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Voting Paradoxes Caused by Dropping Candidates in an Election

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Abstract

Voting theory is plagued by seemingly contradictory results, called *voting paradoxes*. For example, different methods of tallying votes can result in different election results; these voting paradoxes give contradictory answers to the question of what the voting population "really" wants.

This paper studies voting paradoxes brought about by considering the effect of dropping one or more candidates in an election after the voting data has already been collected. Dropping a candidate may change the election results for the remaining candidates. This paper adopts an algebraic framework to approach this voting theoretic problem.

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Chapter 1

Introduction

1.1 The Problem

The question of how to analyze voting data is a fascinating voting theoretic problem. Consider an election where each voter is asked to rank the candidates from favorite to least favorite (or, in a more complicated problem, we might allow the voters to give a so-called "partial ranking" of the candidates, where ties are allowed). Once all of the voting data has been collected, we must find the "best" method of tallying the voting data to declare a winner. As we will see, there are a multitude of ways to tally the voting data: several of which seem quite reasonable. Surprisingly, the application of two different methods may lead to different conclusions, so the choice of voting method has an impact on the result of the election.

Furthermore, consider a situation where one or more candidates in the election drop out after the voters have submitted their voting preferences. Then, the election outcomes for the remaining candidates may change as the data is reinterpreted. Situations such as these may cause us to wonder what the voters *really* prefer. These seemingly contradictory results are called *voting paradoxes*, and are the subject of much interesting mathematical study.

This paper adopts an algebraic framework in order to understand the voting paradoxes caused by dropping candidates in an election.

1.2 History

The concept of voting has existed at least since the creation of democracy by the Greek city-state of Athens in the 6th century BC. In one early instance of voting, Athenian citizens could vote to force a fellow citizen into exile for a period of ten years (**?**). The voting method in use then (and almost exclusively in the early history of democracy) is now referred to as *plurality voting*: each Athenian citizen could vote for one person during the election period. Then, the votes were tallied, and the citizen with the most votes was forced to leave Athens.

The voting method described above is now understood to be a specific kind of *positional voting method*. In a positional voting method, each voter ranks all of the candidates from favorite to least favorite, creating a so-called "full ranking" of the candidates. Then, each candidate in the election receives a number of "points" based on the voters' rankings. The candidate with the most points tallied wins the election.

We can describe each positional voting method with a so-called *weighting vector*, which determines how many points to give to each candidate based on his or her ranked position (**?**). To tally a set of voting data using the weighting vector $[a_1, a_2, a_3, \ldots, a_n]$, consider each voter's ranking of the candidates and award the first ranked candidate with a_1 points, the second with a_2 points, and so on. Thus each weighting vector gives the rule for tallying the votes and finds a winner. For example, the weighting vector for the plurality voting method described above is

$$\mathbf{w} = [1, 0, 0, \ldots, 0],$$

since a voter's favorite candidate receives one point and all other candidates receive zero points.

Although we do not make any assumptions about weighting vectors in this paper, there are several reasonable assumptions which can be made. Since the jth entry in a weighting vector corresponds to the number of points given to the jth candidate in the ranking, many voting theorists assume that the entries in the vector are weakly decreasing (i.e., $a_1 \geq a_2 \geq \cdots \geq a_n$) so that more popular candidates receive more points (and also that $a_1 > a_n$ to avoid a voting method which would always result in a tie between all of the candidates). Also, note that scaling all of the entries in a weighting vector by a constant, or adding a scalar multiple of the vector

$$[1, 1, \ldots, 1]$$

to a given weighting vector does not effect the outcome of an election. Thus many voting theorists assume without loss of generality that $a_1 = 1$ and $a_n = 0$ for a given weighting vector $[a_1, a_2, a_3, \ldots, a_n]$.

The topic of voting theory did not receive much academic interest until the time of the French Revolution. In the late 18th century, Charles de Borda and the Marquis de Condorcet suggested two methods of interpreting voting data. Although both methods were quite reasonable, they were very different (**?**).

Charles de Borda proposed a method now called the *Borda Count*. Borda recommended that each voter rank all n candidates according to preference. Then, for each such ranking, he assigned $n-1$ points to the favorite candidate, $n-2$ points to the second, $n-3$ points to the third, and so on. This method is a positional voting method, so we can associate the vector

$$[n-1, n-2, n-3, \ldots, 0]$$

with the Borda Count. Equivalently, the Borda Count can be described by the weighting vector $[1, \frac{n-2}{n-1}, \frac{n-3}{n-1}, \ldots, 0]$. Note that there are infinitely many weighting vectors which correspond to the Borda Count; we can characterize them all by noting that all such weighting vectors form an arithmetic sequence:

$$a_i - a_{i+1} = a_j - a_{j+1} \neq 0$$

for $\{i, j\} \in \{1, 2, \ldots, n-1\}$.

In contrast to a positional voting method, Condorcet suggested a *pairwise* voting method in which a winner is declared based on considering each pair of candidates in turn. For example, if a candidate is favored relative to each of his opponents in one-on-one comparisons, then he should win the election (this requirement is called the *Condorcet Criterion*). Unfortunately, this condition does not always apply. A set of voting data may rank candidate c_1 over c_2, c_2 over c_3, and c_3 over c_1. In that case, we conclude that there is no Condorcet winner (**?**).

The question of which is the "perfect" voting method could not be answered. In fact, Kenneth Arrow proved in 1951 that there is *no* perfect voting method which satisfies several reasonable requirements (**?**). This theorem silenced the debate over which voting method is "perfect," but it awakened a discussion of comparisons between different positional and pairwise methods, and a search for explanations as to why these voting paradoxes occur.

1.3 A Short Example

In order to illustrate three of the voting methods mentioned above (plurality, pairwise, and the Borda Count) and some of the possible paradoxes involved, let us examine a small example.

Consider an election of candidates c_1, c_2, and c_3, and assume that each voter casts his or her ballot by ranking the candidates from favorite to least favorite. Then the collection of all of the voters' preferences, called a *voting profile*, might be represented using the following table:

Ranking	Number of Votes
$c_1 \succ c_2 \succ c_3$	6
$c_1 \succ c_3 \succ c_2$	0
$c_2 \succ c_1 \succ c_3$	6
$c_2 \succ c_3 \succ c_1$	3
$c_3 \succ c_1 \succ c_2$	9
$c_3 \succ c_2 \succ c_1$	1

.

Here, we write $c_1 \succ c_2$ if candidate c_1 is favored over candidate c_2 in a given vote. For example, the last row in the above table indicates that 1 voter liked candidate c_3 most, followed by candidate c_2, and then candidate c_1. In this election, there are only 25 voters.

Then, using the plurality voting method (defined by the weighting vector $[1, 0, 0]$), we can conclude that the result is

$$c_3 \succ c_2 \succ c_1$$

with a 10:9:6 tally. However, the Borda Count (with weighting vector $[2, 1, 0]$) yields the opposite ranking:

$$c_1 \succ c_2 \succ c_3,$$

with a 27:25:23 tally. Finally, pairwise analysis compares the candidates in pairs only. For example, candidate c_1 was ranked above candidate c_2 by 15 voters, while c_2 was ranked over c_1 by only 10 voters; thus the voters prefer candidate c_1 over candidate c_2. Similar analysis for the other pairs yields the statements

$$c_1 \succ c_2, c_2 \succ c_3 \text{ and } c_3 \succ c_1,$$

showing that the pairwise rankings are not transitive, and there is no Condorcet winner. Thus, applying different voting methods to this voting profile can produce different voting outcomes.

Furthermore, note that "dropping candidates" can also be problematic. For example, recall that the plurality voting method returns

$$c_3 \succ c_2 \succ c_1.$$

However, if candidate c_3 drops out of the election after the votes have been collected, then the resulting voting profile for the remaining candidates would be

Ranking	Number of Votes
$c_1 \succ c_2$	15
$c_2 \succ c_1$	10

.

Considering this profile with the plurality voting method (which has weighting vector $[1, 0]$), we can conclude that

$$c_1 \succ c_2,$$

with a 15:10 tally. Note that the relative ranking between c_1 and c_2 was reversed when c_3 dropped out of the election even though the same voting method was used both times.

Thus we can see that different voting methods achieve different results, and that dropping candidates may completely change the results of an election for the remaining candidates. As the following pages will show, there are several ways to consider these voting paradoxes using mathematics.

