

Love Games: A Game-Theory Approach To Compatibility

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Love Games: A Game-Theory Approach to Compatibility

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Abstract

In this note, we present a compatibility test with a rigorous mathematical foundation in game theory. The test must be taken separately by both partners, making it difficult for either partner alone to control the outcome. To introduce basic notions of game theory we investigate a scene from the film “A Beautiful Mind” based on John Nash’s life and Nobel-prize-winning theorem. We recall this result and reveal the mathematics behind our test. Readers may customize and modify the test for more accurate results or to evaluate interpersonal relationships in other settings, not only romantic. Finally, we apply Dyson’s and Press’s “zero-determinant payoff strategies” to this setting and explore the existence of such strategies and the corresponding implications for relationship dynamics.

1. Introduction

“Are you ready to settle down?” “Are you in love or forcing it?” These are the titles of compatibility quizzes found in Cosmopolitan magazine [8]. The “Love Calculator” featured in Glamour magazine [2] claims to determine whether “you and he add up” or are “destined for a long division?” These and most other compatibility quizzes we have seen in popular culture do not appear to have a rigorous scientific basis and are often easy to cheat to control the test results.

We have created a compatibility test based on sound mathematical principles and John Nash’s notion of equilibrium strategy. The test, which comprises the following section, must be taken separately by both partners, making it difficult for either partner to control the outcome alone. We gradually reveal the mathematics behind the test in §3. A discussion of zero-determinant payoff strategies and their existence as well as implications for relationship dynamics comprises §4, and we offer concluding remarks in §5.

Due to the potentially high diversity of readers, we offer the following organizational tips. Readers may use this note as an introduction to non-cooperative game theory: no pre-requisite knowledge is required, there are several exercises and opportunities for active participation, and there are many possibilities for readers to take the general principles and ideas offered here and apply them in creative and novel ways! In this case we recommend reading all sections in order but advise that §4.1 is mathematically more advanced and may be skipped.

For those who are already game theory experts, the test in §2 can be either taken or merely skimmed passing directly on to the mathematics behind the test in §3.2. The beginning of §4 as well as §4.1 may be skimmed for notational purposes proceeding on to §4.2 and in particular Conjecture 1 in §4.3. For readers primarily interested in applications §2, §§3.3–3.4 and §4.2 will likely be of greatest interest. Whatever your particular focus may be, we hope you enjoy the read!

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2. The Compatibility Test

2.1. Instructions

The first thing to do is print out two copies of the following two pages in this section. This is important so that you and your partner can complete the test *separately, without communicating to each other*.

Next, instead of simply deciding on one of three options as is typical for love quizzes, for each question you choose three numbers between 0 and 10, which add up to 10. Each question will have three different possible outcomes, and the three numbers between 0 and 10 for each of the slots (A, B, C) should be chosen as follows.

1. For the A slot, choose a number from 0 (never/completely disagree) to 10 (always/completely agree) to describe how likely Outcome A is.
2. For the B slot, choose a number between 0 (never/completely disagree) to 10 (always/completely agree) to describe how likely Outcome B is.
3. For the C slot, choose a number between 0 (never/completely disagree) to 10 (always/completely agree) to describe how likely Outcome C is.

A	<input type="text"/>	Outcome A
B	<input type="text"/>	Outcome B
C	<input type="text"/>	Outcome C

For example, if you know that Outcome A always happens, then you would fill in (10, 0, 0). If Outcomes, A, B, and C, are equally likely to happen, then you would fill in (10/3, 10/3, 10/3). The numbers do not have to be whole numbers, fractions are also allowed. They just cannot be negative, and they must add to 10.

2.2. The Test Questions

Question 1. *You have decided to spend the weekend together. How does it go?*

- | | | |
|---|--------------------------|---|
| A | <input type="checkbox"/> | <i>It's a blast!</i> |
| B | <input type="checkbox"/> | <i>It's not my first choice, but it's nice.</i> |
| C | <input type="checkbox"/> | <i>Well, at least my partner is happy.</i> |

Question 2. *How do you feel about your sex life?*

- | | | |
|---|--------------------------|---------------------------------------|
| A | <input type="checkbox"/> | <i>Sex is just the way I like it!</i> |
| B | <input type="checkbox"/> | <i>I'm satisfied.</i> |
| C | <input type="checkbox"/> | <i>I am sexually frustrated.</i> |

Question 3. *How do you and your partner manage your careers and chores?*

- | | | |
|---|--------------------------|---|
| A | <input type="checkbox"/> | <i>Career and chores are just the way I like them.</i> |
| B | <input type="checkbox"/> | <i>I've had to make some compromises.</i> |
| C | <input type="checkbox"/> | <i>I've made significant sacrifices for my partner.</i> |

