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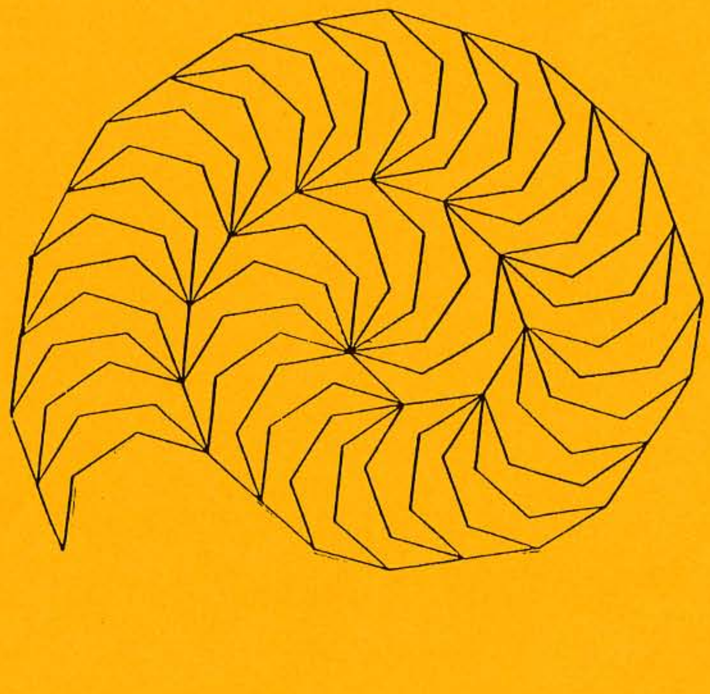

Invitation to Authors

Essays, book reviews, syllabi, and letters are welcomed. Your essay should have a title, your name and address, and a brief summary of content. Your telephone number (not for publication) would be helpful. If possible, avoid footnotes; put references and bibliography at the end of the text, using a consistent style.

Two copies of the paper, double-spaced and preferably laser-printed, should be sent to:

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Essays and other communication may also be transmitted by electronic mail to the editor at AWHITE@HMC.EDU, or faxed to (909) 621-8366. The editor may be contacted at (909) 621-8867 if you have further questions.

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Note to Librarians

Humanistic Mathematics Network Journal issues #8, #9, #10, #11, and #12, ISSN#1065-8297 are the successors to the *Humanistic Mathematics Network Newsletter*, ISSN#1047-627X.

Cover

This is an illustration taken from James Hall's article "Tilings in Art and Science", in which the spiral, formed from enneagons by Heinz Vodeberg, serves as an artistic example of a tiling.

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EXXON EDUCATIONAL FOUNDATION.

Humanistic Mathematics Network Journal #12
October 1995

From the Editor

The Exxon Education Foundation is celebrating its 40th anniversary in 1995. The Mathematics Education Program of the Foundation, begun in 1987, has supported, encouraged, and made possible so many activities that the landscape of mathematics has been transformed and beautified.

Allyn Jackson was asked by the Foundation to listen to the "voices from the reform movement" and to tell what she had learned. Our journal is pleased to reprint her essay. We read about the K-3 Mathematics Specialist Program, the MSEB and other programs started or helped by the EEF; it is tempting to believe that because of their support, mathematics is flourishing and improving almost everywhere. The spirit of their support has certainly improved mathematics over a broad swath at all levels.

This journal began the same year as the EEF Mathematics Education Program, and both the readers and the contributors have benefited from the program's support.

❖ ❖ ❖

Abe Shenitzer has treated us to a translation of a lecture by Hermann Weyl that appears in *The American Mathematical Monthly* Vol.102, No.5, May 1995. I quote the early part of the lecture because, I believe, it describes a part of the Humanistic Mathematics Network.

Weyl quotes Klein about the importance of intuition. He also describes the true Dirichlet principle: to conquer problems with a minimum of blind concentration and a maximum of insightful thoughts.

Weyl continues: "The great art is in the first, analytic, step of appropriate separation and generalization.... Perhaps the only criterion of the naturalness of a severance and an associated generalization is their fruitfulness."

Humanistic Mathematics is in the spirit of Weyl, Wilder, Whitehead.... We who associate with the spirit and the network have a rich heritage from which to draw encouragement.

Thanks again to Abe Shenitzer and the MAA.

❖ ❖ ❖

Excerpts from "...Two Roads of Mathematical Comprehension" by Hermann Weyl

We are not very pleased when we are forced to accept a mathematical truth by virtue of a complicated chain of formal conclusions and computations, which we traverse blindly, link by link, feeling our way by touch. We want first an overview of the aim and of the road; we want to understand *[article continues on back cover]*

Voices from the Reform Movement

by Allyn Jackson

To listen to the voices from the mathematics education reform movement is to hear from a deeply committed group of people, people who are putting all their ingenuity, energy and insight to work on the tough problem of radically changing how mathematics is taught in this country. Over the past decade, this group of professionals – including classroom teachers, education specialists and mathematicians – has built a common vision of mathematics education around richer content, better pedagogy and deeper student understanding. One important supporter has been the Exxon Education Foundation (EEF), whose Mathematics Program, begun in 1987, has made a crucial contribution to the reform effort by providing focused grants to key individuals and organizations.

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The Exxon Education Foundation is celebrating its 40th anniversary in 1995.

The Foundation has been privileged to assist the work of many educators in those 40 years, including the recipients of almost 400 grants for work devoted to improving mathematics instruction. Those awards comprise our Mathematics Education Program which was established in 1987 because we believed in the importance of mathematics for all students, its necessity for serious work in other disciplines and its critical role in business.

To express our high regard for the mathematicians, mathematics educators, and teachers who are the growing source of energy for the movement toward improving mathematics instruction, we asked Allyn Jackson to listen to Voices from the Reform Movement and tell us what she learned.

These "Voices" are engaged in a vital "conversation" about teaching and learning mathematics. The *Humanistic Mathematics Network Journal* is an important element of this conversation. We are pleased to have been able to assist with the Journal's creation and with its growth, and we are grateful for the efforts of its editor, Professor Alvin White, its many contributing authors, and for its readers who sustain and extend this "conversation."

This article first appeared in the program for the National Council of Teachers of Mathematics' 75th anniversary meeting, held in Boston in April 1995. It has been the Foundation's privilege to assist with each of the projects you will find in Ms. Jackson's story. We hope you will find your "voice" here and continue your contributions to these conversations.

Edward F. Ahnert, President
Exxon Education Foundation
Irving, Texas
August 1995

> *"So many people have little respect for schoolteachers. People ask me what I do, and if I tell them about my business, they're curious and interested. But if I tell them, 'I'm a schoolteacher,' I get bored looks. But I still say I'm a schoolteacher, because I want people to know that teachers have deep, deep passion about their work."*
>
> *Marilyn Burns, Educational Consultant, Marilyn Burns Education Associates*

Voices from the Reform Movement

2

A Vision for Reform

> *"We at the NCTM had a unique opportunity to do things ourselves. When we didn't get funding to do the* Standards, *we said, 'Okay, we're going to invest our own money and do this', which was good because we weren't in anybody's pocket. But the fact that, once we got started, we were able to garner the support of groups like the Exxon Education Foundation really made it a wonderful opportunity to make an impact on the thousands and thousands of teachers out there. I think we were pretty gutsy."*
>
> *Shirley M. Frye, Educational Consultant and President of the NCTM 1988-1990*

The Exxon Education Foundation is "taking a very courageous kind of position," says Kathleen Martin, professor of mathematics education at Texas Christian University. "They have a long-term view which does not demand immediacy. So many people want an immediate bang for their buck, and you can't get that in a complex setting [like education]. The Foundation seems to respect that."

> *"People would say, 'But don't the kids still need to learn their 'math facts' and their multiplication tables?' I'd say, 'Of course!' Obviously, there are some things that you have to know, that are basic tools. It's just that the mathematics that's been taught in the schools – and to a certain extent in the colleges – in this century has been focused so exclusively on technique. I mean, all those integration techniques – who needs that now?"*
>
> *Marcia Sward, Executive Director, Mathematical Association of America*

This country is accustomed to hearing about how lousy American kids are at mathematics. Opinions may differ on the standardized tests that reinforce this message, but few believe that the mathematics education kids get today prepares them well for tomorrow's world. With technology allowing business and industry to bring more quantitative mathematical approaches to bear on what they do, employers are finding that the young people they hire don't have the necessary skills. The mathematical sciences are reaching into every field of human endeavor, and kids are easily locked out of opportunities because of deficient mathematical preparation. Consensus has grown that math class should prepare students to tackle a wide variety of mathematical tasks, not just page after page of arithmetic drill.

In 1989, the National Council of Teachers of Mathematics published *Curriculum and Evaluation Standards for School Mathematics*, which provided an ambitious vision for change. At the time, talk of national standards was seen as a threat to the autonomy of local school districts. In addition to the political risk, NCTM also took a financial risk, sinking over \$1 million of its own money into producing the *Standards*. "If it had been a colossal failure, it would have had serious ramifications for the NCTM as an organization," says Glenda Lappan, a professor of mathematics at Michigan State University, who worked on the *Standards* project.

As it turned out, the *Standards* have been a

tremendous success. Not only have the *Standards* revolutionized thinking about mathematics teaching and learning, they have also paved the way for educational standards in other areas such as science, social studies and the humanities. The *Standards* have gained international attention as well – even in Japan, where mathematics test scores leave the U.S. in the dust, interest in the Standards is high. "When we were initially involved in this, I thought there was a real possibility that this would be a document that would be produced, put on a shelf and occasionally looked at by a school curriculum developer," Lappan remarks. "And it has in fact been a document around which there has been a worldwide conversation."

Two years after the *Curriculum and Evaluation Standards* appeared, the NCTM issued a second set, *Professional Standards for Teaching Mathematics*. A third set, on assessment, appeared in 1995. EEF has supported these efforts by providing funding for dissemination as well as for other projects, such as an examination of the impact of the *Curriculum and Evaluation Standards*. The basic thrust of the *Standards* is to put the student at the center of learning, with the teacher acting more as a coach than as an authority figure who has all the right answers. The idea is to make mathematics a living, breathing subject that students can talk about, play with and use. Don't sentence kids to solitary confinement with a page of arithmetic drill, say the *Standards*; give them problems that mean something to them, get them talking to each other and discovering their own mathematical ideas. The *Standards* recognize that this can only happen if teachers have a solid understanding of mathematics and are treated as professionals who are responsible for creating a rich mathematical environment where learning can take place.

> *"Industry, especially small industries, are just not able to get an adequately trained workforce to do what they need done...What we've seen over the last ten years is a major change in the use of technology in the workplace, especially computers, but not just computers. The forklift is obsolete now. Robots are in."*
>
> *Phillip Griffiths, Director, Institute for Advanced Study, Princeton*

Building a Community

> *"One of the major accomplishments of the mathematics education reform movement has been to create a 'mathematical community' from among the many diverse professional societies in the mathematical sciences and to unify their attention on the importance of educational quality at all levels. Although many fissures remain within this community, individuals at all levels now speak with each other and are beginning to take responsibility for mathematics education as a single seamless system."*
>
> *Lynn Arthur Steen, Executive Director, Mathematical Sciences Education Board*

The people working on mathematics education reform have been remarkably successful at drawing into their fold a variety of groups having an interest in seeing mathematics education improve. The Mathematical Sciences Education Board (MSEB) has been responsible for much of the success of this alliance. Created at the National Research Council in 1985, MSEB serves as a broker for mathematics education reform, bringing together not only teachers and curriculum supervisors, but also people whom one would not ordinarily expect to see discussing mathematics education reform – researchers from national laboratories, CEOs from high tech industries, scientists, mathematicians and representatives from the media. With EEF as one of its major supporters, MSEB has worked in close collaboration with the NCTM to broaden consensus about the *Standards*.

EEF has also supported one of MSEB's most successful outreach programs, the State Coalitions. Begun in 1989 and now operating in most states and the District of Columbia, the Coalitions are like mini-MSEBs operating at the state level. Activities vary from organizing teacher in-service programs to working with politicians to change educational policies at the state level. In 1995, the

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State Coalitions reached a milestone by creating their own national organization, the National Alliance of State Science and Mathematics Coalitions. Some of the Coalitions have established strong links to business and industry, while others have joined forces with groups working in science education reform. Together, the State Coalitions provide the mathematics community with a crucial link to local reform efforts.

> *"We're at the stage where the vision of the* Standards *defines pretty much what we want to accomplish. In some sense, we have a picture of what the ideal looks like. And then the question is implementation. That totally dwarfs the problem of defining the vision. The idea of making that vast a change, at the level of ambition of the* Standards, *all around the country, is almost unthinkable And a lot of people think it's naive and it can't be done in this country."*
>
> *Hyman Bass, Professor of Mathematics, Columbia University, and Chair, MSEB*

Bringing in the voices of mathematicians has been an important development in the reform effort. As the group with the deepest understanding of the subject and its connections to other areas, mathematicians clearly have an important contribution to make to discussions of what it is about mathematics that is most important for students to learn. There have been calls for more emphasis on probability and statistics, on shape and geometry, on combinatorics and discrete mathematics – which is the right path to take? Answers to such questions must grow out of conversations between mathematicians and mathematics teachers.

Each summer, the Park City/IAS Geometry Institute, sponsored by the Institute for Advanced Study, brings together research mathematicians, graduate students, undergraduates and high school mathematics teachers for a three- or four-week session. "You get the best cooperation when these people are talking about common interests, and their common interest is education," says John C. Polking, a professor of mathematics at Rice University and director of the Park City Institute. "The teachers have a lot to offer the researchers because they are the experts on teaching, much more so than the typical researcher. And the researchers can offer a lot about mathematics that may not be known to the high school teachers." Polking says that his contact with teachers at the Park City Institute has inspired changes in his own teaching at Rice.

Bringing Vision to Life

The influence of the *Standards* is everywhere. People interviewing for jobs as mathematics teachers are routinely asked about the *Standards*. Just about every mathematics education proposal to the National Science Foundation mentions the *Standards* somehow. Across the country, groups working on mathematics curriculum frameworks and testing programs at the state level have made serious efforts to align their work with the *Standards*. Most secondary school teachers and many middle school and elementary school teachers are aware of the *Standards*.

But are educational practices really changing? "When you move to how many teachers are systematically trying to analyze their practice and move towards the *Curriculum and Evaluation Standards*, then the numbers go down," says Lappan. "There's an awareness level, there are people who are trying to get on board, trying to think hard about what they're doing. But clearly this is a reform that's going to take a very long time."

One of the toughest jobs is at the elementary level, where many kids turn off to mathematics and where teachers often have weak mathematical backgrounds. EEF is investing in this area, through its K-3 Mathematics Specialist program, which is the most extensive and the longest running of the Foundation-funded mathematics programs. Seventy-five K-3 Specialist projects in twenty-eight states have received EEF grants. All of the projects are guided by the *Curriculum and Evaluation Standards*, but "EEF is not interested in pushing any particular program," explains Pat Hess, the facilitator for the K-3 Specialist projects. "You couldn't go to the store and purchase the 'Exxon Education Foundation Project' program."

This diversity is the key to the success of the K-3 Specialist program: Teachers begin to dis-

Reclaiming Intuition in Mathematics

"For me, math is like falling down a giant waterfall. Going down is fun and easy, but once you hit the bottom, you are thrown about by the force of the water. It's a challenge to see if you can 'beat' the water and survive or whether you will just give up and let the water get the best of you. Eventually if you stick it out you move on to calm waters and everything is fine... until the next fall."

"For me, math is like climbing a huge mountain. It is terribly difficult going up, yet when you get there, you know you've accomplished something."

"For me, math is most like a fog because you cannot see or understand where it came from, but you just have to keep moving forward. You can only move forward slowly, otherwise you'll get lost."

"For me, math is a never-ending list of rules that don't really mean anything. Math seems to be the most indecisive subject there is. In the beginning we were taught basic math, like, you can't subtract four from one. That seemed logical to me. Then they told us that you could do that, but it equals a negative number. A negative number is less than zero. Zero is nothing. How can anything be less than nothing?"

These "mathematics metaphors," with their vivid expressions of the exhilaration, confusion, and frustration of learning mathematics, were written by high school students. They appear in a book which was edited by Dorothy Buerk, a mathematics education professor at Ithaca College, and published in 1994 by the NCTM.

The book grew out of a project, funded by the Exxon Education Foundation, in which Buerk met with teachers to discuss classroom strategies for implementing the NCTM *Standards*. Entitled "Empowering Students by Promoting Active Learning in Mathematics: Teachers Speak to Teachers," the book provides ideas for cooperative learning and using writing in the mathematics classroom. Out of the project grew CLAM, the Cooperative Learning Alliance for Mathematics, through which Ithaca-area teachers in grades five through college meet once a month.

"My belief is that one of the major problems for students is their perception of mathematics as something that's rote, something in which they have to shut off their own thinking and try to reproduce somebody else's thinking, without having it make sense to them," says Buerk. "The metaphors give clues about the students' learning strategies and their conceptions of mathematics. This helps the teacher deal with them individually in different ways."

cover the conditions that allow reform to happen in their schools and districts. They are given the means to develop their own ideas for how to improve teaching and learning in mathematics. For many of them, improving their own mathematical backgrounds becomes a natural next step. In addition, the teachers have formed a community in which they can share ideas. Hess puts out a newsletter that describes activities of the projects and, each September, the K-3 program brings the teachers together for a conference. Some of these teachers have become leaders of change at the regional or state level.

