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An Exploration in Subtropical Algebra

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Abstract

This paper explores some properties of subtropical arithmetic, which is the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ considered under the binary operations $\min(\cdot, \cdot)$ and $\max(\cdot, \cdot)$. We begin by examining some results in tropical polynomials. We then consider subtropical polynomials and subtropical geometry, drawing on tropical geometry for motivation. Last, we derive a complete classification of subtropical endomorphisms up to equivalence with respect to the coarsest topologies making these endomorphisms continuous.

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Acknowledgments

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Chapter 1

Introduction

Subtropical arithmetic is an arithmetic system on the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$. The operations consist of the two binary operators $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$, which are defined for all $a, b \in \overline{\mathbb{R}}$ in the usual way. We shall refer to the abstract study of this arithmetic as *subtropical algebra*.

The reason for extending $\mathbb{R}$ to $\overline{\mathbb{R}}$ is completely technical. By appending the element ∞, we gain an identity with respect to the operation $\wedge$, since for all $a \in \overline{\mathbb{R}}$, $a \wedge \infty = \infty \wedge a = a$. Similarly, by appending $-\infty$ we gain an identity with respect to $\vee$. We note, however, that this structure does not afford any inverses with respect to $\wedge$ or $\vee$, since the only $a, b \in \overline{\mathbb{R}}$ such that $a \wedge b = \infty$ are $a = b = \infty$, and similarly $a \vee b = -\infty$ implies $a = b = -\infty$.

Example 1.1 $5 \vee 9 = 9,\ 3 \wedge 6 = 3.$

The adjective "subtropical" is derived from the use of "tropical" in the study of *tropical algebra*. Our motivation for studying the subtropical arithmetic system is primarily game-theoretic, as many of the most celebrated results in game theory come to us in the form of statements about minimums of maximums or maximums of minimums. However, it seems likely that subtropical research could prove useful to mathematicians working in algebraic geometry and other fields.

In this paper we shall explore a few results in tropical algebra before attempting to consider the possibility of similar results in subtropical algebra. We will be largely unsuccessful in our attempts to define subtropical polynomials and subtropical lines, briefly discussing the failings in light of what may be the larger structural issues with the space itself. We will then focus on the connections between the abstract algebraic structure of the space in connection with induced topologies, obtaining a nice result concerning the classification of lattice endomorphisms on $\overline{\mathbb{R}}$.

Since much of our preliminary work in subtropical algebra will greatly mirror that of tropical algebra, we first turn our focus there.

Chapter 2

Tropical Mathematics

2.1 Survey

The term “tropical” is an homage to the Brazilian mathematician Imre Simon, one of the first mathematicians to work in the field [12]. In recent years, the study of the tropical semiring has seen resurgence due to applications to the study of algebraic geometry, plane curves, combinatorics, phylogenetic trees, and various other fields. The tropical semiring $(\mathbb{T}, \oplus, \odot)$ has two equivalent variants, $(\mathbb{R} \cup \{\infty\}, \min, +)$ and $(\mathbb{R} \cup \{-\infty\}, \max, +)$, the inclusion of ∞ or $-\infty$ serving to provide an identity for the $\oplus$ operation min or max, respectively. In yet another version [6], the tropical ring is extended further to include an identical copy of $\mathbb{R}$ whose elements interact with those in the original copy under tropical operations. For the remainder of this paper, we will assume the unextended variation of tropical arithmetic where addition is defined as minimum. That is to say, let $(\mathbb{T}, \oplus, \odot) = (\mathbb{R} \cup \{\infty\}, \min, +)$. Arithmetic in this system is then as in the following example.

Example 2.1 $5 \odot 9 = 14,\ 3 \oplus 6 = 3.$

Through the work of Bernd Sturmfels and many others, coherent notions of many traditional algebraic objects have been developed in a tropical setting. These include polynomials [12, 7], linear spaces [13], varieties [8, 6], ideals [6], matrices and their rank [5, 1], Nullstellensatz [10, 6], and Grassmannians [11]. Perhaps most oddly of all, we have also seen the development of a tropical geometry [8], which has applications to the study of genomics.

As one of our goals will be mimicking some of these developments in a subtropical setting, we begin by examining tropical polynomials. Defined analogously to traditional algebraic polynomials, the tropical polynomials in one variable are of the form

$$p(x) = \bigoplus_{k=0}^{n} a_k \odot x^k, \text{ with } a_i \in \mathbb{T},$$

where exponentiation is defined tropically. For the sake of translation, in a traditional algebraic setting this corresponds to

$$p(x) = \min\{nx + a_n, (n-1)x + a_{n-1}, \ldots, x + a_1, a_0\}. \tag{2.1}$$

The graph of a tropical polynomial then looks like the minimum of a sequence of lines of decreasing, nonnegative integer slope.

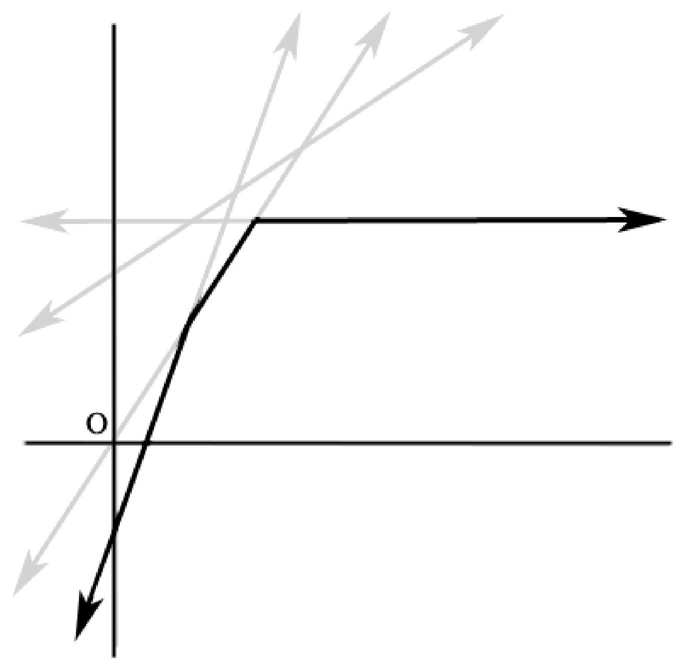

Figure 2.1: An example graph of a tropical polynomial.

In considering the roots of a tropical polynomial, it no longer makes much sense to consider the values of x where the expression achieves zero. First we remark that zero lacks the property $0 \odot a = 0$ for all $a \in \mathbb{T}$. This was the property that made roots of polynomials also roots of a product polynomial under multiplication in a traditional algebraic setting, but it no longer applies tropically. Instead we desire to define a root as something that propagates through tropical multiplication such that each tropically factorable polynomial has as its roots the roots of its divisor polynomials.

The distinguishing features of the tropical polynomial in Figure 2.1 are decidedly the "kinks" in the graph where the lines meet. Noting that if we were to add one graph of this form to another graph of this form in a traditional algebraic setting, the kinks of each graph would be preserved. This is due to the fact that the slopes of the lines defining the graph are strictly decreasing, since each kink is provided no means to "cancel" by tropical multiplication. At any given kink, multiplication by another tropical polynomial will only force the slopes on either side have the same or an even greater difference. Thus, the kinks of a tropical polynomial are preserved under tropical multiplication. It is with this in mind that the roots of the tropical polynomial $p(x) = \bigoplus_{k=0}^{n} a_k \odot x^k$ are defined as follows.

Definition 2.2 *The roots of a polynomial $p(x)$ are defined to be all $r \in \mathbb{T}$ where*

$$a_i \odot r^i = a_j \odot r^j = \min_{1 \le k \le n} \{a_k \odot r^k\} \quad \textit{for some } i \textit{ and } j \textit{ such that } 0 \le i < j \le n.$$

In other words, the roots are the values of x where the minimum value of Equation 2.1 is achieved in at least two of its components. Since these values r are precisely the places where the lines

$$y = nx + a_n,\ y = (n-1)x + a_{n-1},\ \ldots,\ y = x + a_1,\ y = a_0$$

have pairwise intersections falling on the graph of $p(x)$, the roots are the kinks on the tropical polynomial, as hinted. The polynomials then have a delightfully familiar property.

Proposition 2.3 *Given any collection of n distinct roots $r_1 > \ldots > r_n$, there exists a tropical polynomial of degree n with these roots.*

Proof: Consider $p(x) = \bigodot_{i=1}^{n}(x \oplus r_i)$. ■

This is equivalent to expanding a bunch of linear terms, each of which we prepare in such a way that it gives the overall polynomial a particular desired root. The next statement is a bit stronger, but would require a stretch of the imagination to make analogous to traditional algebraic polynomials.

Proposition 2.4 *Given a collection of $n+1$ distinct natural numbers $k_0 < \ldots < k_n$ and n distinct roots $r_1 > \ldots > r_n$, we may construct a polynomial with root a_i occurring as the kink between line segments of slope k_{i-1} and k_i.*

Proof: Such a polynomial is given by

$$\bigoplus_{i=1}^{n} s_i \odot x^{k_i},$$

where

$$s_j = c + \sum_{i=1}^{j} a_i(k_{i-1} - k_i), \text{ with } c \in \mathbb{T}.$$

■

The constant c here merely shifts the graph vertically, leaving the roots and slopes untouched. This can be thought of as an analogue to multiplying a real polynomial by a nonzero constant.

It is not too much of a leap to see that this system of designating roots and slopes can be used to produce the curve defined by any polynomial, since such a curve is a piecewise combination of segments of decreasing nonnegative integer slope, intersecting at precisely the roots.

We next note that two tropical polynomials may describe precisely the same curve in the plane.

Example 2.5 *Consider the polynomials $f(x) = x^2 \oplus 1 \odot x \oplus 2$ and $g(x) = x^2 \oplus 2 \odot x \oplus 2$. These describe the same curve, since the x^1 term is greater than at* *least one of the x^0 or x^2 terms for all x in both f and g. Since the x^0 and x^2 terms then define the polynomials and are identical in f and g, f and g are identical.*

We can then create equivalence classes of polynomial expressions based on equality. We might note an interesting fact about our equivalent polynomials f and g in Example 2.5: f is easily factored while g is not. With a little tropical manipulation we can obtain

$$f(x) = x^2 \oplus 1 \odot x \oplus 2 = (x \oplus 1) \odot (x \oplus 1).$$

You can, in fact, demonstrate that g has no such factorization! Now that we have a moderate understanding of the mechanics of tropical arithmetic, we turn our attention to subtropical algebra.

Chapter 3

Fundamentals of Subtropical Algebra

Since $\overline{\mathbb{R}}$ is an ordered set with the natural ordering $<$, we note that de Morgan's Duality Law applies. That is, for every statement about $<$, a roughly equivalent statement about $>$ holds. In our case, we are most interested in how statements about $\wedge$ correspond to statements about $\vee$. The reader should bear this in mind while working through this chapter.

3.1 Commutativity, Associativity, and Bidistributivity

Perhaps the easiest observation to make about the operations $\wedge$ and $\vee$ is that each is commutative and associative. Furthermore, each is distributive over the other, a property we shall call *bidistributivity*. While the commutative and associative properties are straightforward to see, the proof of bidistributivity is a little less intuitive. In terms of our operations the proof is a little cumbersome, so in order to more elegantly demonstrate bidistributivity we shall reformulate $\wedge$ and $\vee$ in terms of an analogue to Dedekind cuts. For each $r \in \overline{\mathbb{R}}$, let the map $\varphi : \overline{\mathbb{R}} \to 2^{\overline{\mathbb{R}}}$ be defined by $\varphi(r) = [-\infty, r]$. We then note that for $a, b \in \overline{\mathbb{R}}$,

$$\varphi(a \wedge b) = \varphi(a) \cap \varphi(b) \quad \text{and} \quad \varphi(a \vee b) = \varphi(a) \cup \varphi(b).$$

Since $\varphi(a) \cap \varphi(b), \varphi(a) \cup \varphi(b) \in \varphi(\overline{\mathbb{R}})$ and φ is injective, we can then safely write

$$a \wedge b = \varphi^{-1}(\varphi(a) \cap \varphi(b)) \quad \text{and} \quad a \vee b = \varphi^{-1}(\varphi(a) \cup \varphi(b)).$$

Proposition 3.1 *The operations $\wedge$ and $\vee$ are bidistributive.*

Proof: Let $a, b, c \in \overline{\mathbb{R}}$. We then observe that since $\cap$ and $\cup$ are bidistributive,

$$\begin{aligned} a \wedge (b \vee c) &= \varphi^{-1}(\varphi(a) \cap (\varphi(b) \cup \varphi(c))) \\ &= \varphi^{-1}((\varphi(a) \cap \varphi(b)) \cup (\varphi(a) \cap \varphi(c))) \\ &= \varphi^{-1}(\varphi(a \wedge b) \cup \varphi(a \wedge c)) \\ &= (a \wedge b) \vee (a \wedge c). \end{aligned}$$

Thus, $\wedge$ distributes over $\vee$. The proof that $\vee$ distributes over $\wedge$ is carried out similarly. ■

It is interesting to note that we could have also defined $\psi : \overline{\mathbb{R}} \to 2^{\overline{\mathbb{R}}}$ by $\psi(r) = [r, \infty]$ and observed that

$$a \wedge b = \psi^{-1}(\psi(a) \cup \psi(b)) \quad \text{and} \quad a \vee b = \psi^{-1}(\psi(a) \cap \psi(b)).$$

More heuristically, we might have anticipated the distributive property of $\vee$ over $\wedge$ as a consequence of the distributive property of $\wedge$ over $\vee$ in conjunction with the Duality Law.

