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Abstract

The dualism between continuous and discrete is relevant in music theory as well as in performance practice of musical instruments. Geometry has been used since longtime to represent relationships between notes and chords in tonal system. Moreover, in the field of mathematics itself, it has been shown that the continuity of real numbers can arise from geometrical observations and reasoning. Here, we consider a geometrical approach to generalize representations used in music theory introducing continuous pitch. Such a theoretical framework can be applied to instrument playing where continuous pitch can be naturally performed. Geometry and visual representations of concepts of music theory and performance strengthen the relationship between music and images: in this way, we can connect a theremin or violin performance with a study on perspective, always through mathematical ideas and paradigms. So can math explain musical concepts, and, on the contrary, can music explain mathematical concepts? Can music and math together give rise to visual arts in ever more innovative ways? In this paper, we try to connect different topics such as musical performance on instruments with continuous pitch, and the paradigm of geometry and continuity.

Keywords

affine geometry, theremin, music theory, perspective

Dense Geometry of Music and Visual Arts: Vanishing Points, Continuous Tonnetz, and Theremin Performance

Maria Mannone, Irene Iaccarino & Rosanna Iembo

Introduction. From Pythagorean Geometry to the Theremin

Let us start 2500 years ago, with the School that Pythagoras and his wife Theanò founded in Crotone, in Italy. Pythagoreans were looking for harmony in science and life. This was based on ratios, (Greek ***λόγοι)*** and on equality of ratios, the proportion (Greek ἀ*ναλογία).* Pythagorean thought contributed to the development of music theory over the centuries: *Bach, Rameau, Tartini, Xenakis, Boulez,* and *Glass*, to name but a few.

In particular, Bach's works show a strong link between mathematics and music. Bach systematically uses geometric transformations that invert, overturn, and expand musical themes. These transformations, basic for polyphony, were then formulated as rules of twelve-tone music (Iaccarino, 2012).

Pythagoras and his students reasoned on the finite, measurable, and on the infinite, immeasurable. This led them to the idea of discrete and continuous. From Pythagorean philosophy to modern Physics, concepts of continuous and discrete have been relevant for both abstract thinking and Nature investigation. We can study molecules through geometry. Geometric structures are also relevant in music theory (Tymoczko, 2011; Mazzola et al., 2016). In music, both continuous and discrete structures are present. The dialogue between discrete notes and continuous pitch is relevant for musical understanding.

Because geometrical reasoning can help introduce continuity (Iembo, 2002), we can deduce one more connection between music and images through mathematics.

Classical geometry was named after Euclid, a follower of Pythagorean philosophical and scientific thinking. This name lasted until the negation of the fifth postulate, the one about parallel straight lines, that never touch. Euclidean geometry can be seen as a local approximation of non-Euclidean one. Non-Euclidean geometries implied essential consequences for physics and arts—e.g., Escher's graphic work. Within a Euclidean framework, by considering simple notions of perspective we can get an intuitive idea of affine geometry and projective geometry — where parallel straight lines intersect in the vanishing point. We can draw segments to connect a point A in the straight-line *a* with a point B in the straight-line *b*. This segment is a portion of a straight-line orthogonal to lines *a* and *b* in Euclidean geometry, and incident on the affine plane.

Now, instead of drawing a segment that is a part of a straight-line, let us draw a curvilinear segment (Figures 3). Let several straight lines intersect in the vanishing point, and let a long curvilinear segment intersect them. This gives an intuitive idea of a central symmetry where we measure the distance from a point, i.e., the vanishing point, let us call it V.

If we distribute our parallel arrows around V, our curvilinear segment will be a circle. Let us suppose to draw more than one circle. The closer we are to V the smaller the circle's radius is. If the V is the center of a physical force, its effect diminishes as 1/r, where r is the distance from V. This represents a system with central symmetry.

We can move forward and intuitively introduce some cylindrical coordinates. Let us imagine building up a vertical axis, let us call it *z*, whose V is the origin. All the distances are invariant concerning the points alongside the *z*-axis.