Marilyn Burns, an educational consultant in Sausalito, California, runs in-service programs in mathematics for elementary school teachers which reach 7,000 to 8,000 teachers each year. She calls the K-3 Specialist program "fabulous," noting that the flexibility EEF allows for the projects "mirrors the kind of flexibility I want teachers to allow for students learning different ways." "To me, teachers are the key," Burns declares. "You can change materials, but a unit is just a unit. Without changing the teachers' belief systems, attitudes, understandings and instructional approaches, you're not doing anything, because they're the ones who are seeing children for six hours a day."

Assessment Issues

"Evaluation has become a political carrot: Standardized tests that children take as often as four or five times a year are used as a way to evaluate teachers and schools, sell real estate and further the ambitions of politicians. Houses are advertised as being in school districts with above-average test scores; secretaries of education point to test score graphs as though they were some sort of Dow Jones average. Lost in this numbers blitz is much concern about how the constant and continual testing in schools affects the day-to-day learning of individual children."

Susan Ohanian, from her book entitled
Garbage Pizza, Patchwork Quilts, and Math Magic

At the same time that many teachers are wading into deeper mathematical waters by using manipulatives, open-ended problems, student groups and other innovative methods, the tyranny of standardized testing remains. "I know for a fact that teaching stops at least three to five weeks before the tests," Marilyn Burns observes. "I can't tell teachers not to prepare their kids for tests if those scores are going to be published in their local paper. Teachers don't lose their salaries, they don't lose their jobs, but it's a pall that hangs over them – 'Oh my God, how are the kids going to do on the test?'"

Standardized testing is a major stumbling block to implementing the *Standards*, because the tests do not measure what the Standards say is important for students to learn. For reform to happen in a lasting way, all three parts of the educational triad of teachers, curricular materials and assessment must change together. The NCTM's *Assessment Standards* will provide some guidance on these issues. This set of standards was carefully developed to align with the other NCTM standards for curriculum and for teaching.

School systems around the country are experimenting with new ways to assess what kids know in mathematics. One such program in Bellevue, Washington, funded by EEF, has an unusual twist. It brings parents and administrators in on the ground floor of the development of the new assessment methods, such as portfolios. Led by Sherry Beard, elementary mathematics specialist for the Bellevue public schools, the program brought together a team of eighty people – parents, teachers and principals – to formulate goals for mathematics in grades K-5 and to explore ways of assessing whether those goals are being met.

With help from EEF, MSEB issued *Measuring Up*, a set of prototypes for fourth-grade assessment. One prototype asks students to look at five different bar graphs and decide, based on the characteristics of the graphs, which one could represent the heights of students in a fourth-grade class, which one could represent the distribution of cavities, which one could represent the distribution of their mothers' ages and so on. A look at the sample student responses shows how these prototypes give a far richer view of what the students do and do not understand than any bubble-answer test could.

Start Pitching In

> *"The textbook people say, 'Well, we're not going to change because of the tests.' And the test people say, 'We're not going to change because of the textbooks.'... Everyone's got the 'because, because', and really, everything has to change at once. 'So don't be waiting around, start pitching in,' is my thought. So the Exxon Education Foundation says, 'We'll pitch in,' and I say, 'Hallelujah.'"*
>
> *Marilyn Burns*

The strength of the Mathematics Program of the Foundation is that it invests in people – savvy, dedicated people, each of whom contributes a unique strength, a unique voice to the tough job of reforming mathematics education. And their voices are being heard. "People say, 'The mathematicians really have their act together,' or 'Mathematics is leading the reform effort,'" notes Marcia Sward. "I think that we have charted out new ways of thinking about the whole educational system."

But more is needed. To get change going in our 16,000 school districts and 3,000 colleges and universities, the entire nation has to join the chorus. "One thing that is going to make a tremendous difference over the next few years is whether or not we can find a way to articulate the intent of this mathematics vision so that parents and community leaders and business and industry buy in and help support it," says Glenda Lappan. "We need to find ways to develop this sense of ownership and support in individual communities across this country." Bringing all of these voices into the melody, to discuss and debate and find common ground, is what it will take.

Listen to the voices of the reform movement. Each is a little different, but they're all singing the same song and rejoicing in the same harmonies.

Allyn Jackson is the staff writer for the American Mathematical Society.

Tilings in Art and Science

James E. Hall
Westminster College

The title "Tilings in Art and Science" is a contraction of one that is longer and more descriptive: "Tilings of the Plane in Mathematics, Science, Nature, Art, and Design: A Personal View." Election to the Henderson Lectureship at Westminster College for 1989-90 was the occasion for the investigation of a topic that has long interested me and that has likewise long awaited the opportunity for deeper study. I was honored to be chosen by my colleagues and grateful to the generosity of Joseph and Elizabeth Henderson, whose endowment of the lectureship made the project possible.

One characteristic of our contemporary culture, viewed with distrust by many, is the increasing mathematization of more and more aspects of our lives. This mistrust or misunderstanding was underlined by the late C. P. Snow in his reference to "two cultures." He lamented the breakdown in communication between scientists and humanists. I attempted, through an illustrated presentation about tilings, to convince my audience that there is, in fact, a positive relationship between the abstract structure of mathematics and the sensory reality of the world in which we live and move.

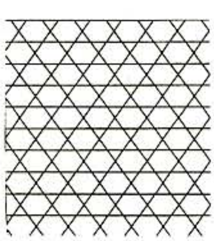
Human beings look for meaning and significance in the multitude of sensory stimuli with which they are bombarded by seeking pattern and order—organizing principles, schemes of classification, necessary relationships. Such abstractions enable them to understand, appreciate, evaluate, predict, and even shape and control certain portions of their surroundings. For example, the diagram below [Fig. 1] abstracts and idealizes the pattern of hexagons and triangles underlying the design on the ninth century Islamic bowl. The elaboration of relationships such as the one between these two pictures, in a wide variety of settings, formed the substance of the "show."

Two recurring themes in the history of thought are change and constancy. They are combined in patterns and tilings, where the theoretical urge to continue the design, and the practical need to curb and regularize it, are in dynamic tension. Sometimes a designer allows random variation to take a hand, as in the facing of a wall or building by irregular pieces of building stone [Fig. 2(a)] or in a random pattern of wood shingles. More often, regularity is imposed: a few basic shapes are chosen and used repeatedly, as in the many patterns that can be created with building bricks of the same shape [Fig. 2(b)]. We thus subject the flowing process of change to regulation, imitating the cyclic behaviors observed in nature.

A bridge to understanding these parallel regularities of pattern in design and nature is the mathematical theory of *tilings* or *tesselations*. Scientist and artist alike attempt to describe important features of the world about them—though the scientist's aim often goes beyond description to prediction and control. These can be as varied as a print by the graphic artist M. C. Escher or a structure diagram from organic chemistry [Fig. 3].

please see p.16

1(a)



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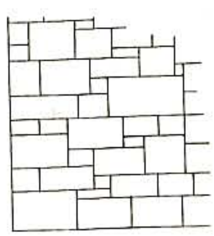

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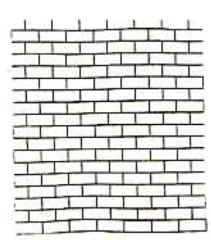
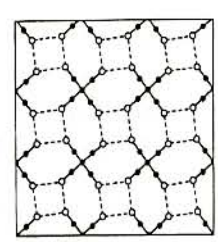


2(b)

please see p.16

3(a)

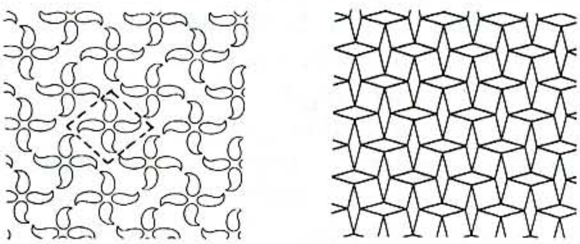


3(b)

Figure 1: Islamic bowl (a) and underlying tiling (b). Figure 2: Random stone (a) and regular brick (b).
1(a) Figure 3: Escher horsemen (a) and chemical diagram (b).

Subjecting our observations to the intellectual discipline of abstraction called mathematics deepens our appreciation of patterns and enables us to participate in creating new and better ones. Mathematical ideas can interact with all aspects of our experience.

Mathematical philosopher Philip J. Davis observes that "mathematics dreams of an order which does not exist. This is the source of its power; and in this dream it has exhibited a lasting quality that resists the crash of empire and the pettiness of small minds. Mathematical thought is one of the great human achievements. The study of its ideas, past and present, can [free] the individual . . . from the tyranny of time and place and circumstance. Is not this what liberal education is about?"



(a) (b)

Figure 4: Tiling (a) and pattern (b).

Let us distinguish between tilings and *patterns;* pattern is the more general word. When Grunbaum and Shephard were collecting materials for their definitive 1987 book on tilings and patterns, they found no rigorous technical definition of pattern in the mathematical literature; their book provides the first textbook treatment of the topic. That most of this material is new is, in their own words, "a surprising fact considering the immense amount of effort that artists and architects have expended in designing and analyzing patterns since time immemorial."

A pattern in the plane is simply a geometric design for which there is a small part whose repetitions create the whole [Fig. 4(a)]. The small prototypic part is called the *motif.* Patterns are studied and classified by considering various motions of the plane, such as rotations, and determining which of them leave the overall pattern indistinguishable from its original state.

Tilings represent a special case in which the plane is partitioned, without gaps or overlaps, into sets called *tiles* [Fig. 4(b)]. We will be mostly concerned with the case in which one or a few distinct shapes, the *prototiles,* are used to generate the entire configuration. Symmetries of these configurations are an important way of describing and classifying them.

There are several classical tiling problems that don't quite fit this description, yet are similar in nature. Though some of these have a "game" or "puzzle" character, they should not be taken lightly. Many mathematical puzzles are the key to understanding significant related applications; many have turned out in the long run to be more useful than their creators imagined. Such a classical puzzle is that of "tiling the crippled chessboard with dominoes." A standard 8 by 8 chessboard can easily be tiled with 32 dominoes, where the size of the domino is just that of two chessboard squares. If two opposite corners of the board are removed, however, it is no longer possible to tile it with the dominoes, even though its 62 square unit area would seem to accommodate precisely 31 of them.

A mathematical proof of this is illuminating and may suggest the appeal of this subject to those with a logical bent of mind. Any domino on the board must cover two adjacent squares, hence one square of each color. The 31 dominoes would thus cover 31 light and 31 dark squares. But the crippled chessboard, because the opposite comers which were removed had the same color, has 30 of one color and 32 of the other!

Like many mathematical demonstrations, this little argument has the virtue of settling the question without recourse to any tedious exploration of a large number of "nearly correct" solutions. For simply failing to find a solution by experiment isn't very convincing—we may just not have tried hard enough!

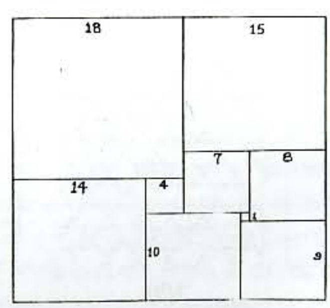


Figure 5: Stein decomposition.

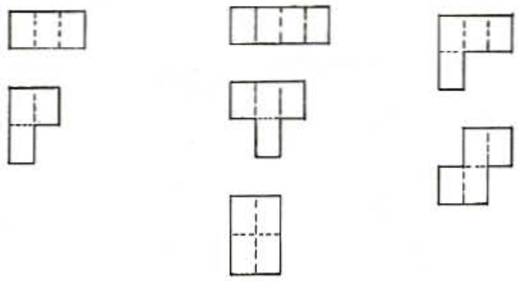


Figure 6: Trominoes (left) and tetrominoes (center and right).

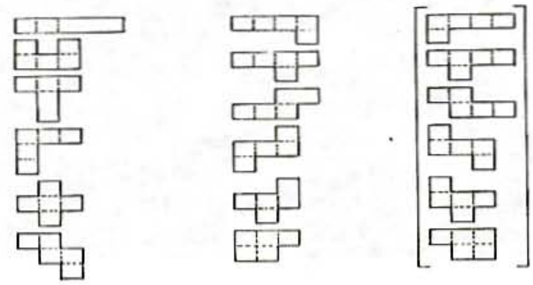


Figure 7: Pentominoes.

A related kind of question is that of tiling a given rectangle with squares that are all of different sizes. This is the antithesis of the notion of regularity mentioned earlier, but is another interpretation of the term tiling. The problem turns out to be difficult: not all rectangles can be tiled this way. There are restrictions on the dimensions of the rectangle as well as on the squares used to tile it. A minimal example is shown in Fig. 5 (from Stein's *Man-Made Universe).* The rectangle is 33 by 32; the number in each square is the length of its side. (These tilings have interpretations as the equilibrium states of certain electrical circuits!)

Mathematical puzzles and recreations, especially those of a geometric nature, have played a significant role in the evolution of the subject. In addition to providing pleasure and diversion, mathematical puzzles and recreations have helped to develop the geometric intuition and insight of many a future geometer. They have stimulated creative and original contributions to the field, not only from mathematical professionals but from "amateurs" as well, that is, those with only modest training in formal mathematics.

For example, an interesting instance of a finite tiling is provided by an innovative jigsaw puzzle, called the "Shmuzzle," all of whose 168 tiles are alike. A typical piece is shaped like a lizard with six extremities: four legs, a head, and a tail. Since the outside border of this tiling is irregular, the puzzle maker provides a border into which to fit the pieces.

Some 25 years ago the mathematician Solomon Golomb generalized the familiar domino, made of two equal squares, to polygonal tiles called *polyominoes.* These can consist of three squares (triominoes or trominoes), four squares (tetrominoes), five squares (pentominoes), and so on. Fig. 6 shows the two trominoes and five tetrominoes; Fig. 7 illustrates the twelve pentominoes.

If we count mirror images as distinct—not allowing the figures to be flipped over—there are then seven "one-sided" or "oriented" tetrominoes and 18 "oriented" pentominoes. (The extra asymmetric figures are shown in brackets.) Higher order polyominoes have been studied, but to date there is no formula known that predicts the number of polyomino forms

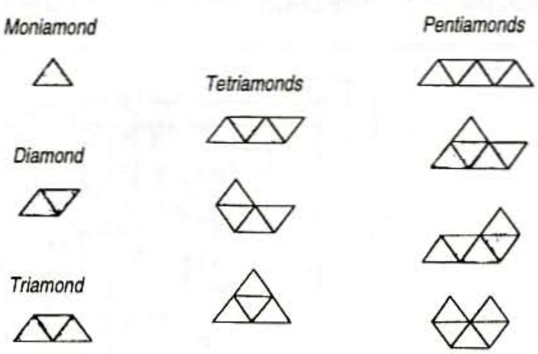


Figure 8: Moniamonds, diamonds, triamonds, tetriamonds, and pentiamonds.

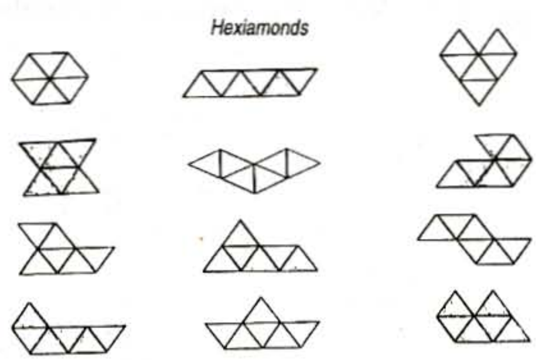
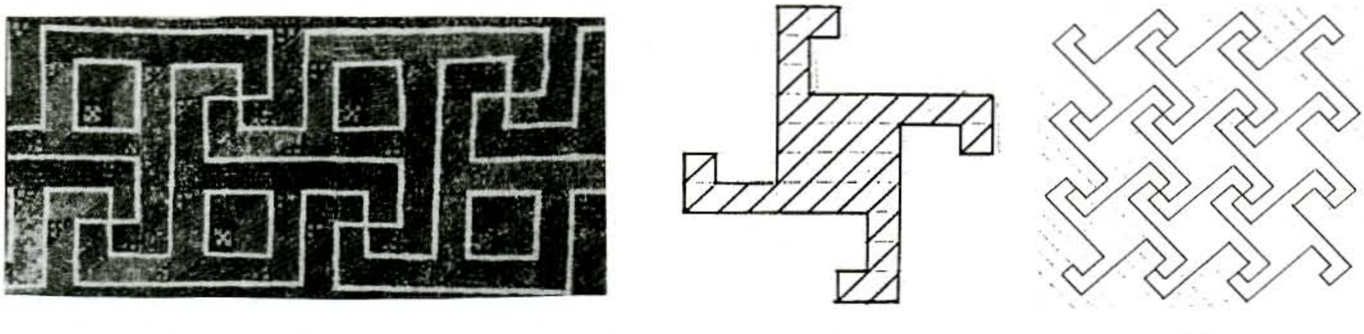


Figure 9: Hexiamonds.



(a) (b) (c)

Figure 10: Ravenna mosaic (a), underlying figure of Ravenna mosaic (b), and tiling of the plane using Ravenna underlying figure (c).

of a given order. *Polyiamonds* are formed in a similar way using equilateral triangles instead of squares. A moniamond is one such triangle; a diamond consists of two. (The double-form is the source of the idea—as well as the terminology—for both polyominoes and polyiamonds!) There is only one triamond, but there are three tetriamonds, four pentiamonds, and a dozen hexiamonds (not counting the oriented or asymmetric forms) [Figs. 8 and 9].

Which polyominoes and polyiamonds are prototiles for plane tilings? Can copies of any one of these figures, laid out appropriately, be used to cover the plane periodically? It turns out that many of these figures do tile the plane, but some do not. There are many unsettled questions!