Chapter 2

Mathematical Background

2.1 A Vector Space Approach to Voting Theory

Donald Saari has added to the field of voting theory using an approach which provides several deep and surprising insights (see **?** and **?**). As usual, Saari considers an election of n candidates in which each voter chooses a ranking of all n candidates. Since there are $n!$ possible rankings, the voting profile may be expressed as a vector in $\mathbb{Q}^{n!}$, where each basis element corresponds to one of the $n!$ rankings. Saari defines the *profile space* to be the vector space $\mathbb{Q}^{n!}$ which contains all such voting profiles. For example, the voting profile from Section **??** may be expressed using the vector

$$\mathbf{p} = [6, 0, 6, 3, 9, 1]^T,$$

assuming the appropriate basis $(c_1 \succ c_2 \succ c_3, c_1 \succ c_3 \succ c_2, \ldots)$. Although voting profiles with fractional or negative entries might seem unintuitive, we still consider such elements of the profile space since a voting profile's outcome does not change if it is scaled by a constant, or added to the vector $[1, 1, \ldots, 1]^T$.

Note that we can associate each possible ranking with an element of S_n. To do this, consider a ranking as a function which maps a number j to the candidate in the jth position. Thus the basis elements for the profile space are the elements of S_n.

By bringing the subject of voting theory into the context of vector spaces, Saari was able to ask: *Which subspaces of the profile space* $\mathbb{Q}^{n!}$ *cause voting paradoxes?* In other words, Saari decomposed the profile space in order to understand it. He was able to identify several subspaces of the profile space which correspond to different kinds of voting paradoxes.

Saari defined several subspaces of the profile space which consist of all of the elements of the profile space satisfying certain characteristics. Although these special subspaces will not be discussed further here, see **?** and **?** for more information.

- There is a surprisingly large *universal kernel*, which consists of all profiles which do not effect the outcome of an election, regardless of positional or pairwise voting method. For $n \geq 3$, the dimension of the universal kernel $\mathcal{UK}^n$ is $n! - 2^{n-1}(n-2) - 2$. For $n \geq 5$, this is more than half the dimension of the profile space (in fact, the universal kernel accounts for over 99% of the dimension of the profile space for $n \geq 9$).
- The *Basic space* is the $(n-1)$-dimensional subspace in which no election outcomes differ. Specifically, all voting paradoxes correspond to profiles with components orthogonal to the Basic space.
- The *Condorcet space* is responsible for all pairwise voting paradoxes, and is related to *Condorcet triplets*, in which three candidates c_1, c_2, and c_3 are ranked

  $$c_1 \succ c_2, c_2 \succ c_3, \text{ and } c_3 \succ c_1.$$

  In fact, all of the paradoxes involving differences between the Borda Count and pairwise counting are due to the Condorcet Space.

Using these subspaces (and many more), Saari was able to prove deep statements about voting theory. In particular, Saari was able to show that election results using different voting methods may be extremely unrelated.

Theorem 2.1 (?). *For $n \geq 4$ candidates, there exists a profile* $\mathbf{p}$ *such that for any choice of candidate c_j and position (first, second, ..., last), there exists a positional voting method which places c_j in the chosen position when it is applied to the profile p.*

Furthermore, choose any ranking for each of the pairs of candidates. There exists a profile where the above conclusion holds and each pairwise ranking is the selected one.

In other words, there is a special profile $\mathbf{p}$ which yields extremely different election results depending on the voting method in use.

Saari was also able to prove that any imaginable voting paradox caused by considering the effect of dropping candidates (in terms of final rankings) can actually be realized for the plurality and pairwise voting methods.

Theorem 2.2 (**?**). *Assume there are $n \geq 3$ candidates. Select, in* ***any*** *manner, a transitive ranking for each of the $2^n - (n+1)$ subsets of candidates (so the rankings of the different subsets need not be related in any manner). Then there exists a profile so that the sincere plurality (or pairwise) ranking for each subset is as selected.*

Note that Saari considers all subsets of candidates with two or more candidates (hence the figure $2^n - (n+1)$). Surprisingly, Theorem **??** applies to many other positional voting methods.

Theorem 2.3 (**?**). *With the exception of a set α^n of weighting vectors for all of the subsets of candidates, all other choices of weighting vectors for all of the subsets of candidates satisfy the same property as described in Theorem* **??***. Furthermore, α^n can be described as the zeroes of a particular set of polynomials.*

These theorems are extremely important for two reasons. First, they state that voting paradoxes caused by dropping candidates in an election are unavoidable for most choices of weighting vectors; any imaginable such paradox can actually be realized (for most weighting vectors).

However, the theorems are also important because they give hope to the voting theorist; there are a few choices of weighting vectors which somehow decrease the possibilities of voting paradoxes for dropping candidates. What weighting vector choices does α^n include (note here that an element of α^n includes a choice of weighting vector for each of the subsets of candidates)? Equivalently, how do we explicitly construct the set of polynomials mentioned in Theorem **??**? What makes these elements of α^n special? Saari does not explicitly answer these questions, but he does state that α^n contains sets of weighting vectors which include Borda and "Borda-like" weighting vectors (**?**). In this paper, we will be able to make an informed conjecture describing how to determine the set α^n by finding the set of polynomials which define it.

2.2 Algebraic Background

Recent student research has shown that an algebraic approach to voting theory can give alternate proofs to Saari's theorems, and even generalize them by taking advantage of the *module* structure inherent in Saari's vector spaces. The following section is adapted from **?** and **?**.

First, let us recall some important definitions from abstract algebra:

Definition 2.4. Let R be a ring with identity. Then a (left) *R-module* or a (left) *module over R* is an abelian group M together with an action of R on M (i.e., a map $R \times M \to M$) denoted by rm, for all $r \in R$ and for all $m \in M$ which satisfies

- $(r+s)m = rm + sm$
- $(rs)m = r(sm)$
- $r(m+n) = rm + rm$
- $1m = m$

for all $r, s \in R$ and $m, n \in M$.

A nonzero R-module M is called *irreducible* if it has no nontrivial submodules.

Definition 2.5. Let R be a ring, and M, N be R-modules. A map $\varphi : M \to N$ is an *R-module homomorphism* if:

- $\varphi(x+y) = \varphi(x) + \varphi(y)$
- $\varphi(rx) = r\varphi(x)$

for all $x, y \in M$ and $r \in R$.

Note that requesting that voters provide full rankings of the n candidates in an election might not actually be feasible because it allows for so many different possible votes. Thus, one might decide to request a *partial ranking*, where voters can place the candidates into ranked sets. Saari's work only touches on such circumstances, but an algebraic perspective sheds new light on the subject (see **?**, **?**).

In order to study partial rankings, we must first introduce a few helpful combinatorial objects.

- $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a *composition* of a positive integer n if each λ_i is a positive integer n such that

$$\lambda_1 + \lambda_2 + \cdots \lambda_\ell = n.$$

  The integers λ_i are called the *parts* of λ.

- $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a *partition* of a positive integer n (written $\lambda \vdash n$) if λ is a composition of n such that

$$\lambda_i \geq \lambda_{i+1}$$

  for $i = 1, 2, \ldots, \ell - 1$.

- Given a composition λ of n, its *associated partition*, denoted $\overline{\lambda}$, is the partition found by ordering the parts of λ in nonincreasing order.

- Given a partition λ of n, a *Young tableau of shape* λ is a left-justified array t^λ containing each of the numbers $1, 2, \ldots, n$ exactly once such that the ith row in the array contains λ_i numbers. For example, for the partition $\lambda = (4, 2, 1, 1)$ of $n = 8$,

$$\begin{array}{|c|c|c|c|}\hline 1 & 4 & 5 & 8 \\ \hline 3 & 6 \\ \cline{1-2} 2 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array}$$

  is a Young tableau of shape λ.

- A tableau is *standard* if its rows and columns are increasing sequences. For example,

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

  is standard.

- We say that two Young tableaux s and t are *equivalent* (written $s \sim t$) if one can be created from the other by rearranging the elements of each row. For example,

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \sim \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

- A *tabloid of shape* λ is the equivalence class

$$\{t\} = \{t_1 | t_1 \sim t\},$$

  where t is a tableau of shape λ. Given a partition λ of n, we define X^λ to be the set of all Young tabloids of shape λ.