This construction becomes helpful while mathematically modeling the theremin playing. The theremin is the first electronic instrument, invented in 1919. It still inspires performers, composers, and scientists (Skeldon et al., 1998). It is not a case that it was invented by a physicist and cellist, after a sequence of experiments on electric circuits that made strange noises. Theremin's sound is electronically generated: the hands of the performers act as the missing plates of a capacitor; changing their position with respect to the two antennas, they produce a change in the frequency of one of the coupled oscillators within the instrument. The frequency is then converted in the domain of audible sounds and then amplified.

Tonnetz and Theremin: thinking geometrically to investigate musical theory and performance

The theremin has two antennas: one for pitch, and the other for loudness. The closer the performer's (left) hand is to the loudness antenna, and the less loud is the instrument. The closer the (right) hand is to the pitch antenna, and the higher is the sound. We can easily schematize the pitch antenna as the z-axis in the described cylindrical system (Figure 1). The theremin allows continuous pitch playing; however, the performer learns how to catch discrete pitch, and how to play precise intervals. This would correspond, once a straight line (in the air!) is selected, to picking up points alongside it. These points would ideally be equally-spaced. However, in an affine/projective framework, the distances would take into account the perspective effect. For the theremin, there is a linear interval preceded and followed by non-linear distribution of pitch (Skeldon et al., 1998). The conceptual analogy still holds. In theremin playing, the performer does not touch anything: he or she feels like 'acting' in an 'empty' space. This may be related to the mathematical concept of fiber.[1] Playing the theremin means 'finding a path' between the infinitely possible paths in the 'empty space' of the air.

Vanishing points, perspective, affine and projective geometry remind us of their stimuli from visual arts, with the 'scientific perspective' versus pre-Renaissance 'intuitive perspective.' With perspective, we can easily study the deformations of visual patterns, in the same way as, in music, we can study motivic variation.

[1] If we refer to categories (Mac Lane, 1975) while analyzing music (Mannone, 2018), we could include fibered categories (Grothendieck, 1971).

These, and other topics of mathematics, such as homotopy theory, give us technical tools to study objects, their spatial deformations under perspective rules, and their deformations in themselves. We can think of a circle that becomes a Möbius strip. Notions of group theory, symmetries, affine and projective geometry, reveal their helpfulness for visual arts and music (Tymoczko, 2011; Gamer and Wilson, 2003). In music, we can both consider performance gestures—and the example of the theremin goes in this direction—as well as some theoretical tool such as the tonnetz. We can also propose some analogies: could the violin's neck be thought as a collection of some parallel tracks, whose V being in the scroll, seen in perspective? The violin allows continuous pitch. However, there are strings, and thus, the paths to follow are restricted and limited to those strings as given tracks. Also, the violinist selects points on strings, and he or she studies how to make intervals, and so on. Both the theremin and the violin are good physical metaphors for the dualism, in music, between discrete pitch versus frequency continuum.

Conversely, piano strings are percussed at fixed points; we have specific strings (or sets of strings) for each note, and thus we have discrete points. If we move from discrete to continuous, we can order these three instruments in the following way: piano, violin, and theremin.

How should we classify human voice? Somehow between violin and theremin: we cannot see where 'hands' are, and we can partially feel inner gestures necessary to sing. We do not have visible references as the left-hand position on the violin's neck (and bow's pressure and position), or the hands' distance from theremin's antennas. Instead, we can directly control musical parameters 'from inside,' even if we are not conscious of all movements happening in the phonatory system. Metaphors of external movements help us learn and sing well; we directly connect brain and voice, and we can immediately "give an A."

Beyond A Music Theory dense in R

Geometry gives analytic tools both to music theory and performance practice. Here, we discuss two examples from music theory: the tonnetz and Lewin's intervals.

Discrete Tonnetz

We can pursue the analogy between mathematical tools and visual representations for music theory, and we can define generalized *tonnetz* (defined in contemporary music theory) with continuous pitch. The *tonnetz* (from the German 'a net(work) of tones') is a visual representation, in 2 or 3 dimensions, of single notes or chords and their relationships. Some representations are animated (Baroin and Seress, 2015).

