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The Case of the Missing Speedometer: The First Day of Calculus

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Synopsis

This article describes the way I've been teaching the first day of Calc I, my single-variable Calculus class. By the end of the hour students have (A) dictated difference quotients for me to write on the board, (B) dictated one example of the limit of difference-quotients definition of derivative of a function at a point, and (C) calculated a few derivatives. The more rigorous definitions of *function*, of *operations* on functions, and of *limits* can wait until later. This approach has been very successful, and students have said they "get it this time around."

Once upon a time I followed the departmental Calculus I syllabus more closely than I do now. And so, in teaching that course, I began by defining function, along with things that get done with functions (operations). After a week or two, I defined and talked about limits of functions at points, gave the usual examples involving quotients of polynomials which have a root in common, and where x (or t) goes to that common root. After another week or two, I finally got to derivatives of functions at points. Derivatives were presented as special kinds of limits (not all of which look like the limits we'd just gone over).

My classes went well. I had not yet used "the missing speedometer approach" to introduce the students to calculus, but throughout the term I'd use other tricks of my trade—my "big plus" drawn on the board before drawing the rest of the Product Rule, my "big minus" for the Quotient Rule, speaking in shorthand—"diff" for differentiate, "deriv" for derivative —and I shared my calculus limericks. For some students, I stayed after class to explain things, as long as they needed me to. My evaluations were great.

"Good quality!" "I hope Temple keeps Dr. Cohen happy." "Most teachers teach things once and that's it. Dr. Cohen goes over things as many times as we need her to [and still gets through the syllabus]." These opinions were unanimous.

However, over several years, some students would, if I asked, confide things like "I still don't get what Calculus IS" and "I know what to DO, and I got an A, but I don't really KNOW Calculus." That's when I decided that I wanted students to know, asap, what Calculus IS. In particular, I wanted them to know what "deriv" and "diff" mean. And that's when I thought, rather suddenly, of a way to do this on the very first day. Over the next few years I zeroed in on what I now think is the very best way, or at least the very best way I know.

Here's what I do—again, on the very first day.[1]

Warming Up

I introduce myself, go over the syllabus, and tell and show the students that I'm friendly and flexible. Then we spend five or ten minutes playing around with the following few questions:

- *When you ride in a car, does the speedometer always read the same?*
- *How long does it usually stay put?*
- *Could it possibly read* 60 *mph at 2:00 PM and* 80 *a minute later?*
- *What about half a minute later?*
- *What about half a second later?*

Moral: There's no end to how frenetic a speedometer can, theoretically, be. Speed can be very inconstant. 60 mph at 2:00 PM says nothing about how many mph at any other time! No velocity is guaranteed to last any amount of time at all.

[1] In the following I use italics to distinguish my teacher voice with students from my collegial voice with readers of this essay.

And so I try to convey to students—or sometimes they convey to me—the feel of "instantaneous velocity" or "instantaneous change." They've possibly heard that that's what a very large part of calculus is about. If so, I gently remind them, and if not, I gently inform them—and not for the last time.

I then zoom in with one more warm-up question:

- *What does* 60 *mph mean, anyway?*

Here are two possible answers, supplied by either me or a student or two:

> *If you keep going at the same speed for an hour, you'll have traveled* 60 *miles by the end of that hour.*

AND

> *If you average that rate over the hour, you'll have traveled* 60 *miles by the end of that hour.*

I then supply the "calculus meaning," with the understanding that this is just a preview, they might not completely understand it; on the other hand, they might get some idea:

> 60 *is the limit, over shorter and shorter periods of time, all starting at, say, 2:00 PM, of the distance traveled divided by the time in which the traveling occurs.*

After the Warm-Up

We then jump right in.

There! That warm-up was fun! Or relaxing! Or a chance to pay not-so-close attention. Now, though, it's time to face the music and investigate The Case of the Missing Speedometer. Here's a typical calculus-type hypothetical scenario: What if your car had no speedometer, but did have an odometer?[2]

[2] An odometer is a device for measuring distance already traveled—if any student knows this, she gets to give a short lecture on it!

And what if you were asked for the car's speed at, say, 2:00 PM? Could you find it, using only the odometer and your watch (or cell phone)? Remember, no speedometer, so you can't just read off the speed. (Hint: Recall the non-calculus formula, $r = d/t$, for the average speed, where $d =$ the distance traveled and $t =$ the length of time of the traveling.)

Is there any way we can at least approximate? That is, give what's known in math as a "first approximation" (before we get a better approximation)?