Question 4. *Your partner has fallen ill. What happens?*

- | | | |
|---|--------------------------|---|
| A | <input type="checkbox"/> | <i>I rarely catch whatever s/he has.</i> |
| B | <input type="checkbox"/> | <i>I take care of him/her and might get sick.</i> |
| C | <input type="checkbox"/> | <i>I stay by his/her side and get sick.</i> |

Question 5. *How often do you see your family and your in-laws?*

- | | | |
|---|--------------------------|--|
| A | <input type="checkbox"/> | <i>Exactly as much as I would like.</i> |
| B | <input type="checkbox"/> | <i>It's a compromise but we manage to get along.</i> |
| C | <input type="checkbox"/> | <i>I don't see my own family enough.</i> |

Question 6. *If you are in a long relationship, your circle of friends often change. Whose friends do you tend to see?*

- | | | |
|---|--|---|
| A | | <i>I spend as much time with my friends as I want.</i> |
| B | | <i>I spend less time with my friends but still keep up.</i> |
| C | | <i>I rarely see my old friends.</i> |

Question 7. *How is your financial situation?*

- | | | |
|---|--|--|
| A | | <i>Great.</i> |
| B | | <i>I can't always spend the way I'd like, but it's fine.</i> |
| C | | <i>We have financial problems.</i> |

Question 8. *Have you and your partner talked about having children?*

- | | | |
|---|--|--|
| A | | <i>Yes and I'm happy with our decisions.</i> |
| B | | <i>We will discuss it eventually.</i> |
| C | | <i>That is up to my partner to decide.</i> |

Question 9. *How do you feel about your lifestyle in terms of health, fitness, and physical appearance?*

- | | | |
|---|--|--------------------------------------|
| A | | <i>I am totally happy.</i> |
| B | | <i>Fine.</i> |
| C | | <i>It is not what it used to be.</i> |

Question 10. *You have agreed to spend a cozy night at home watching TV. How does it go?*

- | | | |
|---|--|--|
| A | | <i>I love what we watch.</i> |
| B | | <i>We compromise on something.</i> |
| C | | <i>My partner chooses what we watch.</i> |

3. The Mathematics behind the Test

Our quiz is based on non-cooperative game theory. You may be thinking

*Why use **non**-cooperative game theory for a **compatibility** test?*

Although it may seem counter-intuitive, there is a quite natural reason. Most everyday decisions are made spontaneously with little or no discussion. People naturally tend to act in their best self-interest. This is similar to non-cooperative games in which each player acts independently to maximize his own payoff without communication with the other players. Incorporating rigorous pure mathematics into real-world situations can be a subtle problem. To illustrate this and to introduce some of the basic ideas of non-cooperative game theory, we consider an example from popular culture.

3.1. *A Beautiful Mind*

Some readers may already be familiar with a rather perplexing scene from the film “A Beautiful Mind” based on Nash’s life and work, which purportedly depicts Nash’s Nobel prize winning theorem. The scene is set in a bar and begins with Nash’s character and his buddies watching a group of attractive women enter. One woman is blond, and she is shown as being considered the most attractive by the men, whereas the other women are shown as being considered averagely attractive. The men begin discussing their best strategies to court the blond. After a moment of reflection Nash’s character realizes that the “best” tactic is for the men to each approach one of the other women, leaving the blond alone in the end. Is this really an equilibrium strategy according to Nash’s Theorem?

This situation can be mathematically described using a non-cooperative game. A *non-cooperative game* has a set of *pure strategies* for each player and a set of *payoffs* corresponding to each combination of pure strategies across the players. The pure strategies are the different things that a player can do. For example, let us consider the scene from the film and for the sake of simplicity assume there are two men, denoted by man 1 and man 2, and 3 women, one denoted by “B” (for blond) and two denoted by “N” (for non-blond). Each man has two pure strategies: B which corresponds to courting the blond, and N which corresponds to courting one of the other women.

A two player game can be expressed in *normal form*, which is also known as a *payoff matrix*, because it is a matrix which lists the payoffs according to each combination of pure strategies. For example:

	B	N
B	(-1, -1)	(1, 0)
N	(0, 1)	(0, 0)

The interpretation is that if both men attempt to court the blond, they are both unsuccessful, indicated by a negative payoff -1 . If man 1 chooses B, and man 2 chooses N, then both men's courtships are successful, and we consider man 1 to be positively rewarded indicated by a score of 1, and man 2 is merely content, indicated by a neutral score of 0. If both men choose N, we presume that they each court a different woman and are both successful, so they are both content, indicated by a score of 0.