By construction, the tonnetz is discrete. However, we may try to extend it to include continuity. We may define a 'tonnetz dense in R,' the field of real numbers. To this aim, we can use the continuity of pitch, where usual pitch is just a discretization, and the Continuity Axiom (Iembo, 2002) to introduce real numbers' continuum in an affine-geometry framework. Real numbers had already entered music theory since longtime: let us think of square root of 2 for equal temperament.

We can approach continuous tonnetz in two ways. First, we can start from the discrete model and 'fill up the empty spaces' with continuous pitch. Second, we can involve the inverse process: we can first think continuously and then recover, 'sampling' some discrete structures, obtaining a vector fiber from the continuum of pitch (Figure 3).

Within the theremin analogy, we could either think of discrete frequencies and find them on the instrument—and this is what experienced performers can do without 'looking for the notes'—or, instead, we can feel the continuum and then recognize and isolate some specific frequencies. Both procedures also happen while singing: we can sing a precise note or we can use continuous portamento or glissando. This reminds of fractals like the Sierpinski carpet with its tridimensional version, the Menger sponge, containing a collection of points with the cardinality of continuous, while visually looking like a discrete structure.

The theremin has an almost cylindrical symmetry: positions in the space, having the same distance from the z-axis (pitch antenna), ‘generate’ the same sounds at the same height (as cylindrical coordinates). Musically-useful positions constitute a circular sector around the z-axis. In all these cases, we can define intervals as the distance between the points-notes. Thereminists localize single notes as hand positions and reach possible intervals through finger displacement. Both theremin and violin easily allow continuous glissando, as well as human voice.[2]

Lewin-style intervals

We used the notion of musical intervals in performance and in musical practice. We can now consider the notion of movement. We can see a musical interval as a starting point, and a movement, represented by an arrow, that leads us to the final point (Figure 3). This concept is included in David Lewin’s studies (Lewin, 1987; Peck, 2015). It also constitutes the starting point for the mathematical definition of musical gestures. A gesture is seen as a mapping from an oriented graph—‘abstract’—and a system of curves in space and time—‘concrete’.— (Mazzola et al., 2016). We can just define a gesture as a way to continuously connect discrete points in space and time. Theremin performance seems a suitable model for such a description: we can choose a pitch by moving the right hand with respect to the right antenna; then, we can reach a new pitch by opening the hand. The arrow describing the hand’s opening also describes the connecting movements between two pitch. Thus, theoretical representations within music theory and performance-related movements may help each other.

[2] Pitch, frequency, and note are not the same thing, but we are using some musical jargon.

This concept may broaden the mathematical theory of musical gestures (Mannone, 2018). Similar gestures may lead to the definition of similar intervals and similar structures in the affine system with cylindrical symmetry.

The notion of infinitesimal displacements may help define intervals *immersed* in the continuum. Starting from a continuum of pitch, the discrete tonnetz can be obtained as a fiber.

From Lewin's work, we can retain the idea of 'jumping' from one point to another, an essential concept for the mathematical definition of musical gestures. This seems relevant because of the concept of directionality—and maybe also of intentionality.

These ideas can be the starting point for a categorical formalism, as well as a starting point for a gestural analysis of music. Finally, some music-theory related concepts can also be effective while describing contemporary musical instruments.

Musical Instruments and Real Numbers

Sound frequencies belong to real numbers: each note of a well-tempered scale has a frequency that is a real number. These scales are a discrete selection of points in the imaginary straight line of continuous pitch — as a finite selection of specific real numbers. Other tuning systems involve rational numbers. Some instruments allow to play continuous intervals between one note and the other, while others only allow discrete frequencies.

Based on the notes we can play on them, we can compare musical instruments; e.g., guitar has little points (where to press and touch the strings) 'corresponding' to notes and chords; violin has strings but no other references; theremin has no references at all, except the ear of the performer. Guitar and violin's strings are approximately parallel, easily described within an affine-geometrical framework where V is at the extremity end of the scroll.

