Humanistic Mathematics Network Journal

Issue 13 Article 19

5-1-1996

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This is a report on my efforts to design a mathematics course for liberal arts students, particularly for those whose principal interest is in the humanities. My college requires a mathematics course of each graduate, but not all students have the background to take Calculus I. Twenty years ago, when I began thinking about this problem, the various "mathematics appreciation" courses developed for such students tended to be a potpourri of shallow topics in discrete mathematics from which neither student nor instructor could derive much satisfaction. I remember saying, perhaps too cynically, that these courses taught students more about appreciating parlor games than about appreciating mathematics.

While on sabbatical leave at Berkeley in 1974 I found myself writing notes on calculus with this audience in mind. I stumbled upon the idea of an historical approach in seeking an excuse to review the necessary prerequisites for calculus. An historical approach has the wonderful feature that you *have* to review the development of basic ideas, so no one can think of it in a demeaning way as being "remedial" work. To accommodate students who remember nothing of trigonometry or logarithms, I decided to deal only with algebraic functions or with functions expressed by graphs already drawn. I decided to emphasize writing skills to compensate for lowering the usual prerequisites and to play up the supposed verbal strength of my clientele. These notes became a blend of calculus, history, and writing that I hesitantly served up to a class of students for the first time in 1976. I had gone through student records and had sent out letters inviting only the weakest mathematics students in the college to enroll. I still remember that nervous first class of 13 students, whose S.A.T. scores in mathematics ranged from 330 to 480.

A few years later, thanks to the interest of Paul Halmos, an augmented version of these notes was published as a textbook [1]. I use portions of this text (chapters 1-6, chapter 10, and appendices) as the basis for the course I am describing.

Emphasis upon writing, it seems to me, is essential in a course like this. By forcing students to try to learn how to write mathematics, they will inadvertently learn how to *read* mathematics. This point is stressed in [2]. The major reason students are so poor in mathematics is that they can't (or won't) read a mathematics text. Once they get to the stage where they will do this, the instructor's job is much easier.

Surprisingly, an historical approach is useful here as well. Making students learn (by rote at first, if necessary) some famous short but historically important proofs not only acquaints them with real mathematics, but helps them learn to write a mathematical argument with some sense of beauty and style. I think that no classroom time is better spent than the time devoted to helping students master the classical proofs of the irrationality of $\sqrt{2}$, the infinity of primes, or the Pythagorean theorem. Concentrating on the irrationality of $\sqrt{2}$ also gives us the chance to talk about how the existence of irrationals may have made the Greeks decide to speak mathematics geometrically rather than numerically, and to speculate about whether this decision helped or hindered the development of calculus. (How crucial to this development was the notion of a function from numbers to numbers?)

Writing is important also because students cannot learn to think like mathematicians until they learn to write like mathematicians. As argued in [2], they understand the theory behind optimization techniques in calculus if and only if they can properly use a small glossary of words like *let, denote, then, when, therefore,* and *attain.* Proper usage of most of these words can be picked up incidentally by knowing—or even by just memorizing—a few classical proofs.

The greatest benefit of an historical approach to calculus, however, is its enabling us to present the fundamental theorem as expressing a marvellous connection between ancient and modern (*i.e.*, 17th-century) mathematics, and thus allowing a semester-length course to close with a satisfying unity. Roughly speaking, the fundamental theorem says that a number calculated by a modern numerical interpretation of a method introduced by Eudoxus in the 4th century B.C. yields the same number as calculated by a method of antiderivatives introduced by Leibniz and Newton in the 17th century. When presented this way, there is no possibility for a student to think—as far too many students of mainstream calculus courses mistakenly do—that the resulting equality

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b$$

is a definition of the integral.

The main historical theme holding such a course together might be described as the principle of elimination of wrong answers, as it manifests itself in the Greek method of exhaustion, yet points to the modern notion of a limit. This principle of elimination can be introduced whenever you please by discussing the ancient Babylonian method of approximating square roots and interpreting it in modern, numerical terms. At each stage we eliminate more rationals that are too large and too small (and with an efficiency that numerical analysts call quadratic convergence). The point to be emphasized is that a search for a numerical value of $\sqrt{2}$ is equivalent to a search for all rationals that are too large and all rationals that are too small. To put it more strikingly, a search for the right answer is equivalent to a search for all the wrong answers.

Studying the Babylonian method also leads, incidentally, to some wonderful research questions the students can do: What about cube roots? What about fourth roots? The Babylonians didn't attack these questions, but I have found that humanities students can make progress on them with just a few hints. (The natural thing to do turns out to be, like the Babylonian method of square roots, simply a special case of Newton's method, which—when later they come to a discussion of computing roots by this method—they may be delighted to compare with their own efforts.) If next you attack the problem of finding a numerical value of p, you find the same sort of approach leads to an elimination of numbers too small and too large. You have then planted an idea in the student's head that may later make the Riemann integral easier to grasp.

You can dwell on this longer if you please. Draw five or six famous ratios of geometrical magnitudes and puzzle about the problem of determining when one ratio is equal to (or greater than) another. The picture on Archimedes' tomb leads to a wonderful question for speculation. We are close to Dedekind cuts here, though I have never been brave enough to mention this to my humanities students. (I suspect, however, that this background might make them take to Dedekind cuts better than my real analysis students do.)

The drawbacks of the principle of elimination and the advantages of the notion of a limit are seen clearly when we approach the problem of finding tangent lines the same way. If you want the slope of a tangent line, the method would have you first find the slopes of secant lines in order to eliminate them from consideration. Of course, this won't work if the tangent line cuts the curve twice. But even here, however, the method serves a pedagogical purpose, *viz.*, to emphasize that the "right answer"—whether it be the numerical value of an integral or a derivative—is the limit of "wrong answers" that approach it ever so closely.