A *strategy* is a list of numbers between 0 and 1 corresponding to the probabilities of doing each pure strategy. These are sometimes known as mixed strategies, but we find it slightly more correct to call them strategies, because they include the pure strategies. A *mixed* or *continuous* game assigns *payoffs* for each strategy. These payoffs are defined using the definition of *expected value* from probability theory. Assuming the game is *non-cooperative* means that players act *independently* without communicating with each other. This simplifies computing the probabilities of various outcomes, because the probabilities are *independent*.

Let x be the probability that man 1 chooses B, and y be the probability that man 2 chooses B. This means that the probability that both men choose B is xy , because these probabilities are independent by the assumption that the game is non-cooperative. We assume that each man courts a woman, and so the probability that man 1 chooses N is $1 - x$, because the sum of the probabilities of choosing B and choosing N must be one. The probability that man 2 chooses N is then $1 - y$. So, the probability that man 1 chooses B while man 2 chooses N is $x(1 - y)$, and the probability that man 1 chooses N while man 2 chooses B is $(1 - x)y$. The payoff to man 1 is the sum of the expected values for each possible outcome. The expected value is the product of the probability of the outcome together with the corresponding payoff. We therefore compute the payoff for man 1 according to the probabilities x and y to be:

$$(-1)xy + (1)x(1-y) + (0)(1-x)y + (0)(1-x)(1-y) = x - 2xy.$$

Exercise 1. What is the payoff for man 2?

In this example the game is *symmetric* which means that all players have the same set of pure strategies, and if players swap strategies, then their payoffs swap accordingly. More precisely,

$$\wp_1(x, y) = \wp_2(y, x).$$

A strategy is known as an *equilibrium strategy* if the following holds: no player can increase his payoff by changing his strategy while the others remain fixed. Nash proved the following [5].

Theorem 1 (Nash). *For any n -person non-cooperative game such that the payoffs are consistent with the definition of expected value, there exists at least one equilibrium strategy.*

Exercise 2. Determine the equilibrium strategy/strategies in this example.

There are various ways to compute equilibrium strategies. Since the second author is a geometric analyst, we propose a geometric analytic approach. The set of strategies for each player is a list of probabilities across that player's pure strategies, which is a list of numbers between 0 and 1 which sum up to 1. Consequently, we can identify the set of strategies with a *strategy space*. Each pure strategy is identified with a standard unit vector, because a pure strategy means doing that strategy with probability one and the others with probability 0. The strategy space is the convex hull of these points. Recall that the convex hull of a set M is the smallest closed convex set which contains M . The total strategy space is the product of all the strategy spaces for each player. The total payoff function is a map from the total strategy space to \mathbb{R}^n , where n is the number of players. Each component of the total payoff function is the payoff function for each player, so in our example this function is

$$(\wp_1(x, y), \wp_2(x, y)) = (x - 2xy, y - 2xy).$$

For any non-cooperative game the payoff functions are of the form

$$\wp_i(s) = \sum_{\text{all combinations } C \text{ of pure strategies}} \text{Prob}_s(C) \wp_i(C),$$

where $\text{Prob}_s(C)$ is the probability according to s of the combination of pure strategies listed in C . This means that if all *other* players' strategies are fixed, then my payoff function varies *linearly* as a function of my strategy alone.

Exercise 3. *Show that in order to be consistent with expected value, each payoff function must be linear in the strategy of the corresponding player.*

Similar to finding the extrema of a function on a domain where usually interior and boundary points are investigated separately, we first think about the interior of the total strategy space. At any such point each player can increase or decrease the probability of each of her pure strategies, because the probabilities must be strictly between 0 and 1, otherwise the point lies on the boundary. If such a strategy is an equilibrium, then increasing or decreasing the probability of each pure strategy while keeping the other players' strategies fixed cannot increase her payoff. Mathematically, this forces the partial derivatives of her payoff function with respect to her pure strategies to all be zero. The converse is also true, because the dependence of her payoff on her own strategy alone is *linear*, and the partial derivatives vanishing implies that changing her strategy alone has no effect on her payoff. Consequently, one way to determine equilibrium strategies is to compute the partial derivatives of each payoff function with respect to the pure strategies of the corresponding player and check whether there are strategies on the interior of the strategy space for which these all vanish.

Next, one can compare the payoffs corresponding to the different pure strategies and see if any of these are maximized across all players. If that is the case, then if any one player changes his strategy, his payoff will not increase. Note that in general an equilibrium strategy need not maximize all players' payoffs.

Finally, one can check for other strategies on the boundary of the total strategy space.

We summarize these observations as follows.