Conclusions and further research

We showed how concepts from different areas, such as music theory, musical performance, geometry, and visual arts, can be joined to get a bird-eye perspective of artistic creative and analytic processes. The composer Salvatore Sciarrino (1998) used mathematical metaphors and strategies to describe variational processes in music and visual arts, included additions and multiplications. Our research may be an ideal prosecution of Sciarrino's pioneering view of processes' comparisons in different fields of arts. Also, the formalism of category theory gives a powerful tool to compare transformations between transformations.

There are several studies in the field of musical similarity (Cont et al., 2011). We can develop our research involving self-similarity, and thus, again, fractals. Among the repetition strategies in music composition, we can consider specific cases, such as recursive musical gestures in composition (Mannone, 2017) as examples of self-similar fractals. We can wonder about musical renditions of self-similar and self-affine fractals—where some symmetry conditions are released, see (Mandelbrot, 1985)—for creative applications. Former studies discussed the possibility of generating music through fractals (Cosenza, 2010), finding melodic sequences and exploring their border with chaotic sequences. Further research should investigate the role of harmony in chaos and fractal-generated music, as well as relationships between real numbers, fractals, and sounds.

Visual representations of fractals may constitute a guide for harmony, and hopefully, the study of 'extreme' cases can help deepen our knowledge of the relationship between music, visual arts, and mathematics. These studies may constitute an innovative resource for teaching music, mathematics, music plus mathematics, and also for enhancing multimedia creativity and expression. May this allow some applications in fields that apparently seem far from these disciplines, from medicine to architecture? A Pythagorean renaissance of inter-trans-disciplinarity is our hope.