3.2 Subtropical Polynomials

Consider a polynomial in $(\mathbb{R}, +, \cdot)$. It has the form

$$p(x) = \sum_{i=0}^{n} a_i \cdot x^i.$$

Polynomials in $(\mathbb{R}, +, \cdot)$ have many useful characteristics that we can ascribe to them such as degree, roots, and factorings. In tropical algebra, $(\mathbb{T}, \oplus, \odot)$, there is the analogous form for a polynomial

$$p(x) = \bigoplus_{i=0}^{n} a_i \odot x^i$$

that also has useful notions of degree, roots, and factorings defined for it. We now wish to define something similar in $(\overline{\mathbb{R}}, \wedge, \vee)$. Note that because of the bidistributivity of $\wedge$ and $\vee$, our allowing $\vee$ to fill the role of multiplication and $\wedge$ to fill the role of addition is arbitrary.

We start by considering the basic form for a polynomial suggested by the other two considered systems,

$$p(x) = \bigwedge_{i=0}^{n} a_i \vee x^i \text{ for } a_i \in \overline{\mathbb{R}}, n \in \mathbb{N},$$

with exponentiation interpreted subtropically: $x^k = \bigvee_{i=1}^{k} x = x$ for $k \geq 1$. We shall use the natural definition $x^0 = -\infty$, since the $-\infty$ is the identity with respect to $\vee$. Let P denote the set of all functions of the form $p(x)$.

Proposition 3.2 *Any element of P may be expressed as $p(x) = (a \vee x) \wedge b$ for $a, b \in \overline{\mathbb{R}}$.*

Proof: Following the traditional order of operations allows us to reduce our polynomial to $p(x) = a_0 \wedge \bigwedge_{i=1}^{n}(a_i \vee x)$. By distributivity we then have $p(x) = a_0 \wedge (x \vee \bigwedge_{i=1}^{n} a_i)$. By letting $a = \bigwedge_{i=1}^{n} a_i$ and $b = a_0$, we then have $p(x) = (a \vee x) \wedge b$ for $a, b \in \overline{\mathbb{R}}$. ■

Using the form given for a polynomial by Proposition 3.2, we may then classify each polynomial $p(x) = (a \vee x) \wedge b$ by considering the cases $a < b$, $a = b$, and $a > b$. Since we are clever, we can handle it in two cases:

- If $a < b$, then
$$p(x) = \begin{cases} a & \text{if} \quad x \in [-\infty, a] \\ x & \text{if} \quad x \in (a, b) \\ b & \text{if} \quad x \in [b, \infty] \end{cases}$$
- If $a \geq b$, then $p(x) = b$.

We graph the more interesting of the two forms in Figure 3.1.

If we wish to develop any useful notion of a root, we would likely want roots to somehow correspond to the kinks in the graph. However, the above classification of all $p(x) \in P$ gives that each polynomial has either zero or two kinks. Since the product or sum of two polynomials again has zero or two kinks, there does not seem to be a useful notion of degree

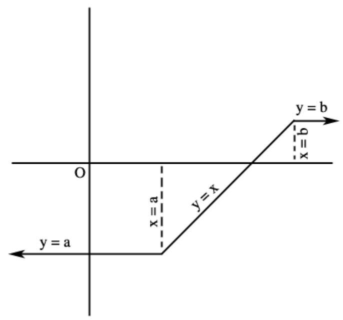

Figure 3.1: $p(x) = (a \vee x) \wedge b$ for $a < b$.

or root to be found this way, shedding doubt on this definition of a subtropical polynomial. Before we finish discussion of this form, however, we shall demonstrate the closure of P with respect to subtropical operations.

Proposition 3.3 *P is closed under $\wedge$.*

Proof: Let $f(x), g(x) \in P$. By Proposition 3.2 we are allowed to write $f(x) = (a \wedge x) \vee b, g(x) = (c \wedge x) \vee d$ for some $a, b, c, d \in \overline{\mathbb{R}}$. We then observe that

$$\begin{aligned} f(x) \wedge g(x) &= [(a \wedge x) \vee b] \wedge [(c \wedge x) \vee d] \\ &= [(a \wedge x) \wedge (c \wedge x)] \vee [(a \wedge x) \wedge d] \vee [(c \wedge x) \wedge b] \vee [b \wedge d] \\ &= [a \wedge c \wedge x] \vee [a \wedge d \wedge x] \vee [b \wedge c \wedge x] \vee [b \wedge d] \\ &= ([[a \wedge c] \vee [a \wedge d] \vee [b \wedge c]] \wedge x) \vee [b \wedge d]. \end{aligned}$$

Letting $r = [[a \wedge c] \vee [a \wedge d] \vee [b \wedge c]]$ and $s = [b \wedge d]$, we have that $f(x) \wedge g(x) = (r \wedge x) \vee s$ for $r, s \in \overline{\mathbb{R}}$, so $f(x) \wedge g(x) \in P$. ■

Proposition 3.4 *P is closed under $\vee$.*

Proof: Let $f(x), g(x) \in P$. We may again write $f(x) = (a \wedge x) \vee b$ and $g(x) = (c \wedge x) \vee d$ for some $a, b, c, d \in \overline{\mathbb{R}}$. It follows that

$$\begin{aligned} f(x) \wedge g(x) &= [(a \wedge x) \vee b] \vee [(c \wedge x) \vee d] \\ &= (a \wedge x) \vee (c \wedge x) \vee b \vee d \\ &= ([a \vee c] \wedge x) \vee [b \vee d]. \end{aligned}$$

Letting $r = [a \vee c]$ and $s = [b \vee d]$, we have that $f(x) \vee g(x) = (r \wedge x) \vee s$ for $r, s \in \overline{\mathbb{R}}$, so $f(x) \vee g(x) \in P$. ■

While closure under both operations is very much a desired property for any potential definition of a subtropical polynomial, we still cannot help but feel that the polynomials afforded by analogy to traditional and tropical algebras are in some sense "too weak," since useful notions of degree and root are very elusive. One alternative form that we may wish to study is the polynomial-type object the form

$$q(x) = a_0 \wedge \bigwedge_{i=1}^{n} a_i \vee (x + b_i) \text{ for } a_i, b_i \in \overline{\mathbb{R}}.$$

The reason for this form is purely because the translations allow for the kinks to propagate more interestingly through subtropical multiplication. However, we do have the philosophical issue of introducing the outside operation of addition. Having failed to come up with an intuitionally satisfying notion of a subtropical polynomial, we instead turn our attention to attempting to construct a coherent subtropical geometry.

Chapter 4

Subtropical Geometry

One of the things we wish to establish within the context of subtropical algebra is a suitable geometry. A natural place to start is with the question of what we might expect to see happen in subtropical space $\overline{\mathbb{R}}^n$.

4.1 Subtropical Spaces

As noted earlier, there are no additive inverses in the subtropical semiring. Because of this, we cannot hope to have a subtropical vector space in any natural way. However, if we choose to ignore this, the subtropical space contructed in analogue to the vector space $\mathbb{R}^n$ may have some interesting properties.

We define the space $\overline{\mathbb{R}}^n$ to be all n-component vectors with entries chosen from $\overline{\mathbb{R}}$. We define addition and scaling as below.

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \wedge \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 \wedge b_1 \\ \vdots \\ a_n \wedge b_n \end{pmatrix}, \qquad k \vee \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} k \vee a_1 \\ \vdots \\ k \vee a_n \end{pmatrix} \text{ for all } a_i, b_i, k \in \overline{\mathbb{R}}$$

However, due to the symmetry of our space it might also make sense to define the vector operations

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \vee \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 \vee b_1 \\ \vdots \\ a_n \vee b_n \end{pmatrix}, \qquad k \wedge \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} k \wedge a_1 \\ \vdots \\ k \wedge a_n \end{pmatrix} \text{ for all } a_i, b_i, k \in \overline{\mathbb{R}}.$$

We begin by considering the notion of an inner product. Constructing the inner product as an analogue to the standard inner product on $\mathbb{R}^n$, we might define

$$\vec{a} \curlyvee \vec{b} = \bigwedge_{i=1}^{n} (a_i \vee b_i).$$

We note that if $a, b \in \overline{\mathbb{R}}$, we simply have $a \curlyvee b = a \vee b$, giving some rationale for our notation.

We next verify that this is almost an inner product, with "almost" in the sense that $\overline{\mathbb{R}}^n$ is almost a vector space.

Proposition 4.1 $\curlyvee$ *is almost an inner product on* $\overline{\mathbb{R}}^n$.

Proof:

Let $\vec{a}, \vec{b}, \vec{c} \in \overline{\mathbb{R}}^n$.

(i) Clearly $\vec{a} \curlyvee \vec{b} = \vec{b} \curlyvee \vec{a}$.

(ii) Define $\vec{a} \wedge \vec{b}$ componentwise. We then have that

$$\begin{aligned}
(\vec{a} \wedge \vec{b}) \curlyvee \vec{c} &= \left(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \wedge \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right) \curlyvee \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \\
&= \begin{pmatrix} a_1 \wedge b_1 \\ \vdots \\ a_n \wedge b_n \end{pmatrix} \curlyvee \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \\
&= \bigwedge_{i=1}^{n} (a_i \wedge b_i) \vee c_i \\
&= \bigwedge_{i=1}^{n} ((a_i \vee c_i) \wedge (b_i \vee c_i)) \\
&= \left(\bigwedge_{i=1}^{n} (a_i \vee c_i) \right) \wedge \left(\bigwedge_{i=1}^{n} (b_i \vee c_i) \right) \\
&= (\vec{a} \curlyvee \vec{c}) \wedge (\vec{b} \curlyvee \vec{c}).
\end{aligned}$$

(iii) Let $k \in \mathbb{T}$. Defining $k \max \vec{a}$ in the natural way one would define scalar action, we then observe

$$\begin{aligned}
(k \vee \vec{a}) \curlyvee \vec{b} &= \left(k \vee \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \curlyvee \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} k \vee a_1 \\ \vdots \\ k \vee a_n \end{pmatrix} \curlyvee \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right) \\
&= \bigwedge_{i=1}^{n} (k \vee a_i) \vee b_i \\
&= k \vee \left(\bigwedge_{i=1}^{n} a_i \vee b_i \right) \\
&= k \vee (\vec{a} \curlyvee \vec{b}).
\end{aligned}$$

(iv) This is the axiom that slightly breaks down. Given that the subtropical additive identity is ∞, demanding $\vec{a} \curlyvee \vec{a} \geq \infty$ with equality iff $\vec{a} = \vec{\infty}$ seems ridiculously. Instead, we might demand $\vec{a} \curlyvee \vec{a} \leq \infty$, with equality iff $\vec{a} = \vec{\infty}$. While the former of these two requirements is trivially true, to see the latter we note that

$$\vec{a} \curlyvee \vec{a} = \bigwedge_{i=1}^{n} (a_i \vee a_i) = \bigwedge_{i=1}^{n} a_i,$$

which may equal ∞ iff $a_i = \infty$ for each a_i, and thus iff $\vec{a} = \vec{\infty}$. ∎

The key hangup in this was that zero no longer holds the same significance in the subtropical setting that it did in traditional arithmetic. “Positive” and “negative” no longer have much significance, either, since subtropically all numbers between $-\infty$ and ∞ are in some sense equivalent in their relations, up to translation.

Example 4.2

$$\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} \curlyvee \begin{pmatrix} 7 \\ 1 \\ 9 \end{pmatrix} = (2 \vee 7) \wedge (5 \vee 1) \wedge (3 \vee 9) = 7 \wedge 5 \wedge 9 = 5.$$

Due to the symmetry (by Duality) of $\vee$ and $\wedge$, however, we note that it might make just as much sense to define the inner sum

$$\vec{a} \curlywedge \vec{b} = \bigvee_{i=1}^{n} (a_i \wedge b_i).$$

We note similarly that if $a, b \in \overline{\mathbb{R}}$, we have $a \curlywedge b = a \wedge b$. We may also prove that $\curlywedge$ has all the desired properties of an inner product if we were to reverse the roles of multiplication and addition. The proof would go as in Proposition 4.1, except we would demonstrate

(i) $\vec{a} \curlywedge \vec{b} = \vec{b} \curlywedge \vec{a}$,

(ii) $(\vec{a} \vee \vec{b}) \curlywedge \vec{c} = (\vec{a} \curlywedge \vec{c}) \vee (\vec{b} \curlywedge \vec{c})$,

(iii) $(k \wedge \vec{a}) \curlywedge \vec{b} = k \wedge (\vec{a} \curlywedge \vec{b})$,

(iv) $\vec{a} \curlywedge \vec{a} \geq -\infty$, with equality iff $\vec{a} = \vec{-\infty}$.

Since we must always have $a \wedge b \leq a \vee b$, we might suspect that $\vec{a} \curlywedge \vec{b} \leq \vec{a} \curlyvee \vec{b}$. For the case where $\vec{a}, \vec{b} \in \overline{\mathbb{R}}$, this obviously holds. However, the next example in $\overline{\mathbb{R}}^2$ shows that we may have $\vec{a} \curlywedge \vec{b} > \vec{a} \curlyvee \vec{b}$.