Mosaic is an art form closely related to tiling. A finite region is covered with small shapes, usually polygonal, but the requirement that there be no gaps between the tiles is not strictly observed. In addition, the tiles are colored. (This chromatic distinction can be made for strict tilings, too; the more colors allowed, the more complicated becomes the problem of classification.) Most mosaics are approximate tilings: the artists' creativity and realism have overruled strict structure.

A mosaic from Ravenna, the fretwork design from an alcove of the Galla Placidia Mausoleum [Fig. 10(a)], represents a very different meeting of creativity and structure. The shape outlined in white resembles a cross between a starfish and a swastika, but each example has only three arms instead of four.

The underlying regular figure [Fig. 10(b)], has four-fold rotational symmetry. Although each instance in the mosaic is missing an extremity in order that the design be confined to a linear border, the full design can easily be extended to a tiling of the plane [Fig. 10(c)]. Note how four tiles cluster together some restrictions commonly adopted when considering tilings of the plane.

The first of these is that the number of distinct prototiles be finite, ordinarily quite few in number. A second is that each tile itself be finite in extent. That these are not necessary can be seen in the examples in Fig. 11. The tiles on the left are unbounded strips. The tiling on the

(a) (b)

Figure 11: Unbounded tiles (a) and infinitely many prototiles(b).

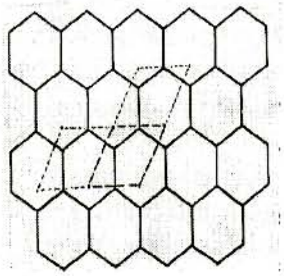


Figure 12: Hexagonal tiling with parallelogram motif.

right consists of infinitely many strips, of differing, ever-increasing size. While such tilings are not without interest, we won't pursue them further here.

As suggested earlier, the symmetries of a tiling and of its constituent prototiles are characteristics important in the description and classification of plane tilings. These are rigid motions of the plane after which the outline of the tiling appears identical to its original form.

One such motion is *translation:* slide the pattern in a fixed direction a specific distance, always parallel to its starting position. After a certain fixed distance, it coincides with its original configuration. Most of the tilings we'll look at are *periodic:* there are two distinct directions and distances in which translations will bring about coincidence. This means we can find a parallelogram whose contents form a motif for the overall pattern. The motif sometimes consists of fragmented copies of the prototiles [Fig. 12].

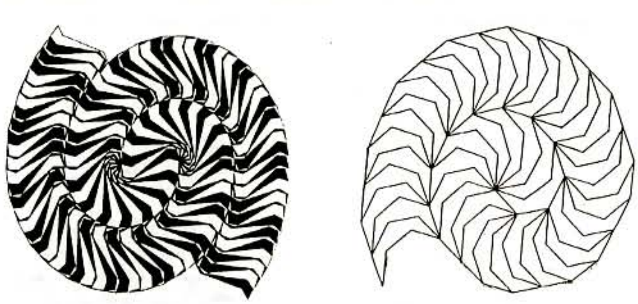
Many interesting tilings are *aperiodic*, however; artistic examples are provided by "spiral" tilings [Fig. 13]. The tiling on the left, constructed from enneagons, is due to Heinz Voderberg. Marjorie Senechal created the tiling on the right from concave heptagons.

Other symmetries are described in terms of *rotations*, *reflections*, and *glide-reflections*. These can be demonstrated most effectively using transparent models and other dynamic visual aids.

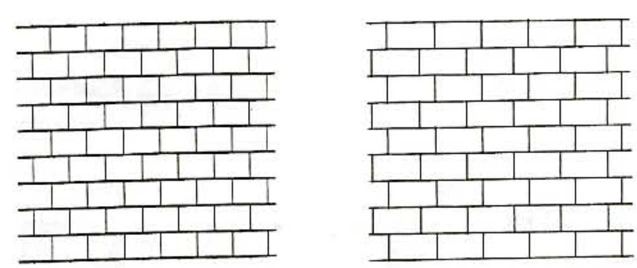
A further characteristic used in classifying tilings is that of being "edge-to-edge." This requirement limits the number of possibilities and makes the mathematical treatment easier. Yet many tiling patterns are not edge-to-edge. A familiar example is the usual pattern in which bricks are laid to face a building. Just by varying the offset, with no change in the shape of the prototile, an infinite number of distinct patterns is possible. In Fig. 14(a) the overlap is half a brick; in Fig. 14(b) it is one-third of a brick.

The examples mentioned so far may begin to convince you that tiling patterns can be found all around. Near-tilings appear in nature, while their imitations and idealizations abound in art and design. Once you begin to look for tilings, you will find them everywhere!

The pattern of patches on the hide of the giraffe provides a crude example of an irregular tiling; it is clearest on the reticulated giraffe. The cracks in drying mud often form similarly reticulated patterns, as do gelatinous preparations of tin oil, rock formations like the Devil's Postpile, packings of soap bubbles, and the pattern of cracks formed by the shrinking of plaque. The scales of fishes form tiling-like patterns. That reptiles provide illustrations of tiling patterns in their scales and skins is not surprising. We may be startled, though, to find similar patterns in the tail of the beaver and the paw of the mole. Additional examples of the

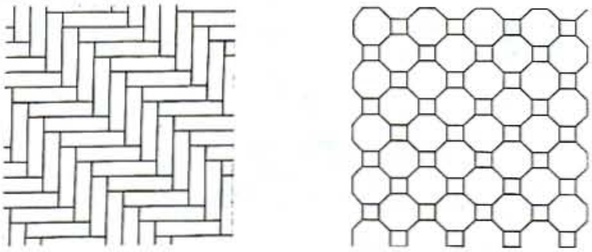


13(a) *13(b)*



14(a) *14(b)*

Figure 13: Vodeberg spiral (a) and Senechal spiral (b). Figure 14: Bricks with one-half brick offset (a) and one-third brick offset (b).



(a) (b)

Figure 15: Herringbone bricks (a) and squares with octagons (b).

regular hexagonal tiling are found in cross-sections of the honeycomb and the nest of the paper wasp.

The study of crystallography has a close relationship with tiling. Although crystals are three-dimensional, suggesting a generalization of the tiling idea into space that is outside the scope of this discussion, their cross-sections and projections lead to configurations with tiling patterns.

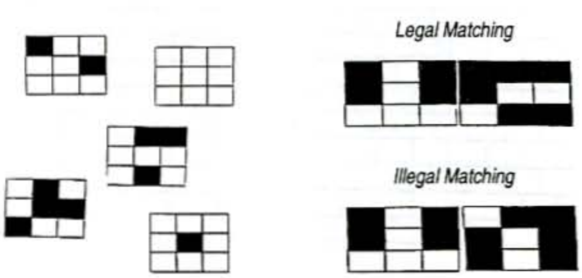
Taking a cue from some of these natural phenomena, but adding creativity and the ability to make accurate copies of the motif to generate precise tiling patterns of any size, the human designer and artist have incorporated such elements into a wide variety of settings, using many different materials. To the underlying geometric pattern are often added the further dimensions of color and texture.

At a practical everyday level we see tilings in the roofs of the buildings in which we live and work, whether in asphalt shingles with an "anvil pattern," slate roofs arranged like diamonds, or in real old-fashioned ceramic roof tiles. The common postage stamp is another source of patterns for the student of tilings, since the sheets of which they are a part are tiled by the stamps [Fig. 27]. By far the most common pattern is that of simple squares or rectangles, but several countries have issued stamps shaped like triangles, trapezoids, parallelograms, and pentagons.

Earlier we mentioned patterns in brick facings. Bricks are used for walks and malls, too, as we see in Fig. 15, with a criss-cross pattern of rectangles on the left, a mixed tiling of octagons and squares on the right.

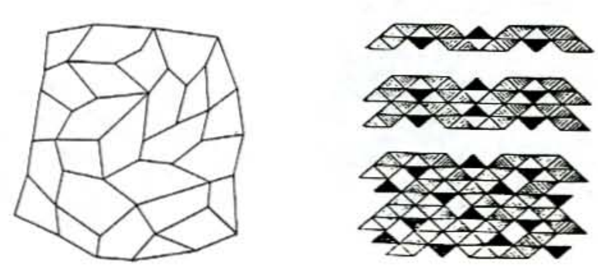
Glimpses of geodesic domes and some of nature's near-tilings may have inspired mathematician Doris Schattschneider, herself an expert on mathematical aspects of Escher's work, to use Escher designs to tile the surfaces of the five regular *or platonic* solids, three-dimensional analogs of regular polygons, having all edges, faces, and angles equal. These imaginative combinations of artistic creativity and mathematical regularity must be seen to be appreciated. Using tilings whose prototiles represent reptiles (is this a "rep-tiling"?), fishes, bats, lizards, shells, and starfish, Schattschneider and her colleague Wallace Walker present us with the tetrahedron, cube, octahedron, dodecahedron, and icosahedron resplendently clothed in their new "Escher prints."

This image of clothing leads naturally to the observation that many textiles are decorated with patterns and tilings. Somewhere between the sublime near-tessellations of a seventeenth century Persian carpet and the mundane (if not ridiculous) diamond tiling on a sweater in a fashion ad, are the colorful designs of quilts. Those of us living in the midst of Amish country, of course, find this no surprise. The designs vary from the relative simplicity of the Bow Tie pattern to the complexity of Pierced Star and Sunburst. The complexity results both from the intricacy of the underlying tiling and the further variations resulting



16(a) 16(b)

Figure 16: Novi tiles (a) and matching rules (b).

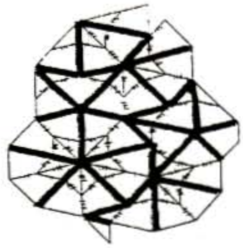


17(a) 17(b)

Figure 17: Irregular "glass" tilings (a) and crystal tilings (b).

18(a)

18(b)



19(a)

19(b)

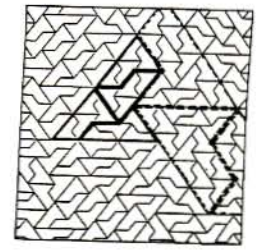


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Figure 18: Penrose kites and darts (a) and Penrose rhombi with matching (b).
Figure 19: Inflation (a) and deflation (b) of Penrose kite and dart tiling. Figure 20: Polyamond sphinxes.

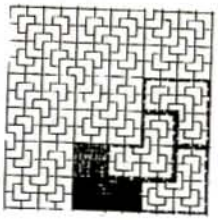
from the use of color or pattern within the tiles. Like the Escher-covered polyhedra, these must be seen in "living color" to be appreciated fully.

Recently a game has appeared called Novi, described by its creators as "a game of visual intelligence." Its 256 tiles, colored on both sides, represent the 512 possible ways of coloring either white or black the nine small squares of a square enneomino [Fig. 16(a)]. The tiles may be used to construct specified figures or played on a game board according to various sets of rules. A feature of all the games and pastimes associated with Novi is that tiles are to be placed next to one another only if their edge colorings match; Fig. 16(b) shows examples of a legal match and an illegal one.

This matching rule is similar to the matching rules imposed on *Penrose tiles*. In the 1970s, British mathematical physicist Roger Penrose was intrigued by tilings that were not quite periodic, not quite random. They exhibited the five-fold symmetry strictly forbidden by the crystallographers (apparently related to the fact that regular pentagons won't tile the plane). It wasn't until the discovery in 1982 of an icosahedral quasicrystalline phase of aluminum-manganese alloy that this pentagonal symmetry appeared convincingly in nature. Poised somewhere between the disorder of a glass, represented by the tiling by random polygons in Fig. 17(a), and the regimentation of a crystal in Fig. 17(b), these quasi-crystalline substances seem right in step with the recent emergence of the study of chaos as a scientific discipline: the remarkable discoveries, as David Eck puts it, of "unpredictability without randomness . . . [and] pattern without determinism."

Penrose's most interesting tiles, the kites and darts, must be matched according to rules enforced by corner labels or by matching colored arcs; these seem to reflect rules of chemical structure in the quasi-crystalline state of matter. Tiling by kites and darts [Fig. 18(a)], and its companion based on two differently shaped rhombuses [Fig. 18(b)], tantalize the physicist, intrigue the mathematician, and stimulate the artist to delight the eyes of all!

Penrose tilings are related to another topic of current interest, fractal geometry and sets of fractal or fractional dimension. Penrose tilings, like many fractals, exhibit *selfsimilarity*—identical structure at different scales of magnification. Tiles of a kite-and-dart configuration can be broken down into smaller versions or combined into larger ones [Fig. 19]. Other tilings have this property, too. The "sphinx-like" tile, repeated in Fig.



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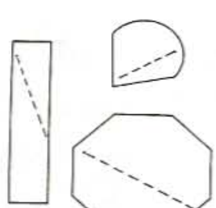
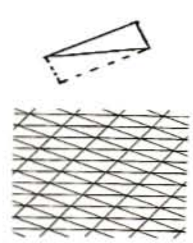
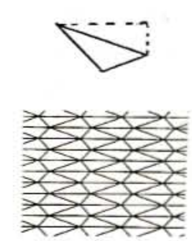


22(a)

22(b)

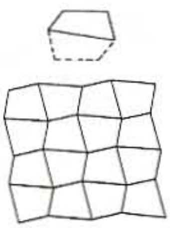


23(a)

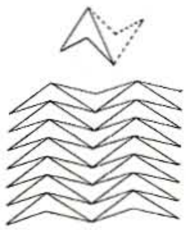


23(b)

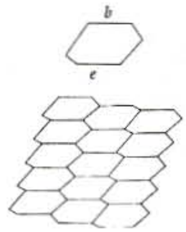
Figure 21: Triamond cells. Figure 22: Convexity (a) and non-convexity (b). Figure 23: Distinct tilings by the same triangle.



24(a)



24(b)



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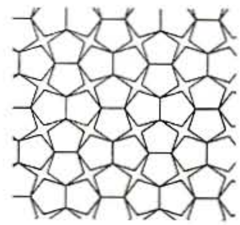
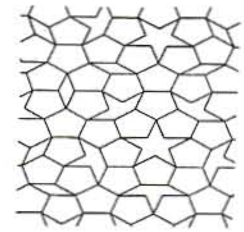


26(a)



26(b)

Figure 24: Quadrilateral tilings, convex (a) and non-convex (b).
Figure 25: Tiling by Type 1 hexagon. Figure 26: Tiling by pentagons (a) and stars (b).

20 at several scales, is just a hexiamond; a self-similar pattern is built from a tromino in Fig. 21.

A question about tilings that has intrigued both professionals and amateurs concerns the possible plane tilings with just one polygonal prototile—with the restriction that the tiles be *convex*. Convexity means that the boundary always bulges outward: there are no "dents" [Fig. 22]. In other words, the straight line path between any two points of the set remains entirely within the set.

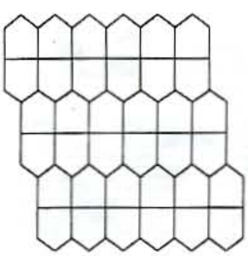
We know, of course, that we can tile with equilateral triangles, squares, and regular hexagons; these are the only regular edge-to-edge tilings. One can, in fact, tile with any triangle, and in more than one way [Fig. 23]. It's also possible to tile the plane with any quadrilateral [Fig. 24]. There are no restrictions on the relative sizes of the sides or on the angles: the quadrilaterals needn't even be convex!

For hexagons the situation is more complex. We've seen several examples of tilings by regular hexagons, but irregular ones work only under special conditions on sides and angles. It was shown by Reinhardt in 1918 that there are only three distinct types of hexagons which tile the plane. Each of these types is characterized by certain restrictions. The first, for example, has top side *b* and bottom side *e* equal in length; in addition, the three angles at the left end must add up to 360 degrees, as must the three at the right end. Fig. 25 illustrates these conditions and shows a tiling by a hexagon of this type.

Tiling by convex heptagons was believed for a long time to be impossible. No formal proof was written anywhere, and the fact was referred to as part of the "folklore" of the subject. An elementary, if tricky, proof was supplied by Niven in 1978. He actually proved the impossibility for 7 or more sides!

The alert reader will have noticed the omission of the case n = 5. Can we tile with pentagons? For regular pentagons the answer is no, though regular pentagons can be combined with other prototiles to produce interesting tilings like those in Fig. 26.

For many irregular pentagons, however, tiling is easy. The typical shape of a house drawn by a young child, for example, tiles as shown in Fig. 27(a), while the lopsided version of this pentagon [Fig. 27(b)] can also be used to tile. Possibly because its form suggests the



27(a)

please see
p. 16

27(b)

28

29(a)

29(b)

Figure 27: Drawings of houses (a); stable-shaped stamps (b).
Figure 28: Tiling by pentagon species 10 (M. Rice). Figure 29: Escher tiling sketches from the Alhambra.

outline of a stable, it was chosen as the shape for a set of Maltese Christmas stamps in 1966.

Experience with hexagons suggests that there may be restrictions on pentagons as well. Indeed, the two examples above are special, since each contains two right angles and a pair of parallel and equal sides. The Reinhardt who classified hexagons also listed five types of pentagons that work and claimed that his list was complete, but in 1968, R. B. Kershner published three more types. He too claimed completeness for the now expanded list, but it was more "folklore"—not formally established. An engineer named Richard James produced a ninth one in 1975, stimulated to study the matter when the taxonomy of Reinhardt and Kershner was published by Martin Gardner in the Mathematical Games column of *Scientific American* earlier that year. This was not the end of the story, however.

Marjorie Rice, a San Diego housewife and mother of five, whose formal mathematics training ended with high school "general math" but whose informal training included years of reading Gardner's columns in *Scientific American*, devised her own scheme of classification and came up with type 10 in February of 1976, types 11 and 12 in December of 1976, and type 13 in December of 1977. The experts were amazed! Fig. 28 shows a tiling by her type 10. Its not quite parallel sides have a rather disconcerting effect, but it is a periodic tiling!