- Given a set S, then

$$\mathbb{Q}S = \{\sum_i a_i s | a_i \in \mathbb{Q}, s \in S\}$$

  is the set of formal sums with coefficients in $\mathbb{Q}$.

- Given a partition λ of n, we define $M^\lambda = \mathbb{Q}X^\lambda$ to be the permutation module corresponding to λ. That is,

$$M^\lambda = \{\sum_i a_i t_i^\lambda | a_i \in \mathbb{Q}, t_i^\lambda \in X^\lambda\},$$

  the set of all linear combinations of tabloids with coefficients in $\mathbb{Q}$. For any partition λ of n, M^λ is a $\mathbb{Q}S_n$-module, where S_n acts on the Young tabloids by permuting the numbers $\{1, 2, \ldots, n\}$ in a tabloid. For example,

$$(1,2,3)\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}.$$

- Note that the definitions for equivalence, tabloids, and permutation modules do not explicitly depend on λ being a partition, so the restriction may be relaxed by some authors to include all compositions.

Note that a tabloid of shape λ corresponds to a partial ranking of candidates $1, 2, 3 \ldots, n$ into sets of size $\lambda_1, \lambda_2, \ldots, \lambda_\ell$. For example, the tabloid of shape $\lambda = (2, 1, 1)$

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1} 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}$$

corresponds to one vote which ranks candidates c_1 and c_4 as tied in first place, followed by candidate c_3, and then finally by candidate c_2. Thus a voting profile corresponds to an element of M^λ. Note that the full ranking case studied by Saari corresponds to the S_n-module

$$M^{(1,1,\ldots,1)} \cong \mathbb{Q}S_n,$$

which is an $n!$-dimensional vector space over $\mathbb{Q}$ as expected. For example, we expressed the voting profile from Section **??** using the vector

$$\mathbf{p} = [6, 0, 6, 3, 9, 1]^T,$$

assuming the appropriate basis $(c_1 \succ c_2 \succ c_3, c_1 \succ c_3 \succ c_2, \ldots)$. Now, we can write the profile as an element of $M^{(1,1,1)}$:

$$6\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} + 6\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 3 \\ \hline\end{array} + 3\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 1 \\ \hline\end{array} + 9\begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline 2 \\ \hline\end{array} + \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}.$$

Now, consider the case where each voter ranks the candidates according to a composition μ. For example, consider the case where each voter is asked to vote for his favorite candidate only; this corresponds to the composition $\lambda = (1, n-1)$ and an example vote may be

$$\begin{array}{|c|ccc}\hline 2 & \multicolumn{3}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \hline \end{array}.$$

Then, since $M^{\mu} \cong M^{\overline{\mu}}$, we can consider each voter's ranking in terms of the partition $\overline{\mu}$. In this example, such a voting profile could be considered as an element of $M^{(n-1,1)}$ even though the shape of the vote is $(1, n-1)$.

Given a voting profile consisting of each voter's partial ranking of the candidates in an election, positional and pairwise analysis can be used to tally partially ranked data. For example, given a voting profile corresponding to the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, we define a weighting vector

$$\mathbf{w} = [a_1, a_2, \ldots, a_\ell]$$

to denote a positional voting method in which the λ_i candidates in the ith set receive a_i points for each voter. In the pairwise case, we need only explain how to assign points for two candidates who are tied; if two candidates c_i and c_j are tied, then we assign t points to the ranking $c_i \succ c_j$ and t points to $c_j \succ c_i$ (here, t is a constant usually chosen to be $1/2$).

In this algebraic setting, the different methods of tallying votes are $\mathbb{Q}S_n$-module homomorphisms (**?**). For example, the pairwise method of tallying votes (given a choice for the constant t) corresponds to a map

$$P_t : M^{\lambda} \to M^{(n-2,1,1)}$$

because it uses the votes of shape λ to consider each pair separately. The positional tally map (for a given weight vector $\mathbf{w}$) corresponds to a map

$$T_{\mathbf{w}} : M^{\lambda} \to M^{(n-1,1)}$$

because it uses the votes of shape λ to assign points to each candidate individually.

Furthermore, these tally maps can be expressed as matrices once bases for M^{λ}, $M^{(n-2,1,1)}$, and $M^{(n-1,1)}$ have been chosen. For example, consider a 3-candidate election where $\lambda = (1, 1, 1)$ (i.e., the voters each provide full rankings of the candidates). Then the map

$$T_{\mathbf{w}} : M^{(1,1,1)} \to M^{(2,1)}$$

can be written in terms of a matrix once the basis

$$\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 3\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\right) \tag{2.1}$$

has been chosen for $M^{(1,1,1)}$ and the basis

$$\left(\begin{array}{|c|c|}\hline 2 & 3\\\hline 1 & \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3\\\hline 2 & \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2\\\hline 3 & \\\cline{1-1}\end{array},\right) \tag{2.2}$$

has been chosen for $M^{(2,1)}$ For example, if $\mathbf{w} = [2, 1, 0]$, then we can express the tally map using the matrix

$$T_{\mathbf{w}} = \begin{bmatrix} 2 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{bmatrix}. \tag{2.3}$$

Here, the rows of the matrix correspond to the candidates c_1, c_2, and c_3 (in that order), and the columns correspond to the 6 possible rankings of the candidates (in the order given by Equation **??** above). This expression of the tally map as a matrix operator can facilitate many computations (at least for small n). For example, the voting profile from the example in Section **??** could be tallied using the Borda Count as follows:

$$T_{\mathbf{w}}(\mathbf{p}) = \begin{bmatrix} 2 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 9 \\ 6 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 27 \\ 25 \\ 23 \end{bmatrix}, \tag{2.4}$$

which agrees with Section **??**. Thus, in practice, we often choose a basis for the candidate rankings and consider the problem in terms of matrices and vectors.

2.3 Algebraic Results

This algebraic framework for studying voting theory has proven successful for both fully and partially ranked data. The following results consider the Borda count as well as generalize it, and assert the existence of voting paradoxes due to differences between voting methods. These theorems were proven using algebraic techniques (although Theorem **??** was not originally proven algebraically).

Theorem 2.6 (**?**). *The Borda Count is the unique positional weighting scheme which minimizes conflict with the pairwise map for fully ranked data.*

Theorem 2.7 (**?**). *The unique analog to the Borda Count for data of shape* $\lambda = (n-k, 1, 1, \ldots, 1)$ *is a positional map which gives the ith place candidate* $1 - \frac{2(i-1)}{n+k-1}$ *points for* $1 \leq i \leq k$ *and 0 otherwise.*

Theorem 2.8 (**?**). *Let* $\{\mathbf{w_1}, \mathbf{w_2}, \ldots \mathbf{w_k}\}$ *be a completely different set of weighting vectors (for either a full or partial ranking), and let* $\{\mathbf{r_1}, \mathbf{r_2}, \ldots \mathbf{r_k}\}$ *be arbitrary results vectors in* $\mathbb{Q}^n$*. Then there exists a profile* $\mathbf{p}$ *such that* $T_{\mathbf{w_j}}(\mathbf{p}) \equiv \mathbf{r_j}$ *for* $j = 1, 2, \ldots k$ *(i.e,* $T_{\mathbf{w_j}}(\mathbf{p})$ *and* $\mathbf{r_j}$ *differ by at most a multiple of the all-ones vector* $[1, 1, \ldots, 1]$*).*

Theorem 2.9 (**?**). *For a given ties vector* $\mathbf{t}$ *(which corresponds to ties in a pairwise analysis of partially ranked data), let* $T_{\mathbf{w}}$ *be a positional map not compatible with the partial pairwise map* $P_{\mathbf{t}}$*. For any pairwise results vector* $\mathbf{a} \in img(P_{\mathbf{t}})$ *and any positional results vector* $\mathbf{b} \in \mathbb{Q}^n$*, there exists a partial profile p such that* $P_{\mathbf{t}} = \mathbf{a}$ *and* $T_{\mathbf{w}} \equiv \mathbf{b}$.

Thus, studying voting theory from an algebraic perspective has allowed us to understand full ranking voting theory better, and to generalize it to the partial ranking case.

2.4 Open Questions

Although previous research has yielded powerful statements regarding voting paradoxes, there are few questions which remain unanswered.