Usually one or two students offer something like:

$$\text{Speed at 2:00 PM equals approximately } \frac{\text{distance traveled between 2:00 and 2:05}}{5}.$$

This actually is the approximate speed in miles per *minute*, but the ideas are the same as if we were working with miles per hour.

But gee, that took a long time to write down! And we can't see it all that easily with a single glance. So this is a good place to begin using a little mathematical notation. "$\sim$" will be short for "equals approximately", and "$D(t)$" will be short for the distance traveled by t o'clock—in our case, for now, $t =$ 2:05. Thus in our shorthand notation we have:

$$\textit{Speed at 2:00} \sim \frac{D(\textit{2:05}) - D(\textit{2:00})}{5}.$$

That looks better! And it gives our "first approximation" to the speed at 2:00 PM. Notice, we didn't use any speedometer. Instead we (hypothetically) used our odometer (for the numerator) and our watch (for the denominator).

Now, can we get a better approximation?

The same two students (and maybe others) often can. How? By taking a shorter time interval than 5 minutes—that is, by reading our odometer at a time closer to 2:00—perhaps 2:02. (The approximation is likely to be better because there is less time for the speed to vary in.) In our shorthand mathematical notation our "second approximation" to the speed at 2:00 might be written:

$$\text{Speed at 2:00} \sim \frac{D(2{:}02) - D(2{:}00)}{2}.$$

2:05 has been replaced by 2:02, and 5 has been replaced by 2.

Can we now get a third approximation? Yes we can! For example:

$$\text{Speed at 2:00} \sim \frac{D(2{:}01) - D(2{:}00)}{1}.$$

Furthermore, we can risk getting a little bored (but acquire more and more converts) and do yet a "fourth approximation." We'd need to use a fraction of a minute, and the denominator would be $l/2$.

Does anybody see just what we've been doing? Yes! Taking shorter and shorter time intervals starting at 2:00 PM, and getting closer and closer to the "true" speed at 2:00 PM, what a speedometer would (theoretically) say.

Then it's time to get less boring, time to make the giant leap to calculus—and the teacher's turn to dictate difference quotients and then some.

We let Δt stand for the length of the time interval—how much time has elapsed since 2:00. This is one of the notations decided upon by mathematicians. Δt is read "delta t", for the Greek letter Δ (delta), and delta anything indicates the change in that anything. Our approximation (corresponding to whatever Δt we choose) would be, using the same idea as before:

$$\textit{Speed at 2:00} \sim \frac{D(\textit{2:00} + \Delta t) - D(\textit{2:00})}{\Delta t}.$$

As we have seen, the smaller the time interval Δt, the closer our approximation is likely to be.

How close can we get? What's the bottom line here? In calculus, the way to phrase that question is: What's the limit? So let's just write "limit", or rather "lim", our math shorthand for "limit":

$$\textit{Speed at 2:00} = \lim_{\Delta t \to 0} \frac{D(\textit{2:00} + \Delta t) - D(\textit{2:00})}{\Delta t}.$$

$\Delta t \to 0$ means that Δt gets closer and closer to 0—that is, Δt gets smaller and smaller. And yes, that's an equal sign, not an approximation sign, because now we're talking about the limit of the "average speeds".

There is, yes, a more mathematically rigorous way of defining "lim", along with that little arrow, but let's be intuitive for now. (If you take Real Variables, you'll get less intuitive and more mathematically rigorous.) We know what we mean!

So, that "lim expression" is exactly what the speed AT 2:00 is. It's what, theoretically, the speedometer would say, or rather it's what the speedometer means. (If we were being practical, we wouldn't worry about the lim. We'd just get reasonably close, perhaps by taking $t = 1$ (1 minute).

Students do not usually ask the following question: Why can't we just take $t = 0$, that is, NO time elapsing between 2:00 PM and 2:00 PM? Why do we have to worry about the limit?

But I bring it up. And I provide, or students provide, the answer:

It wouldn't work. It wouldn't get us anywhere. Here's what would happen:

$$\textit{Speed at 2:00} = \frac{D(\textit{2:00}) - D(\textit{2:00})}{0} = \frac{0}{0} = ????????????????????$$

$\frac{0}{0}$ *is, as it's often put and as many of you know, "undefined." It could be any number, since* 0 *times any number is* 0*. Thus taking* $\Delta t = 0$ *gives no information.*

Getting a Little More Rigorous

Now, we come back to reality for just a moment: *The above formula, without knowing anything else (namely, calculus), wouldn't exactly solve our missing speedometer problem (since it involves an infinitude of numbers). But it is extremely profound and important, and useful once we learn how to handle limits, that is, once we learn some calculus. (We'd have to know* $D(t)$ *as a function of time* t*, and we'd have to plug in some numbers.)*

Next comes the important point: This limit formula provides the essence of calculus. It's the definition of what we'll soon call the derivative of the function D, or D(t), at the "time" t = 2:00, and again, it's what the speedometer means by the speed "at 2:00 PM", the "instantaneous velocity" at 2:00 PM.