Proposition 1. *For all strategies on the interior of the total strategy space, a necessary and sufficient condition to be an equilibrium is that the partial derivatives of each payoff function with respect to the corresponding pure strategies of that player are all zero. If a strategy which is a pure strategy for all players maximizes the payoffs of all players, then it is an equilibrium strategy.*

Let us now compute the equilibrium strategies for the payoff functions

$$\wp_1(x, y) = x - 2xy, \quad \wp_2(x, y) = y - 2xy.$$

First, we compute the partial derivatives with respect to each player's pure strategies,

$$\frac{\partial}{\partial x}\wp_1(x, y) = 1 - 2y, \quad \frac{\partial}{\partial y}\wp_2(x, y) = 1 - 2x.$$

These vanish precisely when $x = y = 1/2$. This corresponds to each man choosing B with probability $1/2$, which can be seen as randomly choosing between B and N. As for pure equilibrium strategies, $(x = 0, y = 1)$ and $(x = 1, y = 0)$ are also equilibrium strategies. The first means that man 1 courts a non-blond, and man 2 courts the blond, each with probability 1. The second is the other way around.

None of these three equilibrium strategies fit with the film! This is rather perplexing. One explanation is that Hollywood just does not understand math. That may very well be the case. However, as mathematicians, we ought to know that things are not always what they seem. Is it possible to resolve the mathematical definition of equilibrium strategy with the scene depicted in the film?

Exercise 4. *What is a payoff matrix for a two-player symmetric game with two pure strategies B and N with the interpretations described above, such that the equilibrium strategy is $x = y = 0$?*

How do we begin? The payoffs are symmetric, which means that $\wp_1(B, N) = \wp_2(N, B)$, $\wp_1(N, N) = \wp_2(N, N)$, and $\wp_1(B, B) = \wp_2(B, B)$. The unknown payoff matrix for the film is:

	B	N
B	(a, a)	(c, d)
N	(d, c)	(b, b)

The payoff functions are

$$\wp_1(x, y) = xya + x(1 - y)c + (1 - x)yd + (1 - x)(1 - y)b,$$

$$\wp_2(x, y) = xya + y(1 - x)c + (1 - y)xd + (1 - y)(1 - x)b.$$

The payoffs for the strategy $x = y = 0$ are $\wp_1(0, 0) = b = \wp_2(0, 0)$. In order for this to be an equilibrium strategy, the following must hold:

$$\wp_1(x, 0) \leq \wp_1(0, 0) \text{ for all } x \in [0, 1]; \quad \wp_2(0, y) \leq \wp_2(0, 0) \text{ for all } y \in [0, 1].$$

This is equivalent to:

$$xc + (1 - x)b \leq b \text{ for all } x \in [0, 1], \quad yc + (1 - y)b \leq b \text{ for all } y \in [0, 1],$$

which holds if and only if

$$c \leq b.$$

This means that the payoff associated to successfully courting the blond is *less than or equal to* the payoff associated to successfully courting a non-blond! This is inconsistent with the film which indicates that each man would be most content if he successfully courted the blond.

This leads us to the same initial conclusion: Hollywood simply does not understand the math. Well, perhaps there is a logical explanation. What if $b = c$? The payoff matrix would be

	B	N
B	(a, a)	(c, d)
N	(d, c)	(c, c)

Now, clearly $a < c$, because successfully courting the blond is a more positive outcome than unsuccessfully courting her. It is also natural to presume that $d < c$, as the film indicates that successfully courting the blond while the other man courts a non-blond is more desirable than the other way around. How can we make sense of the fact that $\wp_1(N, N) = \wp_2(N, N) = \wp_1(B, N) = \wp_2(N, B) = c$?

Instead of thinking about the relationship between the men and the women, let us think about the relationship *between the men*. They are shown as buddies in the film. If man 1 courts the blond while his buddy, man 2, courts a non-blond, then although man 1 might be very happy about the outcome, man 2 might be jealous. Jealousy can have unpleasant consequences. Now, on the other hand, if both men court a non-blond, then nobody will be jealous! So, perhaps jealousy explains why the above payoff matrix with $d < a < c$ is what the filmmakers had in mind.

There is still a glitch with this explanation. You will see what we mean if you do the following exercise.

Exercise 5. *Are there any other equilibrium strategies?*

Well, yes. The strategy (B, B) is also an equilibrium strategy, and this is clearly not the best outcome since both men end up alone. So, it is still not entirely clear whether or not the filmmakers understand the mathematics.

3.2. A Game to Evaluate a Relationship

We would like to use a game to evaluate two parameters in a relationship:

1. overall happiness, and
2. balance.

We consider the following model.

