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The Disciplinarity of Mathematical Practice

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Cover Page Footnote

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The Disciplinarity of Mathematical Practice[1]

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Abstract

Despite an extensive literature on the nature and origins of mathematical truth, few if any studies exist of the everyday practices through which the adequacy of mathematical argumentation is cultivated and assessed. The work of a novice prover afforded insight into these practices and, in particular, into the disciplined character of discovering and proving mathematical theorems.

Keywords: *disciplinarity, mathematical practice, mathematical proof.*

1. Background

In the late 1970s and early 80s, a number of publications—among them, Latour and Woolgar [15], Knorr [11], Garfinkel et al. [7] and Lynch [22]—marked the rise of what came to be called social studies of laboratory science, focusing on the laboratory as a lived-environment and on laboratory scientists as embodied actors and practical reasoners. Influenced to some extent by Kuhn [12], these studies examined aspects of scientific work often implicit, unacknowledged, and ignored in research publications as well as in the more reflective writings of scientists, and provided a contrast with then current philosophical writings, especially those seeking to cultivate an appreciation of the scientific method in the social sciences. The research that

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followed might best be characterized as part of a social movement, on the one hand turning away from the received view of scientific methods and from traditional studies in the sociology of science and, on the other, emphasizing more anthropological, ethnographic, and participant observer approaches to the study of scientific practice.

Consider, for example, the volumetric pipet, one of the most precise instruments in the traditional chemistry laboratory [19]. The pipet is used to transfer liquid in exact amounts, and a large technology surrounds the cleaning, calibration, and preparation of the instrument as well as the transfer and draining of liquid. The practical problem, however, is lifting the index finger to lower the level of the liquid to the calibration mark. If the finger is actually lifted, the pipet quickly empties. Attempts to rapidly raise and lower the finger at best result in the liquid level dropping in staccato increments, far from approximating a precise measurement.

The problem is well known: using a volumetric pipet is one of the most difficult manual techniques in the laboratory. The finger is not actually lifted. Instead, tension in the finger is reduced allowing the experimenter to watch and time the slow, steady flow of the fluid. For experimental chemists, if, at first, the technique is difficult to learn, in the end, it is ordinary, mundane, and trivial. Yet if they were not able to do it, their measurements would vary; their results would differ; there would be no chemical science. The development of chemistry depended on chemists' ability to reduce the tension in their index fingers. Multiplied by the numerous, interrelated procedures of experimental chemistry, the example affords insight into the "lived-work" of the laboratory and the laboratory as a "lived environment."

Studies often identified as the "philosophy of mathematical practice," as "constructivism" in mathematics education, and, more generally, in terms of an interest in "mathematics as a human activity, a product, and a characteristic of human culture and society" [8, page *xi*] seem to have developed at roughly the same time as the laboratory studies. Lakatos's *Proofs and Refutations* [13] is frequently and perhaps oddly cited as a progenitor—oddly because Lakatos advocated a "rational reconstruction" of scientific practice, less strangely because his book suggests mathematical clarity as an historical and possibly even a social process. The books by Davis and Hersh [2], Lakoff and Núñez [14], and Byers [1], and the articles in Ernest [3], Hersh [9], and Kerkhove and Bendegem [10] reflect the range and diversity of these studies, loosely drawn together in part by their opposition to abstract, idealist

philosophies of mathematics, in part by alternative approaches to mathematics pedagogy, and in part by their interest in what mathematicians actually do and actually think, and in the study of mathematics "in the process of being invented" [24, page *vii*].

A difficulty that I have with this literature is the intended referents of expressions like "mathematical practice" and "mathematical activity." What do mathematicians do and think? For myself, this is an open question, but I have come to see the "doing of mathematics" in terms of the concrete details of collaborative proving at the blackboard, the ways that provers work out and write mathematical argumentation, the dynamic features of seminar presentations, the specifics of collegial discussions, the recognition of mistakes, and the pursuit and utility of ultimately (or seemingly) futile lines of inquiry. I am interested in notes written on paper napkins, blackboard erasures and repairs, and the material that ends up in the wastebasket. Mathematicians are engaged in the work of mathematics, often preoccupied by it, and it is this work that, for me, makes up mathematical activity and practice.[2]

Such considerations provide the context for the present paper. Similar to chemists at work in the laboratory conducting experiments, mathematicians, on a daily basis, for teaching and in their research, are continually proving mathematical theorems. Yet, despite extensive literatures on the nature of mathematics and mathematical proofs, on what mathematicians do and think, the question that this massive production of mathematical proofs raises seems to have attracted little attention: how are the proofs of ordinary mathematical practice, as a practical matter and as a matter of mathematical practice, recognized by other mathematicians as legitimate or "practically rigorous" proofs?