Now—we make another giant leap—*reflect that other things change besides distance. In other applications of calculus, not only the distance traveled by cars, rockets, and particles, but electrical charge, too, can change, populations of countries and of bacteria cultures can change, and speed itself can change (that's what we mean by acceleration— change in speed). That is, the function* D*—or whatever you choose to call it—doesn't have to be distance.*

And there's more. t*—or whatever you chose to call it—doesn't have to be time. For example, in business it could be the quantity produced. (Profit and cost are both functions of the quantity produced, called "level of production".) There are other examples of what* D *and* t *could be, other applications of calculus besides those involving distance and speed.*

Perhaps you now have some idea of what "function of a variable" means (or perhaps you already knew). But here's a quick summary, an actual (though not mathematically rigorous) definition of "function": A function f is a rule which assigns, to every number x (or sometimes only certain numbers) another number $f(x)$. $f(x)$ is called the value of f at x. We'll hear more about functions and limits in a couple of weeks. We don't need to hear more yet!

Students have seen functions before, they will give examples such as $3x$, x , $x+1 \dots.$ It is fine at this stage not to worry about their imprecise ways of describing them. In fact, you can even accept that a short, rather visual, *almost*-correct description of "function" is "anything with x in it" (or t, or whatever we call the "independent variable"). The exception in this almost-correct description is constant functions — such as 2. The function 2 assigns, to every number, the value 2. (Students might remember that the graph of this function 2 is a horizontal straight line 2 units above the x-axis.)

Next we move on to the derivative.

Let us now do the main thing of calculus, which is define what "derivative" means. It will look a lot like the limit formula from "the case of the missing speedometer;" in fact, it's a generalization of that formula. t is replaced by x, Δt by Δx, D by f, and 2:00 by x_0:

Definition. *Given a function f and a point (number) x_0, the derivative of f at x_0 is defined to be:*

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

This "deriv" is denoted, for short, by $f'(x_0)$ and read "f-prime of x-naught."

I make a comment about notation next.

Sometimes, Δx is replaced by h. h takes less time to write when we actually compute derivs via this limit formula (and we'll be doing that very soon). So the deriv of f at x_0 could also, equivalently, be defined as:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

I recapitulate and clarify: *x is the abstraction of 2:00 and $f(x)$ is the abstraction of the distance function $D(t)$. Also, $f'(x_0)$ is always a number, and can always be found when f and x_0 are given.*

I also point out: *We can use $x - x_0$ instead of h, we can talk about Δf, or Δy instead of $f(x) - f(x_0)$, there are actually about six or seven different shorthand notations for the deriv of a given function at a given point—and why do we need to keep writing x_0? We could just write x, and then we can see that the deriv is itself a function.*

Actually Calculating Derivatives

We usually then calculate, together, the derivs of the functions $2x$ and of x^2. And if it's an hour and twenty-minute class, we have time to go on to higher powers and speculate on what the pattern might be. At least one student usually sees this pattern, and can express it via the formula which is The Power Rule. Sometimes there's even time to go over a short proof of this rule. At any rate (so to speak...), by the end of the first day I can tell students, excitedly, "You're now doing Calculus. And next class we'll do more calculus." And we do!

Departing Thoughts on The Case of the Missing Speedometer

This "case of the missing speedometer" is, in and of itself, not new. However, the time and energy that I give to it seems to be, along with the generalizing to all functions of all variables, not only distance and time, as well as using it, not only as an *introduction* to the course, but as an actual beginning.

This approach seems to be quite successful. Students learn and feel comfortable with calculus; their body language reveals happy campers. And many (most) get A's and B's. One course I taught was a section of what my department called "trailer calculus," for students who'd previously failed two or three previous calculus courses. At some point, during the second or third week—we had already studied the Sum, Product, and Quotient Rules, at the very least—I asked the class what the other calculus section was doing. I knew, of course, that they were probably learning about functions and limits—as per "calculus reform"—and I wondered what my students had heard from their friends about this other section.