So the list stands at thirteen, but the question is still open. At this point not even the experts claim to know whether the list is complete. Perhaps some reader of this account will discover species number fourteen!

The artistic imagination leads one to seek more complex and interesting prototiles than just polygons.The Dutch graphic artist M. C. Escher was fascinated by tilings. In 1936 he made sketches of a number of Islamic designs from Moorish Spain [Fig. 29].

Ernest Ranucci and Joseph Teeters were so intrigued by Escher's drawings that they wrote a book entitled *Creating Escher-type Drawings*. While they produced some amusing efforts (for example, tilings by football players and by St. Bernard dogs), they didn't seriously challenge Escher, the master. From drawings of those Andalusian tiles in the Alhambra, Escher went on to lizards, angels and devils, fish and birds, and many other memorable designs. New York art publisher Harry Abrams has created a stunning collection of wrapping paper designs, colorfully continuing Escher's tilings into works of art too lovely to use on any package.

The art of Islam has been rich in geometric design because of Mohammed's prohibition against representing the human figure. The variety and inventiveness of these can be sampled by leafing through books on Islamic art. One will find such examples as a tiling by three species of octagon (two of them non-convex!) in a panel from a mosque at Isfahan and an intriguing design of interlocking arrows in a column from a tomb in Maragha. (A similar pattern appears in a nineteenth century French graphic by Cahier and Martin.)

The diversity and beauty with which these many tilings are executed in the world about us, both natural and man-made, together with the ingenuity of their neatly dovetailed designs, can provide both stimulation to the intellect and refreshment to the spirit. As you move about in your own world, be alert for tilings all around you—and enjoy!

POSTSCRIPT

In collaboration with my photographer/student assistant Mark Tanner, I assembled a library of approximately five hundred slides illustrating tilings in architecture, science, philately, Islamic art, and other areas. Eighty pairs of these formed the central visual vehicle for the Westminster Henderson Lecture and for subsequent versions of the lecture given in the US, New Zealand, and Australia. The narrative above conveys the structure of the presentation within the limitations of much simpler illustrations.

FOR FURTHER READING

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Fig. 1(a): Plate VII,B (Fig. 246), *Early Islamic Ceramics,* Helen Philon (Islamic Art Publishers, 1980; ISBN 0-85667-698-7).

Fig. 3(a): Plate 137, *World of M. C. Escher,* J. L. Locher, ed. (NY: Abrams, 1971; ISBN 0-81090107-2).

Fig. 3(b): adapted from Fig. 99, p. 223, *Organic Crystals and Molecules,* J. Monteath Robertson aca: Cornell Univ. Press, 1953).

Fig. 4(a): adapted from Fig. 4, p. 8, of *Contemporary Crystallography,* Martin J. Buerger (NY: McGraw-Hill, 1970).

Fig. 10(a): from "Fifteen Centuries Later, Ravenna's Mosaics Still Glow," Robert Warnick, in *Smithsonian,* Jan. 1990, p. 65.

Fig. 13(a): Fig. 37 p. 4, *Penrose Tiles to Trapdoor Ciphers,* Martin Gardner (NY: Freeman, 1989; ISBN 0-7167-1986-X).

Fig. 13(b): Fig. 16, p. 20, of "A Brief Introduction to Tilings," Marjorie Senechal, in *Introduction to the Mathematics of Quasicrystals,* Marko Jaric, ed. (San Diego: Academic, 1989; ISBN 0-12-040602-0).

Fig. 16: author; *Novi is* a registered trademark of the R/L Group, Cambridge, Massachusetts.

Fig. 17(b): Fig. 1, p. 80, of *Crystal Chemistry of Large-Cation Silicates,* A. N. Belov (NY: Consultants Bureau, 1963).

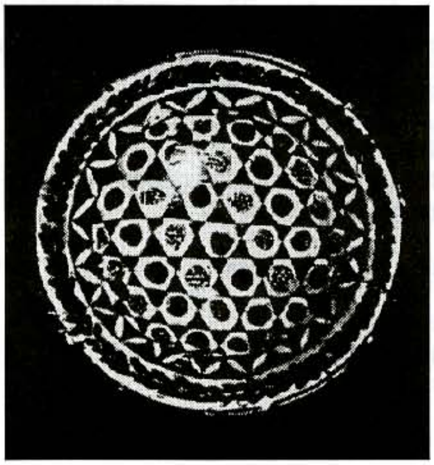
Fig. 18(b): adapted from Fig. 10.3.18, p. 543, of *Tilings and Patterns,* Grunbaum and Shephard (NY: Freeman, 1987; ISBN: 0-7167-1193-1).

Fig. 26(a): Fig. 2.5.4(q), p. 85, of *Tilings and Patterns,* Grunbaum and Shephard (NY: Freeman 1987; ISBN: 0-7167-1193-1).

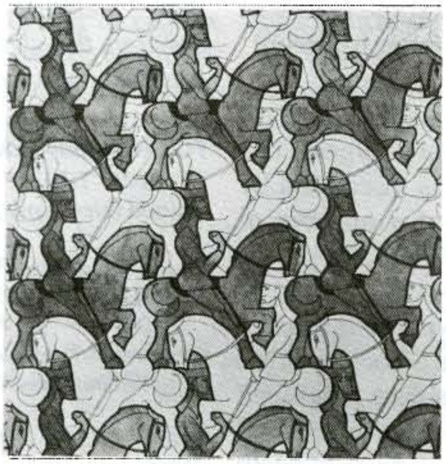
Fig. 26(b): adapted from Fig. 10.3.3, p. 532, of *Tilings and Patterns,* Grunbaum and Shephard (NY: Freeman, 1987; ISBN: 0-7167-1193-1).

Fig. 28: adapted from Fig. 6, p. 148, "In Praise of Amateurs," Doris Schattschneider, in *The Mathematical Gardner,* David A. Klarner, ed. (Boston: PWS, 1981; ISBN: 0-534-98015-5).

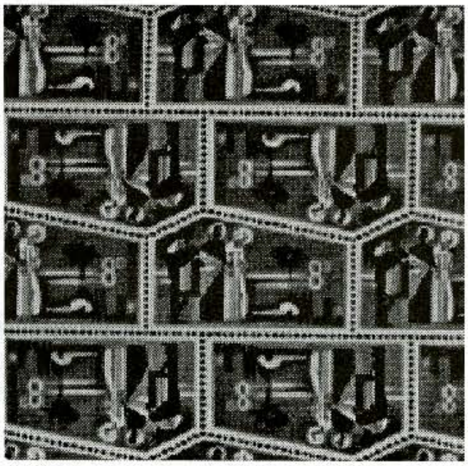
Fig. 29(a): tri-arrows from Plate 86 in *World of M. C. Escher,* J. L. Locher, ed. (NY: Abrams, 1971; ISBN 0-8109-0107-2).



1(a)



3(a)



27(b)

Figure 1(a): Islamic bowl. Figure 3(a): Escher horsemen. Figure 27(b): Stable-shaped stamps.

Mathematics as an Aesthetic Discipline

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INTRODUCTION

This brief paper offers a defense of the study of mathematics. It is intended for those people who are convinced either that mathematics is not worth studying or that mathematics is "just not for them."

This paper is especially intended for those humanists and the literati who thrive in the world of art, music, and literature, but who think that mathematics is a mechanical, cold, unimaginative discipline, suitable only for unartistic, uncreative "computer-types." This paper will suggest that these humanists have confused mathematics with the discipline that went by that name in their schooling. In short, this paper will suggest that the literati who think that the study of mathematics needs defending are completely unaware of the meaning of the word "mathematics." And thus they are really demanding a defense of something else; namely, the memorization of formulae and equations and the mechanical manipulation of numbers that was forced upon them in school. They will find no such defense here.

The reader is alerted to a caveat: this paper is not intended for those who find the entire academic enterprise in need of defending. Those who demand a defense of the study of music, poetry, philosophy, biology, chemistry, and mathematics are advised to look elsewhere. They will not find it here.

THE COMMON DEFENSE

Usually, the study of mathematics is defended almost exclusively along the lines of its effectiveness as an instrument. Legions of so called "mathematics" teachers attempt to sell mathematics to their students as nothing more than a manipulative and a practical tool.

Of course mathematics is useful and practical as a utensil, but only to the professional scientist and engineer. Almost everyone else will use no more "mathematics" in their everyday life than the simplest of grammar school arithmetic: balancing a check book, counting change. One needs little more. The notion that anyone other than a scientist will ever use even the most elementary trigonometry or algebra is laughable. Imagine the absurdity of being in a car or on a plane when suddenly the need arises to solve a quadratic equation or to graph a trigonometric function. But this is precisely the scenario that the traditional defense has coerced us into accepting as realistic. Clearly this is absurd. And so is our complicity.

Of course, students realize this. They become apathetic or openly hostile towards this "mathematics." And who can blame them? Why should anyone care about mathematics if its only value is its practicality, a practicality relegated either to the simplest of childish arithmetic or to the arcanely out of reach, complex world of the professional scientist's mathematics? If this is mathematics, then something is wrong with the student who likes mathematics!

"My early teachers chanted the notion of practical value like a litany. It was repeated at each level, in each course, from grade one through high school. They meant to justify mathematics on the basis of its utility in the conduct of one's daily life.

There is nothing wrong with this except they went too far and claimed too much. Mathematics is useful in this sense. But, with this narrow connotation of 'value,' a little goes a long way. Counting change, measuring carpet, or balancing one's checkbook requires only the slimmest knowledge of mathematics. From early on, I wondered why such pedestrian activity required so much schooling.

The true value of mathematics lies outside commonplace activity." J. P. King [Ki].

THE NEW DEFENSE

The new defense of the study of mathematics does not rely on the utility of mathematics. The cornerstone of

this new defense is the beauty of mathematics, a notion singularly alien to the general public.

AESTHETICS

We study mathematics for the same reasons we study poetry or music or painting or literature: for aesthetic reasons. Simply put, we study mathematics because it is one of the loveliest disciplines known to man.

"A mathematician, like a painter or a poet, is a maker of patterns.... The mathematician's patterns, like the painters or the poet's, must be beautiful; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics." G. H. Hardy [Ha].

"...this character of beauty and elegance [in mathematics is] capable of developing in us a sort of aesthetic emotion." Henri Poincare [Po].

"There is, first of all the motivating force for mathematics which is beauty." J. P. King [Ki].

The fashioners of this sublime beauty, artists indeed, must possess a rare creativity and an imagination of the highest order.

"The moving power of mathematical invention is not reasoning but imagination." Augustus de Morgan [HA].

"There is an astonishing imagination even in the science of mathematics... We repeat, there is far more imagination in the head of Archimedes than in that of Homer." Voltaire [HA].

"The essence of mathematics is its freedom." Georg Cantor [HA].

"The science of pure mathematics, in its modern developments, may claim to be the most original creation of the human spirit." A. N. Whitehead [Wh].

One of the most compelling aesthetic features of mathematics is its refined austerity. Its unadorned gracefulness is unique among the arts. In fact, part of the very essence of mathematics is its precision. People are referring to this quality when they suggest that mathematics teaches "clear thinking." Mathematics' precision does not lie in any claims of universal truth. But rather this precision, and hence power, lie in the acknowledgment of exactly the points at which mathematics consciously and deliberately abandons claims of universal truth. Mathematics is the only discipline that I am aware of that does this. And this precision and austerity allow for an elegant economy, an economy that comes from the elimination of the cluttering mire of imprecision.

"Strange as it may sound, the power of mathematics rests on its evasion of all unnecessary thought and on its wonderful saving of mental operations." Ernest Mach [Be].

"Mathematics is precise or it is nothing." J. P. King [Ki].

"Mathematical knowledge adds vigour to the mind, frees it from prejudice, credulity, and superstition." John Arbuthnot [Mo].

"One cannot escape the feeling that these mathematical formulae have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than we originally put into them." Heinrich Hertz [HA].

"Calculus is the most powerful weapon of thought yet devised by the wit of man." N.W.B. Smith [Mo].

The mathematician, however, is not merely an ascetic, cold and austere. He or she is an expressive artist involved in the richly human struggle to create and to discover.

"...a mathematician experiences in his work the same expression as an artist; his pleasure is as great and of the same nature." Henri Poincare [Be].

"I have heard myself accused of being an opponent, an enemy of mathematics, which no one can value more highly than I, for it accomplishes the very thing whose achievement has been denied me." Goethe [Be].

"A mathematician who is not also something of a poet will never be a complete mathematician." Karl Weierstrass [Mo].

"Other qualities of a far more subtle sort, chief among which in both cases is imagination, go to the making of a good artist or a good mathematician." Maxime Bocher [Mo].

Sadly, most people, including the otherwise sensitive and culturally sophisticated, are completely unaware of the intrinsic aesthetic features of mathematics.

"The useful combinations are precisely the most beautiful, I mean those best able to charm. This charm is the special sensibility that all mathematicians know but of which the profane are so ignorant as often to be tempted to smile." Henri Poincare [Po].

"Nothing lives further from the intellectual experience of members of the educated public than the notion that mathematics can have aesthetic value." J. P. King [Ki].

The common defense is not, however, supplanted by the new defense, but rather it is subsumed by it. This subsumption takes the unexpected form of an appreciation for the utility of mathematics. By this I mean that to most students of mathematics, the utility of mathematics should be presented in something like the same fashion as music is presented to students of music history, namely as a marvel to be appreciated, not an instrument to be operated. Those students interested in actually creating music (i.e., in becoming musicians or composers) are advised to study performance or composition. Similarly, those students interested in actually harnessing the utilitarian powers of mathematics (i.e., in becoming engineers, scientists, and mathematicians) are advised to study engineering and applied mathematics. But for the vast majority of mathematics students, a simple, honest appreciation of the remarkable utility of mathematics should be seen as the ultimate "real-world" goal. In short, the sense of agency developed in most students regarding the utility of mathematics should be of an appreciative nature, not an instrumental nature. And since "appreciation" is an aesthetic term, not a scientific term, for most students, the traditional defense of the study of mathematics as a tool is subsumed by the aesthetic perspective of the new defense.

"There is no branch of mathematics, however abstract, which may not someday be applied to phenomena of the real world." Nicolai Lobachevsky [HA].

"The mathematician, carried along on his flood of symbols, dealing apparently with purely formal truths, may still reach results of endless importance for our description of the physical universe." Karl Pearson [Be]

"Algebra is the intellectual instrument which has been created for rendering clear the quantitative aspects of the world." K. N. Whitehead [HA].

"Mathematics is the queen of the sciences." Carl Fredrich Gauss [Be].

"It is mathematics that offers the exact mathematical sciences a certain measure of security which, without mathematics, they could not obtain." Albert Einstein [Be]

"A book on the new physics, if not purely descriptive of experimental work, must be essentially mathematical." P. A. M. Dirac [Di].

"The great book of nature can be read only by those who know the language in which it was written. And this language is mathematics." Galileo [Be].

GREAT THINGS

The study of great things, including the study of great ideas, needs no defense. And many of the greatest of human thoughts have taken the form of mathematics.

"...not the mere fact of living is to be desired but the art of living in the contemplation of great things." Bertrand Russell [Ru].

"This therefore is Mathematics, she reminds you of the invisible forms of the soul; she gives life to her own discoveries; she awakens the mind and purifies the intellect; she brings light to our intrinsic ideas; she abolishes oblivion and ignorance which are ours by birth." Proculus Diadochus [HA].

"Mathematics is the only good metaphysics." Lord Kelvin [Be].

"To create a healthy philosophy you should renounce metaphysics but be a good mathematician." Bertrand Russell [Be].

"Number rules the universe." Pythagoras [Be].

"God ever geometrizes." Plato [Be].

"The Great Architect of the Universe now begins to appear as a pure mathematician." J. H. Jeans [Je].

Mathematics is created by human beings. It was not carved on tablets and handed down by a god. The most brilliant members of our species have exerted, and continue to exert, the most noble effort to give us this mathematics.

When school children study analytic geometry, they should be made aware that this seemingly trivial and esoteric subject exists to us only because of the heroic efforts of a succession of brilliant minds, culminating in the work of Descartes. Its depth, originality, and profundity are lost on students. It has been carefully polished and refined so exquisitely, presented so elegantly and simply, that students myopically receive it as a trifle.

"Though the idea behind it all is childishly simple, yet the method of analytic geometry is so powerful that very ordinary boys of seventeen can use it to prove results which would have baffled the greatest of the Greek geometers-- Euclid, Archimedes, and Apollonius. The man, Descartes, who finally crystallized this great method had a particularly full and interesting life." E. T. Bell [Be].

"(Analytic geometry), far more than any of his metaphysical speculations [which include, "Cogito ergo sum."] immortalized the name of Descartes and constitutes the greatest single step ever made in the progress of the exact sciences." John Stuart Mill [Be].

When calculus students give a sleepy, disinterested yawn during the discussion of the fundamental theorem of calculus, they should be told that the most outstanding human minds struggled for over two millennia to find this seductively simple formula. Until Newton and Liebnitz finally uncovered it for us, no human eyes had ever gazed upon it, although the greatest intellects had searched for it.

Today, we present this masterpiece to teenage students in a ten-minute lecture. And students receive it in the same spirit that it's presented: as just another boring, god-given, inhuman formula to memorize. Clearly this is unacceptable. Students must learn that mathematics is the most human of endeavors. Flesh and blood representatives of their own species engaged in a centuries long creative struggle to uncover and to erect this magnificent edifice. And the struggle goes on today. On the very campuses where mathematics is presented and received as an inhuman discipline, cold and dead, new mathematics is created. As sure as the tides.