As will become clear, I am far from being a professional mathematician. Nevertheless, and possibly because of this, the present study gives a disconcerting perspective on how mathematical argumentation is cultivated and recognized as a mathematical proof. My work as a novice prover begins to reveal that mathematicians are continually monitoring their practices of

[2] In earlier publications I have tried to characterize proofs in terms of how they describe the practices of proving ([16, 19]) and to examine how the work of discovering a proof provides the context for the proof discovered through, but later divorced from it ([18]). [20] outlines the general research program as I currently see it; [21] attempts to understand perceptions of mathematical structure in terms of the local work of proving theorems.

proving theorems. In effect, unremittingly, they are "disciplining" themselves and each other to work in ways that, while appearing ordinary, habitual, and uninteresting, are critical to the recognizability of mathematical proofs as proofs.

2. The Origins of this Course of Inquiry

Although a study was unanticipated at the time, an opportunity to examine how mathematical argumentation is cultivated and recognized as a mathematical proof developed from asking a mathematician, late in the summer of 2012, for advice concerning a theorem familiar from first-year calculus:

Theorem 1. *If* $f\colon [a,b] \to \mathbf{R}$ *is continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $f'(x) > 0$*, then* f *is a strictly increasing function.*

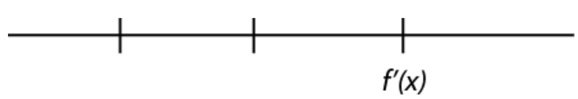
In calculus texts this theorem is proved using the Mean Value Theorem. As part of the consultation, the mathematician suggested that the theorem might be proved directly from the definition of the derivative and sketched a proof of the conjecture.

The proof began by giving the definition of the derivative:[3]

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x) > 0\colon \Longleftrightarrow$$

$$\forall\, \epsilon > 0, \exists\, \delta > 0 \text{ s.t. } \left| \frac{f(x+\Delta x) - f(x)}{\Delta x} - f'(x) \right| < \epsilon \text{ for } \Delta x < \delta$$

Drawing and contemplating a figure similar to the one below,



the mathematician then made the following assertions:

$$\Longrightarrow \frac{f(x+\Delta x) - f(x)}{\Delta x} > 0 \quad \Delta x > 0$$

$$\Longrightarrow f(x+\Delta x) - f(x) > \Delta x > 0$$

"Hence," $f(x+\Delta x) > f(x)$ for $0 < \Delta x < \delta$.

[3]The displayed lines are given verbatim: the mathematician implicitly assumed that $0 < \Delta x < \delta$ with the argument for $0 < -\Delta x < \delta$ to follow similarly.

The argument is elaborated in Section 3 below. Even with this elaboration, it shows only that $\forall x \in (a,b), \exists\, \delta > 0$ such that $\forall p \in (x-\delta, x)$ and $\forall q \in (x, x+\delta), f(p) < f(x) < f(q)$, not that $\forall p, q \in (x-\delta, x+\delta), p < q$ implies $f(p) < f(q)$.

Unsatisfied with the argument, I eventually sought the assistance of a second mathematician whose response was to define a function g:

$$g(x) = \begin{cases} x+1, & \text{if } -1 < x < 0; \\ x-1, & \text{if } \quad 0 < x < 1. \end{cases}$$

Although this function has a positive derivative everywhere on its domain, it is not strictly increasing. The problem is that the domain is not connected. Since the first mathematician's argument made no use of the connectivity of the interval (a,b), that argument would apply to g as well. Something must be wrong with the suggested proof of the theorem.

Following this interchange, a collaboration developed with the second mathematician (hereafter, the "professional prover"). Defining the "pointwise increasing property"[4] of a function, we were able to prove Theorem 2 and Theorem 3:

Definition 1. A function $f\colon D \subset \mathbf{R} \longrightarrow \mathbf{R}$ is said to be *pointwise increasing* at $x \in A$ if $\exists\, \delta_x > 0 \ni\colon \forall p \in (x-\delta_x, x) \cap D$ and $\forall q \in (x, x+\delta_x) \cap D, f(p) < f(x) < f(q)$.