	H	O	N
H	(a, a)	(b, c)	(d, e)
O	(c, b)	(f, f)	(g, h)
N	(e, d)	(h, g)	(i, i)

The pure strategies are H (happy), O (okay), and N (not happy).¹ It would be possible to use *any game* to create a test in the spirit of ours, so we would like to describe how we came to choose this particular game to evaluate our test. We assume the game is symmetric, and we also assume some amount of empathy between the partners. Ideally, both partners are happy, and so the corresponding payoff a should be the highest. If one partner is happy, and the other partner is okay, then we consider this to be less optimal than the pure strategy (H, H) , and certainly better for the happy partner than for the okay partner, so we assume

$$a > b > c.$$

We also assume that being happy while the other partner is unhappy is worse than being happy while the partner is okay, and of course being unhappy is not better than being okay, so

$$a > b > d \geq c.$$

¹A common term of endearment used in American couples is “hon.”

We consider both partners being okay to be a worse outcome than if one partner is happy and the other partner is okay, so

$$c > f.$$

If one partner is okay, and the other is unhappy, then this is worse than both partners being okay, but it is not as bad as being unhappy, so

$$f \geq g > h.$$

Finally, the absolute worst is if both partners are unhappy so

$$a > b > d \geq c > f \geq g \geq e \geq h > i. \quad (1)$$

If we remove the assumption of empathy between partners and assume instead that each partner is only concerned with his or her own happiness, then the game should have

$$a = b = d > c = f = g > e = h = i. \quad (2)$$

Under either of these assumptions (1) or (2) on the payoffs, the “best” outcome corresponds to the pure strategy in which both players do *H* with probability 1. This strategy uniquely maximizes the payoff to each player, and it is an equilibrium strategy. So we see that there is a whole range of payoffs one could assign to the game in order to customize it to evaluate a relationship. For the sake of simplicity, we choose

$$a = 3, b = 2, c = 1 = d, f = 0 = g, e = -1 = h, i = -3,$$

and so the game in normal form is

	H	O	N
H	(3, 3)	(2, 1)	(1, -1)
O	(1, 2)	(0, 0)	(0, -1)
N	(-1, 1)	(-1, 0)	(-3, -3)

3.3. Evaluating the Test Results

By now you probably have an idea how to evaluate the test. We will use the answers to the test questions to compute the strategy of each partner,

corresponding to the cumulative probability of each strategy H, O, N for each partner. The A column tells how likely strategy H is; the B column tells how likely strategy O is; the C column tells how likely N is. Since we assumed that for each question these numbers sum to 10, and there are 10 questions,

$$\begin{aligned}\text{The probability of A, } P(A) &= \frac{\text{sum over column A}}{100}, \\ \text{The probability of B, } P(B) &= \frac{\text{sum over column B}}{100}, \\ \text{The probability of C, } P(C) &= \frac{\text{sum over column C}}{100}.\end{aligned}$$

Of course, it is easier to work with whole numbers rather than fractions, so rather than dividing by 100, you can simply compute

$$\begin{aligned}P_i(A) &= \text{sum over column A}, & P_i(B) &= \text{sum over column B}, \\ & & \text{and } P_i(C) &= \text{sum over column C},\end{aligned}$$

for $i = 1, 2$ corresponding to partners 1 and 2.

Next, we use these to compute the (re-scaled) payoff to each partner according to the HON game.

$$\begin{aligned}\wp_1 &= 3P_1(A)P_2(A) + 2P_1(A)P_2(B) + P_1(A)P_2(C) \\ &\quad + P_1(B)P_2(A) - P_1(C)P_2(A) - P_1(C)P_2(B) - 3P_1(C)P_2(C). \\ \wp_2 &= 3P_2(A)P_1(A) + 2P_2(A)P_1(B) + P_2(A)P_1(C) \\ &\quad + P_2(B)P_1(A) - P_2(C)P_1(A) - P_2(C)P_1(B) - 3P_2(C)P_1(C).\end{aligned}$$

Well, these are not quite the payoffs because the actual payoffs for the HON game would be these numbers divided by 100^2 . We find it more useful to normalize the payoffs as follows

$$S = \frac{1}{200} (\wp_1 + \wp_2).$$

1. The overall happiness of the couple is given by S . The maximum possible is 300, and the minimum possible is -300.
2. If $\wp_1 > \wp_2$, then the first partner is more dominant. If $\wp_2 > \wp_1$, then the second partner is more dominant.

We interpret S to be the overall happiness of the couple because it corresponds to the total payoff to both partners, and we have used payoff to quantify happiness in the definition of the HON game. If $\wp_1 > \wp_2$, then the first partner is happier, and according to the test questions this means that the couple is generally tending to make decisions which favor the interest or wishes of the first partner, whereas the interest or wishes of the second partner are met less often. It is the same conclusion with the roles reversed if $\wp_2 > \wp_1$. For this reason we characterize the first partner as more dominant if $\wp_1 > \wp_2$, or the second partner as more dominant if $\wp_2 > \wp_1$.

