2006

An Exploration of Riemann's Zeta Function and Its Application to the Theory of Prime Distribution

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May, 2006

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Abstract

Identified as one of the 7 Millennium Problems, the Riemann zeta hypothesis has successfully evaded mathematicians for over 100 years. Simply stated, Riemann conjectured that all of the nontrivial zeroes of his zeta function have real part equal to 1/2. This thesis attempts to explore the theory behind Riemann's zeta function by first starting with Euler's zeta series and building up to Riemann's function. Along the way we will develop the math required to handle this theory in hopes that by the end the reader will have immersed themselves enough to pursue their own exploration and research into this fascinating subject.

Acknowledgments

I would like to thank Professor Yong, Professor Ward, and especially Professor Raugh for helping this thesis come into being. I could not have accomplished this without their input, their faith and their nearly unending patience. I would also like to thank Libby Beckman for being my late night thesis buddy on many occasions.

Chapter 1

History

Every great story has a beginning, and like many adventures in mathematics, this story begins with Leonard Euler. In 1737 Euler proved that the infinite sum of the inverses of the primes

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots$$

diverges [Edw]. Euler no doubt realized that this observation not only supported the already known fact that the primes are infinite, but also hinted that the primes are fairly dense among the integers. In addition to this, Euler also proposed what is now known as the Euler product formula,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\text{prime } p} \frac{1}{1 + \frac{1}{p^s}},$$

although it was not rigorously proven until 1876 by Leopold Kronecker [Dun]. Amazingly, this formula links the worlds of number theory and analysis and as a result allows tools from each field to combine and mingle like never before. These two facts represent the gateway and the key to one of the most intriguing and mathematically diverse theories, the theory of the Riemann zeta function.

Sometime later, Gauss found himself working on a related problem, although he probably didn't know it. It was a habit of Gauss' to keep a table of prime number counts which he continually added to. At some point Gauss noticed an interesting pattern. "Gauss states in a letter written in 1849 that he has observed as early as 1792 or 1793 that the density of prime numbers appears on average to be $1/\log x$."[Edw] This observation would later become known as the prime number theorem.

Theorem 1.0.1 (Prime Number Theorem)**.**

$$\pi(x) \sim \frac{x}{\ln x} \quad as \quad x \to \infty,$$

where $\pi(x)$ is the number of primes less than or equal to x (where 1 is not considered to be prime).

Gauss later refined this to the following statement.

Theorem 1.0.2 (Prime Numer Theorem II)**.**

$$\pi(x) \sim Li(x) = \int_2^x \frac{dt}{\ln t} \quad as \quad x \to \infty.$$

Over the years, many notable mathematicians worked to prove this conjecture. In 1800 Legendre published his *Theorie des Nombres* in which he proposed an explicit formula for $\pi(x)$ that basically amounted to the primes having a density of $1/\ln x$. Unfortunately, his proof was not rigorous, leaving the problem just as open as it was before.

In 1850 Chebyshev proved some major results, notably that

$$(0.89) \int_2^x \frac{dt}{\ln t} < \pi(x) < (1.11) \int_2^x \frac{dt}{\ln t},$$
$$(0.89)\text{Li}(x) < \pi(x) < (1.11)\text{Li}(x),$$

for all sufficiently large x. He was also able to show that *if* the ratio of $\pi(x)$ to Li(x) approached a limit as x goes to infinity, then the limit must be 1.

Then Bernhard Riemann attempted to work on the problem, and he was the first to see the potential in Euler's product formula. He used it as a jump off point to create his zeta function by extending the product function not only to the entire real line, but to the entire complex plane with the exception of a singularity at $s = 1$. This of course was only natural given that he was right on the frontier of complex variable analysis. In 1859 he published his monumental paper *On the Number of Primes Less Than a Given Magnitude*.

In this paper Riemann attempted to show that $\prod(x) \sim \pi(x)$ where

$$\prod(x) = Li(x) - \sum_\rho Li(\rho) - \ln 2 + \int_x^\infty \frac{dt}{t(t^2-1)\ln t},$$

and ρ runs over the non trivial zeros of his zeta function. Unfortunately he was unable to show (or at least he did not show in the paper) whether

Li(x) was the dominating term or even if the series converged. The real importance of his paper lay in the novel tools he developed and utilized. For the first time a mathematician was using tools like Fourier transforms and complex analysis to say something about the nature of prime number distribution. It truly was revolutionary, and it is not surprising that there was no more substantial work in the field for 30 years after his publication, almost as if it "took the mathematical world that much time to digest Riemann's ideas."[Edw]

Then in 1896 Jacques Hadamard and C. J. de la Vallee Poussin simultaneously and independently proved the prime number theorem using many of the tools that Riemann had developed and used. Their proofs showed that Riemann's formula for $\prod(x)$ was in fact correct and that the largest term was Li(x). Even though the main ideas that Riemann proposed were now rigorously proved, there was still one more conjecture that remained unverified.

In his 1859 paper, Riemann mentioned that he considers it "very likely" that the nontrivial zeros of $\zeta(s)$, which will be defined during the course of this thesis, all lie on the line $\Re(s) = 1/2$. This statement would later become known as the Riemann zeta hypothesis and has managed to stump mathematicians for over a century. The purpose of this thesis is to explore some of the different facets of Riemann's zeta function and introduce some of the math that will assist us. Chapter 2 will start by introducing Euler's zeta series, which is essentially the predecessor of Riemann's zeta function. The chapter will also introduce concepts such as convergence, Barenoulli numbers, and the gamma function in order to construct a solid base to launch from. Chapter 3 is concerned with the values of Euler's zeta series. It will present 2 proofs for the values of the function at the even integers as well as provide a method to approximate the values elsewhere on the positive real line. In order to accomplish the proofs presented, the chapter also introduces Fourier series and the Euler Maclaurin summation formula. Chapter 4 uses a transformational property of Jacobi's theta function to help derive the functional equation in order to extend the Euler zeta series to the complex plane. This extension of the series then becomes Riemann's zeta function. Finally, Chapter 5 examines Riemann's famous hypothesis and the many consequences of its truth including Lindelöf's hypothesis.

Chapter 2

Fundamentals and Euler's Zeta Series

2.1 Introduction

This chapter will be concerned with the Euler zeta series, which is the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where s is a real number greater than 1. All of the ideas and facts that are shown in this chapter will continue to be true after the function is extended to the complex plane because when Riemann's zeta function is restricted to real $s > 1$, it becomes Euler's zeta series. For this reason, we shall discover as much as we can using the Euler's zeta series rather than the more complicated Riemann zeta function.

Before anything else, it behooves us to ask whether the function even makes sense, but before we can answer that we need to establish what an infinite series is and what is required for an infinite series to make sense. Therefore we will have a short section which introduces infinite series and some characteristics that will prove useful throughout our exploration.

2.2 Infinite Series and Convergence

Quite simply, a *series* is an infinite sum. The interesting part of the theory surrounding series is whether they sum to something besides infinity.

Clearly if we sum an infinite number of 1s we will end up with infinity:

$$\sum_{k=1}^{\infty} 1 = \infty.$$

What about the following infinite sum (known as the harmonic series)

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots ?$$

Even though there are an infinite number of terms, we can see that the terms get smaller and smaller, and toward the 'end' of the sequence the terms are practically zero. It seems like this would imply that the sum is finite, but consider the following grouping

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k} &= (1) + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \\ &\geq \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\ &= \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \cdots \end{aligned}$$

With this grouping, we can see that the harmonic series sums to something that is greater than an infinite number of halves whose sum is clearly infinite. Therefore the harmonic series shows that even if the terms of a series approach zero, this is not sufficient for a series to sum to a finite number. The question then becomes, when will an infinite series sum to a finite number? To answer that we introduce the notion of convergence.

Given a sequence $\{a_n\}$ of real numbers. We say that the series $\sum a_n$ is *convergent* and *converges* to S as $n \to \infty$ if for all $\epsilon > 0$ there exists an N such that

$$\left| \sum_{k=1}^{n} a_k - S \right| < \epsilon \quad \text{for all} \quad n > N$$

If no such S exists, then we say that the series diverges [Kno].

In other words, our series converges to S if the partial sums become arbitrarily close to S as n goes to infinity. Note that the *partial sum* of a series is simply the sum of the first n numbers, where n is finite. Another way to think about convergence is that the partial sums of the series are all bounded.

Theorem 2.2.1. *The infinite series $\sum a_n$ converges if and only if there exists a real M such that*

$$\left|\sum_{n=1}^{N} a_n\right| \leq M \quad \textit{for all} \quad N \geq 1.$$

From the divergent harmonic series, we can see that convergence is not always as simple as we would hope. Therefore mathematicians have developed several methods for determining convergence without directly showing that a N exists for every ϵ as the definition demands. We already surreptitiously used the following method when we showed that the harmonic series was divergent.

Theorem 2.2.2. *Given infinite series*

$$\sum_{k=1}^{\infty} a_k \qquad \textit{and} \qquad \sum_{k=1}^{\infty} b_k$$

of non-negative terms (meaning $a_k \geq 0$ and $b_k \geq 0$), with $a_k \geq b_k$. Then we have

- *If $\sum a_k$ converges, then $\sum b_k$ converges.*
- *If $\sum b_k$ diverges, then $\sum a_k$ diverges.*

Another useful convergence test is known as the integral test, which will help us to show that Euler's zeta function does in fact make sense.

Theorem 2.2.3. *Given the an infinite series*

$$\sum_{n=1}^{\infty} a_n$$

of non-negative terms, if $f(x)$ is decreasing and continuous for $x \geq 1$ and $f(n) = a_n$ for all $n \geq 1$, then

$$\int_1^{\infty} f(x)dx \qquad \textit{and} \qquad \sum_{n=1}^{\infty} a_n$$

either both diverge or both converge. In other words, if the integral is finite then the series converges. Otherwise the series diverges.

Using the integral test we can now finally prove that $\zeta(s)$ converges for $s > 1$ and thus makes sense.

Proof. If we let

$$f(x) = \frac{1}{x^s}$$

where s is taken to be a constant greater than 1, then clearly

$$\zeta(s) = \sum_{n=1}^{\infty} f(n).$$

Note also that $f(x)$ is continuous, positive and decreasing (given that $s > 1$) and that

$$\int_1^{\infty} f(x)dx = \frac{-1}{s-1}\left(\lim_{x\to\infty}\frac{1}{x^{s-1}} - \frac{1}{1^{s-1}}\right) = \frac{-1}{s-1}(0-1) = \frac{1}{s-1}$$

Thus the integral test tells us that since $\int_1^{\infty} f(x)dx$ converges then $\sum_1^{\infty} f(n)$ converges as well for $s > 1$. □

Note that the requirement that s be greater than 1 makes sense because $\zeta(1)$ is the harmonic series which we showed diverges. What's remakable is that as soon as s crosses that threshold, than the series instantly becomes finite.

Although they will not be useful until the next chapter, the Bernoulli numbers and Bernoulli polynomials play a very important role in the theory behind both Euler's zeta series and Riemann's zeta function and thus merit mention in the chapter on Fundamentals. The next two sections will be devoted to their descriptions and properties.