Theorem 2. *If $f\colon [a,b] \to \mathbf{R}$ is continuous and pointwise increasing on (a,b), then f is strictly increasing.*

Theorem 3. *If $f\colon [a,b] \to \mathbf{R}$ is pointwise increasing for all $x \in [a,b]$, then f is strictly increasing.*

3. A Way of Reading

Before turning to my work proving these theorems, the distinctive way that mathematicians read mathematical texts—reflected in the commonplace instruction that students should read mathematics with pencil and paper at hand—needs clarification.

[4] The "pointwise strictly increasing property" would have been a more precise description. The terminology used during the collaboration has been retained in the body of this paper.

Consider the following lines of argumentation:

> Let f be defined on an open interval (a, b) with $f'(x) > 0$ for some $x \in (a, b)$. Then
>
> $$\forall \epsilon > 0, \exists\, \delta > 0 \text{ s. t. } \left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| < \epsilon \text{ for } 0 < h < \delta$$
> $$\implies \exists\, \delta > 0 \text{ s. t. } \frac{f(x+h) - f(x)}{h} > 0 \text{ for } 0 < h < \delta$$

There is a gap in the reasoning. Although the observation of such a gap may reflect inadequacies of the person working through the argument, it is experienced first, phenomenally, as a failure of the text.

With some work, the argument can be justified. In the inequality

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| < \epsilon,$$

the value of ϵ can be chosen arbitrarily. In order to compare the two sides, let $\epsilon = f'(x)$ and let $\delta > 0$ be chosen such $0 < h < \delta$ implies

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| < f'(x). \tag{1}$$

If $f(x+h) - f(x) < 0$, the left side of (1) is the sum of two negative quantities, $\frac{f(x+h)-f(x)}{h}$ and $-f'(x)$. The absolute value of that sum will be greater than the absolute value of either of its components. Thus, $\left| \frac{f(x+h)-f(x)}{h} - f'(x) \right| > f'(x)$, a contradiction. On the other hand, if $f(x+h) - f(x) = 0$, we get $|f'(x)| < f'(x)$, another contradiction. The conclusion is that $f(x+h) - f(x) > 0$ or $f(x+h) > f(x)$. A similar argument for $0 < -h < \delta$ shows, in this case, that $f(x+h) < f(x)$.

Depending on the circumstances, such work can be seen as a process of understanding the textual description and establishing the cogency of its detail—of rectifying the text and setting the argument right—or, in the failure to do so, of finding the argument specious and lacking mathematical "rigor." If the written text of a mathematical argument is called "the material text," and if the associated work of finding its descriptive cogency (or lack thereof) is a part of "the lived-work of proving," reading mathematics seems to always involve the pairing of these two objects [16, 19]:

the material text / the lived-work of proving

When professional provers read mathematical argumentation, they seem to always, unavoidably, seek to find and maintain the association of the text with proving's lived work. If they are not doing this, they are not fully engaged in the professional practice of doing mathematics.

Finding the adequacy of the textual description or, on the other hand, finding its inadequacies and failure to provide such a pairing situates the discipline at the heart of mathematical practice. In reading for themselves, provers are also reading for the discipline: they are establishing that the details of a text provide a practically adequate description of the work of its own reading, not just for themselves but for anyone competent to read the text—that is, for the community of theorem provers.

4. The Development of a Proof

Documents 1-4 are a verbatim textual record of my work trying to find and articulate a proof of a theorem. They were sent to the professional prover in email correspondence. In considering and writing mathematical argumentation, I was looking into the practices of proving to find a proof of a theorem while, at the same time, I was looking to find or clarify a theorem that I might prove.

My project began with a slight variant of Theorem 1 where the domain of the function is an open interval:

Theorem 4. *If $f\colon (a,b) \to \mathbf{R}$ is differentiable and $f'(x) > 0$ for all $x \in (a,b)$, then f is a strictly increasing function.*

Spurred by the inadequacies of the first mathematician's argument, my initial aim was to prove that mathematician's conjecture that the theorem followed directly from the definition of the derivative.

Although, in retrospect, this appears to be what I was doing, I did not have quite this understanding (certainly not clearly) as I was trying to prove whatever theorem I was trying to prove. I was working toward such a conceptualization of my work. Documents 1-4 help situate ourselves inside this project.

Document 1

`[There exists a left neighborhood` $(\mathtt{x}-\lambda, \mathtt{x})$ `such that if]`[5] $\lambda \in (x-\xi, x), f(\lambda) < f(x)$, and there exists a right neighborhood $(x, x+\xi)$ such that if $\lambda \in (x, x+\xi), f(x) < f(\lambda)$.