My experience has been that this indirect approach of finding wrong answers, in order to eliminate them from consideration, is attractive to students, particularly to students in the humanities who have never before realized that we are doing good mathematics if we have a method for proving that an answer is wrong. Once the idea becomes familiar, it can be seen in unsuspected places, such as in the fundamental optimization principle, *viz.*, that one need only consider endpoints and critical points when searching for the extreme values of a differentiable function on an interval. How many students of Calculus I can explain well the main idea behind this principle? Most of my humanities students can tell you that the curve is either rising or falling as it passes through noncritical points, so we may eliminate such points from consideration unless we are at an endpoint.

Introducing the principle of elimination and then seeing limits as a generalization of the principle results in significantly less confusion among students about limits. In contrast to my students of years ago, these students rarely ask naive or nonsensical questions about whether the secant line ever gets to the tangent line or whether two points can ever become one. Yet this principle is no more difficult than Sherlock Holmes's familiar observation: "When you have eliminated the impossible, whatever remains, however improbable, must be the truth." It is surprising that a reasoning device so simple and so useful is not ordinarily taught in grade school.

I wish to emphasize that the course I describe is first and foremost a course in calculus (though it is restricted to the calculus of algebraic functions). All my students know (because I remind them once every few weeks) that calculus is the study of the interplay between functions and derivatives. Discovering and experiencing the richness of this interplay is all-important. By the end of the course they are expected to demonstrate knowledge of five aspects of this interplay by being able to work simple problems in optimization, in geometric interpretations of the first and second derivatives, in rates of change, in approximating solutions of equations by Newton's method, and in areas and volumes. All my students know that there will be five problems on the final examination testing knowledge of these five aspects of calculus. They also know there will be a few historical questions, and a few proofs to be given, chosen from ones we have concentrated upon.

When I first began to teach this course I never dreamed that students of this caliber could sustain an argument that lasted more than a few lines. Yet I slowly discovered they were capable of writing coherent three-paragraph arguments when they set up optimization problems, calculated critical points, and justified their answers. This has emboldened me to push them a little further in recent years.

My most pleasant surprise has been to learn that these students—whose skills at algebra, inequalities, *etc.* are very low—nevertheless are fully capable of stating precisely the fundamental theorem of calculus and writing, in a style that indicates understanding, a convincing argument for it when the theorem is interpreted as expressing a connection between areas and antiderivatives. My expectation now is that each student understand the meaning of the fundamental theorem in historical context, state it precisely, and present a convincing argument for it. They all know that they will be expected to demonstrate this ability on the final hour test and again on a comprehensive final examination.

I have tried to develop for humanities students a one-semester course in mathematics that is within their ability to learn, that they could be proud to study, and that I could be proud to teach. It is a course that is not a shallow jumble of unrelated topics, but has a unity about it, and builds upon itself to show the depth of the discipline. I try hard to get the students to become engaged in mathematics, to know the spirit of delight in the discovery of unexpected connections between things and to acquire a sense of beauty and style in a mathematical argument—*i.e.,* to know why mathematics is appealing in itself; but I try also to help the students see mathematics as a significant element in the history of thought that has played a role in our understanding of nature, in the rise of philosophy, and in the development of the liberal arts—*i.e.,* to know how mathematics has interacted with areas outside itself. Whenever I have a little spare time, I remind them of such things. Sometimes I have them read Hardy's *Mathematician's Apology* and write a paper on beauty versus utility in mathematics. Sometimes I even pass out reprints of [3].

More than a few students come into the course with the fantastic notion that *liberal arts* means "a lot of arts" (and that consequently mathematics, to them, is not a part of the liberal arts). They are surprised to find out that in this context *liberal* means "liberating" and that mathematics has been part of the liberal arts for nearly 2500 years. I hope that a course like this helps humanities majors to understand the real nature of mathematics, and helps to bridge the gap that separates students in the humanities from those in the sciences. I hope they will appreciate the centrality of mathematics in education by seeing mathematics as a bridge between the arts and the sciences.

Overall, the students' responses have been pleasing. The rate of withdrawals and failures in this course has been lower than in my regular Calculus I course or in Finite Mathematics. It is a joy to teach calculus leisurely for its own sake, to try to transfuse into the

students an intuitive understanding of the fundamentals of the subject, rather than to rush through a pressure-packed semester of Calculus I, emphasizing manipulative skills and multifarious applications to mostly plug-and-chug students picking up calculus only as a tool.

This is not say there are no drawbacks. Any mathematics appreciation course can be frustrating to teach because it is bound to draw some initially recalcitrant students. Yet even after teaching it often, it still excites me because I sense that, after a while, it begins to excite some—perhaps even most—of the students, especially when they realize they are not in a frivolous "mathematics for poets" course. Being treated as grownups in a serious mathematics course is a behavior-modifying experience for many of them. Occasionally, an exceptional student comes into the course with a fine background in mathematics and is able to follow it successfully with Calculus II. Generally, however, it serves as a terminal course in mathematics.

My experience over the years has convinced me that this approach is likely to succeed with humanities students. I am also convinced that if I had caught the brightest of these students when they were younger, it would have marked the beginning, rather than the end, of their involvement with serious mathematics. It might be worthwhile, therefore, to try to adapt this course for use in secondary school, following courses in algebra and geometry. A different adaptation might prove valuable to prospective teachers enrolled in mathematics education programs. I know very little about teaching in secondary school and nothing about mathematics education, but I believe that if someone were able to fit this approach into either of these settings, it would make a real difference.

A revision of a talk presented at an AMS/CMS/MAA Meeting, Vancouver, BC, August 18, 1993.

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