2.3 Bernoulli Numbers

Bernoulli was well aware of the formulas for the partial sums of the powers of the integers, the first few being:

$$\begin{aligned}\sum_{k=1}^{n} k &= \frac{n(n+1)}{2}\\ \sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6}\\ \sum_{k=1}^{n} k^3 &= \frac{n^2(n+1)^2}{4}.\end{aligned}$$

Given the parallel nature between increasing exponents in the summands and the exponents in the explicit formulas, it is natural to search for some generalization. The result gave rise to an interesting sequence of numbers, called the *Bernoulli Numbers*; named by Euler in honor of his teacher. Viewed as a formal expression (meaning that B^k is actually B_k) we have the general formula

$$\sum_{k=0}^{n-1} k^m = \frac{1}{m+1}\left[(n+B)^{m+1} - B_{m+1}\right], \tag{2.1}$$

where B_k is the kth Bernoulli number. Using the binomial formula on (2.1) yields

$$\sum_{k=0}^{n-1} k^m = \frac{1}{m+1}\left[\sum_{j=0}^{m}\binom{m+1}{j} n^{m+1-j} B_j\right]. \tag{2.2}$$

From here, we can produce the recursive relationship between the Bernoulli numbers by plugging $n = 1$ into equation (2.2), resulting in

$$0 = \sum_{j=0}^{m}\binom{m+1}{j} B_j. \tag{2.3}$$

Using the recursive formula and $B_0 = 1$ we have a fairly straight forward, although not very speedy, method for computing the Bernoulli numbers. For reference, the first couple Bernoulli numbers are:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0.$$

Notice that both the third and fifth Bernoulli numbers are zero, and in fact $B_k = 0$ for all odd $k > 1$. This fact will be proved rigorously momentarily.

Another important aspect of the Bernoulli numbers is that they are generated [IK] by the Taylor series approximation of

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}. \tag{2.4}$$

This relation is very important since the frequent appearance of this generating function amid formulas involving Riemann's zeta function will be an indication of the involvement of the Bernoulli numbers.

Theorem 2.3.1. *For odd $k > 1$, $B_k = 0$.*

Proof. After moving the $k = 1$ term of the sum over to the other side, equation (2.4) becomes

$$\frac{x}{e^x - 1} - \frac{B_1 x}{1!} = \sum_{k=0, k\neq 1}^{\infty} \frac{B_k x^k}{k!} \tag{2.5}$$

Focusing on the left hand side of the equation we have

$$\begin{aligned}
\frac{x}{e^x - 1} - \frac{B_1 x}{1!} &= \frac{x}{e^x - 1} + \frac{x}{2} \\
&= \frac{x}{2}\left(\frac{2}{e^x - 1} + \frac{e^x - 1}{e^x - 1}\right) \\
&= \frac{x}{2}\left(\frac{e^x + 1}{e^x - 1}\right) \\
&= \frac{x}{2}\left(\frac{e^{x/2}}{e^{x/2}}\right)\left(\frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}}\right).
\end{aligned}$$

Now notice that substituting $-x$ for x in the last line leaves the expression unchanged. Therefore the right hand side of equation (2.5) must also remain unchanged by the same substitution. This means that

$$\sum_{k=0, k\neq 1}^{\infty} \frac{B_k x^k}{k!} = \sum_{k=0, k\neq 1}^{\infty} \frac{B_k (-x)^k}{k!} \tag{2.6}$$

and therefore that $B_k = (-1)^k B_k$ for all $k \neq 1$. While this is trivially true for even k, this forces $B_k = 0$ for all odd $k \neq 1$. $\square$

2.4 Bernoulli Polynomials

Next after the Bernoulli numbers we have the *Bernoulli polynomials.* In short, these are the polynomials (for $n \geq 0$) uniquely defined by the following three characteristics:

(i) $B_0(x) = 1$.

(ii) $B'_k(x) = kB_{k-1}(x)$, $k \geq 1$.

(iii) $\int_0^1 B_k(x)dx = 0$, $k \geq 1$.

Using these facts, we can easily compute the first couple Bernoulli polynomials:

$$\begin{aligned} B_0(x) &= 1 \\ B_1(x) &= x - \frac{1}{2} \\ B_2(x) &= x^2 - x + \frac{1}{6} \\ B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \\ B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}. \end{aligned}$$

Euler found a generating function [IK] for these functions as well:

$$\frac{xe^{zx}}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k(z)x^k}{k!}. \tag{2.7}$$

You may have noticed that the Bernoulli polynomials that we listed take on their corresponding Bernoulli numbers at $x = 0$. This is in fact true for all Bernoulli polynomials, as evidenced by the result of substituting $z = 1$ into the generating function,

$$\sum_{k=0}^{\infty} \frac{B_k(0)x^k}{k!} = \frac{xe^{(0)x}}{e^x - 1} = \frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}.$$

Here are some other useful properties of the Bernoulli polynomials.

Theorem 2.4.1. $B_n(1 - x) = (-1)^n B_n(x)$

Proof. Using the (2.7) we have

$$\sum_{k=0}^{\infty} \frac{B_k(1-z)x^k}{k!} = \frac{xe^{(1-z)x}}{e^x-1} = \frac{(e^x)}{(e^x)}\frac{xe^{-zx}}{1-e^{-x}} = \frac{(-x)e^{z(-x)}}{e^{-x}-1} = \sum_{k=0}^{\infty} \frac{B_k(z)(-x)^k}{k!}$$

Equating the coefficients then yields $B_n(1-x) = (-1)^n B_n(x)$. □

Corollary 2.4.2. *$B_n(1) = (-1)^n B_n(0)$. Furthermore, since $B_n(0) = B_n$, and $B_n = 0$ for all odd $n > 1$, then $B_n(1) = B_n(0) = B_n$ for all $n \neq 1$.*

Theorem 2.4.3. *$B_n(x+1) - B_n(x) = nx^{n-1}$.*

Proof.

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(B_k(x+1)-B_k(x))t^k}{k!} &= \frac{te^{(x+1)t}}{e^t-1} - \frac{te^{xt}}{e^t-1} \\
&= \frac{te^te^{xt} - te^{xt}}{e^t-1} \\
&= \frac{te^{xt}(e^t-1)}{e^t-1} \\
&= te^{xt} \\
&= \sum_{k=0}^{\infty} \frac{t(xt)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(k+1)x^kt^{k+1}}{(k+1)!} \\
&= \sum_{k=1}^{\infty} \frac{(kx^{k-1})t^k}{k!}
\end{aligned}$$

The index difference between the beginning and the end is not a problem since the first term of the left hand side is zero because

$$B_0(x+1) - B_0(x) = 1 - 1 = 0.$$

Thus $B_k(x+1) - B_k(x) = kx^{k-1}$. □

2.5 Gamma Function

Another useful function when dealing with the theory surrounding Euler's zeta series is the *gamma function*. Essentially, the gamma function is an extension of the factorial function to all real numbers greater than 0 [Sto].

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} \mathrm{dx} \text{ for } s > 0. \tag{2.8}$$

While Euler was the first to introduce this function, he defined it slightly differently and Legendre was the first to denote it as $\Gamma(s)$.

We should note that this function can actually be extended to all complex numbers so that it only has simple poles at the negative integers (and zero). And it is for this reason that we can continue to use $\Gamma(s)$ even when we begin to work with Riemann's zeta function in the realm of complex numbers.

Be careful when researching this subject using different sources because Gauss [Edw] also defined this function, except that he denotes it by $\prod(s)$ where

$$\prod(s) = \Gamma(s+1).$$

This thesis will be using the more popular notation of $\Gamma(s)$ as defined in (2.8).

The rest of this section will derive (or sketch the idea of the derivation) of several identities which will prove useful as we manipulate $\zeta(s)$ and $\Gamma(s)$ over the course of this thesis.

What is known as the functional equation of $\Gamma(s)$,

$$\Gamma(s+1) = s\Gamma(s), \tag{2.9}$$

can easily be obtained by applying integration by parts on $s\Gamma(s)$ and using the definition given in (2.8). Using this as a recursive relation and seeing that $\Gamma(1) = 1$ shows us that for any positive integer n we have

$$\Gamma(n) = (n-1)!,$$

confirming that $\Gamma(s)$ is in fact an extension of the factorial function.

We mentioned earlier that $\Gamma(S)$ can be extended over the entire complex plane with the exception of the negative integers. This is accomplished by redefining the function as

$$\Gamma(s+1) = \lim_{N\to\infty} \frac{1 \cdot 2 \cdot 3 \cdots N}{(s+1)(s+2)(s+3)\cdots(s+N)} (N+1)^s. \tag{2.10}$$

Since

$$(N+1)^s = (1)^{-s}(1+1)^s(2)^{-s}(2+1)^s \cdots (N-1)^{-s}(N-1+1)^s(N)^{-s}(N+1)^s$$

Then we can write (2.10) as an infinite product instead of a limit, yielding

$$\Gamma(s+1) = \prod_{n=1}^{\infty} \frac{nn^{-s}(n+1)^s}{(s+n)}.$$

A little manipulation of this then gives us

$$\Gamma(s+1) = \prod_{n=1}^{\infty} \left(1+\frac{s}{n}\right)^{-1} \left(1+\frac{1}{n}\right)^{s}.$$

Now, if s is not an integer we have

$$\begin{aligned}
\Gamma(s+1)\Gamma(-s+1) &= \prod_{n=1}^{\infty} \left(1+\frac{s}{n}\right)^{-1} \left(1+\frac{1}{n}\right)^{s} \prod_{n=1}^{\infty} \left(1+\frac{-s}{n}\right)^{-1} \left(1+\frac{1}{n}\right)^{-s} \\
&= \prod_{n=1}^{\infty} \left(1+\frac{s}{n}\right)^{-1} \left(1+\frac{1}{n}\right)^{s} \left(1+\frac{-s}{n}\right)^{-1} \left(1+\frac{1}{n}\right)^{-s} \\
&= \prod_{n=1}^{\infty} \left(1-\frac{s^2}{n^2}\right)^{-1}. \qquad (2.11)
\end{aligned}$$

Although it won't be proved until section 3.3, let us take for granted that we have the product formula for sine,

$$\sin \pi x = \pi x \prod_{n=1}^{\infty} \left(1-\frac{x^2}{n^2}\right).$$

Plugging (2.11) into the sine formula then yields the identity

$$\sin(\pi s) = \frac{\pi s}{\Gamma(s+1)\Gamma(1-s)}. \qquad (2.12)$$

Finally, a special case of the Legendre relation gives us the additional identity

$$\Gamma(s+1) = 2^s \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s-1}{2}\right) \pi^{-1/2}. \qquad (2.13)$$

Chapter 3

Evaluation of Euler's Zeta Series

3.1 Introduction

Now that we have defined and verified that $\zeta(s)$ converges for real $s > 1$ it would be nice to figure out what the series converges to. This goal divides itself nicely into two parts: the values at the even integers and the values everywhere else.

Two proofs will be presented of the formula for the values of Euler's zeta series at the even integral points, but before we proceed we need to introduce Fourier series and prove that we can integrate an infinite series because both of which are required for the proofs that will be presented. Therefore the next section will introduce infinite series of functions and integration while the following section will introduce the theory of Fourier series and help establish a useful identity which will serve as the starting point for the first proof.

3.2 Integration of Infinite Series

Several proofs in this chapter require that we swap an infinite sum and an integral. In other words, we want to know whether we can integrate an infinite sum term by term and still arrive at the integral of the entire sum. Before we present that theorem we should reintroduce convergence since we are now talking about infinite sums of functions, which are subtly different then infinite sums of numbers. All of the definitions and theorems in this section are discussed in more detail in [Kno].

Given a sequence of functions $f_n(x)$, we say that the infinite sum

$$\sum_{n=1}^{\infty} f_n(x)$$

is *point-wise convergent* to $f(x)$ in a region D if given any $\epsilon > 0$ and $x \in D$ there exists n_0 (which is dependent on ϵ and x) such that

$$|f_n(x) - f(x)| < \epsilon \quad \text{for all} \quad n > n_0$$

We say that the infinite sum

$$\sum_{n=1}^{\infty} f_n(x)$$

is *uniformly convergent* to $f(x)$ in a region D if given any $\epsilon > 0$ there exists n_0 (which is *only* dependent on ϵ) such that

$$|f_n(x) - f(x)| < \epsilon \quad \text{for all} \quad n > n_0$$

for all $x \in D$.