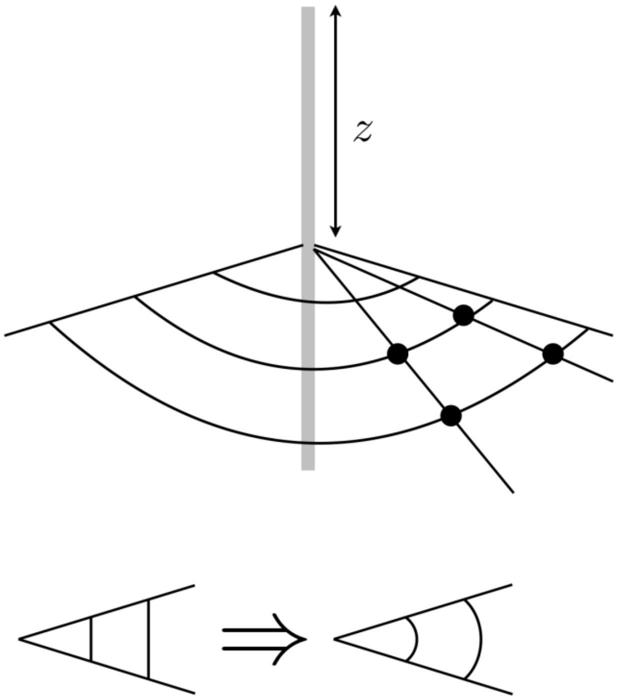

Figure 1

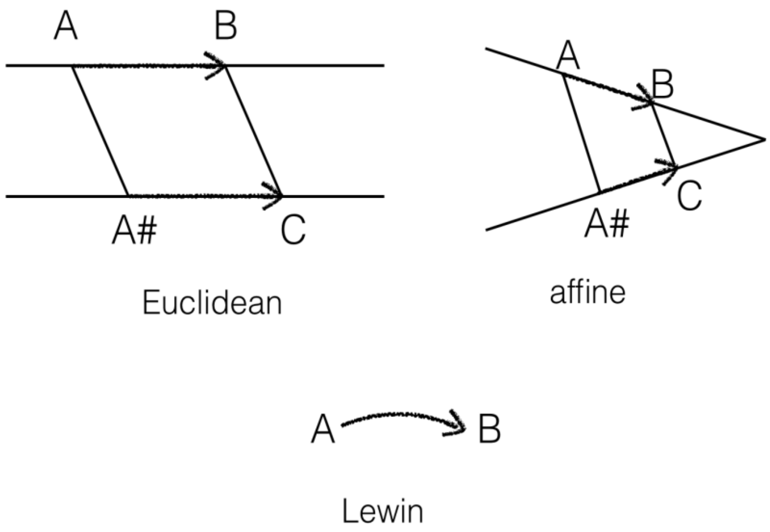

Figure 2

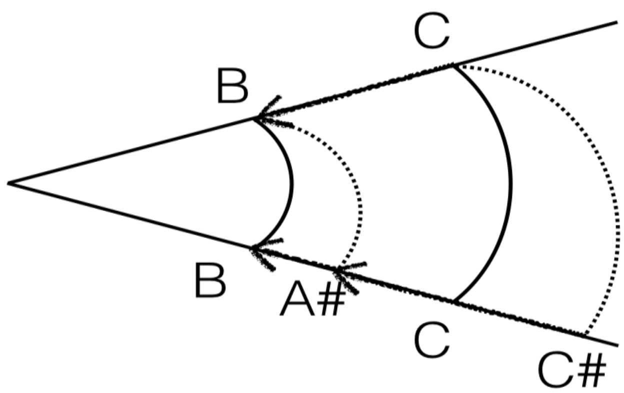

Figure 3

References

Baroin, G., and Seress, H. 2015. The Spinnen-Tonnetz: New Musical Dimensions in the 2D Network for Tonal Music Analysis: Using Polarization and Tonal Regions in a Dynamic Environment. *Mathematics and Computation in Music*, Lecture Notes in Artificial Intelligence, 2015, 5th International Conference, MCM 2015, London, UK, June 22-25, 2015

Cont, A., Dubnov, S., and Assayag, G. 2011. *On the information geometry of audio streams with applications to similarity computing.* Transactions on Audio, Speech, and Language Processing, 19 (4), pp. 837-846

Cosenza, B. 2010. *Applicazione della Teoria del Chaos alla Composizione Musicale*. Master thesis, Music Conservatory of Palermo, Italy

Gamer, C., Wilson, R. 2003. *Microtones and projective planes*, chapter from *Music and Mathematics: From Pythagoras to Fractals* (ed. J. Fauvel, R. Flood and R. Wilson), Oxford Univ. Press, pp. 148-161

Grothendieck, A. 1971. *Catégories fibrées et descente.* Within *Revêtements étales et groupe fondamental*, Springer Verlag

Iaccarino, I. 2012. *Numbers in music. Harmonies, numbers, notes: the values of the Pythagorean School*, Degree dissertation, Department of Arts, Music, Entertainment, University of Calabria (Italy); supervisor: Jacque Guenot, mathematician.

Iembo, R. 2002. *Costruzione Geometrica del campo dei numeri reali.* Serie "Quaderni della Mathesis" N.1 del periodico Mathesis, Teramo. 10

Kenneth D. Skeldon, Lindsay M. Reid, Viviene McInally, Brendan Dougan, and Craig Fulton. 1998. *Physics of the Theremin.* American Journal of Physics, 66 (11)

Lewin, D. 1987. *Generalized Music Intervals*. Oxford University Press

Mac Lane, S. 1975. *Categories for the Working Mathematician*. Springer

Mandelbrot, B. 1985. *Self-Affine Fractals and Fractal Dimension.* Physica Scripta, 32 (4)

Mannone, M. 2018 (to appear). *Introduction to Gestural Similarity. An Application of Category Theory to the Orchestra*. Journal of Mathematics and Music

Mannone, M. 2017. *Musical Gestures between Scores and Acoustics: A Creative Application to Orchestra.* Ph.D. dissertation, University of Minnesota

Mazzola., G., Mannone, M., and Pang, Y. 2016. *Cool Math for Hot Music*. Springer

Peck, R. 2015. "All-Interval Structures." *Mathematics and Computation in Music*, Lecture Notes in Artificial Intelligence, 2015, 5th International Conference, MCM 2015, Eds. Tom Collins et al., London, UK, June 22-25, 2015

Sciarrino, S. 1998. *Le Figure della Musica*. Ricordi

Tymoczko, D. 2011. *A Geometry of Music*. Oxford University Press