3.4. Modifying the Test

We chose questions to assess sex-life, family, friends, health, lifestyle, career, and finances. For more accurate results, the test can be altered to consist of as many questions on as many topics as desired. The test can be modified in a virtually infinite number of ways.

3.4.1. Personalization

One way to make the test more accurate is to allow each partner to choose the total of the answer column for each question. The default total is 10, but if a certain question is more important to the partner, s/he can make that question's column have a higher sum like 50. On the other hand, if a certain question is really not so important to him or her, then s/he could assign the column's total to be a smaller number like 5. We call these numbers the *weights* of the questions.

To evaluate such a test, first compute the sums of each of the A, B, and C, columns, respectively for each partner. Next, for each partner, add up all the weights. Then, to compute $P_1(A)$, the sum of the first partner's A column is divided by the total sum of the first partner's weights. Similarly for the other columns and for the second partner. These values $P_i(A)$, $P_i(B)$, and $P_i(C)$ are then substituted into the definition of \wp_i for $i = 1, 2$ as above. This time the calculation yields the payoffs according to the HON game, so the payoffs are each between 3 and -3. Consequently, to obtain S as a number between -300 and 300, it is therefore necessary to multiply $\wp_1 + \wp_2$ by 50. The overall happiness of the couple is given by S , and if $\wp_1 > \wp_2$ then the first partner is more dominant, whereas if $\wp_2 > \wp_1$, then the second partner is more dominant.

3.4.2. A different game

More generally, you could define a *different game* to create a test. Each test question has n possible answers, corresponding to the n pure strategies for each player, and you would need to define the payoff matrix for your game. You would then analogously compute the expected payoff for each player (=test taker) determined by the results of both players. With a suitable choice of test questions, our test could be used to evaluate relationships between business partners, employees, political partners, or any situation involving two (or more) people who behave in their best self interest. It is also possible to create tests based on iterated games as in [1].

4. The Iterated Dating Dilemma and Manipulation

Have you been in a relationship for a length of time? In any relationship there is give and take. We can describe this mathematically using the *Prisoner's Dilemma*. Two crooks have committed a crime and have gotten caught! They are locked in a prison cell together before they are taken separately for questioning. They agree in the prison cell that they will not rat each other out. This is known as *cooperation* (C). However, when each prisoner goes off for questioning, he has the chance to *defect* (D) by claiming he is innocent, and his partner is solely responsible for the crime.

The payoff matrix seen below in Figure 1 means that if both prisoners cooperate, they both receive an equal payoff of W . In terms of give and take within a couple, we identify the strategy C with *giving*, and the strategy D with *taking*. So if both partners give, then they both receive an equal payoff W . In the Prisoner's Dilemma, if one prisoner cooperates but his partner in crime defects, then the prisoner who cooperated gets slapped with a longer prison sentence than the defector. So the payoff for cooperating Z is smaller than Y , the payoff for defecting. Similarly, if one person in a relationship gives, corresponding to strategy C , while the other person in the relationship

	C	D
C	(W,W)	(Z,Y)
D	(Y,Z)	(X,X)

Figure 1: Payoff matrix for the Prisoner's Dilemma and the Iterated Dating Dilemma.

takes, corresponding to strategy D, then the payoff to the giver is lower than the payoff to the taker. Finally, if both prisoners defect, then they both get a longer sentence than if they cooperate, so the payoff for mutual defection X is smaller than the payoff for mutual cooperation W . Similarly, if both people in the relationship take, then their payoff is lower than if they both give. Consequently, the payoffs satisfy

$$Y > W > X > Z.$$

In the Prisoner's Dilemma it is also customary to assume that the expected value of cooperating is higher than the expected value of defecting. If a prisoner defects, a neutral expectation of his partner in crime is $1/2$ probability defecting and $1/2$ probability cooperating. Hence, the expected payoff of defecting is

$$(Y + Z)/2,$$

and we assume the payoff for cooperating is better than the expected payoff of defecting so

$$W > (Y + Z)/2.$$

In the *Iterated Prisoner's Dilemma* the “game” is repeated. We can use the dynamics of the Iterated Prisoner's Dilemma to study the dynamics of give and take in a relationship, which we call the *Iterated Dating Dilemma*. Of course, we are simplifying things a bit here, for example by assuming the payoffs are symmetric and remain constant over time. Still analyzing the long-term dynamics of give and take in a relationship using the Iterated Prisoner's Dilemma leads to interesting results. For instance, at each round of the game we assume that each person remembers what happened in the previous round. In [6], Press and Dyson proved that it does not matter whether one person can remember more than the other, which game-theoretically means that a long-memory player has no advantage over a memory-one player (someone who can only remember the previous date). This in itself is rather interesting and a bit of a relief! It means that to ensure our best possible payoff over time, we just need to remember what happened last time.