As a side note, clearly uniform convergence is stronger than point-wise convergence since any function which converges uniformly obviously converges at each point.

Just like in the case of convergence, strictly using the definition to check for uniform convergence can be a daunting task, therefore mathematicians have developed several tests to help spot and show uniform convergence. Here are two of these tests.

Theorem 3.2.1 (Abel's Test for Uniform Convergence)**.** *Given a series*

$$f(x) = \sum_{n=1}^{\infty} u_n(x)$$

with $u_n(x) = a_n f_n(x)$*. If*

- *$\sum_{n=1}^{\infty} a_n$ is convergent.*
- *$f_{n+1}(x) \leq f_n(x)$ for all n.*
- *$f_n(x)$ is bounded for $x \in [a, b]$, meaning $0 \leq f_n(x) \leq M$.*

then $\sum_{n=1}^{\infty} u_n(x)$ uniformly converges to $f(x)$ in $[a, b]$.

Theorem 3.2.2 (Weierstrass' Test for Uniform Convergence)**.** *Given a series*

$$f(x) = \sum_{n=1}^{\infty} f_n(x),$$

if there exists a positive γ_n for each n such that $f_n(x) \leq \gamma_n$ in the interval $[a, b]$ (same interval for all γ_n) and $\sum \gamma_n$ is convergent, then $\sum f_n(x)$ is uniformly convergent in $[a, b]$. Essentially, this theorem says that if you can bound the sequence of functions by a convergent sequence then the sequence of functions converges uniformly.

Now that we have established convergence in terms of infinite sums of functions and introduced some tests for convergence we can present a very useful theorem regarding the integration of these series.

Theorem 3.2.3. *If a given infinite sum*

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

is uniformly convergent in the region D and $f_n(x)$ is integrable in $[a, b]$ (where $[a, b] \subseteq D$) for all $n \in \mathbb{N}$, then the integral of $f(x)$ on $[a, b]$ may be obtained by the term by term integration of the series. In other words

$$\int_a^b f(x)dx = \int_a^b \left[\sum_{n=1}^{\infty} f_n(x) \right] dx = \sum_{n=1}^{\infty} \left[\int_a^b f_n(x)dx \right].$$

3.3 Fourier Series

The theory and implementation of Fourier series is a very useful tool in mathematics. For the purposes of this thesis, this section will provide the background needed.

The whole idea behind Fourier series is to write a periodic function $f(x)$ as an infinite sum of sines and cosines:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right), \tag{3.1}$$

where the coefficients of the sines and cosines are given by Euler's formulas:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \tag{3.2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx. \tag{3.3}$$

The previous formulas can be derived by multiplying equation (3.1) by either $\cos(mx)$ or $\sin(mx)$, integrating x over $(-\pi, \pi)$, and then using the orthogonality relations:

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\, dx = \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\, dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \neq 0 \end{cases}$$

$$\text{and} \qquad \int_{-\pi}^{\pi} \sin(nx)\cos(mx)\, dx = 0 \qquad \text{for integral } m, n.$$

Note that if $f(x)$ is an odd function (meaning that $f(-x) = -f(x)$) then the integrand of equation (3.2) is an odd function (because $f(-x)\cos(-nx) = -f(x)\cos(nx)$) which means that integrating over a symmetric interval will yield 0. Therefore $a_n = 0$ and thus the Fourier series of $f(x)$ is a sine series. By similar logic, if $f(x)$ is an even function (meaning that $f(-x) = f(x)$) then its Fourier series consists only of cosines.

Now that we have defined what a Fourier series is, we are naturally driven to ask which functions can be represented as Fourier series? Given that we would be representing the function as a sum of functions that all have periods that divide 2π, it is clear that $f(x)$ must not only be periodic, but must also have a period that divides 2π. Furthermore, in order for the series to have any chance of converging, the sequence $\{a_n \cos(nx) + b_n \sin(nx)\}$ must tend to 0 as n goes to infinity. This last property is guaranteed by the following lemma.

Lemma 3.3.1. *If a function $k(x)$, and its first derivative $k'(x)$ are sectionally continuous in the interval $[a,b]$, then the integral*

$$K_\lambda = \int_a^b k(x)\sin(\lambda x)\,dx$$

tends to zero as $\lambda \to \infty$.

A proof of this Lemma, as given by [CJ], is easily obtained through integration by parts of K_λ. To clarify, by *sectionally continuous* we mean that $f(x)$ can be broken up into a finite number of subintervals such that $f(x)$ is continuous on the open subintervals and approaches definite limits at the endpoints. This essentially means that $f(x)$ can only have a finite number of discontinuities on the interval $[a,b]$. If both $f(x)$ and $f'(x)$ are sectionally continuous we say that $f(x)$ is *sectionally smooth*. In addition to this we require that the value of $f(x)$ at any discontinuity be the average of the limiting values from each side.

To summarize, a function $f(x)$ can be represented as a Fourier series provided the following properties hold:

- $f(x) = f(x+2\pi)$
- $f(x)$ is sectionally smooth.
- At every point x we have

$$f(x) = \frac{1}{2}\left(\lim_{a\to x^+} f(a) + \lim_{a\to x^-} f(a)\right)$$

As an exercise we will now derive the product formula for sine, which will be useful later on in another proof. This derivation comes from [CJ].

To start off our derivation we will find the Fourier series of $\cos(\mu x)$ on the interval $(-\pi,\pi)$ where μ is a real number that is not an integer. Note that all three properties are satisfied, so our quest for a Fourier series is feasible. Since $\cos(\mu x)$ is an even function we immediately know that our Fourier series will be a cosine series, thus $b_n = 0$. Then using equation (3.2) we have

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(\mu x)\cos(nx)dx \\
&= \frac{1}{\pi}\int_{0}^{\pi}\cos((\mu+n)x)+\cos((\mu-n)x)dx \\
&= \frac{1}{\pi}\left[\frac{\sin((\mu+n)\pi)}{\mu+n}+\frac{\sin((\mu-n)\pi)}{\mu-n}\right] \\
&= \frac{1}{\pi}\left[\frac{(\mu-n)\sin((\mu+n)\pi)+(\mu+n)\sin((\mu-n)\pi)}{\mu^2-n^2}\right] \\
&= \frac{1}{\pi}\left[\frac{2\mu\sin(\mu x)\cos(nx)}{\mu^2-n^2}\right] \\
&= \frac{2\mu(-1)^n}{\pi(\mu^2-n^2)}\sin(\mu\pi)
\end{aligned}$$

Just to make the arithmetic clear, we used the trigonometric identities

$$\begin{aligned}
\sin(x\pm y) &= \sin(x)\cos(y)\pm\cos(x)\sin(y) \quad \text{and} \\
\cos(x\pm y) &= \cos(x)\cos(y)\mp\sin(x)\sin(y)
\end{aligned}$$

in the previous series of steps. Now that we have found our coefficients, we have the Fourier series representation

$$\cos(\mu x)=\frac{\sin(\mu\pi)}{\mu\pi}+\frac{2\mu\sin(\mu\pi)}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{\cos(nx)}{\mu^2-n^2}.$$

Since our function satisfied the property that the value at every point (specifically the endpoints $-\pi$ and π) was the average of the limits from both sides, we can plug $x=\pi$ into our Fourier expression without any trouble. After doing this and dividing by $\sin(\mu x)$ (which we can do since μ is not an integer) we get

$$\cot(\mu\pi)=\frac{1}{\mu\pi}+\frac{2\mu}{\pi}\sum_{n=1}^{\infty}\frac{1}{\mu^2-n^2}.$$

Moving terms around yields

$$\cot(\mu\pi)-\frac{1}{\mu\pi}=-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\mu}{n^2-\mu^2}. \tag{3.4}$$

From here we can use Abel's test to verify that the series is uniformly convergent, so that we can integrate the series term by term. We can write the previous series as $\sum a_n f_n(\mu)$ where

$$a_n = \frac{1}{n^2} \qquad \text{and} \qquad f_n(\mu) = \frac{\mu}{1 - \frac{\mu^2}{n^2}}.$$

We know that $\sum a_n$ converges because $\sum a_n = \zeta(2)$ and in section 2.2 we showed that $\zeta(s)$ converges for all real $s > 1$. Inspection shows us that

$$f_{n+1}(\mu) = \frac{\mu}{1 - \frac{\mu^2}{(n+1)^2}} < \frac{\mu}{1 - \frac{\mu^2}{n^2}} = f_n(\mu)$$

for all n, and inspection also shows us that for $\mu \in [0, 1/2)$ we have

$$f_n(\mu) = \frac{\mu}{1 - \frac{\mu^2}{n^2}} < \frac{\mu}{1 - \mu^2} < \frac{\mu}{1 - 1/4} < \frac{1/2}{3/4} = 2/3.$$

Since $\sum a_n$ is convergent, $f_n(\mu)$ is a monotonically decreasing sequence, and $f_n(\mu)$ is bounded on $\mu \in [0, 1/2)$ then by Abel's theorem, the function

$$\sum_{n=1}^{\infty} a_n f_n(\mu) = \sum_{n=1}^{\infty} \frac{\mu}{\mu^2 - n^2}$$

converges uniformly on $[0, 1/2)$. After multiplying by π and integrating μ from 0 to x we have

$$\pi \int_0^x \left(\cot(\mu\pi) - \frac{1}{\mu\pi} \right) d\mu = \int_0^x \sum_{n=1}^{\infty} \frac{-2\mu}{n^2 - \mu^2}\, d\mu. \tag{3.5}$$

On the left hand side of (3.5) we have

$$\begin{aligned} \pi \int_0^x \left(\cot(\mu\pi) - \frac{1}{\mu\pi} \right) d\mu &= \log \frac{\sin(\pi x)}{\pi x} - \lim_{a \to 0} \frac{\sin(\pi a)}{\pi a} \\ &= \log \frac{\sin(\pi x)}{\pi x}. \end{aligned}$$

On the right hand side of (3.5) we can swap the sum and integral because we showed that equation (3.5) was uniformly convergent. Therefore we end up with

$$\begin{aligned}
\int_0^x \sum_{n=1}^{\infty} \frac{-2\mu}{n^2-\mu^2}\, d\mu &= \sum_{n=1}^{\infty} \int_0^x \frac{-2\mu}{n^2-\mu^2} \\
&= \sum_{n=1}^{\infty} \log\left(1-\frac{x^2}{n^2}\right) \\
&= \lim_{\alpha\to\infty} \sum_{n=1}^{\alpha} \log\left(1-\frac{x^2}{n^2}\right) \\
&= \lim_{\alpha\to\infty} \log \prod_{n=1}^{\alpha} \left(1-\frac{x^2}{n^2}\right) \\
&= \log \lim_{\alpha\to\infty} \prod_{n=1}^{\alpha} \left(1-\frac{x^2}{n^2}\right) \\
&= \log \prod_{n=1}^{\infty} \left(1-\frac{x^2}{n^2}\right).
\end{aligned}$$

This gives us

$$\log \frac{\sin(\pi x)}{\pi x} = \log \prod_{n=1}^{\infty} \left(1-\frac{x^2}{n^2}\right).$$

After exponentiating both sides and multiplying the πx over, we end up with the product formula for sine:

$$\sin(\pi x) = \pi x \prod_{n=1}^{\infty} \left(1-\frac{x^2}{n^2}\right). \tag{3.6}$$

3.4 Differentiation of Infinite Series

In the next proof we will need to differentiate an infinite series, and in order to do that, we need to verify that we can differentiate term by term and that the resulting series will converge to the derivative of the initial series.