As discussed earlier, voting paradoxes may occur when one or more candidates drop out of the election. How can we characterize the behavior of different voting methods when this occurs? Are some voting methods "better" than others? These questions serve as the topic for this paper.

Note that a positional tally map has a nontrivial kernel (for example, consider that the tally map in Equation **??** has a 3-dimensional kernel). By writing the tally map as a matrix and using linear algebra to compute the kernel of the linear transformation, we can easily determine the kernel of a positional tally map. However, the reverse question remains unanswered: given the kernel of a positional tally map, how can we determine which weighting vector characterizes that map?

There are voting methods which cannot be explained as either pairwise or positional voting methods. How could this algebraic approach be applied to these other voting methods?

Chapter 3

An Algebraic Approach to Dropping Candidates

3.1 Dropping Candidate Maps

Let us consider the effect of dropping candidates by merely "erasing" the dropped candidates to create a ranking for the remaining candidates. For example, if candidate c_2 drops out of an election, we have

$$\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 2 \\ \hline 4 \\ \hline \end{array} \mapsto \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \end{array},$$

since a sincere voter would rank candidate c_1 over candidate c_3, and candidate c_3 over candidate c_4 as shown above. This procedure defines a map

$$D_2 : M^{(1,1,1,1)} \to M^{(1,1,1)}.$$

In general, when a set I of m candidates drop out of an election, we can define the map

$$D_I : M^{(1^n)} \to M^{(1^{n-m})}$$

as described above to understand the situation.

Note, however, that there are two important facts to consider regarding such dropping candidate maps. Let D_I be the map as described above which corresponds to the set I of m candidates dropping out of an n-candidate election. First, D_I is not a $\mathbb{Q}S_n$-module homomorphism. However,

$$D_I : M^{(1^n)} \to M^{(1^{n-m})}$$

is clearly an $S_{\{1,2,\ldots,n\}\setminus I}$-module homomorphism. For example, one might rearrange the candidates 1, 3, and 4 before *or* after dropping candidate 2, with no change in the final result. Here, S_A denotes the group of permutations of a set A.

Also, this approach does not generalize easily to partial rankings. For example, consider an election of 5 candidates, where each voter gives a partial ranking according to the partition $(2,2,1)$. Consider the map D_1 in which candidate 1 drops out of the election. Then note that

$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 5 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \mapsto \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 5 \\ \hline \end{array}$$

and

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \mapsto \begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 3 & \multicolumn{1}{c|}{4} \\ \hline 5 & \\ \cline{1-1} \end{array}.$$

While the above statements do represent the desires of the sincere voter, they do not correspond to the same partition. Thus, given partial rankings of shape λ, then the appropriate μ such that the dropping candidate map is of the form

$$D_I : M^\lambda \to M^\mu$$

does not necessarily exist. Thus we will consider the fully ranked case in this report.

3.2 Generalizing Important Theorems

Recall from the previous chapter that Saari has already proven several statements regarding the subject of dropping candidates in an election. Most notably, Saari proved Theorem **??** and Theorem **??**.

Given these two theorems, several important questions come to mind:

- Can these theorems be proven easily using algebraic methods?
- Can these theorems be proven to apply to all possible choices of final tallies for each subset (rather than rankings)?
- What is the algebraic set α^n?
- Can these theorems be generalized to partial rankings?

This chapter discusses the answers to some of these natural questions. First, I conjecture that the theorems can be applied easily to arbitrary choices of *final tallies* for the subsets of candidates:

Conjecture 3.1. *Assume there are $n \geq 3$ candidates. Select, in **any** manner, a final tally for each of the $2^n - (n+1)$ subsets of candidates such that the sum of voters for each subset is the same. Then there must exist a profile so that the sincere plurality (or pairwise) tally for each subset is as selected.*

Furthermore, with the exception of an algebraic set α^n of weighting vectors for all of the subset of candidates, all other choices of weighting vectors for all of the subsets of candidates satisfy the same property as described above.

Note that this conjecture contains an extra restriction which Saari's Theorem did not require. Since the dropping candidate maps described above are defined by mapping each vote for all of the candidates to a new vote for the remaining ones, the sum of voters for all of the subsets must be the same if they are derived from the same voting profile. In other words, the number of voters should remain constant. Without loss of generality, we assume that the sum of voters is 0 (this is possible because one can always add a multiple of the all-ones voting profile $[1, 1, \ldots, 1]^T$). Thus we assume that the final tally for each subset is "sum-zero," i.e., that the sum of the entries in the result vector equals 0.

In the next section, I will give a constructive proof of the first statement of Conjecture **??**.

3.3 Special Case: Plurality

Let us first consider the plurality voting method. We can choose a tally for each of the subsets of candidates, and we must find a voting profile which yields those tallies using the dropping candidate maps described above and the plurality voting method for all of the subsets of candidates. In fact, such a voting profile can easily be calculated. Let us consider the case where $n = 3$.

Example 3.2. *Let there be $n = 3$ candidates (call them c_1, c_2, and c_3). Note that there are 4 subsets of candidates to consider. Assume that the tally for:*

$$\begin{array}{ccc} \{c_1, c_2\} & \textit{is} & [a, -a]^T \\ \{c_1, c_3\} & \textit{is} & [b, -b]^T \\ \{c_2, c_3\} & \textit{is} & [c, -c]^T \\ \{c_1, c_2, c_3\} & \textit{is} & [d, e, -d-e]^T \end{array}$$

where a, b, c, d, e are constants. Then we can calculate a voting profile corresponding to these tallies by noting that:

- *The element*

$$v_1 = \begin{array}{|c|}\hline 3\\\hline 1\\\hline 2\\\hline\end{array} - \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}$$

corresponds to a tally of $[1, -1]^T$ for the subset $\{c_1, c_2\}$, but a tally of all zeroes for all other subsets of candidates.

- *The element*

$$v_2 = \begin{array}{|c|}\hline 2\\\hline 1\\\hline 3\\\hline\end{array} - \begin{array}{|c|}\hline 2\\\hline 3\\\hline 1\\\hline\end{array}$$

corresponds to a tally of $[1, -1]^T$ for the subset $\{c_1, c_3\}$, but a tally of all zeroes for all other subsets of candidates.

- *The element*

$$v_3 = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} - \begin{array}{|c|}\hline 1\\\hline 3\\\hline 2\\\hline\end{array}$$

corresponds to a tally of $[1, -1]^T$ for the subset $\{c_2, c_3\}$, but a tally of all zeroes for all other subsets of candidates.

- *The element*

$$v_4 = \begin{array}{|c|}\hline 1\\\hline 3\\\hline 2\\\hline\end{array} - \begin{array}{|c|}\hline 3\\\hline 1\\\hline 2\\\hline\end{array} - \begin{array}{|c|}\hline 2\\\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline 1\\\hline\end{array}$$

corresponds to a tally of $[1, 0, -1]^T$ for the subset $\{c_1, c_2, c_3\}$, but a tally of all zeroes for all other subsets of candidates.

- *The element*

$$v_5 = -\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline 1\\\hline\end{array} - \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}$$

corresponds to a tally of $[0, 1, -1]^T$ for the subset $\{c_1, c_2, c_3\}$, but a tally of all zeroes for all other subsets of candidates.

Thus, the voting profile

$$av_1 + bv_2 + cv_3 + dv_4 + ev_5$$

yields the desired results for each subset of candidates.

Thus, the above conjecture is true for 3-candidate elections which use the plurality voting method only. Note that the voting profile above was created by considering each subset of candidates of size k and defining $k-1$ special voting profiles (i.e., elements of $M^{(1^n)}$) for each. Each such profile v_i influenced the tally for only one subset of candidates. This method can be generalized to any value of n.

Theorem 3.1. *Assume there are $n \geq 3$ candidates. Select, in **any** manner, a final sum-zero tally for each of the $2^n - (n+1)$ subsets of candidates. Then there must exist a profile so that the sincere plurality tally for each subset is as selected.*

Proof: Let N be the set of all n candidates in an election. For each k-candidate subset S of candidates, it suffices to find $k-1$ voting profiles such that:

- the $k-1$ profiles map to $k-1$ linearly independent tallies for the subset S, and
- the $k-1$ profiles map to the tally of all zeroes for any other subset of candidates.

