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A Subjective Comparison Between a Historical and a Contemporary Textbook on Geometry[1]

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Synopsis

In order to investigate how a 19th century mathematical textbook (in contrast to a contemporary one) would be experienced by a novice reader, we embarked on the following project: In the summer of 2013, a student with no previous training in college-level mathematics (the first author) set out to learn projective geometry from Pasch's 1882 textbook *Lectures on Modern Geometry*. Afterwards, he studied the same material from Coxeter's 1994 popular undergraduate textbook *Projective Geometry*. We report here some of his experiences and impressions contextualizing them along the way.

1. Introduction

It is generally accepted that mathematics underwent a radical transformation around the turn of the 20th century: it became more rigorous, abstract, structural, or simply *modern*. For a general account of this development, see [5]; for the developments in algebra and set theory, see [1] and [4]. In some

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disciplines, the turning point can be identified fairly clearly. For example, in geometry it happened in 1899. Gray writes "Modernism in geometry arrives with Hilbert, and the difference between his *Grundlagen der Geometrie* (*Foundations of Geometry*) and the earlier work of Pasch" [5, page 19].

The rise of modern mathematics is frequently described in terms of a change in the content of mathematics, its objects and our knowledge about them. However, this development also accompanied a change in the way mathematics is presented, for instance, in textbooks. Long and verbose explanations that meander towards specific results were replaced by crisp expositions following more or less a definition-theorem-proof structure. At the same time, the use of formal symbolisms, such as set-theoretic notation, increased. As a result, we take it that the reader will have no difficulties in determining whether a mathematical textbook was published in 1850 or 1950 by just flipping through and quickly glancing at the pages.

The previous observation poses certain difficulties for a contemporary reader who is interested in learning about the historical development of mathematics directly from the sources. This is particularly relevant for the use of historical sources in mathematics education; see [3, 13]. If one already has some mathematical background, then the modern form of presentation is the familiar one that one has grown up with, and the historical exposition will almost inevitably feel tedious and cumbersome. We deliberately describe this reaction as a feeling, because it is spontaneous, natural, and unconscious. However, such learned attitudes can unwittingly distort one's understanding of unfamiliar representations. Discussions of computations with Roman numerals can serve to illustrate this point. To support his claim that addition with Roman numerals is extremely cumbersome, Hankel in 1874 invited his readers to "just imagine what it would take" to perform such a calculation [6, page 51]; and almost a century later in 1969, Menninger echoes this sentiment: "The fairly simple multiplication [...] looks clearly impossible to the uninitiated reader" [8, page 294]. Had they relied less on their preconceptions, they would have found that both addition and multiplication are indeed not more complicated with Roman numerals than with our decimal place-value system [12]. The years of rote learning, instruction, and practice that it took us to become proficient in using our system of numerals have been internalized and, in contrast, it has become very difficult to even imagine how computations could be done differently, even for historians of mathematics. The situation of being faced with a 19$^{\text{th}}$-century mathematics textbook is analogous. Only with conscious effort can we overcome our biases, but even then we will retain the modern perspective in the back of our minds.

The above considerations led us to the following question: How would someone experience learning geometry in Moritz Pasch's classroom in the late 19$^{\text{th}}$ century and following his *un-modern* lecture course on projective geometry? We reasoned that the situation would be somewhat similar to that of a contemporary student, with no formal training in college-level mathematics, who would set out to learn geometry from Pasch's 1882 textbook *Lectures on Modern Geometry* [9]. And so, this became an undergraduate summer project in 2013: For a student without prior knowledge of higher-level mathematics to work through the first twelve chapters of Pasch's book and, afterwards, to study the same material from a contemporary presentation, namely Coxeter's popular undergraduate text *Projective Geometry* from 1994 [2], while paying careful attention to one's initial reactions to the two presentations.

2. Reading Notes

In the following we describe some aspects in which Pasch's and Coxeter's presentations differ and how these differences affected the learning and understanding of the material from the point of view of a mathematically untrained reader. Notice, however, that these comments are not the result of extended deliberations on these issues; instead they reflect the first impressions of approaching the texts.

2.1. Presupposed background knowledge

Both Pasch and Coxeter claim explicitly that no mathematical background is required for understanding their texts. Coxeter does not merely say that he does not presuppose a prior acquaintance with geometry (and mathematics, generally), but he insists that it is actually best that the reader lack this familiarity. He suggests that "the best possible advice to the reader is to set aside all his previously acquired knowledge (such as trigonometry and analytic geometry) and use only the axioms [presented in *Projective Geometry*] and their consequences" [2, page 15]. He explains that attempts to understand the content of *Projective Geometry* with methods previously inculcated in the reader will lead to either cumbersome calculations or to downright failures. Pasch formulates similar expectations for his reader. He writes that the lectures do "not presuppose any kind of knowledge that is customarily gained first in geometry" [9, page 7]. Rather, Pasch asks only for vigilance in dutifully neglecting "in thought those things which one is familiar with and to return to a point of view from which one has distanced oneself considerably" [9, page 7]. Pasch's attitude adopted in these lectures in consistent with comments he made later in his essay *On the educational value of mathematics* from 1894 [10], where he observes that the "language in which we find mathematics presented nowadays lacks the clarity by which the ancients distinguish themselves and remain our role models" [10, page 12]. Accordingly, one finds in Pasch's work an attempt to re-introduce the clarity demonstrated first by the ancients, i.e., Euclid. Thus, both Pasch and Coxeter present the material in an axiomatic fashion and seem to think that previous exposure to related concepts may make learning the new concepts introduced in their works more difficult.[2]