Theorem 3.4.1. *If*

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

is point-wise convergent and $f_n(x)$ has continuous derivatives in a region D for all $n \in \mathbb{N}$ and

$$\sum_{n=1}^{\infty} f_n'(x)$$

is uniformly convergent in D then the derivative of $f(x)$ can be obtained by the term by term differentiation of the series. In other words,

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left[\sum_{n=1}^{\infty} f_n(x)\right] = \sum_{n=1}^{\infty}\left[\frac{d}{dx} f_n(x)\right]$$

for all $x \in D$ [Kno].

As an exercise in using this theorem we will differentiate

$$\sum_{k=1}^{\infty} \log\left(1 + \frac{u^2}{4k^2\pi^2}\right) \tag{3.7}$$

term by term and verify uniform convergence. The derivative of this series will help us in a later proof.

Proof. First we need to confirm that this series is point-wise convergent. Note that

$$\sum_{k=1}^{\infty} \log\left(1 + \frac{u^2}{4\pi^2}\frac{1}{k^2}\right) = \log \prod_{k=1}^{\infty}\left(1 + \frac{u^2}{4\pi^2}\frac{1}{k^2}\right). \tag{3.8}$$

From Knopp, we know that an infinite product

$$\prod_{k=1}^{\infty}(1 + a_k)$$

converges if and only if the series $\sum_{k=1}^{\infty} a_k$ converges. Returning to (3.8), we see that

$$\sum_{k=1}^{\infty} \frac{u^2}{4\pi^2}\frac{1}{k^2}$$

converges because this is simply a constant multiple (because u is a constant when talking about point-wise convergence) of $\zeta(2)$ which we have already proved converges. Therefore the infinite product converges and thus (3.8) is point-wise convergent.

Since the summand has continuous derivatives in the region $0 < u < 2\pi$ we consider the derivative of (3.7):

$$\sum_{k=1}^{\infty} \frac{2u}{4k^2\pi^2 + u^2}. \tag{3.9}$$

Now we just need to verify uniform convergence of this sum in order to finish the proof. To accomplish this we will use Weierstrass' test (theorem 3.2.2). Consider the following line of logic:

$$\begin{aligned}
0 < u < 2\pi \quad &\Rightarrow \quad \frac{4\pi^2}{u} > 1 \\
&\Rightarrow \quad \frac{4\pi^2 k^2}{u} > k^2 \\
&\Rightarrow \quad \frac{4\pi^2 k^2}{u} + u > k^2 \\
&\Rightarrow \quad \frac{1}{\frac{4\pi^2 k^2}{u} + u} < \frac{1}{k^2} \\
&\Rightarrow \quad \frac{u}{4\pi^2 k^2 + u^2} < \frac{1}{k^2} \\
&\Rightarrow \quad \sum_{k=1}^{\infty} \frac{u}{4\pi^2 k^2 + u^2} < \sum_{k=1}^{\infty} \frac{1}{k^2}.
\end{aligned}$$

We know that $\sum 1/k^2$ converges because it is $\zeta(2)$, therefore by Weierstrass' test we know that

$$\sum_{k=1}^{\infty} \frac{u}{4\pi^2 k^2 + u^2}$$

converges uniformly and therefore (3.9) converges uniformly as well. Thus we can differentiate term by term and equation (3.9) does indeed converge to the derivative of (3.7). $\square$

3.5 Values of Euler's Zeta Series at the Even Integers

In 1644 Pietro Mengoli proposed a problem to all mathematicians. Determine the value of the infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This problem, which is known as the Basel problem, was eventually solved by Euler in 1735 after multiple renowned mathematicians had failed. Euler's solution granted him instant fame in the mathematical community, but that was hardly a reason for Euler to stop there. Almost as if to rub it in their faces Euler derived and proved the general formula for which the Basel problem is a special case [Dun]. The purpose of this section is to present this formula, which happens to be the value of Euler's zeta series at the even integers, as well as two proofs.

Theorem 3.5.1. *For $k \geq 1$*

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} B_{2k} (2\pi)^{2k}}{2(2k)!}.$$

3.5.1 Proof Through Equation of Coefficients

We start with the product formula for $\sin(x)$, which was produced at the end of Section 3.3,

$$\sin(x) = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right).$$

After taking the natural logarithm of both sides and substituting $x = -iu/2$ we have

$$\log\left(\sin\left(-\frac{iu}{2}\right)\right) = \log\left(-\frac{iu}{2}\right) + \sum_{k=1}^{\infty} \log\left(1 - \frac{\left(-\frac{iu}{2}\right)^2}{k^2\pi^2}\right),$$

and after some simplification we get

$$\log\left(\sin\left(-\frac{iu}{2}\right)\right) = \log\left(-\frac{iu}{2}\right) + \sum_{k=1}^{\infty} \log\left(1 + \frac{u^2}{4k^2\pi^2}\right).$$

We can then differentiate both sides with respect to u. Note that we can differentiate the infinite sum term by term because we proved that we could at the end of section 3.4. This yields

$$\left(-\frac{i}{2}\right)\frac{\cos\left(-\frac{iu}{2}\right)}{\sin\left(-\frac{iu}{2}\right)} = \frac{1}{u} + \sum_{k=1}^{\infty}\frac{2u}{4k^2\pi^2+u^2}. \tag{3.10}$$

Focusing on the left hand side and using the following identities

$$\cos(z) = \frac{1}{2}\left(e^{iz}+e^{-iz}\right) \quad \text{and} \quad \sin(z) = \frac{1}{2i}\left(e^{iz}-e^{-iz}\right)$$

gives us

$$\begin{aligned}
\left(-\frac{i}{2}\right)\frac{\cos\left(-\frac{iu}{2}\right)}{\sin\left(-\frac{iu}{2}\right)} &= \left(-\frac{i}{2}\right)\frac{\left(\frac{e^{-u/2}+e^{u/2}}{2}\right)}{\left(\frac{-e^{-u/2}+e^{u/2}}{2i}\right)} \\
&= \frac{1}{2}\frac{\left(e^{-u/2}+e^{u/2}\right)}{\left(-e^{-u/2}+e^{u/2}\right)} \\
&= \frac{1}{2}\frac{\left(e^{-u/2}\right)}{\left(e^{-u/2}\right)}\frac{\left(e^{u}-1+2\right)}{\left(-1+e^{u}\right)} \\
&= \frac{1}{2}+\frac{1}{e^u-1}.
\end{aligned}$$

Then, after substituting this back into equation (3.10), multiplying by u and moving a term over we end up with

$$\frac{u}{e^u-1}+\frac{u}{2}-1 = \sum_{k=1}^{\infty}\frac{2u^2}{4k^2\pi^2+u^2}. \tag{3.11}$$

Now for the homestretch, focusing on the left hand side we have

$$\begin{aligned}
\frac{u}{e^u-1}+\frac{u}{2}-1 &= \sum_{k=0}^{\infty}\frac{B_k u^k}{k!}+\frac{u}{2}-1 \\
&= 1-\frac{u}{2}+\sum_{k=2}^{\infty}\frac{B_k u^k}{k!}+\frac{u}{2}-1 \\
&= \sum_{k=2}^{\infty}\frac{B_k u^k}{k!} \\
&= \sum_{k=1}^{\infty}\frac{B_{2k}u^{2k}}{(2k)!} \qquad (3.12)
\end{aligned}$$

In the last step we were able to relabel the indices because all the odd terms were zero since $B_k = 0$ for odd $k > 1$ (from theorem 2.3.1). The right hand side of equation (3.11) becomes

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{2u^2}{4k^2\pi^2 + u^2} &= \sum_{k=1}^{\infty} \frac{2u^2}{(2k\pi)^2} \left(\frac{1}{1 + \left(\frac{u}{2k\pi}\right)^2} \right) \\
&= \sum_{k=1}^{\infty} \frac{2u^2}{(2k\pi)^2} \sum_{j=0}^{\infty} (-1)^j \left(\frac{u}{2\pi k}\right)^{2j} \\
&= \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} 2(-1)^j \left(\frac{u}{2\pi k}\right)^{2j+2} \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} 2(-1)^{j-1} \left(\frac{u}{2\pi k}\right)^{2j} \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{2(-1)^{j-1}}{(2\pi)^{2j}} \left(\frac{1}{k^{2j}}\right) u^{2j} \\
&= \sum_{j=1}^{\infty} \frac{2(-1)^{j-1}}{(2\pi)^{2j}} \sum_{k=1}^{\infty} \left(\frac{1}{k^{2j}}\right) u^{2j} \\
&= \sum_{j=1}^{\infty} \frac{2(-1)^{j-1}}{(2\pi)^{2j}} \zeta(2j) u^{2j}. \qquad (3.13)
\end{aligned}$$

Finally, changing the index from j to k and substituting equations (3.12) and (3.13) back into equation (3.11) yields

$$\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} u^{2k} = \sum_{k=1}^{\infty} \frac{2(-1)^{k-1}}{(2\pi)^{2k}} \zeta(2k) u^{2k}. \qquad (3.14)$$

Finally, we equate the corresponding coefficients in the sums to arrive at our goal:

$$\frac{B_{2k}}{(2k)!} = \frac{2(-1)^{k-1}}{(2\pi)^{2k}} \zeta(2k)$$

$$\Downarrow$$

$$\frac{(-1)^{k-1} B_{2k} (2\pi)^{2k}}{2(2k)!} = \zeta(2k).$$

3.5.2 Proof Through Repeated Integration

We start with the Fourier expansion of the first Bernoulli polynomial

$$B_1(x) = x - \frac{1}{2} = -\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2\pi xk)}{k}.$$

We then multiply by 2 and integrate both sides from 0 to u,

$$\int_0^u 2B_1(x)dx = -\frac{2}{\pi}\sum_{k=1}^{\infty}\int_0^u \frac{\sin(2\pi xk)}{k}dx.$$

Then we clump the constants on one side resulting in

$$C + \int_0^u 2B_1(x)dx = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2\pi uk)}{2\pi k^2}.$$

After switching x and u we define

$$P_2(x) \triangleq C + \int_0^x 2B_1(u)du = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2\pi xk)}{2\pi k^2}.$$

Note that $P_2'(x) = 2B_1(x)$ and since

$$P_2(x) = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2\pi xk)}{2\pi k^2}$$

then $P_2(0) = P_2(1)$. These are the characteristics of Bernoulli polynomials! Because they are uniquely defined by these characteristics, we know that $P_2(x) = B_2(x)$, giving

$$B_2(x) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2\pi xk)}{\pi k^2}.$$

Plugging in $x = 0$ yields

$$1/6 = B_2(0) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\cos(0)}{\pi k^2}dx \qquad \Rightarrow \qquad \zeta(2) = \pi^2/6,$$

which is the solution to the Basel problem.

From here we can keep integrating, and due to the properties of Bernoulli polynomials, we will always end up with a Bernoulli polynomial on the right hand side.

After a couple of iterations you notice the following general pattern:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^{2k}} &= (-1)^{k-1}\frac{(2\pi)^{2k}B_{2k}(x)}{2(2k)!}\\ \sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n^{2k+1}} &= (-1)^{k-1}\frac{(2\pi)^{2k+1}B_{2k+1}(x)}{2(2k+1)!}\end{aligned}$$

for any integer $k \geq 1$. If we let $x = 0$ in the first equation, we arrive at

$$\zeta(2k) = \frac{(-1)^{k-1}B_{2k}(2\pi)^{2k}}{2(2k)!}.$$

Setting $x = 0$ in the second equation reaffirms that the odd Bernoulli numbers (besides B_1) are zero.