Thus, we need only find voting profiles which, for any two distinct candidates $c_i, c_j \in S$, allot c_i with 1 vote, c_j with -1 votes, and all other candidates with 0 votes when considering the voting profile for subset S. Furthermore, these profiles must yield the all zero tally for any other subset of candidates.

To define such profiles, we will need to temporarily abuse our tabloid notation to include tabloid whose entries may be sets of candidates. In that case, simply place the candidates in the tabloid in any order (as long as the same order is always used). For example, if $X = \{c_1, c_2\}$, then

$$\begin{array}{|c|}\hline X \\ \hline 3 \\ \hline 4 \\ \hline \end{array} - \begin{array}{|c|}\hline X \\ \hline 4 \\ \hline 3 \\ \hline \end{array}$$

might be written

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} - \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline 3 \\ \hline \end{array}.$$

Now, let

$$R = N - S,$$

and

$$T = S - (X \cup \{c_i, c_j\})$$

given a set $X \subseteq S - \{c_i, c_j\}$ and consider the voting profile

$$v = \sum_{X \subseteq S - \{c_i, c_j\}} (-1)^{|S-X|} \left[\begin{array}{|c|}\hline R \\ \hline X \\ \hline i \\ \hline j \\ \hline T \\ \hline \end{array} - \begin{array}{|c|}\hline R \\ \hline X \\ \hline j \\ \hline i \\ \hline T \\ \hline \end{array} \right].$$

In order to show that the voting profile v satisfies the requirements, note that a term in the sum of v results in a tally of zeroes for a subset of candidates if and only if the first entry in the tabloids (after the appropriate candidates are dropped) is the same.

- Consider the tally of v for the subset S. Note that the set of candidates $R = N - S$ is dropped from each tabloid when the corresponding candidates drop out of the election. Note that for each term in the sum where $X \neq \emptyset$, the tally for all candidates is 0 (for the plurality voting method). Thus the only term which results in a nonzero tally is the term corresponding to the case where X is the empty set,

$$(-1)^{|S|} \left[\begin{array}{|c|}\hline i \\ \hline j \\ \hline T \\ \hline \end{array} - \begin{array}{|c|}\hline j \\ \hline i \\ \hline T \\ \hline \end{array} \right]$$

(where in this case $T = S - \{c_i, c_j\}$). Thus v results in the desired tally for the subset S.

- Consider the tally of v for a subset of candidates S_1, where S_1 contains a candidate which is not an element of S. In each term of the sum, note that at least one candidate $c_k \in R$ will remain in the top portion of the tabloids because $S_1 \cap R \neq \emptyset$. Thus the tally of v for S_1 is

$$[0, 0, \ldots, 0]^T,$$

the tally of zeroes for each candidate in S_1.

- Consider the tally of v for a subset of candidates S_2, where $S_2 \subsetneq S$. That is, there exists an element $c_\ell \in S$ such that $c_\ell \notin S_2$. Then

$$v = \sum_{\substack{X \subseteq S - \{c_i, c_j\} \\ c_\ell \notin X}} (-1)^{|S-X|} \left[\begin{array}{|c|}\hline R \\ \hline X \\ \hline i \\ \hline j \\ \hline T \\ \hline \end{array} - \begin{array}{|c|}\hline R \\ \hline X \\ \hline j \\ \hline i \\ \hline T \\ \hline \end{array} \right] + \sum_{\substack{X \subseteq S - \{c_i, c_j\} \\ c_\ell \in X}} (-1)^{|S-X|} \left[\begin{array}{|c|}\hline R \\ \hline X \\ \hline i \\ \hline j \\ \hline T \\ \hline \end{array} - \begin{array}{|c|}\hline R \\ \hline X \\ \hline j \\ \hline i \\ \hline T \\ \hline \end{array} \right]$$

$$= \sum_{X \subseteq S - \{c_i, c_j, c_\ell\}} (-1)^{|S-X|} \left(\left[\begin{array}{|c|}\hline R \\ \hline X \\ \hline i \\ \hline j \\ \hline T \\ \hline \end{array} - \begin{array}{|c|}\hline R \\ \hline X \\ \hline j \\ \hline i \\ \hline T \\ \hline \end{array} \right] - \left[\begin{array}{|c|}\hline R \\ \hline Y \\ \hline i \\ \hline j \\ \hline T \\ \hline \end{array} - \begin{array}{|c|}\hline R \\ \hline Y \\ \hline j \\ \hline i \\ \hline T \\ \hline \end{array} \right] \right),$$

where $Y = X \cup \{c_\ell\}$. Note that each term in this new sum will map to zero when the appropriate candidates are dropped to consider the subset S_2.

Thus, by calculating v for appropriate values of c_i, c_j, and S, we can easily define a set of voting profiles which satisfy the desired characteristics.

□

This proves Conjecture **??** for the plurality voting method. Furthermore, the proof of the theorem is a constructive one, unlike Saari's existence proofs. Thus, we now have the ability to easily create profiles which exhibit any desired paradox of this form, although the paradoxical profiles we create are not unique. A simple Maple program can easily calculate any voting paradox you can imagine regarding dropping candidates and exclusively using the plurality voting method.

3.4 Finding α^n

It remains to find the choices of weighting vectors for the different subsets of candidates for which Theorem **??** will *not* apply. For example, if the Borda Count is chosen for any subset of 3 or more candidates, the theorem will not be true. The theorem fails because the Borda Count awards each candidate with a point value equal to the number of times he or she "beats" another candidate in a one-on-one comparison. Thus the Borda Count is inextricably related to the tally results for the "pairs" of candidates (i.e., subset of size two). However, these Borda weighting vectors are not the only possible weighting vectors in α^n. The construction of these choices of weighting vectors remains to be found.

Chapter 4

Decompositions of the Profile Space

In order to better understand the voting paradoxes involving the dropping of a candidate in an election, it useful to understand the algebraic structure of the profile space by decomposing it into submodules.

This chapter explores two different decompositions of the profile space: the isotypic decomposition and the inversion decomposion, as well as their respective contributions to voting theory.

4.1 Isotypic Decomposition

Since M^λ is a $\mathbb{Q}S_n$-module, it can be decomposed into its so-called *isotypic subspaces*:

$$M^\lambda \cong a_1N_1 \oplus a_2N_2 \oplus \cdots \oplus a_mN_m,$$

where the N_i are m distinct irreducible submodules of M^λ and the a_i are positive integers. Furthermore, this decomposition is unique (up to ordering).

Fortunately, the decomposition of M^λ into its isotypic subspaces can be understood easily because the representation theory of $\mathbb{Q}S_n$-modules is well-known in terms of algebraic objects called *Specht modules* and associated numbers called *Kostka numbers*. Although the constructions of the Specht modules S^λ and the Kostka numbers $K_{\lambda\mu}$ are not included here, they are well-understood objects which can be computed fairly easily. Each partition $\lambda \vdash n$ corresponds to a Specht module S^λ. Furthermore, every irreducible $\mathbb{Q}S_n$-module is isomorphic to a Specht module S^λ for some $\lambda \vdash n$.

Thus, we can generate the isotypic decomposition (up to isomorphism) of M^{μ} using the equation

$$M^{\mu} \cong \bigoplus_{\lambda} K_{\lambda\mu} S^{\lambda},$$

where the Kostka number $K_{\lambda\mu}$ is the multiplicity of S^{λ}.

In the case where $\mu = (1, 1, \ldots, 1)$, $K_{\lambda\mu}$ is quite easy to compute. Here,

$$K_{\lambda\mu} = f^{\lambda}$$

and

$$\dim S^{\lambda} = f^{\lambda},$$

where f^{λ} is the number of standard tableaux of shape λ. For example,

$$M^{(1,1,1)} \cong S^{(3)} \oplus 2S^{(2,1)} \oplus S^{(1,1,1)}.$$

For more information, see (**?**).

This isotypic decomposition of the profile space (and, similarly, the space of all result vectors) has proven extremely useful to better understand the tally maps from an algebraic point of view. We can understand these maps by decomposing the domain and range of these tally maps according to their isotypic subspaces. For example, considering the kernels of a positional tally map

$$T_{\mathbf{w}} : M^{\lambda} \to M^{(n-1,1)}$$

and the pairwise tally map

$$P : M^{\lambda} \to M^{(n-2,1,1)}$$

allows us to compare the two maps and understand which subspaces of M^{λ} cause differences in voting outcomes. The following theorem helps explain how:

Theorem 4.1 (Schur's Lemma)**.** *Let M and N be irreducible R-modules, and let $\varphi : M \to N$ be a homomorphism. Then φ is either an isomorphism, or the zero map.*

This theorem states that a given tally map, when restricted to an irreducible submodule of the domain M^{λ}, is either the zero map or an isomorphism. Thus, by considering the isotypic decomposition of the permutation modules and using Schur's Lemma, we can deduce statements about the *kernel* of different pairwise and positional maps.