Students deserve the truth: Mathematics is vibrant and dynamic, an incredibly rich and human discipline, a liberal art, and a humanity in the purest sense.

"...the mathematics of a mathematician is profoundly personal." Seymore A. Papert [Pa].

"Although mathematics itself is 2,500 years old, more has been created in the last fifty years than in all the previous ages combined..." Jerry King [Ki].

"In mathematics alone each generation builds a new story to the old structure." Hermann Hankel [Kl].

"(Arithmetic) is one of the oldest branches, perhaps the very oldest branch, of human knowledge; and yet some of its most abstruse secrets lie close to its tritest truths." H. J. S. Smith [Be].

Educated men and women, from the dilettante to the cognoscente, must be at least modestly literate in all fields of intellectual inquiry. Imagine the literate who is not acquainted with the theories of evolution, relativity or quantum mechanics. Imagine the sophisticate who is unfamiliar with the works of Shakespeare, Picasso, or Mahler. But most so-called educated people know nothing of mathematics.

"One's intellectual and aesthetic life cannot be complete unless it includes an appreciation of the power and the beauty of mathematics. Simply put, aesthetic and intellectual fulfillment requires that you know about mathematics." J. P. King [Ki].

"What is there about mathematics that compels so many men and women to work at it with the fervor of dedicated artists and yet keeps it simultaneously outside the experience of the rest of intellectual society?" J. P. King [Ki].

"Outside of the closed circle of professional mathematicians, almost nothing is known of the true nature of mathematics or of mathematics research." J. P. King [Ki].

Most people, at least most 20th century Americans, are interested in the lives of public figures. Even the lives of some intellectuals are of interest to the average citizen: Einstein is a pop icon. Amadeus, a movie about Mozart, was a popular success. There was a recent movie about the physicist Stephen Hawking shown in American popular movie houses. The rank and file recognize references to artists and thinkers as diverse as Heisenberg, Schroedinger, Beethoven, Picasso, Stravinsky, Monet, Plato, Aristotle, Freud, Jung, Camus, and Sartre. But almost no one knows

even the names of the most important mathematicians. Who but the mathematician has heard of Gauss, Galois, Cantor? They are thinkers of the first rank. But unlike their counterparts in every other discipline, their names are completely unfamiliar. Clearly, if the masses were aware of the humanness of the mathematics enterprise, natural human curiosity would demand that mathematicians be included in the class of thinkers worth knowing.

"Those who have never known a professional mathematician may be rather surprised on meeting some, for mathematicians as a class are probably less familiar to the general reader than any other group of brain workers. The mathematician is a much rarer character in fiction than his cousin the scientist." E. T. Bell [Be].

The human essence includes an amazingly robust sense of wonder. If students realize that they have been banned access to a tremendously rich body of knowledge (mathematics), this natural wonder, if properly cultivated, will transform the "banned" into the "tempting." And students will demand to know of it. Bertrand Russell perfectly captured this refined sense of wonder in his autobiography. It is a fitting epigram for this paper.

"There was a footpath leading across fields to New Southgate, and I used to go there alone to watch the sunset and contemplate suicide. I did not, however, commit suicide, because I wished to know more of mathematics." Bertrand Russell [Ru].

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Mathematizing

Lee Goldstein

Mathematics begins
Upon a denominative
Foundation,
Where the anteceding nonverbality
Is in place,
And when a verbal undifferencing
Is eliminative,
Then the symbolic shift
Does take.

The Questionable Probability Theory Behind The Strange Story of *The Bell Curve*'s Bell Curve

Miriam Lipschütz-Yevick
formerly of Rutgers University, now retired

The mathematical underpinnings of Hernstein and Murray's *The Bell Curve* are to be found in the appendices. In the first of these we see a diagram of a few bellshaped (normal) distribution curves with the (scientifically fuzzy) explanation:

"...a common way in which natural phenomena arrange themselves approximately."

The title of the book and the various statistical techniques used do in fact indicate that the authors' interpretation of the observed data assumes that I.Q. is normally distributed in the population. The applicability of many of their statistical methods necessitates that the bellshaped curve prevail. The discussion below explains why a theoretical model based on the conclusions the authors draw from the observed data *will not bring about a bellshaped distribution.*

The normal distribution, even if very prevalent, does not however fall out of the sky. In fact the mathematical criteria needed to produce a normal distribution are not satisfied in the case of the population the authors of *The Bell Curve* hypothesize—a non-homogeneous group in which there is a significant difference between the mean I.Q. of the two groups. The authors cannot have it two ways: either the two population groups—black and white; poor white and middle and upperclass white—are sufficiently homogeneous to generate a bellshaped curve with a common mean, or we are dealing with two distinct populations and the various statistical tests based on the model of a bellshaped curve simply do not apply.

A large number of small, independent, random effects (say, those that combine to generate I.Q.s) may, under certain circumstances, combine to display a collective (statistical) regularity. In particular the sum of a large number of such small random fluctuations may combine into what we call a "stable" limiting distribution law, to which family the bellshaped curve belongs. A good example of when this does happen is the example discussed in *The Bell Curve* of the distribution of the body heights in a class of schoolboys. Similarly a close to bellshaped frequency curve will be observed for the physical sizes in a homogeneous adult population of one gender. There is a reason for this. (For a more detailed discussion see Miriam Lipschütz-Yevick, "Probability and Determinism," *American Journal of Physics*, 1957; a classical and beautiful exposition can be found in the early work *Théorie des Probabilités*, Gauthier-Villars, Paris, 1925,page 103, by the great French mathematician Paul Lévy.)

It so happens that the physical stature of an individual is determined by the sum of the sizes of some two hundred bones making up the skeleton. In a large population of males, say, the small, accidental differences from the mean size—which are caused by a host of environmental and genetic factors—over the whole population for a particular bone fluctuate randomly from individual to individual and quite independently from bone to bone. Some of the bones will be larger than average, some smaller, so that winners are more or less matched equally by losers. Yet even the *largest deviation* from the mean will contribute a negligible part—i.e., be statistically negligible—to the sum of all the individual differences which together determine how physical sizes are statistically distributed over the whole population.

In view of the sloppy theoretical underpinnings of Murray and Hernstein's book, it is doubtful that the measure of these two scholars' achievements would be located at the extreme upper end of such a nonnormal stable distribution curve.

These exactly are the necessary and sufficient conditions—the individual and uniform (collective) smallness of the variations compared to their sum—for the normal distribution to evolve when a large number of

small independent random effects, or "errors" conspire together, i.e., sum up, to produce a statistically regular distribution of some "phenotype."

Clearly these conditions would not be satisfied if our population were composed of, say, American males and Japanese females—for the deviations from the mean would not be uniformly small. The result in this case would, most likely, be twopeaked, a bimodal distribution for physical size. And by the same token, *The Bell Curve*'s conclusion that intelligence quotient is distinctly different for the two subpopulations hypothesized, cannot yield a normal distribution with the one subpopulation squeezed into the lower ten percentile. We are, from a theoretical point of view, not in the domain of the normal distribution.

A bellshaped distribution for a phenotype can then be ascribed to a genetic factor only if this factor operates *randomly* and *independently* on each of a large number of genes which conspire together to produce the particular phenotype. And the measure of the factors must be such that the fluctuations in the values of each component are individually and uniformly (i.e., no component deviation is *overwhelmingly* large) negligible against their sum. Once again *The Bell Curve*'s conclusions preclude that these theoretical (mathematical) conditions be satisfied for the distribution of I.Q.s. For *The Bell Curve* concludes that the subpopulation is such that its genotype will systematically land

The normal distribution, even if very prevalent, does not however fall out of the sky. In fact the mathematical criteria needed to produce a normal distribution are not satisfied in the case of the population the authors of* The Bell Curve *hypothesize.

the measure of its intelligence in the lowest ten percentile. The small individual genetically induced components which are summed in this case are neither independent nor randomly distributed in a uniformly negligible manner over the whole population. A bellshaped curve would hence not be statistically generated and empirically observed.

Yet we *do* empirically observe a normal distribution for I.Q.s as well as many other test results. This is compatible with the hypothesis that the normal distribution evolved from a large number of random, independent environmental and genetic fluctuations, whose differences from the mean were individually and uniformly negligible against their total in a single population. Fluctuations whose values lie mainly to the left of the mean (reflecting negative environmental factors) will so sum statistically and similarly for positive variations—collectively producing a bellshaped curve.

The Bell Curve's asssumptions (or conclusions as the case may be) could more easily be fitted into another model, that of a non-normal stable distribution. The graphs in the book showing the high values for measurements of achievement for a small group of elite college graduates, etc., are compatible with this model. To wit, when a few of the measures of the component terms contribute a sizable fraction of the sum (so that

***...a theoretical model based on the conclusions the authors draw from the observed data* will not bring about a bellshaped distribution.**

the components are statistically not uniformly negligible) a highly skewed distribution will evolve. The distribution of the sum will reflect the distribution of its largest term(s) and a sizable part of the total distribution will be concentrated in the upper tail end of the curve . Such, for instance, is the distribution of wealth and income in most present-day societies. Such too is the distribution of scientific, intellectual, or artistic achievements, where a minute fraction of practitioners makes most of the major contributions.

In view of the sloppy theoretical underpinnings of Murray and Hernstein's book, it is doubtful that the measure of these two scholars' achievements would be located at the extreme upper end of such a nonnormal stable distribution curve. Let us remember that it has been the hallmark of contemporary authoritarian and racist theory-inspired governments to eliminate the true intellectual elite (those at the upper end of the distribution) and their creations in short order (vide Nazi Germany, Stalinist Russia, Cambodia, Bosnia, Rwanda...).

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A University Mathematician's View Of What's Wrong With University Mathematics Education

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I will consider mathematics education only at the university level, where the habits and values of the university mathematician are decisive. I start with a couple of little "anecdotes".

When members of the mathematics department at University X get together to talk about course Y in their Undergraduate Committee, their agenda does not include quality of teaching or student difficulties. The only point concerning course Y is the syllabus.

Practically all mathematics courses descend from an ancient tradition. Even modern courses like linear algebra have now been going on for decades. So they all, including course Y, already have a well-established syllabus, as can be seen from the table of contents of any respectable textbook on the subject. For calculus and pre-calculus courses, that syllabus is dictated (so it is believed) by the needs of the succeeding courses. Intermediate algebra must prepare the student for college algebra, college algebra must prepare the student for calculus 1, calculus 1 for calculus 2, calculus 2 for calculus 3, and so on. There comes a moan, "It's too much material to cover in a semester. Half of them always flunk!" And the familiar answer: "There's nothing we could take out without messing them up for the next semester." Other reforms besides tinkering with the syllabus are neither proposed nor considered.

What about constraints imposed by other departments? A Mathematical Emissary walks over to the Engineering School to talk to a committee of engineering professors about the syllabus for some course in engineering mathematics. To every possible topic, the engineers cry, "Yes! Very good! They should know that too!" The M.E. secretly suspects that not all her engineering colleagues around the table "know that too." Never mind, they want their students to know it. When the M.E. gets back to the math department, her colleagues quickly decide that the engineering profs are "out of their gourds," and cut the swollen syllabus back to traditional size.

Such troubles with syllabi are semi-trivial. Much more serious are the troubles that come from self-defeating teaching styles, and from the teacher's false conception of the nature of his subject.

Nearly all U.S. university professors, including math professors, have been shaped by a shared trauma: their graduate training. They have survived an intensive apprenticeship as aspiring Ph.D.'s, struggling for years to win their advisor/supervisor's approval.

They would learn that treating mathematics students as if they were human beings ("humanistic mathematics education") is the way to avoid mathematics avoidance.

For many, this ordeal has permanently imprinted upon their thinking their advisor's way of thinking and teaching. (Sometimes, it is true, the imprint is reversed. After a "stormy" advisership, the student may seek to teach and think in a way opposite to the advisor's.)

In research, this tendency is well known. An experienced reader recognizes the writing, not only of Professor X, but also of X's students. It is not surprising that somethinng similar happens in teaching style. This tendency is mentioned less often because teaching, unlike publication of research, is a private performance. Not totally private, of course, since it is done in the presence of students. But so far as the math professor's colleagues and fellow mathematicians are concerned, it is definitely private. If mathematicians A, B, and C are asked about the teaching of mathematician D, generally none of them will have any knowledge of it, except what they could conclude from hearing D talk at a research seminar or at a meeting of the American Mathematical Society. D's performance in the classroom will be unknown to any of the three, unless some stray student once commented about it. What the university mathematician does in

the classroom is virtually unknown to colleagues, even in her department. And it's strongly influenced by her experience as a graduate student.

What then is the character of graduate mathematics teaching, which indirectly determines the character of all university math teaching, graduate and undergraduate? The purpose of graduate mathematics teaching is to produce new mathematicians. If enough students from University A get Ph.D.'s, publish papers, and become recognized mathematicians, then University A's graduate mathematics program is a success. If some students in the program fail to follow the lectures, fail to do the exercises, fail to complete the program, that is of little consequence.

Nearly all U.S. university professors, including math professors, have been shaped by a shared trauma: their graduate training.

An influential graduate mathematics professor is of necessity deeply embedded in his research. In his teaching he uses the same language, assumptions, viewpoints as in conversation with his colleagues. These may well be unfamiliar to the graduate student. The student who succeeds has to overcome the disorientation of the lecture room and somehow leap into the gestalt of research level talk.

There is a connection between teaching style and writing style. One vicious trait of the periodical research literature in mathematics is the exclusion of motivation or heuristics. An author is not usually permitted to tell the reader about the false leads and blind alleys which led ultimately to success. Neither is she encouraged to tell why the problem in question is interesting or useful.

In the classroom, the graduate professor is not constrained by journal editorial policy. Nevertheless, his lectures are usually as barren of heuristics or motivation as are his research articles. From a certain so-called "rigorous" point of view, all that is necessary in mathematics is to state the theorems accurately and prove them correctly (rigorously). Where they come from and what they are good for are not considered to be part of the mathematics. Indeed, the graduate professor himself may not have much of a clue where his subject came from, or what it's good for. He may well have been educated in the same abstract, dogmatic style he now perpetuates.

The university mathematicians who are educated this way tend naturally to teach this way. They start as teaching assistants while still attending graduate classes, so the influence from graduate class to precalculus teaching is immediate and direct. They usually are given no training in teaching or lecturing, no obervation or criticism by more experienced teachers. Instead, they are just handed a textbook, a classroom number, and a meeting time.

Later, as assistant professors, they persist in the habits acquired as teaching assistants. After all, nobody ever told them to do differently. Their main concern now is the struggle for tenure, which means—not teaching, but publication. True, there are student evaluations. But students don't usually explain very well what they like or don't like. Anyhow, their evaluations don't matter much, unless they are catastrophic.

Despite all this, some teaching assistants are good natural teachers. And some who aren't naturally good teachers learn after a while to listen to their students, and achieve communication with them. This is a personal matter. Nothing in the university system requires it or rewards it.

Given teachers indoctrinated in this manner, it is no surprise what happens in the undergraduate classroom. As in the graduate class, the lecture method is supreme. Interruptions are not desired. Students are there to take notes, not to engage in dialogue. The important thing is for the professor to give a correct statement of the facts ("theorems"). Failure to mention an exception or a condition is considered "dishonest." If he can possibly do it, he should prove everything he says. ("Prove" means "prove rigorously," leaving nothing out.) This ideal is seriously struggled for in upper-division and graduate courses. In calculus and pre-calculus, everyone admits that it's impossible. That is part of the reason why math faculty dislike teaching these courses. Someone has to teach them, of course. It is done by teaching assistants, part-timers from local high schools, and a few full faculty members forced to take a turn at this sub-mathematical chore.

Most university mathematicians are "pure" (not "applied"). They are ill at ease teaching concepts from biology or physics. If the textbook contains such applications, such teachers prefer to quickly pass over them. Calculus students don't hear the names of Copernicus or Galileo or Kepler. No one expects them to understand the part that calculus played in the scientific revolution that created the modern world. Calculus is just something you do with formulas ("functions") and graphs.

I hasten to say that not all graduate mathematics courses are unmotivated piles of dogma hurled at the heads of hapless graduate students. Some great mathematics researchers are natural teachers. Some are eager to explain the heuristic behind their work. The inspiration from such professors can be carried forward in the teaching of their students, just as the smug dogmatism of other professors can be carried forward in the teaching of their students.

No one will be surprised to hear that the inspiring teachers and graduate classes are not the majority. To be appointed to a graduate faculty of mathematics it is not necessary that one's teaching be brilliant, or even passable. Whether a graduate professor of mathematics does or doesn't take pains with his teaching, his colleagues won't be delighted and won't be upset. What he does with his classes is his business, not theirs.

So the most important disciplinary constraints on the reform of mathematics teaching are the teaching style mathematicians learn in graduate school, and the institutional values which neglect teaching quality in deciding tenure and promotion.

Every newspaper-reader knows that recently a lot of talking, meeting-going, and some Federal money are being spent to reform mathematics education. Should we expect this activity to bring significant results? I would like to think so. But it is my impression that the reform being promoted is curriculum reform, especially increasing the use of computers. The problems of teaching style and of a mistaken idea of the nature of the subject are hardly mentioned. Why so? To answer that question would take a whole separate paper. But so long as it is so, what changes do come will be little more than reshuffling what we already have, not creating anything essentially new or different. (Readers who are curious about what I mean by "mistaken idea of the nature of the subject" can look up references 1, 3, or 4.)

How can we change this lamentable situation, where bad teaching of mathematics is propagated down from one generation to the next?

I can imagine two different ways. One is <u>segregation</u>. Impose a sharp separation between research and teaching in mathematics.