Example 4.3

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} \curlywedge \begin{pmatrix} 2 \\ 4 \end{pmatrix} = (1 \wedge 2) \vee (3 \wedge 4) = 1 \vee 3 = 3$$

and

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} \curlyvee \begin{pmatrix} 2 \\ 4 \end{pmatrix} = (1 \vee 2) \wedge (3 \vee 4) = 2 \wedge 4 = 2,$$

so

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} \curlywedge \begin{pmatrix} 2 \\ 4 \end{pmatrix} > \begin{pmatrix} 1 \\ 3 \end{pmatrix} \curlyvee \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

4.2 Subtropical Lines (Euclidean Analogy)

In developing our subtropical geometry, we might next attempt to develop the notion of a line. Since the easiest way to start is by analogy to known arithmetics, we begin by considering analogues to standard Euclidean geometry. Analytically, we are accustomed to seeing lines expressed in standard form: $ax + by + c = 0$. With this expression we may modify a, b, c appropriately to describe any line in the plane. However, the presence of additive inverses is what allows for this canonical representation and allows us to equate the line $5x + 3y + 9 = 0$ with the line $10x + 5y + 4 = 5x + 2y - 5$. Due to a lack of subtropical additive inverses, we might then guess that the form for a line would be equations of the form

$$(a \vee x) \wedge (b \vee y) \wedge c = (d \vee x) \wedge (e \vee y) \wedge f.$$

We shall now explore this system thoroughly. We begin by examining the left side.

To induce a symmetry for the sake of generality, we will instead consider

$$(a \vee x) \wedge (b \vee y) \wedge (c \vee z) \quad \text{for } a \leq b \leq c.$$

Since the names of the variables are arbitrary, we will homogenize the system this way and hold whichever variable corresponds to the "constant term" at a constant value of $-\infty$. For example, the expression $(5 \vee u) \wedge (3 \vee v) \wedge 4$ corresponds to $(3 \vee x) \wedge (4 \vee y) \wedge (5 \vee z)$ with $3 \leq 4 \leq 5$ and $y = -\infty$.

In considering $L = L(x, y, z) = (a \vee x) \wedge (b \vee y) \wedge (c \vee z)$ for $a \leq b \leq c$, we will start by examining cases.

- If $x \leq a$, then we have $L = a$, since $(a \vee x) = a$ and each of $(b \vee y)$ and $(c \vee z)$ is at least a.
- If $a \leq x \leq b$, then we have $L = x$, since $(a \vee x) = x$ and each of $(b \vee y)$ and $(c \vee z)$ is at least x.
- If $y \leq b \leq x$, then we have $L = b$, since $(b \vee y) = b \leq x = (a \vee x)$ and $(c \vee z)$ is at least $c \geq b$.
- If $b \leq x \leq y \leq c$, then we have $L = x$, since $(a \vee x) = x$ and each of $(b \vee y)$ and $(c \vee z)$ is at least y, which is greater than or equal to x.

- If $b \leq y \leq x \leq c$, then we have $L = y$, since $(b \vee y) = y \leq x = (a \vee x)$ and $(c \vee z)$ is at least c, which is greater than or equal to y.
- If $b \leq x \leq c \leq y$, then we have $L = x$, since $(a \vee x) = x$ and each of $(b \vee y)$ and $(c \vee z)$ is at least c, which is greater than or equal to x.
- If $b \leq y \leq c \leq x$, then we have $L = y$, since $(b \vee y) = y$ and each of $(a \vee x)$ and $(c \vee z)$ is at least c, which is greater than or equal to y.
- If $z \leq c \leq x \leq y$, then $L = c$.
- If $z \leq c \leq y \leq x$, then $L = c$.
- If $c \leq z \leq x \leq y$, then $L = z$.
- If $c \leq z \leq y \leq x$, then $L = z$.
- If $c \leq x \leq z \leq y$, then $L = x$.
- If $c \leq x \leq y \leq z$, then $L = x$.
- If $c \leq y \leq z \leq x$, then $L = y$.
- If $c \leq y \leq x \leq z$, then $L = y$.

So, in summary:

$$L(x,y,z) = \begin{cases} a & \text{if} \quad x \leq a \\ b & \text{if} \quad y \leq b \leq x \\ c & \text{if} \quad z \leq c \leq x \leq y, \quad z \leq c \leq y \leq x \\ x & \text{if} \quad a \leq x \leq b, \quad b \leq x \leq y \leq c, \quad b \leq x \leq c \leq y, \\ & \quad\;\;\; c \leq x \leq y \leq z, \quad c \leq x \leq z \leq y \\ y & \text{if} \quad b \leq y \leq x \leq c, \quad b \leq y \leq c \leq x, \\ & \quad\;\;\; c \leq y \leq z \leq x, \quad c \leq y \leq x \leq z \\ z & \text{if} \quad c \leq z \leq x \leq y, \quad c \leq z \leq y \leq x \end{cases}$$

The right hand side of the equation is handled similarly, except we that we now no longer have the ability to treat the variables as arbitrary, since the left hand side forces properties onto the right hand side. The solution to this system of equations would then be all values $(x, y) \in \overline{\mathbb{R}}^2$ such that equality holds. We shall see with the next examples, however, that this creates a very unintuitive line.

Example 4.4 $(2 \vee x) \wedge (3 \vee y) \wedge 4 = 3.$

We see that a necessary condition for equality would be $x = 3$ a minimum, $y = 3$ a minimum, or 3 (the coefficient of y) a minimum. By our formula for $L(x, y, z)$ we see that this happens when $x = 3$ and y is any value or when $x \geq 4$ and $y \leq 3$. We draw observe this "line" below, in Figure 4.1.

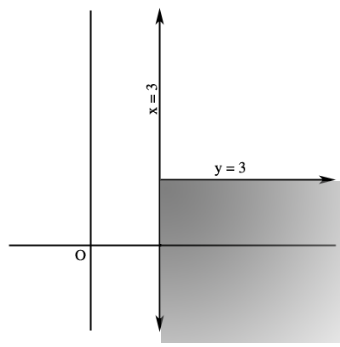

Figure 4.1: $(2 \vee x) \wedge (3 \vee y) \wedge 4 = 3.$

This "line" is actually a sheet in the plane. The next example is a little more satisfying, but simplistic in form.

Example 4.5 $x \wedge y = 1.$

The solutions to this equation are $x = 1$ and $y \geq 1$ or $y = 1$ and $x \geq 1$. It appears in Figure 4.2.

But next, an even more degenerate "line."

Example 4.6 $x \wedge y = x \wedge 2.$

The solutions to this equation are $x \leq y$ and $x \leq 2$ or $y = 2$ and $x \geq 2$. This region can be seen in Figure 4.3.

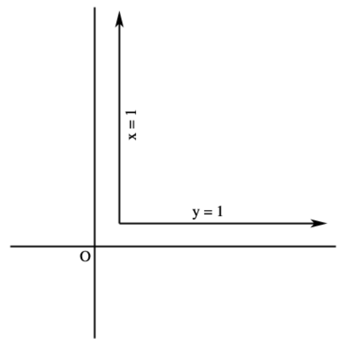

Figure 4.2: $x \wedge y = 1$.

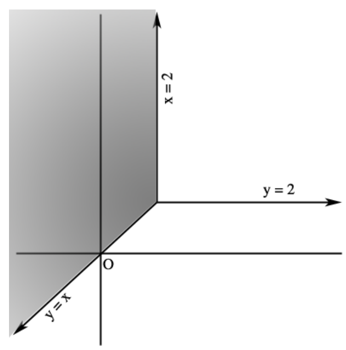

Figure 4.3: $x \wedge y = x \wedge 2$.

The "line" in Figure 4.3 is a sheet in the plane attached to a ray. While the line in Figure 4.2 nearly satisfies our geometric intuitions for what a line might be in terms of dimension, intersections, and so forth, the general form for a line as suggested by analogy to analytic geometry in a traditional algebraic setting yields far more exceptions to this intuition than it does satisfactory "lines."

4.3 Subtropical Lines (Tropical Analogy)

As we have grown accustomed to, the next step would then be to attempt to formulate a notion of a line by analogy to tropical geometry. In tropical geometry, a line is defined in terms of ideals of tropical polynomials. Since the subtropical polynomials we explored in an earlier section never proved fruitful for most meaningful criteria of analysis, defining a subtropical line analogously to a tropical line will not yet have much rationale other than wishful thinking. In that spirit, we consider the solutions to

$$(a \vee x) \wedge (b \vee y) \wedge c, \text{ with } a, b, c \in \overline{\mathbb{R}},$$

where the minimum of $(a \vee x), (b \vee y), c$ is achieved at least twice. We note that one of three conditions must hold for this to happen:

- $(a \vee x) = (b \vee y) \leq c$,
- $(a \vee x) = c \leq (b \vee y)$, or
- $(b \vee y) = c \leq (a \vee x)$.

We begin by assuming $c = \infty$ so that we need only examine the first case. Consider the equation $a \vee x = b \vee y$, we observe that the solutions are as follows.

- If $a < b$, then we have $x = b$ and $y \leq b$ or $y \geq b$ and $y = x$.
- If $a = b$, we have $x = y \geq a$ or $x \leq a$ and $y \leq a$.
- If $a > b$, then we have $y = a$ and $x \leq a$ or $x \geq a$ and $y = x$.

We graph these cases below.

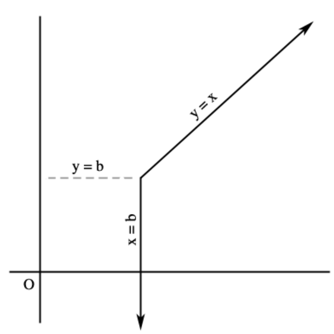

Figure 4.4: $(a \vee x) \wedge (b \vee y)$ for $a < b$.

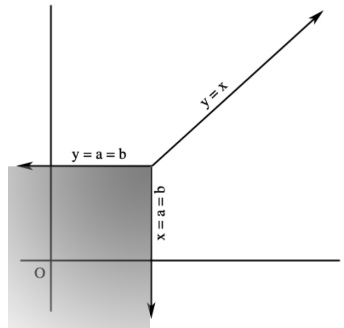

Figure 4.5: $(a \vee x) \wedge (b \vee y)$ for $a = b$.

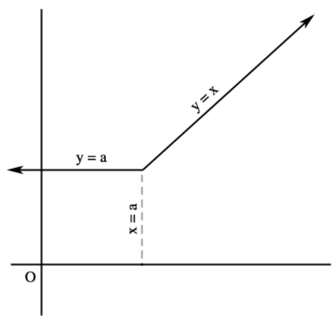

Figure 4.6: $(a \vee x) \wedge (b \vee y)$ for $a > b$.

With these lines, the only degenerate case which involves a sheet of the plane is when $a = b$. For the other lines we have the nice properties that the lines are in some sense "one-dimensional" and intersect each other never, once, or infinitely many times. Using similar methods, we may classify all varieties of the form $(a \vee x) \wedge (b \vee y) \wedge c$. For lines of this slightly more general form, we will also find that the degenerate cases similarly hold in the case of pairwise equalities among a, b and c. While we do not wade through the analysis, we do include diagrams that the reader may verify in Figures 4.7– 4.16 in the next section.

Due to the large number of degenerate forms, were we to adopt this definition of a subtropical line, we would need to justify in what circumstances we may ignore the degenerate cases or somehow explain why they do not qualify as "lines." Since this is something we lack the patience to explore and since we are furthermore given little pressure or motivation to explore it other than idle curiosity, we shall turn our attention elsewhere. In the next chapter, we shall explore subtropical space in a more abstract algebraic and topological setting.

4.4 Categorization of Varieties of the Form $(a \vee x) \wedge (b \vee y) \wedge c$

In this section we simply enumerate the forms of such varieties through diagrams. The remaining cases will be $c < a < b$, $c < b < a$, and $c < a = b$, all of which have an empty solution.

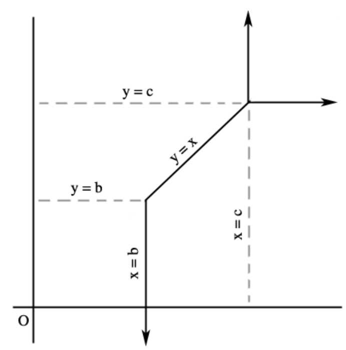

Figure 4.7: $(a \vee x) \wedge (b \vee y) \wedge c$ for $a < b < c$.

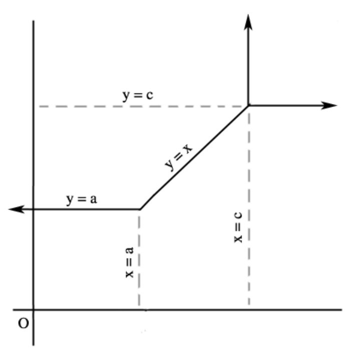

Figure 4.8: $(a \vee x) \wedge (b \vee y) \wedge c$ for $b < a < c$.

