must immerse in a volume of water sufficient for his whole body to enter at one time. And how much water is this? The volume of an *amah* (cubit) by an *amah* (cubit) by the height of three *amos* (cubits)." Thus a column of water $1 \times 1 \times 3$ cubits is the volume of water of the minimal size *mikvah* that will completely cover an average person. This is difficult at the lower estimate for the size of the cubit, but quite reasonable at the upper estimate, where the 3 cubits high column would be some 68 inches. Also let us not forget that people were apparently quite a bit shorter two thousand years ago. And, of course, the *mikvah* is not structured in the shape of a column; this is simply the volume of water in a larger pool.

We now have 40 *seah* based on a *baraita* which is equivalent to 3 cubic standard cubits based on a calculation. But the scholars of the *Gemarah* were still not satisfied; they wanted to show a direct correlation between the 40 *seah* and 3 cubic standard cubits. We will not introduce this calculation as it requires an understanding of a number of new (to us) volumetric measures, such as *log*, *reviis*, etc. A discussion respecting this calculation appears on page 109b in Talmud Tractate Pesachim. The point is that it is a settled fact that the volume of Solomon's pool was 450 cubic standard cubits, and this is the same as 150 ritual pool volumes at the minimum required volume of 40 *seah* each.

Having arrived at a volume of 450 cubic standard cubits, the Talmud now considers the shape of the pool so as to yield the required volume. At first they consider a $10 \times 10 \times 5$ rectangular prism in line with the $1 \times 1 \times 3$ shape of one *mikvah* volume. This yields a volume of 500 cubic standard cubits. This idea is rejected because the volume is too large, and "this is only in a square. But the pool of Solomon was round." They now compute the volume for a cylindrical pool of diameter 10 and set π equal to 3. This yields a volume of 375 cubic standard cubits; which is not enough. They now arrive at a combination of the two above versions. "Rami bar Yechezkel taught the following *baraita*: The pool that Solomon made — the three lower *amos* (cubits) were square and the two upper *amos* (cubits) were round." This computes to a volume of 300 cubic standard cubits for the lower, square, section and 150 cubic standard cubits for the upper, round, section based on π equals 3. The total is 450 cubic standard cubits, which are 150 *mikvah* volumes. Everything fits perfectly. Nevertheless questions continue to be asked about this *baraita*. We have the following among these questions: "But why not say that only one *amah* (cubit) of the pool was round?" The answer: "This possibility should not enter your mind, since it is clear that the pool held 150 *mikvahs* of water. For it is written: It could hold two thousand *bath*." Apparently the *baraita* is not focused on the shape; rather it is the volume that is the primary factor. This appears to provide a bit of wiggle room to change the shape if called for by volume considerations. We will take advantage of this possibility.

There is one more matter respecting volume that we will introduce here. We have an almost identical description of this pool in II Chronicles 4:2-5, but the measuring rope (*kav*) has a different spelling. This matter was dealt with when we discussed the alleged secret encoded in the description. The other difference is that the volume is given as a "capacity of three thousand *bath*-measures." Eruvin 14b explains this discrepancy on the basis that one measure uses liquid units *bath*, while the other is based on dry units *bath*, which differ by 1/3 from the total. That is: $2000 = (1 - 1/3) \times 3000$. Or as Rabbi Steinsaltz notes: a "50% overflow." 50% of 2000 is 1000, and we get 3000 when the overflow is added to the original 2000.