For example, consider the tally map

$$T_{\mathbf{w}} : M^{(1,1,\ldots,1)} \to M^{(n-1,1)}.$$

Now,

$$M^{(1,1,\ldots,1)} \cong S^{(n)} \oplus (n-1)S^{(n-1,1)} \oplus \cdots$$

and

$$M^{(n-1,1)} \cong S^{(n)} \oplus S^{(n-1,1)}.$$

Thus, for a given tally map $T_{\mathbf{w}}$, the *effective space*, defined to be the orthogonal complement to the kernel, is always isomorphic to $S^{(n)}$, $S^{(n-1,1)}$, or $S^{(n)} \oplus S^{(n-1,1)}$.

In other words, any profile in the profile space which is orthogonal to the $S^{(n)}$ and $S^{(n-1,1)}$ isotypic subspaces must be in the kernel of every tally map $T_{\mathbf{w}}$ (i.e., the "$\cdots$" portion of the decomposition of the profile space above is contained in the kernel of every tally map $T_{\mathbf{w}}$).

Thus the isotypic decomposition has proven helpful in past research (i.e., by Daugherty, Eustis, and Minton) to help understand what the kernel and effective space of a tally map actually are. In fact, Schur's Lemma and the isotypic decomposition play a central role in the previous research regarding voting theory from an algebraic perspective (see **?**, **?**).

4.2 Inversion Decomposition

Although the isotypic decomposition is extremely important, an alternate decomposition may also be useful when considering the question of dropping candidates. The inversion decomposition is based on examining different sized subsets of candidates (created by dropping different numbers of candidates). In the following discussion, consider the fully ranked case only. The following definitions are based on **?**.

In order to define the inversion decomposition, we must first define the indicator vectors $x^{(s,r)}$ used to define it. For any subset $s \subseteq \{1, \ldots, n\}$, $r \in S_s$, and $y \in S_n$, let

$$x_y^{(s,r)} = I[D_s(y) = r],$$

where D_s is the "dropping candidate map" described in the beginning of Chapter **??**. That is, to define the indicator vector $x^{(s,r)}$, we set the component of the vector corresponding to an element $y \in S_n$ to be 1 if y ranks the candidates in s according to r, and 0 otherwise (see example below). While the above definition is phrased in terms of the profile space as a vector space, we could alternately define $x^{(s,r)}$ in terms of the group algebra:

$$x^{(s,r)} = \sum_{\{y|D_s(y)=r\}} y.$$

For example, given the basis for S_3

$$\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 3\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \right),$$

letting $s = \{1,2\}$, and $r = \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}$, we have

$$x^{(s,r)} = \begin{bmatrix} 0\\0\\0\\1\\1\\1 \end{bmatrix} = \begin{array}{|c|}\hline 2\\\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline 1\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}$$

since the elements of S_3 which rank the candidates of s according to r are

$$\begin{array}{|c|}\hline 2\\\hline 1\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline 1\\\hline\end{array}, \text{ and } \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}.$$

Then, for $h = 1, \dots, n$, let

$$W_h^* = \text{span}\{x^{(s,r)} | s \subseteq \{1, \dots, n\}, |s| = h, r \in S_s\}.$$

In other words, W_h^* is spanned by all of the indicator vectors corresponding to all of the permutations of the candidates for all of the subsets of candidates of size h. Thus W_h^* is the subspace of the profile space which determines the rankings for the subsets of candidates of size h. Note that the vectors which define W_h^* are not necessarily linearly independent.

Note that W_h^* is indeed a submodule of the profile space since it is closed under addition and acting on an indicator vector $x^{(s,r)}$ with an element $\pi \in S_n$ will only change s or r (or both) as follows: for any $\pi \in S_n$,

$$\pi x^{(s,r)} = \pi \sum_{\{y \in S_n | D_s(y)=r\}} y$$

$$= \sum_{\{y \in S_n | D_s(y) = r\}} \pi y$$
$$= \sum_{\{y \in S_n | D_{\pi(s)}(\pi y) = \pi r\}} \pi y,$$

because $D_s(y) = r$ if and only if $D_{\pi(s)}(\pi y) = \pi r$,

$$= x^{(\pi(s), \pi r)}.$$

Furthermore,

$$W_i \subseteq W_j$$

if $i \leq j$ since any of the vectors spanning W_i can be written as an element of W_j as follows: for $s \subseteq \{1, \ldots, n\}$, $|s| = i$, and $r \in S_s$, choose $s' \subseteq \{1, \ldots, n\}$ such that $|s'| = j$ and $s \subseteq s'$. Then

$$x^{(s,r)} = \sum_{\substack{r' \in S_{s'} \\ D_s(r') = r}} x^{(s',r')}.$$

To see this, note that each permutation y in the sum

$$x^{(s,r)} = \sum_{\{y | D_s(y) = r\}} y$$

satisfies

$$D_{s'}(y) = r'$$

for *some* $r' \in S_{s'}$ (where $D_s(r') = r$). It follows that

$$x^{(s,r)} = \sum_{\{y | D_s(y) = r\}} y$$

$$= \sum_{\substack{r' \in S_{s'} \\ D_s(r') = r}} \left(\sum_{\{y | D_{s'}(y) = r'\}} y \right) = \sum_{\substack{r' \in S_{s'} \\ D_s(r') = r}} x^{(s',r')}.$$

Note that W_1 is spanned by

$$[1, 1, \ldots, 1]^T$$

and W_n is spanned by the standard basis for $M^{(1,1,\ldots,1)}$, so $M^{(1,1,\ldots,1)} = W_n^*$.

Finally, in order to define the inversion decomposition, let

$$W_h = W_h^* \bigcap (W_{h-1}^*)^{\perp}.$$

This creates the *inversion decomposition*

$$M^{(1,1,\ldots,1)} = W_1 \bigoplus W_2 \bigoplus \cdots \bigoplus W_n.$$

Note that this decomposition has been constructed such that W_h is the subspace of the profile space which defines the profiles for the subsets of candidates of size h *but not* size $h-1$.

Chapter 5

The Inversion Decomposition

5.1 Example Calculations

In order to better understand the inversion decomposition, let us calculate the decomposition for a few small subsets of candidates. First, let us consider $n = 3$ candidates (c_1, c_2, and c_3).

To calculate the inversion decomposition for 3 candidates, let us use the familiar basis

$$\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 3\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \right).$$

First recall that $W_1 = W_1^*$ is spanned by the vector

$$[1, 1, 1, 1, 1, 1]^T$$

since each permutation of 3 candidates is the same when dropped to a subset of candidates of size 1.

In order to calculate W_2^*, we must consider each two-element subset of $\{c_1, c_2, c_3\}$. For $\{c_1, c_2\}$, the ranking $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$ yields

$$[1, 1, 1, 0, 0, 0]^T$$

and the ranking $\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}$ yields

$$[0, 0, 0, 1, 1, 1]^T.$$

Similarly, the subset $\{c_1, c_3\}$ gives the profiles

$$[1, 1, 0, 1, 0, 0]^T$$

and

$$[0,0,1,0,1,1]^T.$$

Finally, the subset $\{c_2, c_3\}$ yields the vectors

$$[1,0,0,1,1,0]^T$$

and

$$[0,1,1,0,0,1]^T.$$

Together, these 6 vectors span W_2^*, a 4-dimensional subspace of the profile space, with W_1 as a 1-dimensional subspace of W_2^*. Thus, W_2 is a 3-dimensional subspace of the profile space.

Finally, W_3^* is calculated by considering all of the rankings of the set $\{c_1, c_2, c_3\}$ (since it is the only subset of size 3). Thus W_3^* is spanned by each of the standard basis vectors. In other words, W_3^* is the entire profile space. Thus W_3 is a 2-dimensional subspace of the profile space.