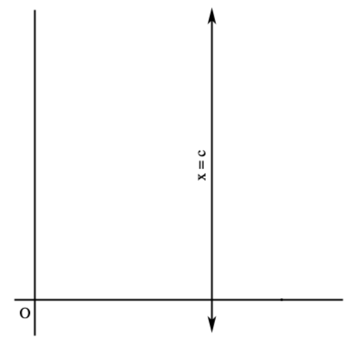

Figure 4.9: $(a \vee x) \wedge (b \vee y) \wedge c$ for $a < c < b$.

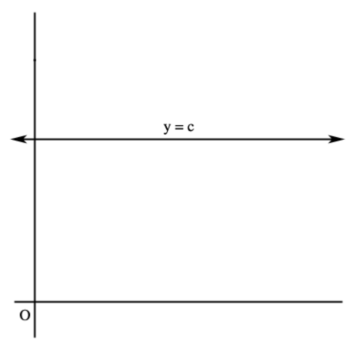

Figure 4.10: $(a \vee x) \wedge (b \vee y) \wedge c$ for $b < c < a$.

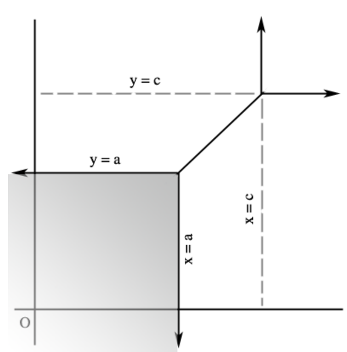

Figure 4.11: $(a \vee x) \wedge (b \vee y) \wedge c$ for $a = b < c$.

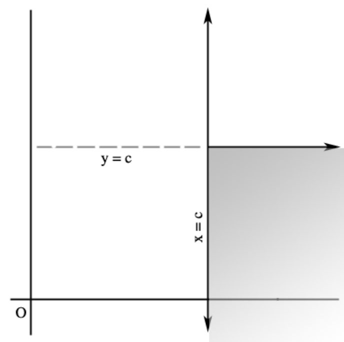

Figure 4.12: $(a \vee x) \wedge (b \vee y) \wedge c$ for $a < b = c$.

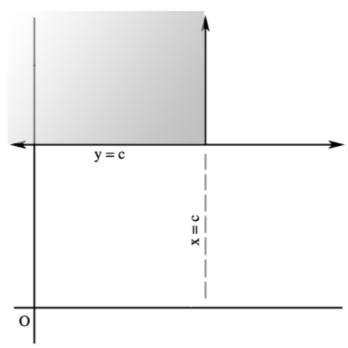

Figure 4.13: $(a \vee x) \wedge (b \vee y) \wedge c$ for $b < a = c$.

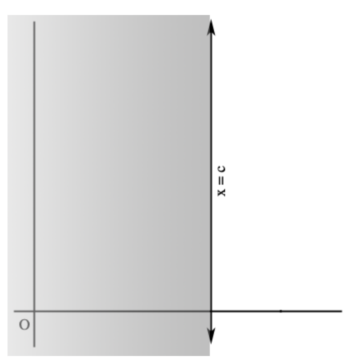

Figure 4.14: $(a \vee x) \wedge (b \vee y) \wedge c$ for $a = c < b$.

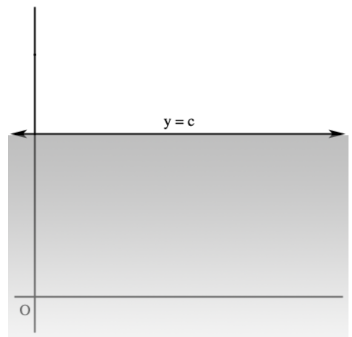

Figure 4.15: $(a \vee x) \wedge (b \vee y) \wedge c$ for $b = c < a$.

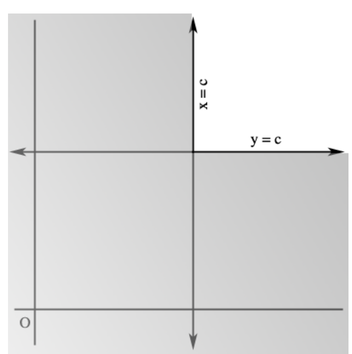

Figure 4.16: $(a \vee x) \wedge (b \vee y) \wedge c$ for $a = b = c$.

Chapter 5

A Topological Approach to Subtropical Algebra

We now turn our attention to the subtropical space $(\overline{\mathbb{R}}, \wedge, \vee)$ in a more abstract algebraic setting. This space does not have inverses with respect to either operation, so there is no way in which we may consider it a ring due to the lack of an underlying group structure with respect to either operation. However, arbitrarily choosing $\wedge$ as our addition and $\vee$ as our multiplication, associativity, commutativity, and distributivity of each operation over the other gives this space the structure of a semiring. Since only associativity of both operations, commutativity of addition, and distributivity of multiplication are required of a semiring, however, we note that this is a semiring with some interesting structural features. First, we note that it is a commutative semiring. Second, we note that it has the strange distributive property of addition over multiplication. Third, it has identities with respect to each operation.

Subtropical space $(\overline{\mathbb{R}}, \wedge, \vee)$, although not quite a ring, does qualify as a lattice. Our notation up to this point has been consistent with the notation in lattice theory. We note that $\wedge$ denotes the *meet* operation in lattice theory, which is precisely the minimum operation in a totally ordered set, such as $\overline{\mathbb{R}}$. Similarly, $\vee$ denotes the *join* operation, which is precisely the maximum operation in a totally ordered set.

5.1 Subtropical Endomorphisms and Automorphisms

Due to its lattice structure, we might next wish to explore the structure of our space with respect to certain algebraic mappings. The natural algebraic mappings to consider would be the lattice homomorphisms, i.e. maps with the properties

$$\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b) \quad \text{and} \quad \varphi(a \vee b) = \varphi(a) \vee \varphi(b)$$

for all $a, b \in \overline{\mathbb{R}}$. In considering such maps, however, the question arises as to what other lattices we might map $\overline{\mathbb{R}}$ into. Our intuition from the first isomorphism theorem, however, suggests that many of the interesting lattices to map into would already live as sublattices of $\overline{\mathbb{R}}$. It is in this spirit that we now examine the substropical endomorphisms.

Definition 5.1 *Let the subtropical endomorphisms be the functions $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ such that $f(a \wedge b) = f(a) \wedge f(b)$ and $f(a \vee b) = f(a) \vee f(b)$ for all $a, b \in \overline{\mathbb{R}}$. We denote the set of subtropical endomorphisms $End(\overline{\mathbb{R}})$. A bijective subtropical endomorphism is called a subtropical automorphism. The set of subtropical automorphisms is denoted $Aut(\overline{\mathbb{R}})$.*

Since we only consider subtropical space, we shall become lazy and refer to subtropical endomorphisms simply as endomorphisms, with the space understood. Likewise, subtropical automorphisms will simply be referred to as automorphisms. The first special property of endomorphisms we notice pertains to the homomorphism axioms themselves. Namely, they become equivalent:

Theorem 5.2 *Consider $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$. Then $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in \overline{\mathbb{R}}$ if and only if $f(a \vee b) = f(a) \vee f(b)$ for all $a, b \in \overline{\mathbb{R}}$.*

Proof: In the case of $a = b$, we obviously have that $f(a \wedge b) = f(a) \wedge f(b)$ forces $f(a \vee b) = f(a) \vee f(b)$. Without loss of generality, assume then that $a < b$ and that $f(a \wedge b) = f(a) \wedge f(b)$. Because $a < b$, we then have

$$f(a) = f(a \wedge b) = f(a) \wedge f(b).$$

Thus, we must have $f(a) \leq f(b)$. Since $a < b$ we must have $f(a \vee b) = f(b)$. Since $f(a) \leq f(b)$, we also have $f(b) = f(a) \vee f(b)$. Thus,

$$f(a \vee b) = f(a) \vee f(b).$$

It follows then that if $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in \overline{\mathbb{R}}$, we have

$$f(a \vee b) = f(a) \vee f(b)$$

for all $a, b \in \overline{\mathbb{R}}$. The backwards direction is similar. ∎

The above proof makes use of the way subtropical space's operations derive from an ordering imposed on the space. In the following theorem, we demonstrate how this relationship is preserved under endomorphisms.

Theorem 5.3 *For $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ and $a < b \in \overline{\mathbb{R}}$, $f(a \wedge b) = f(a) \wedge f(b)$ if and only if $f(a) \leq f(b)$.*

Proof: Suppose $f(a \wedge b) = f(a) \wedge f(b)$. Since $a < b$, we note that

$$f(a) = f(a \wedge b) = f(a) \wedge f(b).$$

Since $f(a) = f(a) \wedge f(b)$, it must be the case that $f(a) \leq f(b)$.

Now suppose $f(a) \leq f(b)$. Since $a < b$, we have that $f(a \wedge b) = f(a)$. By our assumption, we also have that $f(a) = f(a) \wedge f(b)$. It follows that

$$f(a \wedge b) = f(a) \wedge f(b).$$

∎

The previous two theorems enable us to fully classify End($\overline{\mathbb{R}}$), which we do in the following theorem.

Theorem 5.4 *The subtropical endomorphisms $\varphi \in End(\overline{\mathbb{R}})$ are precisely all non-decreasing functions $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$.*

Proof: Let $\varphi \in \mathrm{End}(\overline{\mathbb{R}})$. By Theorem 5.2 we have that $\varphi \in \mathrm{End}(\overline{\mathbb{R}})$ iff $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$ for all $a, b \in \overline{\mathbb{R}}$. By Theorem 5.3, we know this condition is met iff $a < b$ implies $\varphi(a) \leq \varphi(b)$ for all $a, b \in \overline{\mathbb{R}}$. Since this is precisely the requirement of a nondecreasing function, the claim follows immediately. ■

Thus, we have classified all subtropical endomorphisms. We now turn our attention to the automorphisms. Since automorphisms are bijective endomorphisms, the following theorems seek to classify the injective and surjective endomorphisms.

Theorem 5.5 *The injective subtropical endomorphisms are precisely all strictly increasing functions $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$.*

Proof: Let φ be an injective endomorphism. From Theorem 5.2 we must have $a < b$ implies $\varphi(a) \leq \varphi(b)$. However, since φ is injective, this forces that $\varphi(a) < \varphi(b)$. Since this happens whenever $a < b$, we have that φ is strictly increasing.

Now let $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be strictly increasing. Since f is then nondecreasing, f is an endomorphism by Theorem 5.4. Consider $a \neq b \in \overline{\mathbb{R}}$. Without loss of generality, assume $a < b$. Since f is strictly increasing, we must have $f(a) < f(b)$, so $f(a) \neq f(b)$. It follows that f is injective. ■

We now introduce a topology on $\overline{\mathbb{R}}$ in order to reasonably describe continuity of a function at $-\infty$ and ∞. Since we shall shortly classify the surjective endomorphisms in terms of continuous functions of $\overline{\mathbb{R}}$, it is important that we specify our notion of continuity. The topology we shall use is the order topology. That is to say, let T be the standard topology on $\mathbb{R}$, generated by all arbitrary unions and finite intersections of the open intervals $\{(a, b)\}_{a<b\in\mathbb{R}}$. Then the topology we set on $\overline{\mathbb{R}}$ will be $\mathcal{T}$, generated by

$$T \cup \{(c, \infty]\}_{c\in\mathbb{R}} \cup \{[-\infty, d)\}_{d\in\mathbb{R}}.$$

Continuity will then be defined with respect to this topology, $\mathcal{T}$. Unless otherwise noted, $\overline{\mathbb{R}}$ will denote $(\overline{\mathbb{R}}, \mathcal{T})$ when considered as a topological space. To remind the reader of the topological definition of continuity, we state it here.

Definition 5.6 *Let X and Y be topological spaces. A function $f : X \to Y$ is said to be continuous if for every open set $U \subseteq Y$, $f^{-1}(U)$ is open in X.*

We say that f is continuous at the point $x \in X$ if for each open set V containing $f(x)$ we may find an open set U containing x such that $f(U) \subseteq V$. *[14]*

We shall also use the Intermediate Value Theorem from topology, which we state here without proof.

Theorem 5.7 ***(Intermediate Value Theorem)*** *Let $f : X \to Y$ be a continuous map of the connected space X into the totally ordered set Y with the order topology. If $a, b \in X$ and r is a point of Y such that $f(a) < r < f(b)$, then there exists a point c of X such that $f(c) = r$.* *[14]*

Theorem 5.8 *The surjective subtropical endomorphisms are precisely the nondecreasing, continuous functions $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ with $f(-\infty) = -\infty$ and $f(\infty) = \infty$.*

Proof: Let φ be a surjective endomorphism. From Theorem 5.4 we have that φ is nondecreasing. By Theorem 5.3 we must have that $\varphi(-\infty) \leq \varphi(x)$ and $\varphi(\infty) \geq \varphi(x)$ for all $x \in \overline{\mathbb{R}}$. However, since φ is surjective, this may only happen if $\varphi(-\infty) = -\infty$ and $\varphi(\infty) = \infty$. To see the continuity of φ, it is enough to show that the preimages of basic open sets are open. Such basic open sets are of the form (a, b), $(c, \infty]$, and $[-\infty, d)$ for $a < b, c, d \in \mathbb{R}$.