3.6 Euler-Maclaurin Summation Formula

The section will concern itself with the derivation of the Euler-Maclaurin summation formula, which is a means to compute the error when approximating a sum with an integral:

$$\sum_{n=M}^{N} f(n) \approx \int_{M}^{N} f(x)\mathrm{dx} + \frac{1}{2}[f(M) + f(N)], \tag{3.15}$$

where f is a continuously differentiable function on $[M, N]$. While the theory and use of this method of approximation appear as early as 1730, it wasn't until 1740 that the process was independently and rigorously proven and published by Euler and Maclaurin [Edw].

In order to correct the inequality in (3.15) I claim that we should add an error term equal to

$$\int_{M}^{N} (x - \lfloor x \rfloor - \frac{1}{2}) f'(x)\mathrm{dx} \tag{3.16}$$

to the right hand side.

Proof. We start by chopping $[M, N]$ into unit intervals and writing (3.16) as a sum of smaller integrals and then changing variables with $t = x - n$

$$\begin{aligned} \int_{M}^{N} (x - \lfloor x \rfloor - \frac{1}{2}) f'(x)\mathrm{dx} &= \sum_{n=M}^{N-1} \left[\int_{n}^{n+1} (x - \lfloor x \rfloor - \frac{1}{2}) f'(x)\mathrm{dx} \right] \\ &= \sum_{n=M}^{N-1} \left[\int_{0}^{1} (t - \frac{1}{2}) f'(n+t)\mathrm{dt} \right]. \end{aligned}$$

We then use integration by parts with

$$\begin{matrix} u = (t - 1/2) \\ \mathrm{dv} = f'(n+t) \end{matrix} \Rightarrow \begin{matrix} \mathrm{du} = \mathrm{dt} \\ v = f(n+t) \end{matrix}$$

resulting in

$$
\begin{aligned}
\int_M^N (x - \lfloor x \rfloor - \frac{1}{2}) f'(x) \mathrm{dx} &= \sum_{n=M}^{N-1} \left[\int_0^1 (t - \frac{1}{2}) f'(n+t) \mathrm{dt} \right] \\
&= \sum_{n=M}^{N-1} \left[[(t - 1/2) f(n+t)] \big|_0^1 - \int_0^1 f(n+t) \mathrm{dt} \right] \\
&= \sum_{n=M}^{N-1} \left[\frac{1}{2} f(n+1) + \frac{1}{2} f(n) \right] - \sum_{n=M}^{N-1} \int_0^1 f(n+t) \mathrm{dt} \\
&= \sum_{n=M}^{N-1} \left[\frac{1}{2} f(n+1) + \frac{1}{2} f(n) \right] - \sum_{n=M}^{N-1} \int_n^{n+1} f(x) \mathrm{dx} \\
&= \sum_{n=M}^{N-1} \left[\frac{1}{2} f(n+1) + \frac{1}{2} f(n) \right] - \int_M^N f(x) \mathrm{dx} \\
&= \frac{1}{2} f(M) + f(M+1) + \cdots + f(N-1) + \frac{1}{2} f(N) - \int_M^N f(x) \mathrm{dx}.
\end{aligned}
$$

By inspection we can see that if we add this final line to the right hand side of equation (3.15) it will cause both sides to be equal. □

The Euler-Maclaurin summation formula is the result of repeated integration by parts on the error term that we just verified. Before we start integrating like mad, it will benefit us to rewrite (3.16) in terms of Bernoulli polynomials. Recall that $B_1(x) = x - 1/2$. So

$$
\begin{aligned}
\int_M^N (x - \lfloor x \rfloor - \frac{1}{2}) f'(x) \mathrm{dx} &= \sum_{n=M}^{N-1} \left[\int_0^1 (t - \frac{1}{2}) f'(n+t) \mathrm{dt} \right] \\
&= \sum_{n=M}^{N-1} \left[\int_0^1 B_1(t) f'(n+t) \mathrm{dt} \right].
\end{aligned}
$$

Recall that the second defining property of Bernoulli polynomials (section 2.4) is that

$$\frac{1}{k} B_k'(x) = B_{k-1}(x).$$

Using this property we integrate by parts with

$$
\begin{array}{c} u = f'(n+t) \\ \mathrm{dv} = B_1(t) \end{array} \Rightarrow \begin{array}{c} \mathrm{du} = f''(n+t)\mathrm{dt} \\ v = \frac{1}{2} B_2(t) \end{array}
$$

resulting in

$$\begin{aligned}\int_M^N (x-\lfloor x\rfloor-\frac{1}{2})f'(x)\mathrm{dx} &= \sum_{n=M}^{N-1}\left[\frac{1}{2}B_s(t)f'(n+t)\right]\Bigg|_0^1+\sum_{n=M}^{N-1}\left[\int_0^1\frac{1}{2}B_2(t)f''(n+t)\mathrm{dt}\right]\\ &= \frac{1}{2}B_2(0)f'(M)-\frac{1}{2}B_2(1)f'(N)-\int_M^N\frac{1}{2}B_2(x-\lfloor x\rfloor)f''(x)\mathrm{dx}\\ &= \frac{B_2(0)}{2}[f'(x)]_M^N-\int_M^N\frac{1}{2}B_2(x-\lfloor x\rfloor)f''(x)\mathrm{dx}.\end{aligned}$$

The first sum telescoped because of corollary 2.4.2 which says that $B_k(1) = B_k(0)$. When we repeat the process on the final integral we obtain

$$\int_M^N (x-\lfloor x\rfloor-\frac{1}{2})f'(x)\mathrm{dx} = \frac{B_2(0)}{2}[f'(x)]_M^N-\frac{B_3(0)}{2\cdot 3}[f''(x)]_M^N+\int_M^N\frac{1}{2\cdot 3}B_3(x-\lfloor x\rfloor)f'''(x)\mathrm{dx}.$$

In general if we repeat this process $k-1$ times we get

$$\begin{aligned}\int_M^N (x-\lfloor x\rfloor-\frac{1}{2})f'(x)\mathrm{dx} = \frac{B_2(0)}{2!}[f'(x)]_M^N-\frac{B_3(0)}{3!}[f''(x)]_M^N+\cdots\\ +\frac{(-1)^kB_k(0)}{k!}[f^{(k-1)}(x)]_M^N\\ +\frac{(-1)^{k+1}}{k!}\int_M^N B_k(x-\lfloor x\rfloor)f^{(k)}(x)\mathrm{dx}.\end{aligned}$$

Since $B_k(0)$ is the kth Bernoulli number, and $B_k = 0$ for odd $k > 1$ this can be simplified and in conjunction with equation (3.15) we end up with the Euler-Maclaurin summation formula:

$$\begin{aligned}\sum_{n=M}^{N} f(n) = \int_M^N f(x)\mathrm{dx}+\frac{1}{2}[f(M)+f(N)]+\frac{B_2}{2!}f'(x)\Bigg|_M^N+\frac{B_4}{4!}f'''(x)\Bigg|_M^N+\cdots\\ +\frac{B_{2\nu}}{(2\nu)!}f^{(2\nu-1)}(x)\Bigg|_M^N+R_{2\nu},\end{aligned}$$

where

$$R_{2\nu} = \frac{-1}{(2\nu+1)!}\int_M^N B_{2\nu+1}(x-\lfloor x\rfloor)f^{(2\nu+1)}(x)\mathrm{dx}.$$

3.7 Approximation of Euler's Zeta Series Elsewhere

First let's try using the Euler-Maclaurin summation formula directly on $\zeta(s)$. Recall that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

which means that $f(n) = 1/n^s$ and $[M, N] = [1, \infty]$. Note that the kth derivative of $f(n)$ is

$$f^{(k)}(n) = \frac{(-1)^k (s)(s+1)(s+2)\cdots(s+k-1)}{n^{s+k}}.$$

Which means that

$$[f^{(k)}(n)]_M^N = \left[\frac{(-1)^k (s)(s+1)\cdots(s+k-1)}{n^{s+k}}\right]_1^{\infty} = (-1)^k (s)(s+1)\cdots(s+k-1).$$

However, this is a problem since the terms of the Euler-Maclaurin summation formula would not converge very quickly which means we would have to calculate a lot of terms in order to have any kind of accuracy. Luckily this problem can be fixed. What if we instead used this formula on

$$\sum_{n=X}^{\infty} \frac{1}{n^s}$$

and added back in the first $X - 1$ terms afterward? Well, then we would still have $f(n) = 1/n^s$ but our interval would change to $[M, N] = [X, \infty]$. Then

$$[f^{(k)}(n)]_M^N = \frac{(-1)^k (s)(s+1)\cdots(s+k-1)}{X^{s+k}}.$$

Now, if X is fairly large then the terms in the summation formula get smaller much faster, and we end up with a better accuracy without calculating too many terms. Keep in mind though that by doing it this way, we still have to calculate the value of

$$\sum_{n=1}^{X-1} \frac{1}{n^s}.$$

Using this method of approximation we have

$$\zeta(s) = \sum_{n=1}^{X-1} \frac{1}{n^s} + \int_X^\infty \frac{1}{u^s} \mathrm{du} + \frac{B_2}{2!} f'(x)\bigg|_X^\infty + \frac{B_4}{4!} f'''(x)\bigg|_X^\infty$$
$$+ \cdots + \frac{B_{2\nu}}{(2\nu)!} f^{(2\nu-1)}(x)\bigg|_X^\infty + R_{2\nu}.$$

After simplification this finally becomes

$$\zeta(s) = \sum_{n=1}^{X-1} \frac{1}{n^s} + \frac{X^{1-s}}{1-s} + \frac{1}{2}\frac{1}{X^s} + \frac{B_2}{2!}\frac{s}{X^{s+1}} + \frac{B_4}{4!}\frac{(s)(s+1)\cdots(s+2)}{X^{s+3}}$$
$$+ \cdots + \frac{B_{2\nu}}{(2\nu)!}\frac{(s)(s+1)\cdots(s+2\nu-2)}{X^{s+2\nu-1}} + R_{2\nu},$$

where

$$R_{2\nu} = \frac{-(s)(s+1)\cdots(s+2\nu-1)}{(2\nu+1)!} \int_X^\infty \frac{B_{2\nu+1}(x-\lfloor x \rfloor)}{u^{s+2\nu}} \mathrm{du}.$$

There are many other methods to approximate the other values of Euler's zeta series at these points. If you are interested in additional methods, Borwein, Bradley and Crandall have an article which discusses several computational processes [BBC].

Chapter 4

Extension of the Zeta Function to the Complex Plane

4.1 Introduction

As discussed in the beginning, one of the most significant achievements of Riemann was his extension of Euler's zeta series to the entire complex plane with the exception of a simple pole at $z = 1$. Although Euler was the one who conjectured the functional equation of the zeta function, which is one way of extending $\zeta(z)$, it was Riemann who rigorously proved it. The subject of this chapter will be to prove the functional equation and describe the evaluation of several other points of what shall now be known as Riemann's zeta function. Before we get to the functional equation we need to first prove a transformational invariant of Jacobi's theta function, which will be the goal of the next section.

4.2 Jacobi's Theta Function

Jacobi's Theta Function is defined for $t > 0$ to be

$$\Theta(t) = \sum_{k=-\infty}^{\infty} e^{-\pi k^2 t}. \tag{4.1}$$

One of the most important theorems related to Jacobi's theta function is that the function $t^{1/4}\Theta(t)$ is unchanged by the substitution of $1/t$ for t. In other words we have the following theorem.