Future teachers of undergraduates would receive a training appropriate for future teachers. That means that along with creative problem-solving and correct calculating, a central place would be reserved for communicating. Both in writing and in speech, both in speaking and in listening. Not only answering questions, but understanding questions. And also knowing mathematics, not only in itself, but also in relation to history and philosophy, to the human sciences as well as the natural sciences. This solution is possible only in theory. The distribution of power in the academic-governmental world makes such a reform inconceivable.

Such troubles with syllabi are semi-trivial. Much more serious are the troubles that come from self-defeating teaching styles, and from the teacher's false conception of the nature of his subject.

Giving up on solution #1, we turn to solution #2: change the thinking of the big-wigs of American math. The top math professors in the top grad schools, the top research managers in the top industrial labs, the top math bureaucrats in the U. S. Office of Education and the 50 State Departments of Education, the top editors of math texts in the top math text publishing companies, the math ed professors in the top Colleges of Education, the top officers and staff of AMS, MAA, and SIAM.

All these people would learn to care how math is taught, not just what math is taught. They would learn that independent work by students is essential in all mathematics classes, K through 20. They would learn that realistic, credible applications of mathematics are indispensable, from K thnrough 20. They would learn

that treating mathematics students as if they were human beings ("humanistic mathematics education") is the way to avoid mathematics avoidance.

That is my "solution."

But what kind of solution do you call that? It's like the solution to the cat problem in Aesop's fable. To be safe from her claws, the mice need only hang a bell round Kitty's neck. But where will they find a mouse willing to bell the cat?

How can we achieve a revolution in the established views on mathematics education? If at all, only by organized effort, by education, lobbying and agitation, continued over years. That is how change is sometimes achieved in the U.S. cultural-political system. Who will do that organizing, educating, lobbying and agitating? Only those who care enough about it to spend the time and effort.

Presently one small organization is engaged in this work (the Humanistic Mathematics Network c/o Prof. Alvin White, Mathematics Department, Harvey Mudd College, Claremont, California, 91711). If enough people care, more organizations will appear. More people will join in their activity. In time, something may happen.

If, on the other hand, we aren't able to turn around mathematical education in the large, at least we should be able to change it in the small. In our own schools, our own departments, our own courses, we can teach students (not merely "teach the material"). We can make sure we understand where the mathematics came from and where it is going, and share this inside information with our students. We can insist on interaction in the classroom, not tolerating a passive audience that merely copies formulas from the blackboard. To change the old saying slightly, we can light a candle or two, even while we curse the dark.

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"Modern Mathematics" at Sonoma State University

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SUMMARY

This essay is a report on a liberal arts course designed to meet the "Quantitative Reasoning" component of the general education requirement as implemented at Sonoma State University. This particular course is also included in the procedure by which the university accommodates students who are certified as being learning disabled in mathematics.

ENROLLMENT

Between Spring '90 to Fall '93 with Summer '92 and '93, I taught fourteen classes of Math 141. There were 445 students in these classes including 136 "special-admission" students who were enrolled through the Sonoma State University Disability Resource Center. The nature of this special-admit program will be discussed later in this paper.

GOALS

Mathematics 141 (Modern Mathematics) is an experimental course originally entitled "Liberal Arts Mathematics" and was designed to provide students with a liberal arts mathematics course which promotes:

- An understanding of logic, generality, abstraction and proofs in mathematics—specifically the construction of indirect proofs.
- An awareness of current significant problems and ideas of interest to the international mathematics community.
- An awareness of the humanity and the limitations of mathematics.
- The development of students' skills in writing about mathematics.

The variety of problems in this course made demands upon the following five skills: Numerical, Algebraic (generalized numerical), Logical, Spatial, and Verbal.

When this course was first developed, there was no single text that could be used to accomplish these goals, so I planned to create an anthology composed of reprints of articles from about twenty-five books and to develop some problem sets to go along with these articles. It turned out that there was considerable difficulty in getting copyright permission for reprinting the articles, so as an alternative to creating an anthology, I gave students a reading list of books containing the articles and a set of corresponding problems, with no explanatory text.

This was not adequate since there were not enough copies of the books for all the students. Eventually, as the course developed over several semesters, I started adding my own descriptive text to the problem set, dropping books from the reading list, and writing my own text. It is now published as *Math Odyssey 2000* [1], Stipes Publishing, 1994.

MEANS OF GOAL ACHIEVEMENT

Here are some sub-goals and specific ways in which these goals were accomplished.

Logic, Generality, Abstractions, and Proofs

- Presentation of logical paradoxes, Aristotelian law of the excluded middle, universal and existential quantifiers, examples of direct and indirect proofs.
- Study of the rational and irrational numbers.
- Euclid's proof of the infinitude of primes.
- The uncountability of the irrationals.
- The concept of measure of sets of reals.
- Presentations of abstract axiomatic systems, and components of a logico-deductive system.
- Examples of abstractions—extending the interpretation of various axiomatic systems to "non-mathematical" entities, such as people, words and oranges.

Current Problems

This included the introduction to long-standing problems either just recently solved or still unsolved. The solved problems include the Four Color Theorem, the Continuum Hypothesis, Gödel's Theorem, and Fermat's Last Theorem. The unsolved problems include Goldbach's Conjecture and the infinitude of twin primes.

Humanity and Limitations of Mathematics

Throughout the text there are small anecdotes and references to various mathematicians; typically, one out of eight homework problems required an answer based upon such a reference (see Note 1). One entire chapter is devoted to biographies of ten interesting mathematicians (five men and five women) with every problem in the exercises asking questions about

I believe that Math Anxiety turns into Math Panic in the summer's faster pace.

the content of these biographies. In the text and in the reading assignments, we studied the events of the early 20th century relating to Frege, Hilbert, Russell, Gödel, and the constructivists. We studied applications of Gödel's Theorem.

Skills in Writing About Mathematics

An outside book of readings is assigned (in the first semester it was a book list; later I used Guillen's *Bridges to Infinity* [2] until it went out of print in 1992. Then I used Paulos' *Beyond Numeracy* [3]) (see Note 2). Students wrote essays based on readings in these books; I rearranged the order of the authors' chapters to coincide with the material in my text as we covered it in class. The essays, each 5 to 7 typewritten pages in length, were to reflect three things:

1. "Critical Thinking"-type thoughtful summaries of the authors' writing, with insights and personal analysis;
2. discussion of the connection between the readings and our class work, or any other experience; and
3. their personal reaction, in the style of a journal, to both the material from our text and that from the reading assignments.

Essays were graded on these three aspects and were corrected in grammar and spelling. Their authors were chastised for "un-cited", not quoted, "lifted" passages. The students quickly learned to use quotation marks when appropriate.

MEASURING GOAL ACHIEVEMENT

In the latest classes I used 4 exams, 3 midterms plus the final exam (counting 65%), 32 homework assignments (counting 10%), and five essays based upon outside reading (counting 25%).

For some of the earlier classes I used the four exams (60%), 15 weekly journals (5%), 4 essays (25%) and 32 homework assignments (10%). One semester, I gave two classes (60 students) 9 essays, which meant that I ended up grading 540 essays that semester! Big mistake.

SPECIAL ADMISSIONS:
"Learning Disabled" (LD) Students

Some students in this class had not completed our prerequisite of Math 50 (intermediate algebra), had not achieved the equivalent score on the ELM and were admitted through SSU's special program for students with learning disabilities, which is as follows:

1. If a student passes Math 40 (Elementary Algebra), but cannot pass Math 50 and cannot attain an equivalent score on the ELM (Entry Level Mathematics exam) even after repeating, then that student may be eligible to be tested for learning disability.
2. The student is interviewed by a learning disability specialist in the Disability Resource Center to determine whether or not testing is appropriate.
3. If the student is tested and certified as learning disabled, then he or she can be admitted to Math 141 as a special admission.

This admission provides for extra tutorial assistance and certain other arrangements specific to the student's needs. These arrangements apply mostly to exams: isolation during exams, or extra time on exams.

Writing Assignments

Because the regular students had made better grades than the special admits (see Table 1), I had the impression that the regular students in Math 141

TABLE 1: Grade Point Averages (A = 4.00, B = 3.00, etc.)

	Regular Students	Special Admits	Total
Number of Students	309	136	445
Grade Point Average	2.61	2.07	2.44

were, in general, better essay writers than the LD students. I thought I would test this hypothesis by examining the data. My opinion was wrong. Table 2 is of the results for six classes for which I had comparable data (prior to Summer '92 I had used a different type of writing assignment).

Why was there this discrepancy between the Academic year grades and the summer grades? Among the regular students in my summer classes I had at least 3 high school math teachers, a professional advertising writer, and an SSU valedictorian. (She was the student speaker at graduation, June '93.) Also, I believe that Math Anxiety turns into Math Panic in the summer's faster pace.

Why Special Admits Succeeded

Of the 136 special admit students, 17 (12%) made A's, 29 (21%) made B's, 48 (35%) made C's; this is evidence of success in this class. Why was there this much success?

- Much of the material was more "verbal" than usually found in math classes.

- The best of the special-admit students were just plain GOOD STUDENTS. They attended every class and always had their homework ready to turn in on time; they could and did read and write well. They often re-read the reading assignments and had rough drafts of their papers written well ahead of the deadline date. They made changes to the drafts as they gained more insights from our classroom discussions. They participated in these discussions, asked questions about what they didn't understand, and formed study groups.

- The Sonoma State University Learning Center supplies a mathematics major as a student assistant (not a "reader" to grade papers) for this class. This student sits in class and then holds review sessions for the special-admits, though other students may also attend. I periodically met with the student assistant to discuss teaching and studying strategies and to indicate which topics needed to be emphasized in the review sessions.

- Some students did request a separate room for taking exams, but not many. Of the 136 "special-admit" students, only about 12 (9%), actually requested a separate room for taking exams (either my office or a supervised room in one of the administrative offices in "The Village"). The maximum amount of extra time needed on any exam was about ten minutes; most of them completed the exams within the regular time.

Why Were There Failures Among the Special Admits?

There were 12 F's (about 9%). Most of these students were absent a lot, usually didn't read the assignments, didn't get their essays written on time or at all, didn't seek tutors, and didn't study for exams. In about half of these cases, they seemed to be students who had given this class their lowest priority. Some (about 4%), however, did do all the work and tried desperately to overcome an extremely severe disability, but still fell short of passing.

Why were there D's among the special admits?

I could detect two reasons. First, there were the C students who had lost their intensity too soon or for some other reason (pressure from other courses, etc.) dropped down to a D ("C's gone sour"). The other D

TABLE 2: Grades on Essays Based Upon Reading Assignments Maximum possible grade=25.

	Special Admits		Regular Admits		Totals	
Class	Number	Average	Number	Average	Number	Average
Fall '93	7	22.5	36	19.4	43	19.9
Spring '93, sec.1	9	21.3	17	19.2	26	19.9
Spring '93,sec.2	9	20.2	21	18	30	18.6
Fall '92	8	20.9	31	21.3	39	21.2
Summer '93	7	19.7	11	22.5	18	21.4
Summer '92	8	18.7	20	23.5	28	22.1
TOTAL	48	20.5	136	20.4	184	20.4
Academic year	33	21.2	105	19.6	138	20.0
Summer	15	19.2	31	23.1	46	21.8

students were "breakthrough F's". They did attend all classes, did all the homework while getting a lot of help from study groups, and tried very hard on every exam, getting most of the verbal problems and almost half of the computational ones. They might not have passed had it not been for the writing assignments.

"Severe" Learning Disabled, a Judgment Call

Through a very subjective (but not pejorative) judgment on my part I thought I was able to identify some of the students as "severe" and some "not severe" in the difficulty they were having with this material. I did not assign these labels to students while they were in my classes, but only while going back over my grade books, after the fact. Table 3 compares the grade distribution among three groups.

"Severity" had very little relationship to numerical or algebraic skills. They could work with calculations involving specific numbers, and they could substitute specific numbers into formulas. They also "knew" (had memorized) the basic rules of algebra. And severity definitely had nothing to do with verbal skills. Their ability to communicate ideas verbally, to insert personal analysis, to write coherent sentences and well organized paragraphs was more than adequate.

It now seems to me that severity is related to logic in the following sense: a failure to appreciate the intricacy of detail and the "literalness" of a mathematical statement, suggesting a "glossing-over" of the fine points of a logical process, perhaps in the sense of "fuzzy logic". Whatever "severity" is, I thought a student's disability was "severe" if I perceived that student to be unsuccessfully struggling with many of the concepts.

By struggling I mean:

- spending a lot of time on these problems,
- getting help from every available source,
- requiring a variety of explanations and completely misunderstanding every one of them (especially the generality of words such as "any" or "every", as in "prove that the square of every odd number is odd").
- turning in a paper for every homework assignment, even if some were just token papers.

A token paper was one that contained all the solutions to those problems the students could work, plus solutions which were either paraphrased or even copied from the answers or the hints given in the text, *Math Odyssey 2000*. Every problem in this text has a hint and more than half of the problems have answers in the back of the book.

I encouraged and even demanded token papers; the justification for this is based upon the theory that "mathematics enters the body through the fingertips". In addition, it gave me one more chance to concentrate on a particular difficulty that might be revealed by an incomplete copy or an inaccurate paraphrase and to write a comment aimed at resolving this particular difficulty. By the way, sometimes, an incomplete copy was one in which a small word such as "not" was omitted!

To put it another way, not struggling means:

- giving up,
- failing to attend class, and
- failing to turn in even token homework papers.

From Table 3, we see some very interesting results. Some non-severe students got F's (12%), usually because they failed to turn in their homework and essays in on time, skipped exams, or failed exams even when they could work the hardest problems. Of course, some severe students also got F's.

TABLE 3: Grade Distribution Among Three Student Groups

Student	A	B	C	D	F	Total
Non-severe	15(18%)	21(25%)	29(35%)	13(15%)	6(7%)	84
Severe	2(4%)	8(15%)	19(37%)	17(33%)	6(12%)	52
LD Total	17(12%)	29(21%)	48(35%)	30(22%)	12(9%)	136
Regular	82(26%)	96(31%)	79(26%)	34(11%)	18(6%)	309
Overall	99(22%)	25(28%)	127(29%	64(14%)	30(7%)	445

TABLE 4: Comparison of Correct Answers on Final Exams by "Regular", "Severe" LD, and "Non-severe" LD Students on Three KeyFinal Exam Problems

Student	Number	MAP	FERMAT	PRIME
Regular	103	49(47%)	29(28%)	56(54%)
LD(Severe)	19	7(37%)	4(21%)	7.5(39%)
LD(Nonsevere)	21	12(57%)	2(9%)	7.5(36%)
LD(Total)	40	19(47%)	6(15%)	15 (37%)
All Students	143	68(47%)	35(24%)	71(50%)

Some severe students (4%) got A's because they got perfect grades on their essays, memorized proofs for the exams, got lots of help on homework, and were perfect on all the other, easier, exam problems.

Key Problems

Intuitively, I thought I could see an easy way to identify "severe" LD students as ones who had trouble with certain key concepts involving logic, generalization and geometry. I noticed that some students had particular difficulties with the following problems:

- Stating denials and contrapositives
- Construction of a direct proof
- Understanding Fermat's Last Theorem
- Coloring a plane map properly (understanding the four color theorem).

I did a grade analysis on the following final exam problems for which I had comparable data (see Table 4):

MAP: A planar map with instructions to color it properly

FERMAT: A pair of problems; prove FLT for (say) n = 15, given it is true for (say) n = 5 and prove FLT among the odds for (say) n = 3. (Here n is the expo nent in the equation x^n + y^n = z^n.)

PRIME: Prove that there is no largest prime number.

Conclusion

My subjective identification as "severe" failed to distinguish between severe and nonsevere because of the results on FERMAT and PRIME. The funny thing is that the nonsevere LD's had more learning ability than the regular students on MAP.

Perhaps map coloring was sufficiently unlike any math problem they had had that they didn't believe that they couldn't do it. Or maybe they were more desperately in need of a correct solution on the exam and did not treat this problem as cavalierly as the "regular" students.

STUDENT REACTION

Here is a sample of the data that I had at hand; it is representative of the student responses to this class. The written comments were from both summer classes and Spring '92; the machine graded responses were from Spring '92 and Summer '92.

Written comments: (N=65) 75.4% positive, 15.4% mixed, and 9.2% negative. Using 1 for positive, 0.5 for mixed, and 0 for negative, we had a weighted rating of 83%. Machine graded: (N=94) (Summer school 7-point scale transformed to 4.0 scale) 3.46 out of 4, or 86%.

ANOTHER APPROACH TO LIBERAL ARTS MATHEMATICS: NLA, THE SLOAN FOUNDATION'S "NEW LIBERAL ARTS"

Between 1982 and 1990 the Alfred P. Sloan Foundation has spent over $21 million to finance curricular changes (in science, mathematics, humanities, fine arts, and social and behavioral sciences) at 36 liberal arts colleges throughout the U.S. by providing faculty leave grants, course development funds, and financial support for the publication of a variety of texts, monographs, pamphlets, and other classroom material. They have also provided financial support of conferences for faculty members from a great many other colleges

...mathematics enters the body through the fingertips.

all over the U.S. The purpose of this massive effort is to redefine the concept of "Liberal Arts" and to institute the "New Liberal Arts" (NLA) as replacement for the traditional liberal arts curriculum (see Note 3).

According to the NLA 1990 report [4] , "...the Sloan Foundation's New Liberal Arts program is based upon the belief that a liberal education for our time should involve undergraduates in meaningful experi-

ences with technology and with quantitative and mathematical approaches to problem solving in a wide range of subjects and fields". They further state that students should understand "...the nature of modern technology, the scientific and cultural settings in which engineers work, and also the impacts (positive and negative) of technology on individuals and society" ([4], Page 1).