- Consider $(a, b) \subseteq \overline{\mathbb{R}}$. We now examine the set $A = \{x : \varphi(x) \leq a\}$. By the nondecreasing nature of φ, we note that if $x \in A$ and $y < x$, then $y \in A$. Consider $r = \sup A$. We may have $r \in A$ or $r \notin A$. If $r \notin A$, then by φ nondecreasing we have $\varphi(r) > a$. Further,

  $$\varphi(s) \geq \varphi(r) > a$$

  for all $s > r$. It follows that we cannot have $p \in (a, \varphi(s))$ be the image of any point under φ, contradicting surjectivity. We then force $r \in A$. Thus, $A = [-\infty, r]$, so

  $$A^c = \{x : \varphi(x) > a\} = (r, \infty]$$

is open. In a similar fashion we may construct the open set

$$B^c = \{x : \varphi(x) < b\}.$$

Noting that

$$\varphi^{-1}(a,b) = A^c \cap B^c,$$

we have that the preimage of (a,b) under φ is open.

- Next we consider $(c,\infty] \subseteq \overline{\mathbb{R}}$. As in the previous case, we may construct an open set $C^c = \{x : \varphi(x) > c\}$. This set is exactly $\varphi^{-1}(c,\infty]$.
- We construct the open set $D^c = \{x : \varphi(x) < d\}$ and note that it is exactly $\varphi^{-1}[-\infty,d)$.

Since the preimage of any basic open set under φ is open, it follows that φ is continuous.

Now let $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a nondecreasing, continuous function with $f(-\infty) = -\infty$ and $f(\infty) = \infty$. From Theorem 5.4 we have that f is an endomorphism. We note that the $\overline{\mathbb{R}}$ is homeomorphic to the topological space $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ with the order topology, with homeomorphic map

$$h : \overline{\mathbb{R}} \to \left[-\frac{\pi}{2},\frac{\pi}{2}\right], \quad h(x) = \arctan x.$$

Since it is well known that $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ is connected, we then have that $\overline{\mathbb{R}}$ is connected. Now suppose $y \in (-\infty,\infty)$. Since the topology on $\overline{\mathbb{R}}$ is the order topology and since $\overline{\mathbb{R}}$ is connected, then the Intermediate Value Theorem gives that there exists a point $p \in \overline{\mathbb{R}}$ such that $f(p) = y$. It follows that f is surjective. ■

It is worthwhile for the reader to note that endomorphisms need not be continuous, in general. The floor and ceiling functions serve as easy counterexamples.

Theorem 5.9 *The subtropical automorphisms $\varphi \in Aut(\overline{\mathbb{R}})$ are precisely all strictly increasing, continuous functions $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ with $f(-\infty) = -\infty$ and $f(\infty) = \infty$.*

Proof: This result follows immediately from Theorems 5.5 and 5.8. ■

We now wish to probe the structures of End($\overline{\mathbb{R}}$) and Aut($\overline{\mathbb{R}}$). We first note that each obeys a nice closure property.

While the next two propositions are clearly true by the properties of homomorphisms as abstract algebraic maps, we delight ourselves with their proofs, regardless.

Proposition 5.10 *End($\overline{\mathbb{R}}$) is closed under composition.*

Proof: Consider $\varphi, \psi \in \text{End}(\overline{\mathbb{R}})$. By Theorem 5.4, φ and ψ are nondecreasing functions. Since we then have $a < b$ forces $\varphi(a) \leq \varphi(b)$, which in turn forces $\psi(\varphi(a)) \leq \psi(\varphi(b))$, we have that $\psi \circ \varphi$ is nondecreasing. Another application of *Theorem* 5.4 gives that $\psi \circ \varphi \in \text{End}(\overline{\mathbb{R}})$. ■

Proposition 5.11 *Aut($\overline{\mathbb{R}}$) is closed under composition.*

Proof: Let $\varphi, \psi \in \text{Aut}(\overline{\mathbb{R}})$. By Theorem 5.9, φ and ψ are strictly increasing, continuous functions each satisfying $-\infty \mapsto -\infty$ and $\infty \mapsto \infty$. Since each is strictly increasing, we observe that $a < b$ forces $\varphi(a) < \varphi(b)$, and further $\psi(\varphi(a)) < \psi(\varphi(b))$. Thus, $\psi \circ \varphi$ is strictly increasing. Since compositions of continuous maps are continuous, $\psi \circ \varphi$ is continuous. Last, we note that $\psi(\varphi(-\infty)) = \psi(-\infty) = -\infty$ and $\psi(\varphi(\infty)) = \psi(\infty) = \infty$. Thus, $\psi \circ \varphi$ satisfies $-\infty \mapsto -\infty$ and $\infty \mapsto \infty$. By Theorem 5.9, $\psi \circ \varphi \in \text{Aut}(\overline{\mathbb{R}})$. ■

Due to these closure properties, it now makes sense to consider the algebraic structures (End($\overline{\mathbb{R}}$),$\circ$) and (Aut($\overline{\mathbb{R}}$),$\circ$).

Proposition 5.12 *(End($\overline{\mathbb{R}}$),$\circ$) is a monoid.*

Proof: Proposition 5.10 gives that End($\overline{\mathbb{R}}$) is closed under compositions and composition is an associative operation. Noting that $id : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ with $id(x) = x$ is in End($\overline{\mathbb{R}}$) and meets the requirements of an identity, the claim follows. ■

Proposition 5.13 *(Aut($\overline{\mathbb{R}}$),$\circ$) is a group.*

Proof: Proposition 5.11 gives that $\mathrm{Aut}(\overline{\mathbb{R}})$ is closed under compositions. Again, composition is an associative operation and $id \in \mathrm{Aut}(\overline{\mathbb{R}})$ gives our structure an identity. Consider some $\varphi \in \mathrm{Aut}(\overline{\mathbb{R}})$. Since φ is bijective, we may consider its inverse, φ^{-1}, a function defined on $\overline{\mathbb{R}}$. Since φ is strictly increasing, we note that $\varphi(a) < \varphi(b)$ forces $a < b$. Thus, φ^{-1} is strictly increasing. By Theorem 5.4, it follows that $\varphi^{-1} \in \mathrm{End}(\overline{\mathbb{R}})$. Since φ is bijective, so is φ^{-1}, and we conclude that $\varphi^{-1} \in \mathrm{Aut}(\overline{\mathbb{R}})$. ■

Corollary 5.14 *The order-preserving homeomorphisms $\varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ are precisely the automorphisms.*

Proof: Theorem 5.9 gives that the automorphisms are precisely the continuous, bjiective functions and Proposition 5.13 gives that their inverses are also continuous, making them homeomorphisms. Since automorphisms preserve order, they are then order-preserving homeomorphisms.

Homeomorphisms are bijective. If a homeomorphism is to preserve order, it must be nondecreasing. Theorem 5.9 then gives that an order-preserving homeomorphism is an automorphism. ■

5.2 Equivalence of Algebraic Maps

Consider the following example.

Example 5.15 *f_1 and f_2 are two endomorphisms, where*

$$f_1(x) = \begin{cases} -1 & \text{if } x \in [-2,2] \\ x & \text{otherwise.} \end{cases}$$

and

$$f_2(x) = \begin{cases} 1 & \text{if } x \in [-2,2] \\ x & \text{otherwise.} \end{cases}$$

We notice that there are striking similarities between these two maps. Each map fixes everything outside the interval $[-2,2]$ in precisely the same manner. Furthermore, each map handles the interval $[-2,2]$ in a similar fashion, in that $[-2,2]$ is mapped to a single point in $[-2,2]$ that is neither the supremum of $[-\infty,-2)$ nor the infimum of $(2,\infty]$. The similarities of the two maps f_1 and f_2 then seem to topological in nature, so we set out to find a precise statement about the topological similarity of two endomorphisms. Our initial guess might be that f_1 and f_2 derive their similarities from the fact that their images are homeomorphic under the subspace topology of $\overline{\mathbb{R}}$. This notion inspires the following definition of equivalence.

Definition 5.16 *We say two endomorphisms f and g are left-equivalent if we may find some $\psi \in Aut(\overline{\mathbb{R}})$ such that $f = \psi g$.*

Proposition 5.17 *Left equivalence is indeed an equivalence relation.*

Proof: This follows from the fact that Aut($\overline{\mathbb{R}}$) is a group. Namely,

$$f = id \circ f$$

gives reflexivity,

$$f = \psi g \quad \Rightarrow \quad \psi^{-1} f = g$$

gives symmetry, and

$$f = \psi g,\, g = \rho h \quad \Rightarrow \quad f = (\rho \circ \psi) h$$

gives transitivity. ■

In the case of Example 5.15, we see that f_1 and f_2 are actually left equivalent if we choose an isomorphism such as ψ:

$$\psi(x) = \begin{cases} 3x + 4 & \text{if } x \in [-2, -1] \\ \frac{x}{3} + \frac{4}{3} & \text{if } x \in (-1, 2] \\ x & \text{otherwise.} \end{cases}$$

This isomorphism ψ gives $f_2 = \psi f_1$. Our intent in defining left-equivalence was that if $f(\overline{\mathbb{R}})$ and $g(\overline{\mathbb{R}})$ are homeomorphic as subspaces of $\overline{\mathbb{R}}$, ψ is some homeomorphism between them. However, we find that the image topology is too weak to give us the form of equivalence we desire in considering the following example.

Example 5.18 *$g_1(x) = \lfloor x \rfloor$ and $g_2(x) = \lceil x \rceil$ are endomorphisms.*

The problem that this example poses is a case where

$$g_1(\overline{\mathbb{R}}) = \mathbb{Z} = g_2(\overline{\mathbb{R}})$$

but g_1 and g_2 are not left-equivalent. To see this, we note $g_1(0) = g_1(1/2)$ while $g_2(0) \neq g_2(1/2)$. Since g_1 cannot distinguish between 0 and 1/2, there is no map ψ such that ψg_1 can distinguish 0 and 1/2. Since g_2 must distinguish between these points, we cannot have $g_2 = \psi g_1$ for any map ψ.

This suggests, then, that we might want to construct a new notion of topological equivalence. Instead of attempting to formulate equivalence of f and g based on whether or not $f(\overline{\mathbb{R}})$ and $g(\overline{\mathbb{R}})$ are homeomorphic under the subspace topology, we work with the following topology:

Definition 5.19 *Let f be an endomorphism. Then we define $\overline{\mathbb{R}}_f$ to be $\overline{\mathbb{R}}$ considered under the coarsest topology that makes f continuous. That is, let the topology of $\overline{\mathbb{R}}_f$ be defined as follows: U is open in $\overline{\mathbb{R}}_f$ if and only if $U = f^{-1}(V)$ for some V open in $\overline{\mathbb{R}}$.*

Proposition 5.20 *$\overline{\mathbb{R}}_f$ actually defines a topology on $\overline{\mathbb{R}}$.*

Proof: We note that $\emptyset = f^{-1}(\emptyset)$ and $\overline{\mathbb{R}} = f^{-1}(\overline{\mathbb{R}})$. Further, let $\{U_i\}_i$ be a collection of open sets in $\overline{\mathbb{R}}_f$. Then $\{U_i\} = \{f^{-1}(V_i)\}_i$ for a collection of open sets $\{V_i\}$ in $\overline{\mathbb{R}}$. We then observe that

$$U_\alpha \cap U_\beta = f^{-1}(V_\alpha) \cap f^{-1}(V_\beta) = f^{-1}(V_\alpha \cap V_\beta),$$

which is open in $\overline{\mathbb{R}}_f$ since $V_\alpha \cap V_\beta$ is open in $\overline{\mathbb{R}}$. Similarly,

$$\bigcup_i U_i = \bigcup_i f^{-1}(V_i) = f^{-1}\left(\bigcup_i V_i\right)$$

is open in $\overline{\mathbb{R}}_f$. ■

Theorem 5.21 *For any map $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, the induced map $f : \overline{\mathbb{R}}_f \to f(\overline{\mathbb{R}})$ is continuous and open.*

Proof: If W open in $f(\overline{\mathbb{R}})$, then by the definition of the subspace topology, $W = V \cap f(\overline{\mathbb{R}})$ for some V open in $\overline{\mathbb{R}}$. By the way we constructed $\overline{\mathbb{R}}_f$, this forces

$$f^{-1}(W) = f^{-1}(V \cap f(\overline{\mathbb{R}})) = f^{-1}(V) \cap f^{-1}(f(\overline{\mathbb{R}})) = f^{-1}(V)$$

to be open in $\overline{\mathbb{R}}_f$, so f is continuous.

Let U be open in $\overline{\mathbb{R}}$. Then we must find V open in $\overline{\mathbb{R}}$ such that $U = f^{-1}(V)$. We then note that

$$f(U) = f(f^{-1}(V)) = V \cap f(\overline{\mathbb{R}}),$$

which is open under the subspace topology. ■

In a heuristic sense, the problem we run into with trying to create a notion of equivalence based on the images of functions as subspaces of $\overline{\mathbb{R}}$ is that of an inherent loss of information. When we consider the image of a function, we lose all sense of how the function acted with respect to its domain. With Example 5.18, the information lost was whether the points' preimages were half-open on the left or half-open on the right. Our hope in constructing $\overline{\mathbb{R}}_f$ is that it will retain enough information about the function's preimages to allow us to make statements of left-equivalence in terms of how functions act with respect to this topology. We run into a slight problem with the strength of left-equivalence, however.