Theorem 4.2.1.

$$\Theta\left(\frac{1}{t}\right) = t^{1/2}\Theta(t) \tag{4.2}$$

Before we begin to prove this theorem, we will prove a couple of small facts which will help us along the way. To prevent myself from reinventing the wheel, I will take for granted that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x.$$

Since e^x is clearly continuous this tells us that the left hand side is also continuous, and therefore given any sequence $\{x_n\}$ whose limit is x as n goes to infinity we have

$$\lim_{n\to\infty}\left(1+\frac{x_n}{n}\right)^n = e^x. \tag{4.3}$$

Make sure to remember this since we will be using it several times over the course of our proof of theorem 4.2.1.

Another small fact, which will provide the starting point for our proof, begins by letting $z \in \mathbb{C}$ and letting m be a positive integer. Using the binomial expansion formula and then substituting $k = m + j$ yields the following series of steps:

$$\begin{aligned}
\left(z^{1/2}+z^{-1/2}\right)^{2m} &= \sum_{k=0}^{2m}\binom{2m}{k} z^{k/2} z^{-(2m-k)/2} \\
&= \sum_{j=-m}^{m}\binom{2m}{m+j} z^{(m+j)/2} z^{-(2m-m-j)/2} \\
&= \sum_{j=-m}^{m}\binom{2m}{m+j} z^{j}.
\end{aligned} \tag{4.4}$$

We are now prepared to prove Theorem 4.2.1. The proof will be broken up into three lemmas, the first of which proves an equality while the second and third prove that the left hand side and right hand side of this equality approach the left and right hand sides of (4.2) respectively. All of the proofs of these lemmas were provided by [Sto].

Lemma 4.2.2. *Let m and l be positive integers and let $\omega = e^{2\pi i/l}$. Then*

$$\sum_{-\frac{l}{2}\leq n<\frac{l}{2}} \left[\frac{\omega^{n/2}+\omega^{-n/2}}{2}\right]^{2m} = \sum_{k=-\lfloor m/l\rfloor}^{\lfloor m/l\rfloor} \frac{l}{2^{2m}} \binom{2m}{m+kl}.$$

Proof. Let m and l be positive integers and let $\omega = e^{2\pi i/l}$. Note that

$$\omega^l = e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1.$$

Now let $z = \omega^n$ and sum equation (4.4) over $-l/2 \leq n < l/2$. Then swap the sums (which we can do because both sums are independent of each other and finite) to yield

$$\begin{aligned} \sum_{-\frac{l}{2}\leq n<\frac{l}{2}} \left(\omega^{n/2}+\omega^{-n/2}\right)^{2m} &= \sum_{-\frac{l}{2}\leq n<\frac{l}{2}} \sum_{j=-m}^{m} \binom{2m}{m+j} \omega^{nj} \\ &= \sum_{j=-m}^{m} \left[\binom{2m}{m+j} \sum_{-\frac{l}{2}\leq n<\frac{l}{2}} \omega^{nj}\right]. \end{aligned} \tag{4.5}$$

Claim:

$$\sum_{-\frac{l}{2}\leq n<\frac{l}{2}} \omega^{nj} = \begin{cases} l & \text{if } j = lk \text{ for some integer } k \\ 0 & \text{otherwise} \end{cases}. \tag{4.6}$$

Proof. Assume that $j = lk$ for some integer k. Then

$$\omega^{nj} = \omega^{nlk} = (\omega^l)^{nk} = (1)^{nk} = 1.$$

Thus we would have l as the result because we would be summing l ones. Now assume that no integer k exists such that $j = lk$. Note that since

$$\omega^{nj} = \omega^{nj}(1)^{zj} = \omega^{nj}(\omega^l)^{zj} = \omega^{nj}\omega^{lzj} = \omega^{(n+lz)j}$$

for any real z, then we can shift our sum over to have

$$\sum_{-\frac{l}{2}\leq n<\frac{l}{2}} \omega^{nj} = \sum_{-\frac{l}{2}\leq n<\frac{l}{2}} \omega^{(n+l/2)j} = \sum_{0\leq n<l} \omega^{nj}.$$

Finally, since $0 \leq n < l-1$ implies that $1 \leq n+1 < l$ and for $n = l-1$ we have

$$\omega^{(n+1)j} = \omega^{lj} = 1 = \omega(0)j$$

then $\omega^{(n+1)j}$ for $0 \leq n < l$ is simply a rearrangement of ω^{nj} for $0 \leq n < l$. Therefore their sums are the same and thus

$$\sum_{0 \leq n < l} \omega^{nj} = \sum_{0 \leq n < l} \omega^{(n+1)j} = \omega^j \sum_{0 \leq n < l} \omega^{nj}.$$

However, $\omega^j \neq 1$ because if $\omega^j = 1$ then

$$1 = \cos(2\pi j/l) + i\sin(2\pi j/l) \;\Rightarrow\; 2\pi j/l = 2k\pi \;\Rightarrow\; j = lk$$

which is a contradiction since we asserted that no such integral k existed for this case. Therefore this forces $\sum_n \omega^{nj} = 0$ and thus our claim is proved. □

Plugging equation (4.6) into equation (4.5) and then changing the limits of the sum so as to only run over the nonzero elements yields

$$\sum_{-\frac{l}{2} \leq n < \frac{l}{2}} \left(\omega^{n/2} + \omega^{-n/2}\right)^{2m} = \sum_{k=-\lfloor m/l \rfloor}^{\lfloor m/l \rfloor} \binom{2m}{m+kl} l$$

Finally, divide both sides by 2^{2m} to finish the proof of lemma 4.2.2. □

Lemma 4.2.3. *If $t > 0$ then*

$$\lim_{m\to\infty} \sum_{-\frac{l}{2} \leq n < \frac{l}{2}} \left[\frac{\omega^{n/2} + \omega^{-n/2}}{2}\right]^{2m} = \sum_{n=-\infty}^{\infty} e^{-\pi n^2/t} = \Theta\left(\frac{1}{t}\right).$$

Proof. Let $t > 0$, and keep it fixed. Up until now the only restriction placed on l has been that it is a positive integer, now let $l = \lfloor\sqrt{\pi m t}\rfloor$. Note that this implies both that

$$l^2 \sim \pi m t \quad \text{and} \quad m \sim \frac{l^2}{\pi t} \quad \text{as } m \to \infty. \tag{4.7}$$

Since $\cosh(x) = \frac{e^x + e^{-x}}{2}$ we have

$$[\cosh(\pi i n/l)]^{2m} = \left[\frac{e^{\pi i n/l} + e^{-\pi i n/l}}{2}\right]^{2m} = \left[\frac{\omega^{n/2} + \omega^{-n/2}}{2}\right]^{2m}.$$

Furthermore, the Maclaurin series for cosh gives us the approximation

$$\cosh(x) = 1 + \frac{x^2}{2} + \mathrm{O}(x^4).$$

Utilizing this approximation yields

$$\left[\frac{\omega^{n/2} + \omega^{-n/2}}{2}\right]^{2m} = \left[1 + \frac{(\pi i n)^2}{2l^2} + \mathrm{O}\left(\left(\frac{\pi i n}{l}\right)^4\right)\right]^{2m}.$$

At this point we can simplify, sum both sides over $-l/2 \leq n < l/2$ and take the limit as m goes to infinity. Note that since $l = \lfloor\sqrt{\pi m t}\rfloor$, m going to infinity forces l to go to infinity as well. Thus

$$\begin{aligned}\lim_{m\to\infty} \sum_{-\frac{l}{2}\leq n<\frac{l}{2}} \left[\frac{\omega^{n/2} + \omega^{-n/2}}{2}\right]^{2m} &= \lim_{m\to\infty} \sum_{-\frac{l}{2}\leq n<\frac{l}{2}} \left[1 + \frac{-\pi^2 n^2}{2l^2} + \mathrm{O}\left(\frac{(\pi i n)^4}{l^4}\right)\right]^{2m} \\ &= \lim_{m\to\infty} \sum_{n=-\infty}^{\infty} \left[1 + \frac{-\pi^2 n^2}{2l^2} + \mathrm{O}\left(\frac{(\pi i n)^4}{l^4}\right)\right]^{2m}.\end{aligned}$$

Since $m \sim l^2/\pi t$ we can rewrite the exponent and rearrange the terms in the parentheses to obtain

$$\lim_{m\to\infty} \sum_{-\frac{l}{2}\leq n<\frac{l}{2}} \left[\frac{\omega^{n/2} + \omega^{-n/2}}{2}\right]^{2m} = \lim_{m\to\infty} \left[1 + \frac{-\pi n^2/t}{2l^2/\pi t}\right]^{2l^2/\pi t}.$$

We were able to drop the big-O term since it becomes insignificant as l goes to infinity. Then, since $2l^2/\pi t \to \infty$ as $l \to \infty$, we can use equation (4.3) to conclude that the right hand side goes to $e^{-\pi n^2/t}$ as l and m go to infinity. Thus we have our intended result of

$$\lim_{m\to\infty} \sum_{-\frac{l}{2}\leq n<\frac{l}{2}} \left[\frac{\omega^{n/2} + \omega^{-n/2}}{2}\right]^{2m} = e^{-\pi n^2/t} = \Theta\left(\frac{1}{t}\right). \tag{4.8}$$

□

Lemma 4.2.4.

$$\lim_{m\to\infty} \sum_{k=-\lfloor m/l\rfloor}^{\lfloor m/l\rfloor} \frac{\lfloor\sqrt{\pi m t}\rfloor}{2^{2m}} \binom{2m}{m+kl} = \sqrt{t} \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = \sqrt{t}\Theta(t).$$

Proof. Just as before, t is still fixed and $l = \lfloor \sqrt{\pi m t} \rfloor$. Therefore

$$\lfloor \sqrt{\pi m t} \rfloor \sim \sqrt{t}\sqrt{\pi m}.$$

Thus

$$\lim_{m\to\infty} \sum_{k=-\lfloor m/l \rfloor}^{\lfloor m/l \rfloor} \frac{\lfloor \sqrt{\pi m t} \rfloor}{2^{2m}} \binom{2m}{m+kl} = \sqrt{t} \lim_{m\to\infty} \sum_{k=-\lfloor m/l \rfloor}^{\lfloor m/l \rfloor} \frac{\sqrt{\pi m}}{2^{2m}} \binom{2m}{m+kl}.$$

If we focus on the summand of the right hand side we have

$$\frac{\sqrt{\pi m}}{2^{2m}} \binom{2m}{m+kl} = \frac{\sqrt{\pi m}}{2^{2m}} \frac{(2m)!}{(m+kl)!(m-kl)!}.$$

Using Stirling's Formula,

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n},$$

on all the factorial parts yields

$$\frac{\sqrt{\pi m}}{2^{2m}} \left(\frac{2m}{e}\right)^{2m} \sqrt{2\pi 2m} \left(\frac{e}{m+kl}\right)^{m+kl} \frac{1}{\sqrt{2\pi(m+kl)}} \left(\frac{e}{m-kl}\right)^{m-kl} \frac{1}{\sqrt{2\pi(m-kl)}}.$$

After canceling out the e's and π's and 2's we are left with

$$\frac{m^{2m}}{(m+kl)^{m+kl}(m-kl)^{m-kl}} \cdot \frac{m}{\sqrt{m+kl}\sqrt{m-kl}}. \tag{4.9}$$

For the right term of (4.9) we have

$$\frac{m}{\sqrt{m+kl}\sqrt{m-kl}} = \frac{m}{\sqrt{m^2-k^2l^2}} = \frac{1}{\sqrt{1-k^2l^2/m^2}} \sim 1 \text{ as } m \to \infty.$$

For the left term of (4.9) we have

$$\frac{m^{2m}}{(m+kl)^{m+kl}(m-kl)^{m-kl}} = \frac{m^{2m}}{(m^2-k^2l^2)^m} \cdot \frac{(m-kl)^{kl}}{(m+kl)^{kl}}. \tag{4.10}$$