4.1. The Mathematics of Zero-Determinant Strategies

This part is somewhat more technical and is suitable for mathematically inclined readers with some background in linear algebra and Markov matrices. Some readers may prefer to skip this part and proceed directly to §4.2 on the implications of the mathematical results demonstrated here.

Let us call the two people in the couple Pat and Gene. At each iteration of the game there are four possibilities for the previous outcome: (C, C) , (C, D) , (D, C) , (D, D) . Pat's strategy is $p = (p_1, p_2, p_3, p_4)$ which corresponds to her probability of cooperating (giving) under each of the previous outcomes as seen from Pat's perspective. Gene's strategy is $q = (q_1, q_2, q_3, q_4)$ (cf. [6, Figure 1]) which corresponds to his probability of cooperating under each of the previous outcomes. Note that the probability p_2 corresponds to the probability of Pat cooperating if last time Pat cooperated but Gene did not, whereas q_2 corresponds to the probability of Gene cooperating if last time Gene cooperated but Pat did not. Similarly, p_3 corresponds to the probability of Pat cooperating if last time Gene cooperated but Pat did not, whereas q_3 corresponds to the probability of Gene cooperating if last time Pat cooperated but Gene did not. These probabilities can be used to define a Markov matrix

$$M = \begin{bmatrix} p_1 q_1 & p_1(1 - q_1) & (1 - p_1)q_1 & (1 - p_1)(1 - q_1) \\ p_2 q_3 & p_2(1 - q_3) & (1 - p_2)q_3 & (1 - p_2)(1 - q_3) \\ p_3 q_2 & p_3(1 - q_2) & (1 - p_3)q_2 & (1 - p_3)(1 - q_2) \\ p_4 q_4 & p_4(1 - q_4) & (1 - p_4)q_4 & (1 - p_4)(1 - q_4) \end{bmatrix}.$$

The first row lists the probabilities of all possible outcomes if in the previous time both Pat and Gene cooperated. Consequently, these sum to one. Similarly, the second row lists the probabilities of all possible outcomes if in the previous time Pat cooperated but Gene did not. The third row lists the probabilities for the other way around, and the last row lists the probabilities of all possible outcomes if both Pat and Gene did not cooperate. By the Perron-Frobenius Theorem this matrix has a unique (up to scaling) stationary vector $v \neq 0$ with all non-negative components, which satisfies

$$Mv = v.$$

Normalizing this vector so that the sum of the components is 1 can therefore be done by dividing by

$$v \cdot \mathbf{1},$$

where $\mathbf{1}$ is the vector with components all equal to one. Since $Mv = v$ and thus $M^2v = v$ and so forth, the stationary vector can be used to determine the limit of the expected payoffs over time, if we assume that the initially defined probabilities for Pat and Gene are always used in the game which is repeated infinitely many times.

We define the payoff vectors for Pat and Gene respectively by

$$P := \begin{bmatrix} W \\ Z \\ Y \\ X \end{bmatrix}, \quad G := \begin{bmatrix} W \\ Y \\ Z \\ X \end{bmatrix},$$

Press and Dyson showed in [6] that the expected payoffs over time to Pat and Gene are then respectively given by

$$\wp_1(p, q) = \frac{v \cdot P}{v \cdot \mathbf{1}}, \quad \wp_2(p, q) = \frac{v \cdot G}{v \cdot \mathbf{1}}.$$

They also showed that these payoffs are equivalently given in terms of the determinants of certain 4×4 matrices. For a vector $f \in \mathbb{R}^4$ denote

$$D(p, q, f) := \det \begin{bmatrix} -1 + p_1 q_1 & -1 + p_1 & -1 + q_1 & f_1 \\ p_2 q_3 & -1 + p_2 & q_3 & f_2 \\ p_3 q_2 & p_3 & -1 + q_2 & f_3 \\ p_4 q_4 & p_4 & q_4 & f_4 \end{bmatrix}.$$

Then the expected payoffs to Pat and Gene are respectively given by

$$\wp_1(p, q) = \frac{D(p, q, P)}{D(p, q, \mathbf{1})}, \quad \wp_2(p, q) = \frac{D(p, q, G)}{D(p, q, \mathbf{1})}.$$

This dependence is linear and so consequently

$$a\wp_1(p, q) + b\wp_2(p, q) + c = \frac{D(p, q, aP + bG + c\mathbf{1})}{D(p, q, \mathbf{1})}.$$

Note that the matrix $D(p, q, f)$ has the second column completely determined by Pat and third column completely determined by Gene. This means that if Pat chooses a strategy \tilde{p} such that

$$\tilde{p} = aP + bG + c\mathbf{1} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad (3)$$

then the second and fourth columns will coincide and consequently the determinant

$$D(\tilde{p}, q, aP + bG + c\mathbf{1}) = 0.$$

Note that there is an error in [6] in the paragraph between Equations [6] and [7] which claims that the strategy \tilde{p} should be chosen equal to $aP + bG + c\mathbf{1}$. The calculation which follows in Equation [8] therein is however consistent with our Equation (3), so it is clear that the correct equation should be (3).