Proposition 5.22 *For a map h and automorphism φ, $\overline{\mathbb{R}}_h$ is homeomorphic to $\overline{\mathbb{R}}_{h\circ\varphi}$.*

Proof: Consider an open set V in $\overline{\mathbb{R}}_h$. Then we must have $V = h^{-1}(U)$ for some open U in $h(\overline{\mathbb{R}})$. Since

$$h \circ \varphi : \overline{\mathbb{R}}_{h\circ\varphi} \to (h \circ \varphi)(\overline{\mathbb{R}}) = h(\varphi(\overline{\mathbb{R}})) = h(\overline{\mathbb{R}})$$

is continuous by Theorem 5.21, we then have

$$(h \circ \varphi)^{-1}(U) = \varphi^{-1}(h^{-1}(U))$$

is open in $\overline{\mathbb{R}}_{h\circ\varphi}$. Thus, V open in $\overline{\mathbb{R}}_h$ forces

$$\varphi^{-1}(V) = \varphi^{-1}(h^{-1}(U))$$

to be open in $\overline{\mathbb{R}}_{h\circ\varphi}$, so

$$\varphi : \overline{\mathbb{R}}_h \to \overline{\mathbb{R}}_{h\circ\varphi}$$

is continuous. Similarly,

$$\varphi^{-1} : \overline{\mathbb{R}}_{h\circ\varphi} \to \overline{\mathbb{R}}_h$$

is continuous, so $\overline{\mathbb{R}}_h$ is homeomorphic to $\overline{\mathbb{R}}_{h\circ\varphi}$ by way of φ. ■

This gives rise to the following example:

Example 5.23 *$\overline{\mathbb{R}}_h$ is homeomorphic to $\overline{\mathbb{R}}_{h\circ\varphi}$, where h and φ are given by*

$$h(x) = \lceil x \rceil \quad \textit{and} \quad \varphi(x) = \lceil x + 1/2 \rceil.$$

In the spirit of Example 5.18, we note that h and $h \circ \varphi$ cannot possibly be left-equivalent, since h distinguishes between 0 and $1/2$, while $h \circ \varphi$ does not. In constructing this example, however, we implicity hinted at a new equivalence relationship:

Definition 5.24 *We say two endomorphisms f and g are right-equivalent if we may find some $\varphi \in Aut(\overline{\mathbb{R}})$ such that $f = g\varphi$.*

The proof that this is indeed an equivalence relation is similar to the proof for left-equivalence. In the previous example, we saw two right-equivalent functions that were not left-equivalent. We shall next see two left-equivalent functions that are not right-equivalent.

Example 5.25 $h_1(x) = \lceil x \rceil$ *and* $h_2(x) = \lceil x \rceil + 1/2$.

In this example we see that $h_1(\overline{\mathbb{R}}) = \mathbb{Z}$, while $h_2(\overline{\mathbb{R}}) = \mathbb{Z} + 1/2$. Since for any function φ we will have $(h_1 \circ \varphi)(\overline{\mathbb{R}}) \subseteq h(\overline{\mathbb{R}})$,

$$h_1(\overline{\mathbb{R}}) \cap h_2(\overline{\mathbb{R}}) = \emptyset$$

forces

$$(h_1 \circ \varphi)(\overline{\mathbb{R}}) \cap h_2(\overline{\mathbb{R}}) = \emptyset.$$

Thus, we cannot possibly have a function φ such that $h_1 \circ \varphi = h_2$, so h_1 and h_2 are not right-equivalent.

Thus, we have two notions of equivalence, neither of which implies the other, and each of which indicates something about a pair of functions with respect to how they act on subtropical space topologically. If f and g are left equivalent, then this is a statement about deforming the image space. With f, we simply map $\overline{\mathbb{R}}$ into itself. With $g = \psi \circ f$, we first map $\overline{\mathbb{R}}$ into itself through f and then continuously deform the image. Alternatively, if f and g are right-equivalent, then we have a statement about how the domain space is deformed. With f, we map $\overline{\mathbb{R}}$ into itself. With $g = f \circ \varphi$, we continuously deform the domain space and then apply f. In order to have both of these actions encapsulated in our equivalence class, we create yet another notion of equivalence.

Definition 5.26 *We say two endomorphisms f and g are equivalent if we may find some $\varphi, \psi \in Aut(\overline{\mathbb{R}})$ such that $f = \psi g \varphi$. We shall denote this $f \sim g$.*

We note that by letting ψ or φ equal $id \in \mathrm{Aut}(\overline{\mathbb{R}})$, f and g being right-equivalent or left-equivalent each imply that $f \sim g$. In the next section, we shall see that this definition of equivalence is precisely what we need to formulate a statement about $\overline{\mathbb{R}}_f$ and $\overline{\mathbb{R}}_g$ in terms of the equivalence of f and g. Namely, $\overline{\mathbb{R}}_f \cong \overline{\mathbb{R}}_g$ in an order-preserving way iff $f \sim g$.

5.3 The Structure Theorem

In this section, we shall demonstrate that $\overline{\mathbb{R}}_f \cong \overline{\mathbb{R}}_g$ in an order-preserving way iff $f \sim g$. The strategy we shall use is making each of the diagrams below force the other:

$$\begin{tikzcd}
\overline{\mathbb{R}}_g \arrow[r, "g"] & g(\overline{\mathbb{R}}) \\
\overline{\mathbb{R}}_f \arrow[u, "h"] \arrow[r, "f"'] & f(\overline{\mathbb{R}})
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\mathbb{R}} \arrow[r, "g"] & \overline{\mathbb{R}} \arrow[d, "\psi"] \\
\overline{\mathbb{R}} \arrow[u, "\varphi"] \arrow[r, "f"'] & \overline{\mathbb{R}}
\end{tikzcd}$$

Here, the left diagram consists of the endomorphisms f, g as they are induced on the spaces as pictured, and $h : \overline{\mathbb{R}}_g \to \overline{\mathbb{R}}_f$ an order-preserving homeomorphism. Thus it is a statement about there being some homeomorphism between $\overline{\mathbb{R}}_f$ and $\overline{\mathbb{R}}_g$ that preserves order. We wish that the right diagram commutes for endomorphisms f, g and automorphisms φ, ψ, which would imply $f \sim g$.

For our first result towards the establishing the Structure Theorem, we will need the fact that $\overline{\mathbb{R}}$ is Hausdorff. This is easily seen by an argument we made previously. Since $\overline{\mathbb{R}}$ is homeomorphic to $\left[\frac{\pi}{2}, -\frac{\pi}{2}\right]$, a Hausdorff space, $\overline{\mathbb{R}}$ is Hausdorff.

Theorem 5.27 *Let f, g be endomorphisms and $\varphi : \overline{\mathbb{R}}_f \to \overline{\mathbb{R}}_g$ be an order-preserving homeomorphism. Then there exists an induced order-preserving bijection $\rho : g(\overline{\mathbb{R}}) \to f(\overline{\mathbb{R}})$ such that $f = \rho g \varphi$. That is, we can find ρ such that the diagrams below commute.*

$$
\begin{tikzcd}
\overline{\mathbb{R}} \arrow[r, "g"] & g(\overline{\mathbb{R}}) \arrow[d, "\rho"] \\
\overline{\mathbb{R}} \arrow[u, "\varphi"] \arrow[r, "f"'] & f(\overline{\mathbb{R}})
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\overline{\mathbb{R}} \arrow[r, "g"] \arrow[d, "\varphi^{-1}"'] & g(\overline{\mathbb{R}}) \\
\overline{\mathbb{R}} \arrow[r, "f"'] & f(\overline{\mathbb{R}}) \arrow[u, "\rho^{-1}"']
\end{tikzcd}
$$

Proof: We will directly construct a function ρ meeting our requirements. Since the maps we have to work with are f, g, and φ, a natural starting point would be $g\varphi f^{-1}$. However, it is yet unclear whether this is actually a function, since it involves the composition of a preimage.

Let $a \in \overline{\mathbb{R}}$ and consider $g\varphi f^{-1}(a)$. Suppose $b, c \in g\varphi f^{-1}(a)$ with $b \neq c$. It is necessarily the case that $\varphi^{-1}g^{-1}(b)$ and $\varphi^{-1}g^{-1}(c)$ are disjoint. Let $x \in \varphi^{-1}g^{-1}(b)$ and $y \in \varphi^{-1}g^{-1}(c)$ such that $x, y \in g^{-1}(a)$. Since

$$f(x) = a = f(y),$$

all open sets in $\overline{\mathbb{R}}_f$ containing x must also contain y, and vice versa, since these open sets must contain all of $f^{-1}(a)$. Since $\overline{\mathbb{R}}$ is Hausdorff, we may separate $b \in U$, $c \in V$ for disjoint open sets U and V. We then observe that $\varphi(x) \in g^{-1}(U)$ and $\varphi(y) \in g^{-1}(V)$, where $g^{-1}(U)$ and $g^{-1}(V)$ are disjoint open sets in $\overline{\mathbb{R}}_g$. Furthermore, $\varphi^{-1}(g^{-1}(U))$ and $\varphi^{-1}(g^{-1}(V))$ are then disjoint open sets in $\overline{\mathbb{R}}_f$ with $x \in \varphi^{-1}(g^{-1}(U))$ and $y \in \varphi^{-1}(g^{-1}(V))$. This contradicts that all open sets containing x must contain y. We therefore cannot have $b, c \in g\varphi f^{-1}(a)$ with $b \neq c$.

We note that if $a \in f(\overline{\mathbb{R}})$, then $f^{-1}(a) \neq \emptyset$, and thus $g\varphi f^{-1}(a) \neq \emptyset$, so $g\varphi f^{-1}$ must take each element of $f(\overline{\mathbb{R}})$ to exactly one element of $\overline{\mathbb{R}}$. Thus, we have that $g\varphi f^{-1} : f(\overline{\mathbb{R}}) \to \overline{\mathbb{R}}$ is a function. Furthermore, since its image is contained in $g(\overline{\mathbb{R}})$, we may restrict the codomain to give the function

$$r := g\varphi f^{-1} : f(\overline{\mathbb{R}}) \to g(\overline{\mathbb{R}}).$$

A similar argument yields the function

$$s := f\varphi^{-1}g^{-1} : g(\overline{\mathbb{R}}) \to f(\overline{\mathbb{R}}).$$

Next consider $a \in f(\overline{\mathbb{R}})$, and suppose $r(a) = b$. Since

$$g\varphi f^{-1}(a) = r(a) = b,$$

we have that

$$\varphi f^{-1}(a) \subseteq g^{-1}(b),$$

and thus

$$f^{-1}(a) \subseteq \varphi^{-1} g^{-1}(b).$$

This gives that

$$a \in f\varphi^{-1} g^{-1}(b) = s(b),$$

so $s(b) = a$. Thus,

$$(s \circ r)(a) = s(r(a)) = s(b) = a,$$

and we have that $s \circ r$ is the identity function on $f(\overline{\mathbb{R}})$. Similarly, we find $r \circ s$ to be the identity function on $g(\overline{\mathbb{R}})$. It follows that

$$s : g(\overline{\mathbb{R}}) \to f(\overline{\mathbb{R}})$$

is a bijection with $s^{-1} = r$. Letting $\rho = s$, we claim that the following diagrams commute.

$$\begin{tikzcd} \overline{\mathbb{R}} \arrow[r, "g"] & g(\overline{\mathbb{R}}) \arrow[d, "\rho"] \\ \overline{\mathbb{R}} \arrow[u, "\varphi"] \arrow[r, "f"'] & f(\overline{\mathbb{R}}) \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathbb{R}} \arrow[r, "g"] \arrow[d, "\varphi^{-1}"'] & g(\overline{\mathbb{R}}) \\ \overline{\mathbb{R}} \arrow[r, "f"'] & f(\overline{\mathbb{R}}) \arrow[u, "\rho^{-1}"'] \end{tikzcd}$$

To see that $f = \rho g \varphi$, let $a \in \overline{\mathbb{R}}$ and consider $\rho g \varphi(a)$. We observe that

$$\begin{aligned} \rho g \varphi(a) &= (f\varphi^{-1} g^{-1}) g \varphi(a) \\ &= f(\varphi^{-1} g^{-1}(g(\varphi(a)))) \\ &\supseteq f(\varphi^{-1}\varphi(a)) \\ &= f(a). \end{aligned}$$

Since $\rho g \varphi$ and f are both functions, this implies $\rho g \varphi(a) = f(a)$, and thus $\rho g \varphi = f$. We may similarly show that $\rho^{-1} f \varphi^{-1} = g$.