Now, when we dissect the right hand side of (4.10), the left term yields

$$\frac{m^{2m}}{(m^2-k^2l^2)^m} = \frac{1}{(1-k^2l^2/m^2)^m} \sim \frac{1}{(1-k^2\pi t/m)^m}$$

because $l^2 \sim \pi m t$ as $l \to \infty$. Furthermore we can use equation (4.3) to obtain

$$\frac{m^{2m}}{(m^2-k^2l^2)^m} \sim \frac{1}{(1+(-k^2\pi t)/m)^m} \sim \frac{1}{e^{-\pi k^2 t}} \quad \text{as} \quad m \to \infty.$$

Now we can focus on the right term of (4.10). Recalling that the Maclaurin series of $(1-x)/(1+x)$ yields the approximation

$$\frac{1-x}{1+x} = 1 - 2x + \mathrm{O}(x^2),$$

we can then apply this to the right term of (4.10) to get

$$\frac{(m-kl)^{kl}}{(m+kl)^{kl}} = \left(\frac{1-lk/m}{1+lk/m}\right)^{lk} = \left[1 - \frac{2lk}{m} + \mathrm{O}\left(\left(\frac{lk}{m}\right)^2\right)\right]^{lk}.$$

Using the fact that $l/m \sim \pi t/l$ gives us

$$\frac{(m-kl)^{kl}}{(m+kl)^{kl}} = \left[1 - \frac{2\pi t k^2}{lk} + \mathrm{O}\left(\left(\frac{\pi t k}{l}\right)^2\right)\right]^{lk} \sim \left[1 - \frac{2\pi t k^2}{lk}\right]^{lk}.$$

We can drop the big-O because it is insignificant as $l \to \infty$. Finally using equation (4.3) yet again will yield

$$\left[1 + \frac{-2\pi t k^2}{lk}\right]^{lk} \sim e^{-2\pi k^2 t} \quad \text{as} \quad m \to \infty$$

because $lk \to \infty$ as $l \to \infty$. To summarize everything we have done in this proof:

$$\lim_{m\to\infty} \sum_{k=-\lfloor m/l \rfloor}^{\lfloor m/l \rfloor} \frac{\lfloor \sqrt{\pi m t} \rfloor}{2^{2m}} \binom{2m}{m+kl}$$

$$
\begin{aligned}
&= \sqrt{t} \lim_{m\to\infty} \sum_{k=-\lfloor m/l \rfloor}^{\lfloor m/l \rfloor} \frac{\sqrt{\pi m}}{2^{2m}} \binom{2m}{m+kl} \\
&= \sqrt{t} \sum_{m=-\infty}^{\infty} \left[\frac{m^{2m}}{(m+kl)^{m+kl}(m-kl)^{m-kl}} \right] \cdot \left[\frac{m}{\sqrt{m+kl}\sqrt{m-kl}} \right] \\
&= \sqrt{t} \sum_{m=-\infty}^{\infty} \left[\frac{m^{2m}}{(m^2-k^2l^2)^m} \cdot \frac{(m-kl)^{kl}}{(m+kl)^{kl}} \right] \cdot [1] \\
&= \sqrt{t} \sum_{m=-\infty}^{\infty} \left[\frac{1}{e^{-\pi k^2 t}} \cdot e^{-2\pi k^2 t} \right] \cdot [1] \\
&= \sqrt{t} \sum_{m=-\infty}^{\infty} e^{-\pi k^2 t} \\
&= \sqrt{t}\Theta(t).
\end{aligned}
$$

Thus we have proven lemma 4.2.4. □

Finally, utilizing lemmas 4.2.2, 4.2.3 and 4.2.4 we have that

$$
\begin{aligned}
\Theta\left(\frac{1}{t}\right) &= \lim_{m\to\infty} \sum_{-\frac{l}{2} \le n < \frac{l}{2}} \left[\frac{\omega^{n/2} + \omega^{-n/2}}{2} \right]^{2m} \\
&= \lim_{m\to\infty} \sum_{k=-\lfloor m/l \rfloor}^{\lfloor m/l \rfloor} \frac{\lfloor \sqrt{\pi m t} \rfloor}{2^{2m}} \binom{2m}{m+kl} \\
&= \sqrt{t}\Theta(t)
\end{aligned}
$$

which proves theorem 4.2.1.

4.3 The Functional Equation for $\zeta(s)$

Theorem 4.3.1 (The Functional Equation for $\zeta(s)$)**.**

$$\pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

In other words if we let $\Omega(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ *then*

$$\Omega(1-s) = \Omega(s).$$

Proof. Consider the following integral

$$\int_0^\infty e^{-\pi n^2 t} t^{s/2} \frac{\mathrm{dt}}{t}.$$

Using the change of variables $x = \pi n^2 t$ (and thus $\mathrm{dx}/x = \mathrm{dt}/t$), where n is an integer, we have

$$\begin{aligned}
\int_0^\infty e^{-\pi n^2 t} t^{s/2} \frac{\mathrm{dt}}{t} &= \int_0^\infty e^{-x} \pi^{-s/2} n^{-s} x^{s/2-1} \frac{\mathrm{dx}}{x} \\
&= \frac{1}{n^s}\pi^{-s/2}\int_0^\infty e^{-x} x^{s/2-1}\mathrm{dx} \\
&= \frac{1}{n^s}\pi^{-s/2}\Gamma(s/2).
\end{aligned}$$

Let $\Re(s) > 0$, and then sum both sides from $n = 1$ to ∞ to get

$$\Omega(s) = \zeta(s)\pi^{-s/2}\Gamma(s/2) = \sum_{n=1}^\infty \frac{1}{n^s}\pi^{-s/2}\Gamma(s/2) = \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 t} t^{s/2}\frac{\mathrm{dt}}{t}.$$

To keep from getting sidetracked we will simply switch the order of the sum and the integral while noting that to prove this is legal is not a trivial procedure. Thus we have

$$\Omega(s) = \int_0^\infty f(t) t^{s/2}\frac{\mathrm{dt}}{t} \quad \text{where} \quad f(t) = \sum_{n=1}^\infty e^{-\pi n^2 t}. \tag{4.11}$$

Recall Jacobi's theta function (section 4.2) and the its corresponding transformation invariant, theorem 4.2.1. Using those facts, we have

$$\begin{aligned}
1+2f(t) &= 1+2\sum_{n=1}^{\infty} e^{-\pi n^2 t} \\
&= e^{-\pi(0)^2 t} + \sum_{n=1}^{\infty} e^{-\pi n^2 t} + \sum_{n=-\infty}^{1} e^{-\pi n^2 t} \\
&= \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} \\
&= \Theta(t) \\
&= t^{-1/2}\Theta(t^{-1}) \\
&= t^{-1/2}(1+2f(t^{-1})).
\end{aligned}$$

Then solving for $f(t)$ in the first and last lines yields

$$f(t) = \frac{1}{2}(t^{-1/2}-1) + t^{-1/2} f(t^{-1}). \tag{4.12}$$

Returning to $\Omega(s)$ and using equation (4.12) we have

$$\begin{aligned}
\Omega(s) &= \int_0^{\infty} f(t) t^{s/2} \frac{\mathrm{dt}}{t} \\
&= \int_0^1 f(t) t^{s/2} \frac{\mathrm{dt}}{t} + \int_1^{\infty} f(t) t^{s/2} \frac{\mathrm{dt}}{t} \\
&= \int_0^1 \left(\frac{1}{2}(t^{-1/2}-1) + t^{-1/2} f(t^{-1})\right) t^{s/2} \frac{\mathrm{dt}}{t} + \int_1^{\infty} f(t) t^{s/2} \frac{\mathrm{dt}}{t} \\
&= \int_0^1 \frac{1}{2}(t^{s/2-3/2} - t^{s/2-1}) \mathrm{dt} + \int_0^1 t^{-1/2} f(t^{-1}) t^{s/2} \frac{\mathrm{dt}}{t} + \int_1^{\infty} f(t) t^{s/2} \frac{\mathrm{dt}}{t}.
\end{aligned}$$

Integrating the first term yields

$$\int_0^1 \frac{1}{2}(t^{s/2-3/2} - t^{s/2-1}) \mathrm{dt} = \frac{1}{2}\left(\frac{t^{s/2-1/2}}{s/2-1/2} - \frac{t^{s/2}}{s/2}\right)\Bigg|_{t=0}^{t=1} = \frac{1}{s-1} - \frac{1}{s}. \tag{4.13}$$

Using the change of variables $x = 1/t$ (and thus $\mathrm{dx}/x = -\mathrm{dt}/t$) on the second term results in

$$\int_0^1 t^{(s-1)/2} f(t^{-1}) \frac{\mathrm{dt}}{t} = -\int_{\infty}^1 x^{-(s-1)/2} f(x) \frac{\mathrm{dx}}{x} = \int_1^{\infty} x^{(1-s)/2} f(x) \frac{\mathrm{dx}}{x}. \tag{4.14}$$

Then after substituting t for x in this, we can recombine and rearrange everything to obtain

$$\begin{aligned}\Omega(s) &= \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty t^{(1-s)/2} f(t)\frac{\mathrm{dt}}{t} + \int_1^\infty f(t) t^{s/2}\frac{\mathrm{dt}}{t} \\ &= \frac{1}{s(s-1)} + \int_1^\infty (t^{(1-s)/2} + t^{s/2}) f(t)\frac{\mathrm{dt}}{t}.\end{aligned}$$

Inspection shows us that in fact $\Omega(1-s) = \Omega(s)$, which finishes the proof. $\square$

4.4 Values of Riemann's Zeta Function at the Negative Integers

Now that we have established the functional equation, we can use it to evaluate the Riemann zeta function in areas that Euler's zeta series was not defined, for example the negative integers.

4.4.1 Negative Even Integers

The functional equation we proved has the form

$$\pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s) = \pi^{-s/2}\Gamma(s/2)\zeta(s),$$

however, we can use identities of the Γ-function to rewrite this in another form. After lots of algebraic manipulation using properties (2.9), (2.12) and (2.13) we eventually come to the equivalent functional form

$$\zeta(s) = \Gamma(-s)(2\pi)^{s-1}2\sin(s\pi/2)\zeta(1-s).$$

In this form we can clearly see that Riemann's zeta function is zero at the negative even integers because of the sine term. These zeros at

$$s = -2, -4, -6, \ldots$$

are known as the trivial zeros of the zeta function.