Hopefully this idea will complete the proof.

Basically we have a continuous function $f\colon (a,b) \longrightarrow \mathbf{R}$ with the following property:

$$(*) \quad \begin{array}{l} \forall x \in (a,b), \text{ there exists a left neighborhood } (x-\xi, x) \text{ such that} \\ \text{if } \lambda \in (x-\xi, x), f(\lambda) < f(x)), \text{ and there exists a right neighbor-} \\ \text{hood } (x, x+\xi) \text{ such that if } \lambda \in (x, x+\xi), f(x) < f(\lambda). \end{array}$$

We want to show that f is strictly increasing on (a,b).

Let $x, y \in (a,b), x < y$. We want to show that $f(x) < f(y)$. Suppose that this isn't true—that is, for some $x, y \in (a,b)$ with $x < y, f(x) \not< f(y)$. By the local condition at each point in the domain, there will be points x^* and $y^*, x < x^*$ and $y^* < y$ such that$f(x^*) > f(x), f(y^*) < f(y)$, where $(x, x^*) \cap (y^*, y) = \emptyset$.

We now have a continuous function $f\colon(x,y) \longrightarrow \mathbf{R}$ such that $x^* < y^*$ and $f(x^*) > f(y^*)$. By continuity of f, *(***I THINK THIS IS TRUE BUT I HAVEN'T FIGURED OUT THE ARGUMENT YET***)* $\exists \gamma \in (x^*, y^*)$ such that $f(\gamma) = f(y)$ and $\exists \delta > 0$ such that $\forall z \in (\gamma, \gamma+\delta), f(z) < f(\gamma)$.

The existence of such a γ contradicts the local property stated at the beginning.

As the argument outlined in §§2-3 shows, if a function $f\colon (a,b) \to \mathbf{R}$ is differentiable and $f'(x) > 0$ for all $x \in (a,b)$, then f is pointwise increasing on (a,b). Document 1 begins by replacing the differentiability condition in Theorem 4 with this property and works toward a statement and proof of Theorem 5:

Theorem 5. *If $f\colon (a,b) \to \mathbf{R}$ is continuous and pointwise increasing on (a,b), then f is strictly increasing.*

[5] Throughout the four documents, bracketed typewriter text indicates editorial elaborations or emendations added in this paper.

In the document, the statement of the theorem repeats the definition of a pointwise increasing function and uses an asterisk to identify it. Later, rather than using the "star," the property is referred to as the "local condition." The document reflects my attempt to separate the definition from the theorem.

Let $x < y$. Instead of trying to prove directly that $f(x) < f(y)$, I assume that $f(x) \geq f(y)$ and try to show that this leads to a contradiction. My immediate concern is that two possibilities need to be considered, one where $f(x) = f(y)$ and the other where $f(x) > f(y)$. The solution is to construct points x^* and $y^*, x < x^* < y^* < y$ such that $f(x^*) > f(y^*)$. Thus, even if we start with $f(x) = f(y)$, we know that there are points $x^* < y^*$ where $f(x^*) > f(y^*)$. The case where $f(x) = f(y)$ need not be treated separately.

Figures 1 and 2 illustrate the basic argument, ignoring the possibility that $f(x) = f(y)$. In Figure 1, because f is pointwise increasing at y, the function must lie below $f(y)$ in some left neighborhood of y. Because f is continuous on $[x, y]$, there must be some $\gamma \in (x, y)$ where $f(\gamma) = f(y)$. The idea is that, as in Figure 2, if we look closely at the behavior of f at γ, f cannot be pointwise increasing.

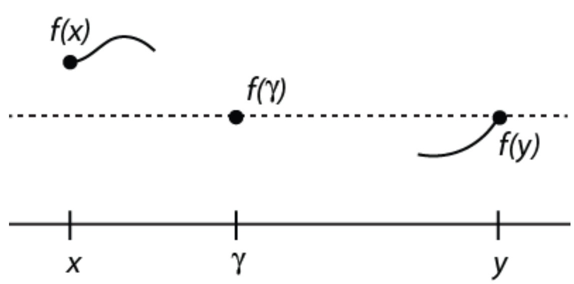


Figure 1: Consequences of the Pointwise Increasing Property

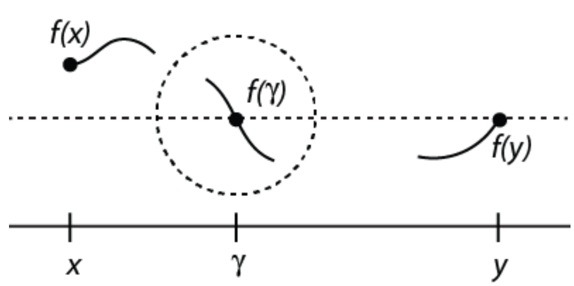


Figure 2: The Prospective Contradiction

Document 1 ends by indicating a rather large hole in the argument, the justification of the existence of such a γ. Document 2 addresses this problem leading, however, to another critical issue.