It seems that Pasch lived up to his promise more than Coxeter did. Let us illustrate this claim by an example. Coxeter tends to leave gaps in his explanations that, presumably, could be filled by one who has a superior (prior) grasp of the material. In Subsection 7.6, called *A self-conjugate quadrilateral*, he introduces Hesse's theorem in one sentence, proves it in four sentences, and then concludes the section. The notion of a self-conjugate quadrilateral is not explained explicitly, but has to be inferred from previous discussions. In the mind of someone without much mathematical background, it may have been preferable to identify more clearly what is happening, as well as, perhaps, some implications of the theorem, such that one could discern precisely why it is in the text.

In general, Pasch proceeds more slowly and introduces new concepts more gradually than Coxeter does. Also Pasch's treatment is significantly more thorough, which makes it much easier for an inexperienced reader to understand. Indeed, what might appear as long-winded from a modern perspective actually appears to be beneficial for the learner. Seventeen pages into *Projective Geometry*, Coxeter says that he has laid out a foundation that suffices "for the erection of the whole system of projective geometry" [2, page 17]. For the inexperienced reader this appears to be too fast. Even if Pasch takes

[2] It is interesting to note that this goes against common educational advice, according to which it is better to connect new material to previous knowledge.

longer and goes more slowly (and thus one dedicates more time to it), the latter also ends up being more successful, at least in the eyes of one (novice) reader. Again, Pasch has, in his work on mathematical education, made it clear that he wanted to avoid some of the problems that beset Coxeter's presentation. He affirms that, sometimes, there are gaps in one's presentation—perhaps just for the sake of concision and elegance—, and that "the less knowledgeable reader is misled into overlooking the gaps or into filling them out incorrectly" [10, page 16]. He makes an effort to avoid this situation in his lectures on projective geometry, much to the student's benefit.

2.2. Organizational structure

The division and order of chapters, too, make Pasch's work more accessible on a first read than Coxeter's. For instance, Pasch's first chapter is called *Of the straight line* and the content of the chapter reflects the title, which made it easier for the student to follow along. This effect was produced in two ways. Firstly, it was possible to anticipate the material in the chapter just after having read the title. Secondly, every time a new concept or theorem was encountered, it could immediately be related to the straight line, which generated a strong (and helpful) sense of coherence and unity. In contrast, Coxeter's work is organized less according to basic geometric objects, but more according to higher-level concepts and principles. For example, Coxeter presents perspectivity at the end of the first chapter, whereas Pasch only presents it in the tenth chapter; whereas Pasch dedicates a chapter to the notion, Coxeter gives it a few pages as a subsection. Another example: The third chapter of Coxeter's book is called *The Principle of Duality*. However, the content of the chapter is not merely an explanation of this principle, but it also features expositions of such notions as the Desargues configuration, a discussion of the invariance of the harmonic relation, and introductions to harmonic nets and trilinear polarity. At the time of the first reading, however, it was not obvious to the (inexperienced) reader what any of these concepts have to do with the principle of duality, or what relation obtains between them in a way that justifies their arrangement together. As such, Coxeter's work ostensibly requires some interpolation and abstraction to figure out how these ideas are united by the principle of duality. This necessary extra effort made it uncomfortably hard for the student to become acquainted with the material, which was not the case with Pasch's presentation.

2.3. Notation

There is a marked difference between the texts of Coxeter and Pasch in the extent to which they use notation. Pasch was sceptical about the usefulness of too formal a presentation [11, page 104]. He knew it would be difficult for the first-time reader to go through the book, when the use of formalisms would be perceived as a needless obstacle. Thus, in [9] he avoids the use of notation entirely (except for the use of variables to label and refer to the diagrams). Coxeter, on the other hand, makes liberal use of it. For instance, he writes some theorems entirely formally; see for instance his Theorem 3.33:

"*If* $ABCF \overset{=}{\barwedge} ABCF$ *and* $H(AB, CF)$*, then* $H(A'B', C'F')$."

What is striking is that some of these theorems are written as early as the third chapter, before the reader can even adjust to the content, let alone the style.

This heavy reliance on notation in Coxeter's book felt like an impediment to developing an understanding of the material for the inexperienced reader. In addition, it probably contributed to the impression that Coxeter moves more quickly through the material than one would have liked. While we acknowledge that, once the mathematical understanding is developed, being able to freely and comfortably use the notation is an advantage, a gentler and more gradual introduction of notation than Coxeter's would have been beneficial. Perhaps, this could also be augmented with some explicit remarks on the role of the notation.