Next, we note that ρ is order-preserving. To see this, we note that if $a, b \in g(\overline{\mathbb{R}})$ with $a < b$ we have $g^{-1}(a) < g^{-1}(b)$, since if $x \in g^{-1}(a)$ and $y \in g^{-1}(b)$ with $x \geq y$, then $g(x) < g(y)$ for $x \geq y$, and g decreases. Similarly $\varphi^{-1}g^{-1}(a) < \varphi^{-1}g^{-1}(b)$, and finally

$$\begin{aligned} \rho(a) &= f\varphi^{-1}g^{-1}(a) \\ &\leq f\varphi^{-1}g^{-1}(b) \\ &= \rho(b). \end{aligned}$$

Thus, $a < b$ forces $\rho(a) \leq \rho(b)$. Since ρ bijective, we then have the strict inequality $\rho(a) < \rho(b)$, and ρ is strictly increasing. ■

We now wish to extend the map $\rho : g(\overline{\mathbb{R}}) \to f(\overline{\mathbb{R}})$ to an isomorphism $\widetilde{\rho} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$. Noting that $g(\overline{\mathbb{R}})$ and $f(\overline{\mathbb{R}})$ are essentially copies of $\overline{\mathbb{R}}$ that are possibly "missing" some elements, our hope is that the missing portions of $f(\overline{\mathbb{R}})$ and $g(\overline{\mathbb{R}})$ somehow "line up" in a way that provides a map between the missing portions. For instance, in Example 5.15, we saw the maps

$$f_1(x) = \begin{cases} -1 & \text{if } x \in [-2, 2] \\ x & \text{otherwise.} \end{cases}$$

and

$$f_2(x) = \begin{cases} 1 & \text{if } x \in [-2, 2] \\ x & \text{otherwise.} \end{cases}$$

For these maps, we have that

$$f_1(\overline{\mathbb{R}}) = [-\infty, -2) \cup \{-1\} \cup (2, \infty]$$

and

$$f_2(\overline{\mathbb{R}}) = [-\infty, -2) \cup \{1\} \cup (2, \infty].$$

The map $\rho : f_1(\overline{\mathbb{R}}) \to f_2(\overline{\mathbb{R}})$ would then be the function

$$\rho(x) = \begin{cases} 1 & \text{if } x = -1 \\ x & \text{if } [-\infty, -2) \cup (2, \infty]. \end{cases}$$

In extending ρ to an isomorphism $\widetilde{\rho}$, the intuitive way to accomplish this would be to have $\widetilde{\rho}$ take the interval $[-2, -1)$ to $[-2, 1)$ and the interval $(-1, 2]$ to $(1, 2]$ in some bijective, order-preserving way. By Corollary 5.14, this could be accomplished through the natural homeomorphism of the intervals.

However, in the general case, it is unclear how to proceed. Perhaps in the previous case we lucked out and simply had two functions whose "missing" portions nicely corresponded. To make matters worse, it is unclear how the "missing" portions of $f(\overline{\mathbb{R}})$ are encoded in the function f. It is straightforward to see that they somehow correspond to the jump discontinuities of the function. However, while nondecreasing functions are fairly nice to analyze, their discontinuities need not be isolated, as seen in the following example.

Example 5.28 *Let (q_n) be some enumeration of $\overline{\mathbb{Q}}$ and let (a_n) be the sequence defined by $a_n = 1/2^n$. We then define the function $q : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ by*

$$q(x) = \sum_{q_n \leq x} a_n.$$

We note that this function is nondecreasing. However, there is no neighborhood in which it is continuous, since it is discontinuous at each rational point. This example suggests that constructing $\widetilde{\rho}$ may be trickier, in general, than was constructing $\widetilde{\rho}$ for our functions f_1 and f_2 of Example 5.15. To accomplish a sensible pairing of missing portions, we will need a result from the analysis of monotonic functions. We state it here without proof.

Definition 5.29 *Let f be a function defined on (a,b), and let $x \in [a,b)$. Then we write $f(x+) = q$ if $f(t_n) \to q$ as $n \to \infty$ for all sequences $\{t_n\}$ in (x,b) such that $t_n \to x$. We similarly write $f(x-)$ for $x \in (a,b]$ by restricting ourselves to subsequences $\{t_n\}$ in (a,x).* *[9]*

Theorem 5.30 *Let f be nondecreasing on (a,b). Then $f(x+)$ and $f(x-)$ exist on every point $x \in (a,b)$. More precisely,*

$$\sup_{t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t} f(t).$$

Furthermore, if $a < x < y < b$, then $f(x+) \leq f(y-)$. Lastly, f is continuous at x iff $f(x+) = f(x-)$. *[9]*

We next prove a technical lemma.

Lemma 5.1 *Let f, g, φ and ρ be as in Theorem 5.27. Then there is an order-preserving bijection $\chi_x : [g(x-), g(x+)] \to [f(\varphi^{-1}(x)-), f(\varphi^{-1}(x)+)]$ for each $x \in \overline{\mathbb{R}}$ with $\chi_x(g(x)) = f(\varphi^{-1}(x))$.*

Proof: Since Theorem 5.30 gives that $g(x-) \leq g(x) \leq g(x+)$ and $f(\varphi^{-1}(x)-) \leq f(\varphi^{-1}(x)) \leq f(\varphi^{-1}(x)+)$, the claim is equivalent to showing

$$g(x-) = g(x) \text{ if and only if } f(\varphi^{-1}(x)-) = f(\varphi^{-1}(x))$$

and

$$g(x+) = g(x) \text{ if and only if } f(\varphi^{-1}(x)+) = f(\varphi^{-1}(x)),$$

since a natural homeomorphism of closed intervals would then follow.

Suppose that $g(x-) = g(x)$. We note that $g(x-) \in g(\overline{\mathbb{R}})$, and thus $\rho(g(x-))$ is defined. We should note the difference between $\rho(g(x-))$ and $(\rho \circ g)(x-)$, where the former is ρ evaluated at the value $g(x-)$ and the latter is the leftward limit of $\rho \circ g$ at x. We observe that by the nondecreasing nature of all functions involved (and thus also of their compositions),

$$\begin{aligned}
\rho(g(x-)) &= \rho\left(\sup_{t<x} g(t)\right) \\
&= \rho\left(\sup_{t<x} \rho^{-1} f \varphi^{-1}(t)\right) \\
&\leq \rho\rho^{-1}\left(\sup_{t<x} f\varphi^{-1}(t)\right) \\
&= \sup_{t<x} f\varphi^{-1}(t) \\
&= \sup_{t<x} \rho g(t) \\
&\leq \rho\left(\sup_{t<x} g(t)\right) \\
&= \rho(g(x-)).
\end{aligned}$$

The result we obtain is that $\rho(g(x-)) = \sup_{t<x} f\varphi^{-1}(t)$. Since φ^{-1} is strictly increasing, $\varphi^{-1}(t) < \varphi^{-1}(x)$ is equivalent to $t < x$, so we may alternatively write

$$\begin{aligned}\rho(g(x-)) &= \sup_{t<x} f\varphi^{-1}(t)\\ &= \sup_{\varphi^{-1}(t)<\varphi^{-1}(x)} f(\varphi^{-1}(t))\\ &= f(\varphi^{-1}(x)-).\end{aligned}$$

A similar argument yields $\rho(g(x+)) = f(\varphi^{-1}(x)+)$. Thus, in the case where $g(x-) = g(x)$ we must also have

$$f(\varphi^{-1}(x)-) = \rho(g(x-)) = \rho(g(x)) = f(\varphi^{-1}(x)).$$

Alternatively, if we start with the condition $f(\varphi^{-1}(x)-) = f(\varphi^{-1}(x))$, we may then show in a similar fashion that $\rho^{-1}f(\varphi^{-1}(x)-) = g(x-)$, concluding that $g(x-) = g(x)$.

Thus, we see that $g(x-) = g(x)$ if and only if $f(\varphi^{-1}(x)-) = f(\varphi^{-1}(x))$. Similarly, $g(x+) = g(x)$ if and only if $f(\varphi^{-1}(x)+) = f(\varphi^{-1}(x))$. It follows that there is an order-preserving bijection

$$\chi_x : [g(x-), g(x+)] \to [f(\varphi^{-1}(x)-), f(\varphi^{-1}(x)+)]$$

for each $x \in \overline{\mathbb{R}}$ given by pasting the natural homeomorphisms

$$\chi_{x-} : [g(x-), g(x)] \to [f(\varphi^{-1}(x)-), f(\varphi^{-1}(x))],$$

$$\chi_{x+} : [g(x), g(x+)] \to [f(\varphi^{-1}(x)), f(\varphi^{-1}(x)+)],$$

where

$$\begin{aligned} g(x-) &\mapsto f(\varphi^{-1}(x)-),\\ g(x) &\mapsto f(\varphi^{-1}(x)),\\ g(x+) &\mapsto f(\varphi^{-1}(x)+).\end{aligned}$$

■

Theorem 5.31 *Let f be nondecreasing. Then if $x < y$ and $f(x+) = f(y-)$, $f(x+) = f(y-) = f(z)$ for any choice of $z \in (x, y)$.*

Proof: Let $f(x+) = p = f(y-)$ and let $z \in (x, y)$. We then note

$$p = f(x+) \leq f(z-) \leq f(z) \leq f(z+) \leq f(y-) = p.$$

■

We now have all of the necessary tools to attempt the extension of ρ to $\widetilde{\rho}$. The strategy will be pasting the maps we created in Lemma 5.1 to our function ρ. We shall soon see that the "missing" portions of $f(\overline{\mathbb{R}})$ and $g(\overline{\mathbb{R}})$ will all fall into intervals having precisely the form $[g(x-), g(x+)]$ and $[f(\varphi^{-1}(x)-), f(\varphi^{-1}(x)+)]$, respectively. Furthermore, there will be a bijective correspondence between these intervals determined by x.

However, we encounter a slight problem if we attempt to extend ρ for arbitrary homomorphisms f, g. Namely, if f and g do not both handle $-\infty$ and ∞ in the same way, it is possible to have homeomorphic spaces $\overline{\mathbb{R}}_f$ and $\overline{\mathbb{R}}_g$ without a way to extend ρ to an isomorphism. The claim is that this happens precisely when, without loss of generality, $g(\infty) < f(\infty) = \infty$ or $g(-\infty) > f(-\infty) = -\infty$. The problem with the first case is that any order-preserving extension of ρ would need to map the entirety of $[g(\infty), \infty]$ into $[f(\infty), \infty] = \{\infty\}$, since we must have $\rho g(\infty) = f\varphi(\infty) = f(\infty)$. However, such a map cannot be injective. Similar problems arise for the case involving $-\infty$. For this reason, we then might choose to impose endpoint restrictions $f(\pm\infty) = g(\pm\infty) = \pm\infty$.

Theorem 5.32 *Let f, g, φ, and ρ be as in Theorem 5.27 with the additional requirement that $f(\pm\infty) = g(\pm\infty) = \pm\infty$. The map $\rho : g(\overline{\mathbb{R}}) \to f(\overline{\mathbb{R}})$ may be extended to an automorphism $\widetilde{\rho} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ such that the diagrams below commute.*

$$\begin{tikzcd}
\overline{\mathbb{R}} \arrow[r, "g"] & \overline{\mathbb{R}} \arrow[d, "\widetilde{\rho}"] \\
\overline{\mathbb{R}} \arrow[u, "\varphi"] \arrow[r, "f"'] & \overline{\mathbb{R}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\mathbb{R}} \arrow[r, "g"] \arrow[d, "\varphi^{-1}"'] & \overline{\mathbb{R}} \\
\overline{\mathbb{R}} \arrow[r, "f"'] & \overline{\mathbb{R}} \arrow[u, "\widetilde{\rho}^{-1}"']
\end{tikzcd}$$

Proof: Define the map $\widetilde{\rho} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ by

$$\widetilde{\rho}(p) = \begin{cases} \rho(p) & \text{if } p \in g(\overline{\mathbb{R}}) \\ \chi_x(p) & \text{if } p \in [g(x-), g(x+)]. \end{cases}$$

We first show that this map is well-defined. Suppose $p \in \overline{\mathbb{R}}$. Then either $p \in g(\overline{\mathbb{R}})$ or $p \in \overline{\mathbb{R}} - g(\overline{\mathbb{R}})$. If $p \in g(\overline{\mathbb{R}})$, then we need only check that if $p \in [g(x-), g(x+)]$ as well, then $\chi_x(p) = \rho(p)$. We shall see that $[g(x-), g(x+)] \cap g(\overline{\mathbb{R}})$ is at most $\{g(x-), g(x), g(x+)\}$. In the case where $g(x-) = g(x+)$, $[g(x-), g(x+)] = \{g(x)\}$. When $g(x-) < g(x)$, suppose $p \in (g(x-), g(x))$. Then there can be no $y \neq x$ such that $g(y) = p$, since if $y < x$,

$$g(y) \leq g(y+) \leq g(x-) < p.$$

Similarly, $g(y) > p$ if $y > x$. Thus, $(g(x-), g(x)) \cap g(\overline{\mathbb{R}})$ is empty. The same reasoning will yield that $(g(x), g(x+)) \cap g(\overline{\mathbb{R}})$ is also empty, so

$$[g(x-), g(x+)] \cap g(\overline{\mathbb{R}}) \subseteq \{g(x-), g(x), g(x+)\}.$$

It follows from the way we defined the map χ_x in Lemma 5.1 that ρ and χ_x agree on $g(x)$. If $g(x-) \in g(\overline{\mathbb{R}})$, then we may consider $\rho(g(x-))$, and from the string of inequalities in Lemma 5.1 we have that

$$\rho(g(x-)) = f(\varphi^{-1}(x)-),$$

so ρ and χ_x agree on $g(x-)$. Similarly, if $g(x+) \in g(\overline{\mathbb{R}})$, ρ and χ_x agree there, as well. It follows that $\widetilde{\rho}$ is well-defined on $g(\overline{\mathbb{R}})$.