They continue to argue that students should be much more comfortable than they currently are with calculations and "reasoning with numbers", and with the application of modern mathematical and physical models not only in the sciences, but in the social sciences and humanities as well.

The report also tells us that a goal of the NLA program is to direct undergraduate curriculum toward "Quantitative Reasoning" which it defines as "engineering thinking", involving "... some numerical, graphical, algebraic and other quantitative or mathematical problem solving technique" [4].

The NLA program strongly promotes mathematical modeling as a crucial part of the curriculum, because they say a mathematical model "... is neutral by its nature and thus the very same piece of mathematics can be variously interpreted to fit applications that at first blush appear quite unrelated".

They speak of "case study" components of the curriculum developed at some places such as at the University of Chicago and at Mount Holyoke College, where the computer modeling software STELLA is extensively used.

Comparison

I like the way the Sloan program is trying to educate our students for today's technological world, but their approach seems to negate the non-technical and non-professional quality that we think of as being the flavor of liberal arts. To me the NLA in the Sloan Program stands for "Non-Liberal Arts", and I think that our Math 141 course in Modern Mathematics is actually closer to the traditional definition of liberal arts and still satisfies even the Sloan Foundation's definition of "Quantitative Reasoning".

NOTES

- You can't expect students to learn anything about the biographies if you don't ask questions about the biographies. In calculus and differential equations, I have tried using texts with biographies sprinkled throughout the text, but these never "worked" in getting students to learn anything about the biographies since they seemed to be off-hand and gratuitously thrown in.
- The two books I used had received negative reviews (Guillen after being in print for 7 years and Paulos after 2 years in print) in the Mathematical Monthly. The criticisms of these books were based upon claims of "errors" and "misleading" oversimplifications and "inaccurate" representation of complex ideas. The critics were saying that students should not be allowed to read these books unless they read them as examples of falsehoods in mathematics.
- The original seven liberal arts (from about 50 BC) were Astronomy, Arithmetic, Geometry, Music, Rhetoric, Grammar and Dialectic– a curriculum for a free people. Today by "liberal arts" we mean intellectual, non-technical and non-professional areas of study such as literature, philosophy, and history.

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Book Review: *Introductory Algebra: A Just In Time Approach*

Textbook author: Alice Kaseberg, Lane Community College, PWS Publishing Co., Boston, MA
Reviewer: Gayle Smith, Lane Community College, Eugene, OR 97405

INTRODUCTION

The community college introductory algebra population (not including the math lab population) comprises approximately 20-25% of the total mathematics student population. Each student in this group can be characterized by one of the following statements: "I have never had an algebra class"; "I have been out of school for many years and need to review forgotten algebra skills"; "I've just graduated high school but need to review my algebra"; or "I failed algebra in high school". What a diverse group of students! It can also be assumed from the above descriptions that there will be variations in age, learning styles, and educational goals.

The question naturally arises: "Is there an introductory algebra text that has a chance of meeting the needs of each of these adult students?" The answer is, "Yes, most definitely." One only needs to consider Alice Kaseberg's *Introductory Algebra: A Just in Time Approach*. This textbook offers each introductory algebra student the opportunity to master needed algebra skills in a way that encourages problem solving, critical thinking, and the use of technology, while motivating the student to study mathematics for its power and usefulness.

BOOK REVIEW

In the title, the phrase "just in time" reflects the author's philosophy of presenting the algebraic principles and procedures as they are needed — and not presenting what is not needed. Also, "just in time" is the phrase chosen by the author to indicate that the standards of the 21st century, set forth by the American Mathematical Association of Two-Year Colleges (AMATYC) and the NCTM Standards, have been met.

Reading through the table of contents, one can see that this is a nontraditional text. All of the necessary topics for introductory algebra are included, but not in the expected order. New topics appear and some old topics are altered — all to present algebra as an active and useful discipline, and bring it up to current standards.

For example, necessary, rich experience with rational expressions is provided in Chapter Four: Ratios, Rates and Proportional Reasoning. A focus on applications, geometry, statistics, tables, calculators, and graphs is the rationale behind the choice and arrangement of topics.

Alice Kaseberg's text keeps the student active and involved. George Polya's four problem solving steps and strategies are important foci of the text. Applications, geometry, and statistics are woven throughout the examples, exercises, and projects.And, as "just in time" indicates, information needed to understand important associated ideas is an integral part of the instruction.

Alice Kaseberg speaks to the reader humanistically, the way she teaches in the classroom. Her presentation is more informal than traditional, and focused on application. Her explanations are clear, detailed, and precise.

For example, Chapter Three: Equations opens with a letter written to the author's grandfather. The question in the letter states:

> Assume a 12 inch section to be fitted into a steel cable that formerly fitted the earth snugly on a great circle. If the cable is now held a uniform distance from the earth's surface, could a mouse go under it?

The solution to this question involves manipulating a literal formula, the circumference of a circle. Chapter Three is the student's first experience with equations. Can an introductory algebra student be expected to understand the question that this letter offers? The answer is yes, since the author very carefully lays the foundation for the student.

First, the concept of equations is developed by having students look at input/output tables, hunt for pat-

terns, recognize and translate patterns symbolically, draw and analyze graphs, and look at the effect of scale on a graph. Next, equations are solved by "working backwards", using tables, algebra tiles, and graphing as well as symbolic means. Attention is given to vocabulary and the distributive property. Solving equations happens informally, then formally. Then, in section 3.5 (Solving Formulas), students are ready to solve the "mouse problem". They are familiar with the concept of circumference from Chapter 2. The formula is manipulated numerically: for each of three radii, students are asked to

1) calculate the circumference,
2) add 12 inches to the circumference,
3) find the resulting radius,
4) compare the beginning and ending radii.

Finally, equation-solving steps, previously developed, are used to prove that the answer to the letter is "yes, an ordinary mouse can go under the cable!"

As I studied the text, the intent of the author became clear to me. I found her style to be friendly, yet challenging. An analogy came to mind of a parent interacting with his/her child. Let me explain. When you interact with your child, your desire is to maximize the richness of the opportunity. You will be verbal but you may also refer to models or pictures, or use manipulatives or technology to aid understanding.

I hope I have given students permission to think for themselves. One of my favorite phrases is "Choose the method that makes the most sense to you." Unfortunately, that kind of choice is going to frustrate those students who want "one way" to do things.

You will capitalize on real-life examples that are meaningful to your child. You will bring in connections of interest (including previous, related sharings) to broaden the scope of your sharing. You will rely on previous learning, but assist in the learning of new skills as needed during your sharing. In her text, the author offers a maximal experience to her readers!

Alice Kaseberg speaks to the reader humanistically, the way she teaches in the classroom. Her presentation is more informal than traditional, and focused on application. Her explanations are clear, detailed, and precise. Manipulatives, drawings, alternate ways to solve a problem, comparisons of equivalent solutions, inclusion of warm-up exercises, and group activities are each presented to respect different learning styles and to keep the student involved. The text is intended to be read completely by both student and instructor.

Intent, important vocabulary, definitions, and summarizing statements are offered as part of the narrative. Examples include complete solutions with narrative. Often, examples are revisited. Examples, exercises, projects, and photographs are the original work of the author unless otherwise indicated. The author shares her professional expertise of mathematics, business administration and engineering, her years of teaching, and herself with the reader. In a few cases, she shares behind-the-scenes, personal history of an exercise with the reader.

For example, in Section 7.5 (Solving Radical Equations and Formulas), Example 6 is an application of square roots, with a personal touch:

> *As a child I dropped a paper-wrapped sugar cube from the open window of the Washington Monument in our nation's capital. Fortunately it was a cold winter day in the 1950's, and no one was at the base of the 555-foot tower.*

The author goes on to offer the radical equation that represents the time for a dropped object to hit the ground. Students are then asked to work through a sequence of related questions. The solution to this example ends with the following note: *P.S. My parents were furious! The Washington Monument windows are now sealed.*

It is impossible to describe the entire text, with its many instances of non-traditional excellence, to the reader! I close with a few reflections, written by the author.

REFLECTIONS BY ALICE KASEBERG

Having taught three years from the manuscript, I was amazed at how different, and how improved, the typeset version is. I appreciate even more the effort made by the students and field-test instructors to use the manuscript version.

There are many more illustrations (tables, graphs, drawings, and pictures) than in other introductory algebra books. I am surprised at the impact they made. The tables and graphs add a visual continuity to the text that those of us who are visual learners have missed in the symbolic algebra presentation.

I have written the book to answer these questions: "Why are we learning math?"; "Where is math used?"; and "What does it mean?". I tried to answer these questions in a spirit of discovery. One of the most frustrating things for me to teach from my own material is that the book now contains all the neat things I used to share with the students in class to make the course more interesting.

I wonder if the students will have the patience to read the book. Both the teacher and the students need to read the book. The student can be taught to scan before class: read the boxes, boldface words, and italic definitions. After class, a more careful reading may make more sense.

I think good teachers get students dependent upon good lectures and forget to teach the student to read the book. It reminds me of the saying "Give a hungry person a fish and they eat for a day. Teach them to fish and they can eat forever." (Sadly, the fishing analogy no longer works because we have wiped out the fish populations, but nevertheless the concept fits in teaching students to read their mathematics books.)

I hope I have given students permission to think for themselves. One of my favorite phrases is "Choose the method that makes the most sense to you." Unfortunately, that kind of choice is going to frustrate those students who want "one way" to do things.

The numeric, symbolic, and graphic approaches are becoming more of an environment. I used to think of them as processes. I think teaching communication skills will help students acquire verbalization needed to make connections among the three approaches. There are lots of questions in the book that ask students to write, to explain in sentences, or to summarize ideas.

I wonder what more can be done to teach the thinking skills needed to do this writing. Without the communication skills the numeric, symbolic, and graphic approaches result in tripling the things the students memorize. Before the next edition I want to learn more about how people teach sentence and paragraph writing.

ADDITIONAL INFORMATION

Features of the Text

- the text is rich with non-traditional topics: interval notation, the binomial theorem, Pascal's Triangle, variations on slope, range and standard deviation, and non-linear graphs
- warm-up exercises are presented at the beginning of each section to act as openers
- mathematical and non-mathematical definitions of important vocabulary are discussed
- tables and graphs are used extensively to connect algebra, geometry, and statistics

George Polya's four problem solving steps and strategies are important foci of the text. Applications, geometry, and statistics are woven throughout the examples, exercises, and projects.

- patterns, their corresponding equations, and graphs emphasize that algebra is a language that connects arithmetic to analysis
- keystrokes for the scientific calculator are presented, as appropriate (the graphing calculator is introduced as an option to students, occasionally; more information is offered to instructors in the Annotated Instructor Edition)
- an index of applications is included for reference
- the Glossary/Index gives all chapter review vocabulary (with definitions) and a full index
- exercises/projects are noted if intended for group work, or if a graphing calculator is required
- each chapter contains a Mid-Chapter Test, a Chapter Review, Review Exercises and Test; a comprehensive Final Exam Review is included

Materials and Availability

- the student edition (ISBN 0-53494-392-2) contains solutions to selected odd exercises
- the student edition sells for $40.00 to bookstores
- in "To the Student", following the Preface, the author offers information on learning styles, independent thinking, alternative approaches and strategies for success, in a friendly and encouraging tone

- the annotated instructor's edition (ISBN 0-53494-395-0) contains annotations in the margin, answers to exercises, and "How to Use" which gives section-by-section teaching hints and strategies and information on course planning
- the carefully written student solution manual (not authored by Ms. Kaseberg) gives solutions for all odd numbered exercises
- assessment materials (ISBN 0-53494-397-7) painstakingly modeled after the author's style contain two versions of tests, a list of other questions, and one project problem for each chapter
- assessment materials are available in hard copy, and on disc for Mac and IBM
- the textbooks and all supplementary materials are due at the distribution center August 21, 1995
- for more information contact International Thompson Publishing Co., 1-800-423-0563