By looking at the algebraic structure of the subspaces, we can deduce the isotypic decomposition of the W_h in terms of Specht modules. Here,

$$W_1 \cong S^{(3)},$$

$$W_2 \cong S^{(2,1)} \oplus S^{(1,1,1)},$$

and

$$W_3 \cong S^{(2,1)}.$$

Now, we can use the inversion decomposition to gather information about different weighting vectors. For example, one can deduce which weighting vectors contain W_i in the kernel of their tally maps, and which weighting vectors have effective spaces which are entirely contained in W_i. Remember, a given weighting vector has an effective space which is isomorphic to either $S^{(n)}$, $S^{(n-1,1)}$, or $S^{(n)} \oplus S^{(n-1,1)}$ and kernel of the tally map consists of the orthogonal complement of this effective space. In this example, we find that a weighting vector $[a, b, c]$ which contains W_1 in the kernel of its tally map must satisfy

$$\begin{bmatrix} a & a & b & b & c & c \\ b & c & c & a & a & b \\ c & b & a & c & b & a \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

using the familiar basis from Equation **??**. Thus we must have

$$a + b + c = 0.$$

Similarly every weighting vector which contains W_2 in the kernel of its tally map must satisfy

$$a - c = 0$$

and every weighting vector which contains W_3 in the kernel of its tally map satisfies

$$a - 2b + c = 0.$$

On the other hand, any weighting vector whose effective space is entirely contained in W_1 must contain W_2 and W_3 in the kernel of its tally map, so it must satisfy

$$a - c = 0$$

and

$$a - 2b + c = 0,$$

so it must be a multiple of

$$[1, 1, 1]^T.$$

This result makes sense because the weighting vector $[1, 1, 1]^T$ merely counts the number of voters in the election; this is exactly the information that one receives by dropping candidates until only one candidate remains. Thus we can easily calculate what equations a weighting vector $[a, b, c]$ must satisfy in order to contain W_k in its kernel, and what the weighting vector must be if its effective space is entirely contained in W_k (up to scalar multiples). This information yields the following table:

W_k	dimension	to have W_k in kernel	effective space $\subseteq W_k$.
W_1	1	$a + b + c = 0$	$[1, 1, 1]$
W_2	3	$a - c = 0$	$[1, 0, -1]$
W_3	2	$a - 2b + c = 0$	$[1, -2, 1]$

Here, the importance of the Borda Count is quite clear, since the vector

$$[1, 0, -1]^T$$

appears in the second row of the above table (indicating that if a weighting vector's effective space is a subset of W_2, it must be a multiple of $[1, 0, -1]$).

In fact, every weighting vector $[a, b, c]$ which corresponds to the Borda Count, i.e.,

$$a - b = b - c \neq 0,$$

can be written as a linear combination of $[1, 1, 1]^T$ and $[1, 0, -1]$ (where the coefficient of $[1, 0, -1]$ is nonzero).

Recall that the Borda Count is the unique positional voting method which agrees most with the pairwise voting method (Theorem **??**). Mathematically, this theorem states that the effective space of the Borda Count tally map is contained in the effective space of the pairwise map. This connection between the Borda Count and the pairwise map helps explain why the Borda Count appeared in the second row; the Borda Count is associated with pairs of candidates, which are determined by W_2^*.

Thinking about this connection algebraically helps express the importance of the Borda Count. The pairwise map has an effective space isomorphic to

$$S^{(1,1,1)} \oplus S^{(2,1)} \oplus S^{(3)},$$

and is equal to W_2^*. Of course, no positional voting method can have the same effective space as the pairwise map because the $S^{(1,1,1)}$ isotypic is in the kernel of every positional voting method. However, the Borda Count's effective space is actually a subspace of the pairwise map's effective space. That is, the Borda Count corresponds to the same $S^{(2,1)}$ submodule of the profile space as the pairwise map does. Furthermore, the Borda Count is unique in this property, so we say that the Borda Count is unique because its effective space "agrees most" with the pairwise map. Thus the Borda Count's weighting vector will always correspond to W_2^*.

Similar computations can easily be made using Maple to calculate the corresponding data for 4 and 5 candidate elections.

For $n = 4$,

W_k	dimension	to have W_k in kernel	effective space $\subseteq W_k$.
W_1	1	$a + b + c + d = 0$	$[1, 1, 1, 1]$
W_2	6	$3a + b - c - 3d = 0$	$[3, 1, -1, -3]$
W_3	8	$a - b - c + d = 0$	$[1, -1, -1, 1]$
W_4	9	$a - 3b + 3c - d = 0$	$[1, -3, 3, -1]$

For $n = 5$,

W_k	dimension	to have W_k in kernel	effective space $\subseteq W_k$.
W_1	1	$a+b+c+d+e=0$	[1,1,1,1,1]
W_2	10	$2a+b-d-2e=0$	[-2,-1,0,1,2]
W_3	20	$2a-b-2c-d+2e=0$	[2,-1,-2,-1,2]
W_4	45	$a-2b+2d-e=0$	[-1,2,0,-2,1]
W_5	44	$a-4b+6c-4d+e=0$	[1,-4,6,-4,1]

In the above tables, the Borda Count appears as expected as the positional voting method whose effective space is entirely contained in W_2^*. That is, the Borda Count is always a linear combination of the first two entries in the last column. For an equivalent characterization of the Borda Count, note that the weighting vector in the ith row of the table is calculated by simultaneously solving all of equations from the other *other* rows of the table (since to be in the effective space of W_i, a weighting vector must be in the kernel of the others). Thus the Borda Count can be found by simultaneously solving the equations from Rows 3 through n. For example, for $n=3$, the Borda Count is characterized by definition by the equation

$$a-b=b-c,$$

or

$$a-2b+c=0,$$

as shown in the table for $n=3$ above.

5.2 Alternate Definition of the Inversion Decomposition

In this section, we will consider an alternate definition of the inversion decomposition.

First note that the calculations in the previous section considered linearly dependent voting profiles. To calculate a basis for W_i^*, we first create a set of linearly dependent voting profiles and then find a basis for the vector space spanned by those vectors. While this is correct, it can be useful to state another definition of the inversion decomposition which avoids the confusion arising from these redundant weighting vectors. For example, the linear dependence of the weighting vectors adds computational difficulty to the calculations, and makes it difficult to predict the dimension of a given W_i.

Previous work by **?** refers to a conjecture that *inversions* are the key to an appropriate alternate definition of the inversion decomposition. Simply stated, an inversion (or derangement) is a permuation with no fixed points (although the definition is slightly different in the case of S_1). Thus we can define the set of inversions on n elements as

$$\mathcal{I}_1 = \{\pi | \pi \in S_1\}$$

and (for $n \neq 1$)

$$\mathcal{I}_n = \{\pi \in S_n | \pi(i) \neq i \text{ for } i = 1, 2, \ldots, n\}.$$

Marden stated a conjecture (originally proposed by P. McCullagh) that for

$$V_i^* = \text{Span}\{x^{(s,r)} | s \subseteq \{1, 2, \ldots, n\}, |s| \leq i, r \in \mathcal{I}_{|s|}\},$$

$V_i^* = W_i^*$.

In order to prove the correctness of the alternate definition, it suffices to show that the elements of the set

$$I_n = \{x^{(s,r)} | s \subseteq \{1, 2, \ldots, n\}, |s| \leq n, r \in \mathcal{I}_{|s|}\}$$

are linearly independent. To see this, assume linear independence and note that V_k^* is a subspace of W_k^* for $k \leq n$, and that the elements spanning V_n^* actually span the entire profile space W_n^* because there are n linearly independent elements. Thus it follows that $V_k^* = W_k^*$.

Example 5.1. *Let $n = 3$. In this case, we have the following basis for the profile space W_3^*:*

$$\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right).$$

Here, the first basis element corresponds to one candidate subsets, the next 3 basis elements correspond to the subsets of candidates of size two, and the last 2 basis elements correspond to the set of all three candidates. Thus W_1^* is spanned by the first basis element, W_2^* by the first four basis elements, and W_3^* by all 6 basis elements. Note all of the elements in the basis are linearly independent.