For such a strategy with $a = 0$, $b \neq 0$, this means that

$$\wp_2(\tilde{p}, q) = \frac{-c}{b},$$

and therefore Pat is controlling Gene's payoff. Similarly Gene can choose a strategy \tilde{q} to control Pat's payoff. These are known as *zero-determinant* strategies. For such a strategy \tilde{p} for Pat which satisfies (3) with $a = 0$, we have:

$$\begin{aligned}\tilde{p}_1 &= bW + c + 1, \\ \tilde{p}_2 &= bY + c + 1, \\ \tilde{p}_3 &= bZ + c, \\ \tilde{p}_4 &= bX + c.\end{aligned}$$

Since this is a system of four equations with six unknowns, we can solve for the strategies in terms of two unknowns, so we choose to solve in terms of \tilde{p}_1 and \tilde{p}_4 . We arrive at the same equation given in Equation [8] of [6]:

$$\begin{aligned}\tilde{p}_2 &= \frac{\tilde{p}_1(Y - X) - (1 - \tilde{p}_4)(Y - W)}{W - X}, \\ \tilde{p}_3 &= \frac{(1 - \tilde{p}_1)(X - Z) + \tilde{p}_4(W - Z)}{W - X},\end{aligned}\tag{4}$$

with \tilde{p}_1 and \tilde{p}_4 chosen freely but with $\tilde{p}_2, \tilde{p}_3 \in [0, 1]$. Then Gene's payoff is:

$$\frac{(1 - \tilde{p}_1)X + \tilde{p}_4W}{1 - \tilde{p}_1 + \tilde{p}_4}.$$

Since the above arguments are identical for any two-player, two-move, symmetric game with payoff matrix as in Figure 1, regardless of the particular choice of the values W , X , Y , and Z , we have shown the following:

Proposition 2 (Press and Dyson). *For any two-player symmetric game with two pure strategies, algebraically there exist zero-determinant strategies. Whether these are actually possible to execute depends on the specific values of the payoff matrix.*

4.2. The Interpretation of Zero-Determinant Strategies

For Prisoner's Dilemma type games, we always have

$$Y > W > X > Z.$$

Consequently, the zero-determinant strategies in Equation (4) yield feasible probabilities when \tilde{p}_1 is close to but less than or equal to 1, and \tilde{p}_4 is close to but greater than or equal to 0. This implies that \tilde{p}_2 is close to but less than or equal to 1, and \tilde{p}_3 is close to but greater than or equal to 0. These probabilities mean that most of the time Pat does the same thing she did last time, regardless of what Gene did. If she cooperated, she will cooperate again, and if she did not, then she will most likely not cooperate again. With this behavioral pattern, it is *mathematically proven above* that Pat is able to unilaterally control Gene's payoff, regardless of what Gene does! If your partner exhibits this pattern, then it may be that she or he is controlling your happiness!

Dyson and Press in [6] showed further that it is impossible for Pat to unilaterally set her own payoff. This might indicate that whereas one can influence his or her partner's happiness by his or her choice of behaviors, one cannot unilaterally control one's own happiness in a relationship. Moreover, since a longer memory player has no advantage over a memory-one player, it suffices to remember what happened "last time" to make one's best decision for "next time."

4.3. Zero-Determinant Strategies for Other Games

Algebraically zero-determinant strategies exist for any two player symmetric game with two pure strategies. Do they exist for games with more than two pure strategies, like the HON game? Do there exist zero-determinant strategies for games with more than two players? We make the following conjecture which could serve as the topic of an ambitious bachelor thesis:

Conjecture 1. *Any two player non-cooperative symmetric game algebraically has zero-determinant strategies, which may or not be feasible depending on the specific values of the payoff matrix.*

5. Concluding remarks

Did you take the test? If so, how did you score?

Even if your score indicates a lack of balance or overall happiness, all is not necessarily lost. We hope that our test is an improvement compared to those contained in popular magazines, but we acknowledge that it is quite simple, and relationships have many facets and subtleties which our test could fail to recognize. The advantages of our approach are the rigorous mathematical and game theoretical basis and the difficulty of manipulating the test because both partners must take it separately. However, there is much more to a relationship than the simple characteristics we have aimed to quantify using our test based on the HON game. In the end, love is qualitative, not quantitative.

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