One student answered, "They say they don't know WHAT they're doing." And my students did know what WE were doing. In fact, I then asked that class whether they were understanding what we were doing and whether they liked doing things this way, and they all nodded. They had also showed by their homework (and perhaps our first test or quiz) that their nods were sincere. One student even joked, "Are you sure this is a Calculus class? It seems more like stand-up comedy to me!"

Also, it got around the department that "you seem to have had a lot of success teaching Calculus" and one of the tenured professors and I held several meetings in which I explained to him the how and why of what I was doing. (This same professor later hired me when he transferred to another school.) True, I was then assigned more calc classes than before, whereas my ideal schedule would have been one Calculus I, Abstract Algebra, and the advanced engineering mathematics sequence. But the students benefited, and my ego did not hurt!

I told the department assistant head what I was doing and she heartily approved. In particular, with the trailer calc class, "obviously what was done before didn't work," she remarked, "so it's a good idea to try something different."

I believe there are at least two reasons why this approach to introducing calculus to newcomers works so well:

1. Functions and limits mean little to students who have not experienced derivatives.

Of course, one could argue that the rigorous definition of the derivative involves both functions and limits, and that with my approach there seems to be a vicious cycle. However, what my classes do on the first day of Calculus I is a *non*-rigorous definition of the derivative—namely, "the case of the missing speedometer." And this non-rigorous definition has been enough.

Actually, nowadays, the mathematically rigorous definitions of both *function* and *limit* (in particular the so-called epsilon-delta definition) often will not appear until more advanced courses. The rigorous definition of derivative is indeed more complicated than that presented in any Calculus I course. Also, still speaking of mathematical rigor, it is not true that, the smaller h becomes, the closer the difference-quotient gets to the actual value of the derivative at x. That happens only *eventually*—once h is small enough. Moreover, take a look at the function below:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

The various difference quotients of this function, at points $\frac{\pm 1}{k\pi}$ tending towards 0, are actually equal to—not only approximately equal to—the derivative of the function at 0. Thus these difference quotients oscillate near 0, rather than tend monotonically to the derivative at 0.

In his book, *What Is Mathematics, Really?*[3] (whose theme seems to be humanism as a philosophy of mathematics), Reuben Hersh writes about how, in a given mathematical theory, there is a difference between the logical sequence of definitions, lemmas, and theorems, and the sequence in which these ideas occur to the person researching the theory, as well as the sequence in which the ideas can be taught or written up. Thus while one must first know what function and limit mean in order to rigorously define the derivative of a function, that is not necessarily how the definition of derivative has to be taught and motivated.

2. Students who have come into my course with calculus anxiety do not develop additional calculus anxiety.

This, I believe, is because they do not wade through days or weeks of functions and limits in prolonged anticipation of the dreaded derivatives (and not quite knowing why they're doing functions and limits). Instead, on the very first day, they actively confront and compute some derivatives—or perhaps watch and absorb as the "class participators" do this computing. Then later, when functions and limits are treated more fully, they're able to understand why, and to feel less alienated.

I recently asked Freda Robbins, a friend and a mathematics professor at New Jersey City University, how she approaches the first day of Calculus I. It turns out that she, too, introduces derivatives very early on but via tangents to curves rather than via cars without speedometers. One reason I do not do things this way is that, in the places where I've taught, students are not necessarily comfortable with analytic geometry, in particular tangents. In general, I never assume that non-math-majors are comfortable with the math they learned in high school or in Pre-Calc.

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[3]Reuben Hersh, *What is mathematics, really?*, Oxford University Press, New York NY, 1997.

"This time I got it," said one trailer calc student. "This is my favorite class this semester," said another. And yes, both received A's.

Marion Deutsche Cohen is the author of *Crossing the Equal Sign*, a poetry collection about the experience of math, and of the recently released memoir, *Still the End: Memoir of a Nursing Home Wife*, the sequel to *Dirty Details: The Days and Nights of a Well Spouse*. Her books total 24, including the recently released poetry collections, *Sizes Only Slightly Distinct*[4] and *Lights I Have Loved*, both of which hinge on math. She teaches math and writing at Arcadia University in Glenside, PA, where she has developed the course *Truth and Beauty: Mathematics in Literature*. Her website is http://marioncohen.net; check out Cohen's "Permission to Add" for her calculus, and other math limericks, and also her "Limericks about Women Mathematicians" both available through links at http://www.marioncohen.net/downloads.php, accessed on January 16, 2015.

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[4] Editor's Note: The poem of this title will appear in a future issue of the *Journal of Humanistic Mathematics*.