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Letter Division

Paul J. Tobias

Test your math logic! Each letter stands for a digit from 0 through 9; the same digit stands for the same letter throughout the problem (answers on p.46).

```
               IN
       ---------
THINK | NEW WAYS
        YWS TSA
        -------
         YN HEAS
         YH INHT
         -------
          H ATIW
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          TO
      ------
EASY | SOLVE
       ZYXL
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           THE
      --------
MATH | CRUSADE
       SMMMD
       -----
        USMED
        UHDDA
        -----
          DUAE
          MATH
          ----
           HUD
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Special thanks to: Sheila Tobias
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Tucson, AZ 85719

The Most Humanistic Mathematician: Florentin Smarandache

Joanne Growney
Bloomsburg University

Florentin Smarandache was introduced to members of the Humanistic Mathematics Network, when two of his poems were published in Issue #7 of this Journal (April 1992).

Recently Smarandache was featured in an article entitled "The Most Paradoxist Mathematician and Philospher of the World." Written by Charles T. Le, it appeared in the *Bulletin of Number Theory*, Vol. 3, No. 1 (March 1995), Number Theory Association, Tucson. The abstract for the article introduces this humanistic mathematician as follows:

Florentin Smarandache, a Romanian mathematician and poet, exiled in the United States, used his talents in wrong directions: poetical skills in mathematics and mathematical skills in poetry! He published in mathematics a collection of "Only Problems, Not Solutions!" A function in number theory related to an infinity of unsolved problems has been called "the Smarandache Function." And in literature, we find his name on a book of poems called "NonPoems," and another called, "The Sense of NonSense." He has established and led a mathematical (but very contradictory) Paradoxist movement in literature.

"Country of Animals," a drama written by Smarandache, was staged at an international science fiction convention in Glasgow in August 1995.

The following poem, by Larry Seagull of Glendale Community College, conveys some of the spirit of Smarandache.

Poem in Arithmetic Space

Larry Seagull
Glendale Community College

There exist some sequences defined as "Smarandache" sequences of numbers.*

Smarandache consecutive sequence: 1, 12, 123, 1234, 12345, 123456, 1234567, 12345678, 123456789, 12345678910, . . . A number in this sequence is called a "Smarandache consecutive number."

Smarandache circular sequence: 1, 12, 21, 123, 231, 312, 1234, 2341, 3412, 4123, 12345, 23451, 34512, . . . A number in this sequence is called a "Smarandache circular number."

Smarandache symmetric sequence: 1, 11, 121, 1221, 12321, 123321, 1234321, 12344321, 123454321, 1234554321, . . . A number that belongs to this sequence is called a "Smarandache symmetric number."

Smarandache deconstructive sequence: 1, 23, 456, 7891, 23456, 789123, 4567891, 23456789, 123456789, 1234567891, . . . A number that belongs to this sequence is called a "Smarandache deconstructive number."

Smarandache mirror sequence: 1, 212, 32123, 4321234, 543212345, 65432123456, 7654321234567, . . . A number that belongs to this sequence is called a "Smarandache mirror number."

THE SMARANDACHE PARADOXIST NUMBERS

A number n is called a "Smarandache paradoxist number" if and only if n doesn't belong to any of the Smarandache defined numbers.

Dilemma

Find a Smarandache paradoxist number sequence.

Solution (?)

If a number k is a Smarandache paradoxist number, then k doesn't belong to any of the Smarandache defined numbers; therefore k doesn't belong to the Smarandache paradoxist numbers either!

If a number k doesn't belong to any of the Smarandache defined numbers, then k is a Smarandache paradoxist number; then k belongs to a Smarandache defined numbers (because Smarandache paradoxist numbers is also in the same category) — a contradiction.

Question

Is the Smarandache paradoxist number sequence empty??

THE NON-SMARANDACHE NUMBERS

A number n is called a "non-Smarandache number" if and only if n is neither a Smarandache paradoxist number nor any of the Smarandache defined numbers.

Dilemma

Find a non-Smarandache number sequence.

> **Question 1**
>
> Is the non-Smarandache number sequence empty, too ??
>
> **Question 2**
>
> Is a non-Smarandache number equivalent to a Smarandache paradoxist number??? (This would be another paradox!! ...because a non-Smarandache number is not a Smarandache paradoxist number).

THE PARADOX OF SMARANDACHE NUMBERS

Any number is a Smarandache number, the non-Smarandache number too. This is deduced from the following paradox**: "All is possible, the impossible too!"

REFERENCES

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[2]** Charles T. Le, "The Smarandache Class of Paradoxes", in <Bulletin of Pure and Applied Sciences>, Bombay, India, 1995, and in <Tempus>, Editor Geo Stroe, Bucharest, No. 2, 1994, and in <Abracadabra>, Salinas, CA, 1993.

[3]* "The Encyclopedia of Integer Sequences", by N. J. A. Sloane and S. Plouffe, Academic Press, 1995; also online, E-mail: superseeker@research.att.com (SUPERSEEKER by N. J. A. Sloane, S. Plouffe, B. Salvy, ATT Bell Labs, Murray Hill, NJ 07974, USA).

Haiku

Frances Rosamond
National University

A haiku is a special form of poetry having three lines, with a 5-7-5 pattern of syllables. In Japan, a haiku master is usually a very learned person who can read the various nuances and innuendos captured in the haiku.

My haiku is about my geometry class. Here are some of the thoughts embedded within the words.
• eager: my geometry students hungry for learning.
• chewing: we have a saying that if we want to think something over, we will "chew" on it. I chose "chewing" to indicate that learning is a process of activity taking place, not something static or confined. I also chose "chewing" because it implies wolves or animals. Animals are not predictable. They may nuzzle up and be willing and ready to work and think, but sometimes they have other things than school on their minds and cannot pay attention. Also, if they do not feel satisfied, they can tear you apart.
• tales: their lessons. A haiku typically includes a season: spring, summer, winter, or fall. I teach in a year-round school so that we do not start school in September, as is often usual. The word "tales" is meant to convey a sort of "old-fashioned" sense that makes the season one that stretches from ancient history to the present.
• giants: a famous mathematician said that he could accomplish much because he "stood on the shoulders of giants" (i.e., previous mathematicians' achievements).
• earth-measure: this is the definition of geometry, and one of the earliest uses to which mathematics was put.
• angles: allusion to Euclidean geometry, to Euclid, and to his predecessors.
• waves: allusion to modern geometry, which includes the study of periodic occurences.
• pattern: allusion to the whole of mathematics, which some conceptualize as a "pattern".

HAIKU

Eagerly chewing
Tales of Giants' Earth-Measure:
Angles, Waves, Pattern.

Attutudes of Students to Independent Learning

S. Kenneth Houston
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Jordanstown, Northern Ireland

INTRODUCTION

First year undergraduate degree students and second year undergraduate diploma students of mathematics at the University of Ulster take a one semester course which is called "Mathematical Modelling" that embraces not just mathematical modelling but also a study of some mathematical models and relevant mathematical methods. It is a second semester module and they will have taken courses in algebra, calculus, statistics, and computing beforehand.

The mathematical methods studied largely relate to the algebraic solution of first- and second-order ordinary differential equations such as are found in most courses of this nature and are covered in chapters ten and eleven of James (1992). The mathematical models studied make use of these differential equations and include such topics as population dynamics, projectile motion, and oscillations. (See, for example, Burghes and Borrie, 1981.)

Students are introduced to the process of mathematical modelling through problem-solving modelling activities carried out in small groups (usually of size four). They are taught, and encouraged to develop, communication skills through the group work and through written reports and oral and poster presentations (Berry and Houston, 1994) and comprehension tests (Houston 1993). They engage, from day one, in the group modelling activities; this continues as a regular weekly activity throughout the semester.

In the past, the methods and models parts of the curriculum have been taught via traditional lectures and tutorial classes. In 1992-93 and 1993-94, the methods part of the curriculum has been studied by "independent learning" with peer-tutor support.

There were several reasons for making this change. There was the desire to respond to pressure to be "doing more with less" i.e. to try to remove the teacher from the classroom for at least some of the time. More importantly there was the belief that students should be encouraged from an early age to take more responsibility for their own learning and to develop peer support groups to help one another learn. There was the belief that students should be encouraged to engage in "active learning", to seek out information for

"What we need are mathematicians who enjoy their work and not think it's a drag, pain or bore!"

themselves and to convert it to knowledge in order to achieve personal understanding. (See, for example, Denicolo et al., 1992, Entwistle and Tait, 1992.) There was the belief that the requirement to tutor someone in a topic enhances learning by the tutor - "To teach is to learn twice". The learning resource most readily available to students is their peer group, and it was considered desirable to set up structures to encourage students to use one another in this way. There was the belief that students who engage in self and peer assessment are better prepared for tutor assessment.

Then the subject matter of the methods curriculum lends itself to learning by independent study. The material is fairly standard and is well presented in many textbooks. (The textbook chosen as the "reader" for this course was James, 1992.) Explanation is supported by worked examples and test questions (with answers).

METHODOLOGY

On day one students were assigned to groups of three or four. These groups were to function both as task groups for the mathematical modelling strand of the course and as peer tutoring groups for the methods strand. These strands and the "study of models" strand ran more or less in parallel through the semester with one two-hour slot per week being assigned to each. It was intended that the lecturer would not be present during the independent learning sessions, but

would deal with questions at other sessions, and that some senior students would be available from time to time to act as tutors and to answer questions. After four weeks the groups were allowed to self-select into new groups. Some stayed the same.

A suggested schedule of readings from the reader was published together with a list of learning objectives - "Having studied this section you should be able to ...". Students were instructed to read the explanation, study the worked examples and attempt a solution of the test questions. They were encouraged to talk to one another in their group about the work and to discuss their solutions to the test questions. A computer algebra package was available to them to use to check answers (which were sometimes wrong at the back of the book). It was suggested that they should attempt

There was the desire to respond to pressure to be "doing more with less", i.e., to try to remove the teacher from the classroom for at least some of the time.

to construct their own examples of the different differential equations, to write out solutions, and to give them to one another to try. Progress through the independent learning strand was not monitored by the lecturer. Students had been requested to be responsible for their own learning and to use all available resources. The following two paragraphs are extracted from a course handout:

"But this module is about more than just three Ms of Mathematics. It is also about growing up, about cutting apron strings or umbilical cords. It is about becoming independent where previously you were dependent and paradoxically, it is about becoming dependent where previously you were independent."

"Previously your teacher was your sole source of knowledge and wisdom. You depended on her to tell you exactly what to learn and how to learn it, and to prepare you for the big tests of life such as GCSE and A-Level. Now the lecturer is just one of many sources of knowledge and wisdom which you will have to access. The lecturer is a scarce resource and you must learn to make best use of the time that she can afford to give to you. Other resources available to you are books and yourselves - yes, yourselves, one another, your peers in this class. You must learn how to use books and how to help each other to learn. Assessment is an important part of learning. Not just the "big test" at the end, but all the little tests as you go along. You must learn how to assess your own work and how to help each other to assess your own work by assessing each others' work."

Furthermore, students were told that they should keep up with the scheduled readings so that they would be in a better state to understand the lectures on models. They were advised that the methods and models strands would be summatively assessed by written examinations at the end of the semester. The modelling strand was summatively assessed by written reports and oral and poster presentations. There was also a comprehension test.

EVALUATION

During the first session (1992-93) our primary interest was in assessing student attitudes to the peer tutoring/peer learning aspects of the programme. This has been reported elsewhere by Houston and Lazenbatt (1994). The students readily accepted the need to work in groups and to support one another in the group project work on the modelling tasks. However, they did not so readily accept the ideas associated with peer support and peer assessment of the independent learning activity. They found the textbook hard to understand, and they preferred to work at their own pace and not to have to meet weekly deadlines. They did not appreciate the value of setting their own questions. The found it difficult to support one another because they themselves had an inadequate knowledge of the subject matter.

A majority of students had an overall negative attitude to the scheme. These students were mostly in the diploma class and this attitude may be understood by recognising that they were coming up to their final examinations, that they had had a fairly "dependent" style of education up to now and consequently were anxious that this new style might prejudice their ability to obtain a good grade.

Having learnt this during 1992-93, it was decided to persevere with the independent learning scheme through 1993-94 but to try to improve it.

Better written support materials were prepared which set clearer and less ambitious goals for each week's learning. It was decided to try to impress upon

students the idea of taking responsibility for their own learning and how this and the peer tutoring aspects of the programme would benefit them.

The outcome of the 1993-94 evaluation indicates that students were very much aware of the benefits of being able to learn independently and of peer support and they were aware of the need to work consistently through the semester. However, while the spirit was willing, the flesh was weak and many students confessed to not keeping up with the weekly reading, to leaving it all to the last minute, and to not being sufficiently well prepared to ask sensible questions of the seniors and the lecturer. It was observed by us that many were not properly prepared to solve the problems arising in the models strand of the course, nor to produce analytic solutions to compare with the numerical solutions to differential equations that they met in a parallel module on Numerical Methods.

There was the belief that the requirement to tutor someone in a topic enhances learning by the tutor– "To teach is to learn twice".

The method used to determine student attitudes in 1993-94 was to include an essay question in the end-of-semester written examination. The question was:

Write an essay of about 1000 words which discusses the statement "Independent learning only gives you a pain in the brain."

This question was one of eight, with the other seven dealing with the methods and models studied during the semester. (Essay questions have been set in the past.) Students had to answer five questions and so were not required to write the essay. However 15 students out of 20 in the diploma class and 16 students out of 23 in the degree class did attempt the essay question. Marks, of course, were awarded for the quality of the essay, and not just for their views (or for telling me what they thought I wanted to hear).

Scores ranged from 5 to 20 out of 20. Of the 31 students who wrote the essay, 19 scored a mark which was greater than 20% of their total mark for the paper (which suggests that it is easier to score marks writing essays on independent learning than it is to answer questions about what should have been learnt independently).

Of those who attempted the essay question and failed the examination (i.e. scored less than 40%), (10 students) the essay marks ranged from 5 to 15. All but one of these students indicated in their essays that they recognised the benefits of independent learning and 6 of them confessed to not having kept up with their reading schedule. In the whole class, of the students who wrote the essay (31 students), all but 3 recognised the benefits of the scheme and 10 confessed to not paying enough attention to it (4 of these students did pass the examination).

The student who scored 20 on the essay obtained only a total of 40%.

The scores obtained on the essay question were not particularly well correlated with the total scores ($r = 0.5$). A more detailed analysis follows in the "Evidence" section below.

THE ESSAY

"Independent learning only gives you a pain in the brain."

The key words in this title are "learning", "independent", "only", and "pain", and answers should have referred to these. Given the discussions we had during the semester and the experiences of the students, I expected answers to define and describe independent learning with reference to how we had arranged it and their own experiences of it. The description should have discussed reading the reader, doing the exercises and checking the solutions, setting test questions, and how independent learning is different from traditional lectures. Comment on the peer support and senior student tutorials should have been included.

There should have been reference to assessment, the reasons for it (feedback, part of learning, etc.) and the methods available (using the answers in the book, each other, tutors, lecturers).

The "pain" in the title should have inspired comment on hard work, self-discipline and time management, and the "only" was intended to evoke a discussion on

the reasons for independent learning (economic, pedagogical, socio-psychological) and the benefits of it (becoming independent, etc).

The marks awarded were determined not only by the breadth of the topics covered but also on the quality of the answers.

EVIDENCE FOR THE CONCLUSIONS

Realising the Benefits

31 students wrote the essay and all but 3 recognised the benefits of the scheme. Of the 10 students who failed the written examination, all but 1 recognised the benefits. Typical comments are:

Personal Understanding

"This gives them a better understanding of the subject in the long run as the student teaches him/her self in a way that they understand."

Peer Interaction

"It is a very important part of learning to be able to listen to other people's points of view and above all be able to give your point of view and back it up with reasons."

Peer Assessment

"Students may, in their peer support group, get together and make up a test for the other members of the group. Tests may be swapped and done by the students. This helps give an understanding as students have to make up questions and so must understand the theory first."

Self Discipline

"Not coming to class prepared with the background reading meant that much of the lecture did not make sense. It is therefore a matter of sitting down and conditioning yourself to do these independent learning sessions, for in the long run it is only yourself that loses out."

Becoming Independent

"The whole reason for the drama of independent learning was to "untie us from the apron strings" of days gone by."

Confessing to Weakness

10 students, including 6 who failed the examination, confessed to not having kept up with the schedule. Perhaps the most telling comment (by a student with poor spelling) was:

"The dangers of independant cannot be over stressed, as I unfortunately can testify. The great hazard is becoming lazy, and avoiding nessicary work through lack of self discipline. This tragic situation of sitting back and letting the chapters pass by you has many unfortunate consequencies, many felt only when it is too late."

Other students commented:

"All the classes I am in I don't seem to want to bother doing any more work for than is necessary."

"So you think, "I think I will just leave that until the exam", well, it isn't going to help my coursework any. So that's it, end of discussion, well that is until a week or two before your exam when you run around the class photocopying notes, and this is the time you realise your mistake."

"We looked back on them [the readings and exercises] as something that didn't have to be done just yet."

OTHER INTERESTING INSIGHTS

Students' responses brought to light a number of interesting points.

We have seen above that some students confessed to not working through their independent learning programme as intended. Laziness was mentioned as a possible cause, but there would appear to be others.

There are a lot of other interesting things to do at University and so students are distracted from their studies to too great an extent.

"The topic of independent learning is a sore one for many students. Having to go off on your own and work is not an appealing idea to most, especially when there are other distractions about."

There is peer pressure to conform and if there is a lazy attitude in at least a section of the class, then it is all too easy to be infected by this.

"Students are under extreme pressure from their

peer group to enjoy themselves and not to seem to worry or even to do any work."

A number of students commented on the lack of motivation and interest in the subject. This was disappointing given that they choose to enroll for a mathematics course, and, it seemed to me, there were lots of interesting things for them to do.

"If I was set the task to independently learn about new fashion or music, the task would not be a pain in the brain. I would find the subject of the learning interesting and would be enthusiastic about the task. The subject again is all important and the motivation behind how it is learnt" but "it is the student's choice to be in that class."

"When it comes to group work...when the lecturer leaves so do many of the students. It is very hard to motivate people outside class time and sometimes in it."

"What we need are mathematicians who enjoy their work and not think it's a drag, pain or bore!"

The whole concept of becoming independent learners was new to them and it frightened them. There had been too much of a dependency culture at school or college.

"This [independent learning] is made harder by coming from a school background when the work is set and it must be done."

"Coming from a Further Education College, which some of us did, we had grown accustomed to being "spoon fed"."

"We all came from schools where the teachers set us work which we had to do or else get punished."
Some students, particularly those in the diploma class (who were in the final year of their studies) were reluctant to get involved with peer-support. They were in a competitive situation with their peers because final rank order had a bearing on how successful they would be at the next stage, whether it is admission to year 2 of the degree course, or employment.

"It is not possible for all of us to get places [on the degree], so we are all trying to do better than everybody else. This means that in our independent learning if somebody could understand something they kept it to themselves rather than enlighten the rest of us."

The students freely gave me advice on how to improve the scheme in future - have regular, frequent, tutor-supervised assessment of the learning programme. They were just too unsure of themselves and needed authoritative reassurance that they were doing things correctly. Because there were mistakes in some answers at the back of the book, they lost confidence in this as a source of reassurance. They did not make use of the computer algebra package even though it was available, they had used it before, and it was suggested that they should.

"I felt that there was not enough checks on us. Maybe if a test had been introduced at the end of each session it would have been beneficial. I know each group had to set each other a test. But when a model project was also underway, there wasn't time."

"It is a good idea as long as it is supervised, i.e. you work towards a test or questions."

"Many people like myself need to be put under pressure to finish things."

"So instead of being the odd one out the general average person wouldn't carry out independent learning sessions unless they were forced to by a higher authority. And we therefore come back to the teacher."

"So maybe they would have been less of a pain in the brain if the book had been easier to follow and had been correct all the time."

CONCLUSIONS

This paper has described the context in which two groups of undergraduate students were introduced to the concept and practice of independent learning. It discussed their approach and attitudes to it and the methodology by which these data were collected (an end of semester essay question).

The students were well aware of the likely benefits to them of developing skills of independent learning and of cooperative learning, including peer assessment.

However, it is a curious trait of human nature that

"The good that I would I do not; but the evil I would not, that I do."

(Paul, c56)

Unless they are very self-disciplined and able to manage their time to good effect, many students put off doing their learning, sometimes until it was too late. They would have liked the tutor to keep them under pressure to do the work. They would also have liked the reassurance of tutor feedback regularly and frequently.

Some disliked the independence culture, first, because they were not used to it, and secondly, because they were unwilling to take so much responsibility for their own learning. Their previous educational experiences had not exposed them to ambiguity or independence.

Some felt (rightly) that they were in a competitive situation with their peers and so were reluctant to share their learning with them. This is an unfortunate consequence of the present economic climate where employment and study opportunities are limited. The name of the game is to get ahead of the other person, rather than for all to move forward together.

It was also disappointing to find that quite a few students were not really fired up with curiosity and enthusiasm for mathematics. Their goal in life was to get a degree as painlessly as possible.

Some valuable lessons for next year's teaching have been learned.

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[article continued from "From the Editor"]

the *idea* of the proof, the deeper context. A modern mathematical proof is not very different from a modern machine, or a modern test setup: the simple fundamental principles are hidden and almost invisible under a mass of technical details. When discussing Riemann in his lectures on the history of mathematics in the 19th century, Felix Klein said:

> Undoubtedly, the capstone of every mathematical theory is a convincing proof of all of its assertions. Undoubtedly, mathematics inculpates itself when it foregoes convincing proofs. But the mystery of brilliant productivity will always be the posing of new questions, the anticipation of new theorems that make accessible valuable results and connections. Without the creation of new viewpoints, without the statement of new aims, mathematics would soon exhaust itself in the rigor of its logical proofs and begin to stagnate as its substance vanishes. Thus, in a sense, mathematics has been most advanced by those who distinguished themselves by intuition rather than by rigorous proofs.

Recently, there have been attempts in the philosophy of science to contrast understanding, the art of interpretation as the basis of the humanities, with scientific explanation, and the words intuition and understanding have been invested in this philosophy with a certain mystical halo, an intrinsic depth and immediacy. In mathematics, we prefer to look at things somewhat more soberly. I cannot enter into these matters here, and it strikes me as very difficult to give a precise analysis of the relevant mental acts. But at least I can single out, from the many characteristics of the process of understanding, one that is of decisive importance.

One separates in a natural way the different aspects of a subject of mathematical investigation, makes each accessible through its own relatively narrow and easily surveyable group of assumptions, and returns to the complex whole by combining by combining the appropriately specialized partial results. This last synthetic step is purely mechanical. The great art is in the first, analytic, step of appropriate separation and generalization.

The mathematics of the last few decades has revelled in generalizations and formalizations. But to think that mathematics pursues generality for the sake of generality is to misunderstand the sound truth that a natural generalization *simplifies* by reducing the number of assumptions and by thus letting us understand certain aspects of a disarranged whole. Of course, it can happen that different directions of generalization enable us to understand different aspects of a particular concrete issue. Then it is subjective and dogmatic arbitrariness to speak of the true ground, the true source of an issue. Perhaps the only criterion of the naturalness of a severance and an associated generalization is their fruitfulness.

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