Although it may seem elementary for this example, let us prove that the above vectors are indeed linearly independent. Let us consider a linear combination

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_{21} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_{31} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + c_{32} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_{312} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_{231} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We wish to show that all of the constants c_i are equal to 0. First note that the first entry (corresponding to the ranking $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$*) helps us conclude that*

$$c_1 = 0.$$

Then, the second entry corresponds to the ranking $\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 2 \\ \hline \end{array}$ *and implies*

$$c_1 + c_{32} = 0 \Rightarrow c_{32} = 0$$

since $c_1 = 0$. The fourth entry corresponds to $\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 3 \\ \hline \end{array}$ *and yields*

$$c_1 + c_{21} = 0 \Rightarrow c_{21} = 0.$$

The sixth entry corresponds to $\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}$ *and says*

$$c_1 + c_{32} + c_{21} + c_{31} = 0 \Rightarrow c_{31} = 0.$$

Next, the fifth entry corresponds to the ranking $\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 1 \\ \hline \end{array}$ *and implies that*

$$c_1 + c_{21} + c_{31} + c_{231} = 0 \Rightarrow c_{231} = 0.$$

Finally, the third entry corresponds to $\begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline 2 \\ \hline \end{array}$ *and yields*

$$c_1 + c_{31} + c_{32} + c_{312} = 0 \Rightarrow c_{312} = 0.$$

Thus we have proven that all of the coefficients equal zero by considering them one by one. Note that at each step (except the first one), we used the fact that we had already proven that some of the other coefficients were zero; by considering the derangements in a particular order, it was easy to show that each one in turn had a coefficient of zero. We conjecture that this method will be effective in the general case.

In the example above, there was a relationship between the coefficient which was proven to be zero at a given step, and the permutation $\pi \in S_n$ (i.e., the entry in the vector) which was considered to prove it was the case. This relationship is given by a function

$$f : I_n \to S_n.$$

To define f, we will define a permutation $f(x^{(s,r)}) \in S_n$ as a function from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$:

$$f(x^{(s,r)})(i) = \begin{cases} i & i \notin s \\ r(i) & i \in s \end{cases}.$$

In other words, $f(x^{(s,r)})$ is a permutation on all n candidates which arranges the candidates in s according to r, and then places all of the other candidates in their "original" positions. For example,

$$f(x^{(\{1,3\},\begin{array}{|c|}\hline 3\\\hline 1\\\hline\end{array})}) = \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}.$$

Because there are $n!$ elements of I_n, and the map is injective, f is a bijection between I_n and S_n.

We conjecture that there is an appropriate ordering of the elements of I_n such that considering each element $x^{(s,r)} \in I_n$ in that order allows us to show that the coefficient of $x^{(s,r)}$ is 0 by considering the entry corresponding to $f(x^{(s,r)})$. Such an ordering is not necessarily unique.

Conjecture 5.2. *(**?**) The elements of the set*

$$I_n = \{x^{(s,r)} | s \subseteq \{1, 2, \ldots, n\}, |s| \leq n, r \in \mathcal{I}_{|s|}\}$$

are linearly independent.

Note that if proven, this conjecture would yield a basis (as given above) for the submodules W_i^*, and a formula for the dimension of the W_i.

5.3 Algebraic Structure of the Inversion Decomposition

The tables from Section **??** assign exactly one weighting vector to each W_i (as the only weighting vector, up to scalar multiples, whose effective space is contained in W_i). Here, we conjecture that this is always the case, and explore some of the consequences if the conjecture is true.

Note that the profile space contains only one $S^{(n)}$ submodule (which is W_1), and $n-1$ orthogonal copies of $S^{(n-1,1)}$. Thus we conjecture that each W_i for $i > 1$ contains its own copy of $S^{(n-1,1)}$ (which then corresponds to a unique weighting vector by **?**).

Conjecture 5.3. *Let $n \geq i > 1$. Then W_i contains an irreducible submodule isomorphic to $S^{(n-1,1)}$.*

If the above conjecture can be proven, then each W_i must correspond to exactly one weighting vector w_i such that the effective space of w_i, $\ker(T_{w_i})^{\perp}$, is contained in W_i. This would provide a set of n weighting vectors which must be linearly independent by construction. Thus, Conjecture **??** implies a decomposition of the space of all weighting vectors according to the inversion decomposition.

5.4 Further Conjectures Regarding the Inversion Decomposition

The inversion decomposition still remains to be understood. Although it has proven difficult to create proofs regarding the properties of the inversion decomposition, there are many conjectures which can be made regarding the inversion decomposition.

- The connection between the inversion decomposition and the set α^n still remains to be investigated. Remember, elements of α^n are choices of weighting vectors for all of the subsets of the candidates (with at least 2 candidates).

  Conjecture 5.4. *A choice of weighting vectors for all of the subsets of candidates is an element of α^n if and only if at least one of the weighting vectors has length i and contains the submodule W_i in the kernel of its tally map.*

That is, we conjecture that we can easily determine if a choice of weighting vectors for each subset is an element of α^n by checking each weighting vector of length i and determining if it satisfies the equation in the last row of the table (from Section **??**) for $n = i$.

For example, if $n = 3$, a choice of weighting vectors for each subset of candidates (say, $\mathbf{w}_{\{1,2\}}, \mathbf{w}_{\{1,3\}}, \mathbf{w}_{\{2,3\}}$, and $\mathbf{w}_{\{1,2,3\}}$) is an element of α^n if and only of

$$\mathbf{w}_{\{1,2,3\}} = [a, b, c]$$

satisifies

$$a - 2b + c = 0,$$

i.e., it is a Borda vector, or a multiple of the all-ones vector $[1, 1, 1]$. This result agrees with **?**.

- Given the decomposition of the profile space for n candidates, the weighting vector corresponding to W_n bears a resemblance to the $(n-1)$th row of Pascal's triangle. We conjecture that the weighting vector is the $(n-1)$th row of Pascal's triangle with alternating sign.

  For example, starting with $n = 0$, the first few rows of Pascal's triangle are

$$\begin{array}{llllll} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ 1 & 5 & 10 & 10 & 5 & 1 \end{array}.$$

  Now, for $n = 3$, every weighting vector whose effective space is contained in W_3 is a multiple of

$$[1, -2, 1].$$

  This weighting vector is merely the second row of Pascals Triangle,

$$1 \quad 2 \quad 1 \;,$$

  with alternating signs added.

  Similarly, for $n = 4$, every weighting vector whose effective space is contained in W_n is a multiple of

$$[1, -3, 3, -1].$$

This weighting vector is merely the third row of Pascals Triangle,

$$1 \quad 3 \quad 3 \quad 1 \ ,$$

with alternating signs added.

Note that if this conjecture were proven as well as the first conjecture (regarding a characterization of α^n based on the inversion decomposition), then we would be able to characterize all elements of α^n with ease.

- One class of weighting vectors is the so-called "reversible" weighting vectors. These weighting vectors have the property that if each voter reverses his/her ranking of the candidates, then the resulting tally (for that weighting vector) is the reverse of the original tally. We conjecture that the weighting vectors corresponding to W_i where i is even are always reversible weighting vectors, and that these form a basis for all reversible weighting vectors.

Chapter 6

Future Work

There are still many questions regarding the topic of dropping candidates in elections, so future work in this area is needed to address several issues.

Most importantly, future work should include clean proofs of Conjectures **??** and **??**. These two conjectures are important because they would help us better understand the structure of the inversion decomposition. In the case of Conjecture **??**, a proof will help us determine the dimension of each subspace W_i, as well as a cleaner way to calculate the inversion decomposition. A proof of Conjecture **??** would shed light on the algebraic structure of the submodules and have consequences in its application to voting theory.

Further work may also include a careful consideration of the three conjectures outlined in Section **??**: the first two of which are especially influential in finding a clean way to calculate the elements of α^n.

There is also a need to complete the generalization of Theorems **??** and **??**. Can these two theorems apply to all final *tallies* of the subsets (instead of final rankings)?

This paper does not consider the case where votes are given in partial rankings (i.e., voters may rank candidates as tied). Previous work (i.e., by **?**) has successfully generalized important results to the partial ranking case. However, there is still a need to consider the problem of dropping candidates in elections which use partial rankings.

Work by Saari considers an important subspace of the profile space called the *Condorcet space* (see Section **??**, **?**, and **?**) and suggests that it is instrumental in understanding the problem of dropping candidates for the Borda Count. One might like to clarify this connection, perhaps using an algebraic approach, and try to make equivalent statements for some other positional voting methods like the plurality voting method (or prove that equivalent statements can not exist).