2.4. Historical comments

Another way in which Coxeter's presentation differs from Pasch's is that the former includes short remarks on the history of mathematics, whereas the latter does not. Pasch only once mentions something that can be characterized as historical, whereas Coxeter's work is crowded with such remarks. The decision to include such details may stem from the thought that the book would be easier to read if there were informal breaks for the reader and that tracing the genealogy of mathematical concepts enriches the reader's understanding. In practice, however, none of these advantages were conferred. The book felt bloated, the presentation suffered in terms of momentum, and nothing important seemed to be added to the discussion.

For example, Coxeter often writes such pithy comments as "in Figure 2.3A, the collineation is a homology (so named by Poncelet) [... and] in Figure 6.2A, it is an elation (so named by the Norwegian geometer Sophus Lie, 1842–1899)" [2, page 53]. These sorts of comments do not obviously add to one's understanding of the mathematics; they disrupt the rhythm instead. As Coxeter broaches Pappus' theorem, he writes that Pappus himself lived in 4^{th} century Alexandria and "used a laborious development of Euclid's methods" [2, page 38]. This detail, and others of its kind, do not make it easier to learn the material, Pappus' theorem in this instance. On the contrary, they might even impede the learning process by unnecessarily bloating the text.

In contrast, the absence of such comments in Pasch's text makes this 19^{th} century presentation easier to follow for the uninitiated reader, since there are no similar interruptions of the momentum. Pasch makes one historical comment, observing that "the empirical origin of the geometric axioms has been investigated in detail by Helmholtz" [9, §1], but this is made in a footnote and Helmholtz was still alive at the time of publication. In any case, it is easier to follow the material when the presentation of that material is not interrupted by lessons in history.

3. Discussion

We have reported above the subjective experiences of one reader of Pasch's and Coxeter's textbooks on projective geometry. Now, what if anything can we conclude from these observations?

First of all, it appears that the modern mathematical presentation style[3] is indeed something that one has to learn and get used to, and that the more abstract and formal presentations are not necessarily more intuitive or easier to follow for the uninitiated learner. Secondly, somebody who is learning mathematics encounters a text from a very different perspective than somebody who is already familiar with the material and the style of presentation. It appears that historical textbooks reflect more closely the way the material was actually presented in class, whereas contemporary textbooks serve more to complement the lectures, rather than to reproduce them (this is also supported by the fact that Coxeter has exercises following each section).

[3] For a discussion of style in mathematics, see [7].

These observations should caution us against approaching and assessing a historical text in the same way as we read a modern presentation, and they teach us that sometimes even an old text can reveal something new when read with fresh eyes.

References

[1] L. Corry, *Modern Algebra and the Rise of Mathematical Structures*, Volume 17 of Science Networks, Historical Studies, Birkhaüser, Basel, 1996.

[2] H. S. M. Coxeter, *Projective Geometry*, 2nd edition, Springer, New York, 1994.

[3] J. Fauvel and J. van Maanen, *History in Mathematics Education*, Kluwer, New York, 2002.

[4] J. Ferreirós, *Labyrinth of Thought: A History of Set Theory and its Role in Modern Mathematics*, 2nd revised edition, Birkhaüser, Basel, 2007.

[5] J. Gray, *Plato's Ghost: The Modernist Transformation of Mathematics*. Princeton University Press, Princeton, N.J., 2008.

[6] H. Hankel, *Zur Geschichte der Mathematik in Alterthum und Mittelalter*, B. G. Teubner, Leipzig, 1874.

[7] P. Mancosu, "Mathematical style," in Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Spring 2010 edition. Available at http://plato.stanford.edu/archives/spr2010/entries/mathematical-style/, accessed on July 4, 2014.

[8] K. Menninger, *Number Words and Number Symbols*, MIT Press, Cambridge, MA, 1969.

[9] M. Pasch, *Vorlesungen über neuere Geometrie*, B. G. Teubner, Leipzig, 1882. Reprinted in 1912 and in a second edition in 1926. English translation by Dirk Schlimm (manuscript).

[10] M. Pasch, "Ueber den Bildungswerth der Mathematik," Rector's Address, Grossh. Hof- und Universitäts-Druckerei Curt von Münchow, Giessen, July 2, 1894. Reprinted in *Mitteilungen des mathematischen Seminars Giessen*, Number 146 (1980), pages 20–39.

[11] D. Schlimm, "Pasch's philosophy of mathematics," *Review of Symbolic Logic*, Volume **3** Issue 1 (2010), pages 93–118.

[12] D. Schlimm and H. Neth, "Modeling ancient and modern arithmetic practices: Addition and multiplication with Arabic and Roman numerals," in V. Sloutsky, B. Love, and K. McRae, editors, *Proceedings of the 30th Annual Meeting of the Cognitive Science Society*, Austin, TX. Cognitive Science Society, 2008, pages 2007–2012.

[13] F. Swetz, J. Fauvel, O. Bekken, B. Johansson, and V. Katz, eds., *Learn from the Masters*, The Mathematical Association of America, Washington, DC., 1995.