4.4.2 Negative Odd Integers

We can arrive at a formula for the values of Riemann's zeta function at the negative odd integers by simply combining the formula we derived in section 3.5 and the functional equation. To review, for $k \geq 1$

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1}B_{2k}(2\pi)^{2k}}{2(2k)!}$$

and the functional equation tells us that

$$\pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Therefore, for $k \geq 1$

$$\begin{aligned} \pi^{-(1-(2k))/2}\Gamma((1-(2k))/2)\zeta(1-(2k)) &= \pi^{-(2k)/2}\Gamma((2k)/2)\zeta(2k) \\ \pi^{-1/2+k}\Gamma(1/2-k)\zeta(1-2k) &= \pi^{-k}\Gamma(k)\zeta(2k). \end{aligned}$$

After substituting $\Gamma(k) = k!$ (since k is a positive integer) and collecting terms on one side we have

$$\begin{aligned}
\zeta(1-2k) &= \frac{k!}{\pi^{2k-1/2}\Gamma(1/2-k)}\zeta(2k) \\
&= \frac{k!}{\pi^{2k-1/2}\Gamma(1/2-k)} \cdot \frac{(-1)^{k-1}B_{2k}(2\pi)^{2k}}{2(2k)!} \\
&= \frac{(-1)^{k-1}B_{2k}2^{2k-1}k!}{\pi^{-1/2}(2k)!\Gamma(1/2-k)}.
\end{aligned}$$

Which finally gives us the formula for the values of the Riemann zeta function at the odd integers:

$$\zeta(1-2k) = \frac{(-1)^{k-1}B_{2k}2^{2k-1}k!}{\pi^{-1/2}(2k)!\Gamma(1/2-k)}.$$

Chapter 5

The Riemann Zeta Hypothesis and the Consequences of its Truth

5.1 The Riemann Zeta Hypothesis

First we define

$$Z = \{s \in \mathbb{C} \mid \zeta(s) = 0 \text{ and } 0 \leq \Re(s) \leq 1\}.$$

In words, Z is the set of nontrivial zeros of the Riemann zeta function. Another helpful definition is that whenever we mention σ or t we are talking about the real and complex components of the variable plugged into $\zeta(s)$. Meaing that if $s \in \mathbb{C}$ then $s = \sigma + it$. The classic Riemann zeta hypothesis then becomes:

$$Z \subset \{s \in \mathbb{C} \mid \Re(s) = 1/2\},$$

which is sometimes also seen as $Z \subset \{s \in \mathbb{C} \mid \sigma = 1/2\}$. As a side note, another useful convention is that ρ usually represents a nontrivial zero of $\zeta(s)$.

There are also other, though more unnatural, ways to express this hypothesis. For example, the following two hypotheses, that were supplied by S. J. Patterson, are equivalent to Riemann's zeta hypothesis.

(i) $\zeta'(s)/\zeta(s) + (s-1)^{-1}$ is *holomorphic* in $\{s \in \mathbb{C} \mid \Re(s) > 1/2\}$.

(ii) $\log\{(s-1)\zeta(s)\}$ is *holomorphic* in $\{s \in \mathbb{C} \mid \Re(s) > 1/2\}$.

Just to be clear, *holomorphic* is just another way to say that the function is analytic, which simply means that the function is complex differentiable at every point in the specified region.

5.2 The Prime Number Theorem

Now that we have clarified what the Riemann zeta hypothesis is, we can start looking at some of the numerous results that pop up if the hypothesis is assumed to be true. The following two theorems, that are proved in Patterson, create a direct link between the Riemann zeta hypothesis and the distribution of the primes.

Theorem 5.2.1. *Suppose there exists $\theta < 1$ such that*

$$Z \subset \{s \in \mathbb{C} \,|\, 1-\theta \leq \Re(s) \leq \theta\}.$$

Then as $X \longrightarrow \infty$

$$\sum_{n \leq X} \Lambda(n) = X + O(X^{\theta}(\log X)^2).$$

Theorem 5.2.2. *Suppose there exists an $\alpha < 1$ such that as $X \longrightarrow \infty$*

$$\sum_{n \leq X} \Lambda(n) = X + O(X^{\alpha})$$

then

$$Z \subset \{s \in \mathbb{C} \,|\, \Re(s) \leq \alpha\}.$$

Just to clarify, $\Lambda(n)$ is Mangoldt's function and is defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and some integral } k > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Currently, these theorems have not been terribly useful because no one has found either a θ or α that satisfies them. However when combined with the validity of the Riemann hypothesis they say a lot regarding the theory surrounding the prime number theorem. Although it is obvious that a proof of the prime number theorem does not require the Riemann zeta hypothesis to be true (because if it did we wouldn't have a proof of the prime number theorem) it does rely on a small part of the hypothesis, mainly that if ρ is a nontrivial zero of $\zeta(s)$, then $Re(\rho) \neq 1$ However, this does not at all say that a proof of Riemann's zeta hypothesis would not improve the prime number theorem. The prime number theorem is usually stated in two equivalent forms, the first is more intuitive while the second is more useful for the purposes of this section. The following two theorems come from [Pat].

Theorem 5.2.3 (Intuitive Prime Number Theorem)**.** *There exists a constant* $c > 0$ *such that*

$$\pi(X) = \int_2^X (\log u)^{-1}\, du \;+\; O(Xe^{-c(\log X)^{1/2}}).$$

Theorem 5.2.4 (Useful Prime Number Theorem)**.** *There exists a constant* $c > 0$ *so that as* $X \longrightarrow \infty$

$$\sum_{n<X} \Lambda(n) = X + O(Xe^{-c(\log X)^{1/2}}).$$

In the second form we can see the relationship between the prime number theorem and theorems 5.2.1 and 5.2.2 that are concerned with the zeta function. In fact, what the validity of the Riemann hypothesis allows us to do is to obtain a tighter bound on the error term in the formula for $\pi(x)$. On the Riemann hypothesis, theorem 5.2.1 would tell us that

$$\sum_{n\leq X} \Lambda(n) = X + O(X^{1/2}(\log X)^2),$$

which would ultimately imply that

$$\pi(X) = \int_2^X (\log u)^{-1}\, du \;+\; O(X^{1/2}(\log X)^2),$$

a much more precise estimate for the number of primes less than a given magnitude.

5.3 Additional Consequences

There are many more consequences of the Riemann hypothesis; I will briefly touch on a few. I will not go into as much detail with these since most have little to do with prime distribution.

Another function that pops up when talking about $\zeta(s)$ is $S(T)$. If $T > 0$ is not the ordinate (where the *ordinate* of a point z is the imaginary component of z) of a zero, let

$$S(T) = \pi^{-1}\arg\zeta(1/2 + iT)$$

obtained by continuous variation along the straight lines joining 2, $2 + iT$, and $1/2 + iT$ starting at 0. If $T > 0$ is an ordinate of a zero then let

$$S(T) = S(T + 0).$$

The Riemann hypothesis provides many characteristics about $S(T)$ and a variant $S_1(T)$ such as big-O and little-o approximations and mean value theorems for them.

It turns out that proving the convergence of

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \qquad (s > 1/2)$$

is not only a necessary condition for the Riemann hypothesis, but also a sufficient one. Recall that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime factorization of n where the p_i's are distinct primes and all $\alpha_i > 0$, then the Möbius function is defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } \alpha_1 = \alpha_2 = \cdots = \alpha_k = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Be careful not to confuse this function with $\mu(\sigma)$ which I will discuss in the next section.

There are also some interesting results surrounding the function

$$M(x) = \sum_{n \leq x} \mu(n).$$

Without any help from the Riemann hypothesis we have

$$M(x) = O\left\{x^{1/2}\left(A\frac{\log x}{\log\log x}\right)\right\}.$$

However, showing that

$$M(x) = O(x^{\frac{1}{2}+\epsilon})$$

for some $\epsilon > 0$ is a necessary and sufficient condition for the Riemann hypothesis.

In addition to this we have the Mertens hypothesis, which is that

$$|M(x)| < \sqrt{n} \qquad (n > 1).$$

It turns out that although this is not implied by the Riemann hypothesis, it does itself imply the Riemann hypothesis.

Titchmarsh notes that the weaker hypothesis, $M(x) = O(x^{1/2})$, is eerily similar to the function

$$\psi(x) - x = \sum_{n \leq x} \Lambda(n) - x,$$

because the Riemann hypothesis does imply that $\psi(x) - x = O(x^{\frac{1}{2}+\epsilon})$.

Now consider the function

$$F(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{(k-1)! \zeta(2k)}.$$

On it's own we can show that $F(x) = O(x^{\frac{1}{2}+\epsilon})$. However showing that $F(x) = O(x^{\frac{1}{4}+\epsilon})$ is another necessary and sufficient condition on the Riemann hypothesis.

You can also derive yet another necessary and sufficient condition on the Riemann zeta hypothesis through the clever use of Farey series [Tit].

5.4 Lindelöf's Hypothesis

Lindelöf's hypothesis is a consequence of the Riemann zeta hypothesis however, the effects of it's veracity are much more accessible and far reaching and thus it merits it own section.

Before we get to the hypothesis we must define a new function. Let

$$\mu(\sigma) = \inf\{a \in \mathbb{R} \mid \zeta(\sigma + it) = O(|t|^a) \text{ as } t \longrightarrow \infty\}.$$

In other words, $|t|^{\mu(\sigma)}$ is the closest upper bound we can get to the zeta function restricted to the line $\{c \in \mathbb{C} \mid \Re(c) = \sigma\}$. To get a feeling for this we can figure out $\mu(\sigma)$ for select areas right off the bat. For example, we know that $\zeta(\sigma + it)$ is bounded for all $\sigma > 1$. Since it is bounded we know that $\zeta(\sigma + it) = O(|t|^\epsilon)$ for all $\epsilon > 0$, therefore $\mu(\sigma) = 0$ for $\sigma > 1$. We can then use this result in conjunction with the functional equation to arrive at $\mu(\sigma) = \frac{1}{2} - \sigma$ for $\sigma < 0$. This only leaves the critical strip in which $\mu(\sigma)$ is unknown, and this is where the Lindelöf hypothesis plays it's part. Since $\mu(\sigma)$ is a downwards convex function we can outline the possible region for the function on the domain $0 \leq \sigma \leq 1$. The Lindelöf hypothesis states that $\mu(\sigma)$ is in fact the extreme convex function within that region ([Tit] and [Pat]). This would mean

$$\mu(\sigma) = \begin{cases} \frac{1}{2} - \sigma & \text{if } 0 \leq \sigma \leq 1/2 \\ 0 & \text{if } 1/2 \leq \sigma \leq 1 \end{cases}.$$

Which works out really nicely with the values we already came up with to become

$$\mu(\sigma) = \begin{cases} \frac{1}{2} - \sigma & \text{if } \sigma \leq 1/2 \\ 0 & \text{if } \sigma \geq 1/2 \end{cases}.$$

Furthermore, because of the convexity of the function, all we really need is that $\mu(1/2) = 0$ and the rest of the function is forced to fix itself as described. Therefore the Lindelöf hypothesis really boils down to showing that

$$\zeta(1/2 + it) = O(|t|^\epsilon) \qquad \text{for } \epsilon > 0.$$

The fact that the Riemann zeta hypothesis implies the Lindelöf hypothesis is probably not easily recognizable in this form. However the following hypothesis is equivalent to that of Lindelöf. First we will define for $T > 0$ and $\sigma > 1/2$

$$N(\sigma, T) = \text{Card}\{\rho \in Z \mid \Re(\rho) > \sigma \text{ and } 0 < \Im(\rho) < T\}.$$

Thus $N(\sigma, T)$ is the number of non-trivial zeroes in the rectangle bounded by σ and 1 on the real axis, and 0 and T on the complex axis. Then the Lindelöf hypothesis is the same as saying for all $\sigma > 1/2$

$$N(\sigma, T+1) - N(\sigma, T) = o(\log T) \text{ as } T \to \infty.$$

With this form, we can see that the Riemann hypothesis forces $N(\sigma, T) = 0$ for all T and $\sigma > 1/2$ and thus would imply the Lindelöf hypothesis.

Hardy and Littlewood continue the work by showing even more statements that are equivalent to this hypothesis.

(i) For all $k \geq 1$ and $\epsilon > 0$

$$T^{-1} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^k dt = O(T^\epsilon).$$

(ii) For all $\sigma > 1/2$, $k \geq 1$ and $\epsilon > 0$

$$T^{-1} \int_0^T |\zeta(\sigma + it)|^k \, dt = O(T^\epsilon) \text{ as } T \longrightarrow \infty.$$

There are 3 more statements by Hardy and Littlewood that are equivalent, however since they deal with Plitz's generalization of Dirichlet's divisor problem (which extends far beyond the scope of this thesis) they will not be discussed [Tit].

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