Now suppose that $p \in \overline{\mathbb{R}} - g(\overline{\mathbb{R}})$. Since p is not in the range of g, a non-decreasing function with $g(\pm\infty) = \pm\infty$, we note that p must correspond to some value x on which g is discontinuous—i.e., $p \in [g(x-), g(x+)]$ for some $x \in \overline{\mathbb{R}}$. To see this, let

$$x := \sup\{r \in \overline{\mathbb{R}} \,:\, g(r) < p\}$$

and

$$x' := \inf\{r \in \overline{\mathbb{R}} \,:\, g(r) > p\}.$$

It is clear that $x \leq x'$. We will show that $x = x'$. Suppose, on the contrary, that $x < x'$. Then we may find some $s \in \overline{\mathbb{R}}$ such that $x < s < x'$. Since, by assumption, $p \notin g(\overline{\mathbb{R}})$, we cannot have $g(s) = p$. If $g(s) < p$, then we have some s such that $g(s) < p$ and $s > x$, contradicting the definition of x.

Similarly, $g(s) > p$ will contradict the definition of x'. Since we may then find no such s, it follows that $x = x'$. We then have

$$\sup\{r \in \overline{\mathbb{R}} : g(r) < p\} = x = \inf\{r \in \overline{\mathbb{R}} : g(r) > p\}.$$

Since $p \notin g(\overline{\mathbb{R}})$, we have that $g(x) \neq p$. Suppose first that $g(x) < p$. Then Theorem 5.30 gives that $f(x-) \leq f(x) < p$. By how we defined x, we note that for any $t > x$ we must have $g(t) > p$. Thus,

$$p \leq \inf_{t>x} g(t) = g(x+),$$

and we have that

$$g(x-) \leq p \leq g(x+).$$

In the case where $g(x) > p$, we observe that

$$g(x-) \leq p \leq g(x+)$$

is similarly forced. Thus, we have shown that if $p \in \overline{\mathbb{R}} - g(\overline{\mathbb{R}})$, then $p \in [g(x-), g(x+)]$ for some $x \in \overline{\mathbb{R}}$, as desired. We also note that g must be discontinuous at x, because otherwise $g(x-) = g(x) = g(x+)$ by Theorem 5.30, forcing $g(x) = p$ and thus contradicting that $p \notin g(\overline{\mathbb{R}})$.

Since $g(x-) < g(x+)$, then if $x < y$, it follows from Theorem 5.30 that the only way we may also find $p \in [g(y-), g(y+)]$ is if

$$p = g(x+) = g(y-).$$

Letting $z \in (x, y)$, by Theorem 5.31, we then have $g(z) = p$, contradicting that $p \notin g(\overline{\mathbb{R}})$. Therefore, if $p \notin g(\overline{\mathbb{R}})$ and $p \in [g(x-), g(x+)]$, we cannot have $p \in [g(y-), g(y+)]$ for $x < y$. This result similarly holds for $x > y$, so the choice of x is unique. It follows that $\widetilde{\rho}$ is well-defined on $\overline{\mathbb{R}} - g(\overline{\mathbb{R}})$.

We next wish to see that $\widetilde{\rho}$ is bijective. To see injectivity, consider $a < b$ with $a \in [g(x-), g(x+)]$, $b \in [g(y-), g(y+)]$. If $x = y$, then a and b are carried by the single map χ_x, which is an order preserving homeomorphism, so

$$\widetilde{\rho}(a) = \chi_x(a) < \chi_x(b) = \widetilde{\rho}(b).$$

Suppose then that $x \neq y$. We then carry a with χ_x and b with χ_y such that $\widetilde{\rho}(a) \in [f(\varphi^{-1}(x)-), f(\varphi^{-1}(x)+)]$, $\widetilde{\rho}(b) \in [f(\varphi^{-1}(y)-), f(\varphi^{-1}(y)+)]$. If $\widetilde{\rho}(a) = \widetilde{\rho}(b)$ and $x < y$, then we note that this forces

$$\widetilde{\rho}(a) \leq f(\varphi^{-1}(x)+) \leq f(\varphi^{-1}(y)-) \leq \widetilde{\rho}(b),$$

so $f(\varphi^{-1}(x)+) = f(\varphi^{-1}(y)-)$. By Theorem 5.31, this means we may find $z \in (x, y)$ for which $\widetilde{\rho}(a) = f(\varphi^{-1}(z)) = \widetilde{\rho}(b)$. It then follows that since

$$g(z) = \rho^{-1}(f(\varphi^{-1}(z))) = \rho^{-1}(\widetilde{\rho}(a)) = \rho^{-1}(\widetilde{\rho}(b)),$$

and ρ^{-1} is inverse to $\widetilde{\rho}$ on $f(\overline{\mathbb{R}})$, we must have $g(z) = a = b$. Similarly, $a = b$ if $\widetilde{\rho}(a) = \widetilde{\rho}(b)$ for $x > y$. Thus, $\widetilde{\rho}$ is injective.

To see that $\widetilde{\rho}$ is surjective, let $q \in \overline{\mathbb{R}}$. If $q \in f(\overline{\mathbb{R}})$, q has $\rho^{-1}(q)$ in its preimage under ρ, and

$$\widetilde{\rho}(\rho^{-1}(q)) = \rho(\rho^{-1}(q)) = q.$$

If $q \in \overline{\mathbb{R}} - f(\overline{\mathbb{R}})$, we may find q in some interval $[f(\varphi^{-1}(y)-), f(\varphi^{-1}(y)+)]$ corresponding to a discontinuity of f analogously to the case of finding the points of $\overline{\mathbb{R}} - g(\overline{\mathbb{R}})$ in intervals of the form $[g(x-), g(x+)]$. We note that these points q then have as their preimages the points in $[g(y-), g(y-)]$ carried to them by the homeomorphism χ_y.

To complete the proof that $\widetilde{\rho}$ is an automorphism, we need only show that it is strictly increasing. Let $a < b$. We noted above that if we have $a \in [g(x-), g(x+)]$, $b \in [g(y-), g(y+)]$ for $x = y$, then $\widetilde{\rho}(a) < \widetilde{\rho}(b)$. In the case where $x \neq y$, we note that if $x < y$, we force $g(x) \leq g(y)$, and thus

$$f(\varphi^{-1}(x)) = \rho(g(x)) \leq \rho(g(y)) = f(\varphi^{-1}(y)).$$

Furthermore, by Theorem 5.30,

$$f(\varphi^{-1}(x)+) \leq f(\varphi^{-1}(y)-).$$

Since $\widetilde{\rho}$ is injective and

$$\widetilde{\rho}(a) \leq f(\varphi^{-1}(x)+) \leq f(\varphi^{-1}(y)-) \leq \widetilde{\rho}(b),$$

we must have $\widetilde{\rho}(a) < \widetilde{\rho}(b)$. If $y < x$, on the other hand, we derive the relation

$$b \leq g(y+) \leq g(x-) \leq a,$$

contradicting that $a < b$. It follows that $\widetilde{\rho}$ is strictly increasing. We then have that $\widetilde{\rho} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is an automorphism, by Theorem 5.9. ■

Theorem 5.33 *Let f and g be two functions and $\varphi, \psi \in Aut(\overline{\mathbb{R}})$ such that the diagram commutes.*

$$\begin{tikzcd} \overline{\mathbb{R}} \arrow[r, "g"] & \overline{\mathbb{R}} \arrow[d, "\psi"] \\ \overline{\mathbb{R}} \arrow[u, "\varphi"] \arrow[r, "f"'] & \overline{\mathbb{R}} \end{tikzcd}$$

Then φ is an order preserving homeomorphism $\varphi : \overline{\mathbb{R}}_f \to \overline{\mathbb{R}}_g$.

Proof: Let U be an open set in $\overline{\mathbb{R}}_g$. Since U is open in $\overline{\mathbb{R}}_g$, we may write $U = g^{-1}(V)$ for some set V open in $\overline{\mathbb{R}}$. Since $\psi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is an automorphism, it is a homeomorphism by Corollary 5.14. Thus, $\psi(V) = W$ is open in $\overline{\mathbb{R}}$, and we therefore have that $f^{-1}(W)$ is open in $\overline{\mathbb{R}}_f$. We then note that since the diagram commutes, we must have

$$\varphi^{-1}(U) = \varphi^{-1}g^{-1}(V) = \varphi^{-1}g^{-1}\psi^{-1}(W) = f^{-1}(W).$$

Thus, the open set U in $\overline{\mathbb{R}}_g$ has the open set $f^{-1}(W)$ in $\overline{\mathbb{R}}_f$ as its preimage under φ. It follows that φ is continuous. It similarly follows that φ^{-1} is continuous, noting that ψ^{-1} would be the homeomorphism to pass through this time. Since $\varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is an automorphism and the underlying sets of $\varphi : \overline{\mathbb{R}}_f \to \overline{\mathbb{R}}_g$ are $\overline{\mathbb{R}}$, it follows that $\varphi : \overline{\mathbb{R}}_f \to \overline{\mathbb{R}}_g$ is a homeomorphism. ■

Since f, g here were arbitrary functions, this theorem is a little stronger than what we actually need. We now prove the structure theorem.

Theorem 5.34 ***(The Structure Theorem)*** *Let f and g be two homomorphisms with $f(\pm\infty) = g(\pm\infty) = \pm\infty$. Then $\overline{\mathbb{R}}_f \cong \overline{\mathbb{R}}_g$ in an order-preserving way if and only if $f \sim g$. In other words, there exists an order preserving homeomorphism $h : \overline{\mathbb{R}}_f \to \overline{\mathbb{R}}_g$ shown in the diagram on the left if and only if there exist $\varphi, \psi \in Aut(\overline{\mathbb{R}})$ such that the diagram on the right commutes.*

$$\begin{tikzcd} \overline{\mathbb{R}}_g \arrow[r, "g"] & g(\overline{\mathbb{R}}) \\ \overline{\mathbb{R}}_f \arrow[u, "h"] \arrow[r, "f"'] & f(\overline{\mathbb{R}}) \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathbb{R}} \arrow[r, "g"] & \overline{\mathbb{R}} \arrow[d, "\psi"] \\ \overline{\mathbb{R}} \arrow[u, "\varphi"] \arrow[r, "f"'] & \overline{\mathbb{R}} \end{tikzcd}$$

Proof:

($\Rightarrow$) Let $h : \overline{\mathbb{R}}_f \to \overline{\mathbb{R}}_g$ be an order-preserving homeomorphism. We first note that if h is an order-preserving homeomorphism, it must be a nondecreasing bijective function on all of $\overline{\mathbb{R}}$ by Corollary 5.14, making it an automorphism by Theorem 5.9. Thus, take $\varphi = h$. Next, we apply Theorem 5.32 to obtain the automorphism $\widetilde{\rho} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$. Letting $\psi = \widetilde{\rho}$, we have found $\varphi, \psi \in \text{Aut}(\overline{\mathbb{R}})$ such that the diagram below commutes.

$$\begin{tikzcd} \overline{\mathbb{R}} \arrow[r, "g"] & \overline{\mathbb{R}} \arrow[d, "\psi"] \\ \overline{\mathbb{R}} \arrow[u, "\varphi"] \arrow[r, "f"'] & \overline{\mathbb{R}} \end{tikzcd}$$

($\Leftarrow$) Let $f = \varphi g \psi$, as in the above diagram. Letting $h = \varphi$, Theorem 5.33 then gives that h is an order preserving homeomorphism $h : \overline{\mathbb{R}}_f \to \overline{\mathbb{R}}_g$, as desired. ■

Chapter 6

Conclusion

Since we were not able to formulate a satisfying notion of a subtropical polynomial, we might be skeptical as to our ability to further explore subtropical geometry, since the tropical linear spaces were derived with respect to ideals of tropical polynomials [8]. Furthermore, tropical lines were examined within a projective space, while the idea of subtropical projection does not make as much sense—when we "scale" subtropically, we end up losing information about our vector in a less than useful manner. This further casts doubt upon our ability to explore subtropical geometry as an analogue to tropical geometry. For future studies, we might wish to explore alternative forms of polynomials or alternative forms of linear spaces, unless we can make sense of the ones presented here in such a way that they become useful. Once the notions of polynomials and lines have been safely established, then perhaps things such as a Fundamental Theorem of Subtropical Algebra would make sense for the polynomials, or a notion of subtropical distance (and possibly even arclength for varieties) would make sense in subtropical space.

In the last section we completely categorized the subtropical endomorphisms, creating equivalence classes and relating those classes to topologically homeomorphic spaces. There are quite a few questions that remain yet unanswered, however. First, there is the question of under which circumstances do left- and right-equivalence become the same. We have seen clear examples of functions that relate through one equivalence class but not the other. Second, there are some unresolved issues concerning when two image spaces $f(\overline{\mathbb{R}})$ and $g(\overline{\mathbb{R}})$ are homeomorphic in an order-preserving way. We saw through the Structure Theorem that equivalence of f and g is enough, but it is a decidedly strong condition. For instance, the ceiling and floor functions have the same image, but one may check that they are not equivalent as endomorphisms. From a more abstract standpoint, there is also the question of which lattices have similar structure theorems. Our Structure Theorem is inherently tied to the total ordering of $\overline{\mathbb{R}}$, but do we need compactness? Can we somehow extend the results to partial orderings? These questions among others may be worth exploring.

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