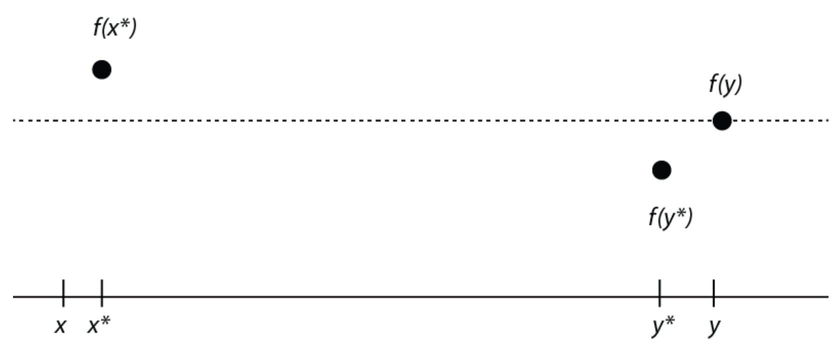
Document 2

$f\colon \mathbf{R} \longrightarrow \mathbf{R}$ continuous

f has a local property: at every point x , there exists a left neighborhood of x such that, for all points p in that neighborhood, $f(p) < f(x)$ and there exists a right neighborhood of x such that, for all points q in that neighborhood, $f(x) < f(q)$.

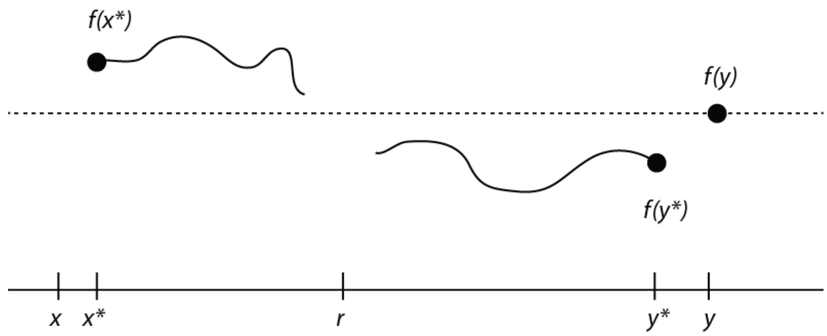
Want to show that f is strictly increasing.

Pick any two points $x < y$. Need to show $f(x) < f(y)$. Assume this is not the case. By the local property, always have points x^* and y^* such that following picture is the case, that is, $f(x^*) > f(y^*)$.



Now we have a continuous function on $[x^*, y^*]$ such that $x < y$ `[correction:`$\mathtt{x}^* < \mathtt{y}^*$`]`and $f(x^*) > f(y^*)$.

By the intermediate value theorem (because $[x^*, y^*]$ is connected), there exists some point r in $[x^*, y^*]$–hence in (x^*, y^*)–such that $f(r) = f(y)$.



> Furthermore, there must be at least one such point r^* in (x^*, y^*) such that $f(r^*) = f(y)$ for which there is a right neighborhood of r^* such that, for any q in that neighborhood, $f(q) < f(y)$. [**I have yet to work out the argument, but the idea is that if this were not the case, we could never get down to** $f(y^*)$**.**]
>
> At this point r^* the local property doesn't hold—that is, there's a right neighborhood of r^* such that, for any p in that neighborhood, $r^* < p$ and $f(r^*) > f(p)$.
>
> This is a contradiction, so the assumption that $f(x) < f(y)$ [`correction:` $\mathtt{f(x)} \geq \mathtt{f(y)}$] can't be true. So, for $x < y, f(x) < f(y)$.

In this document, the Intermediate Value Theorem is now used to justify, with a change in notation, the existence of an $r^* \in (x^*, y^*)$ such that $f(r^*) = f(y)$. Orienting to the professional prover's critique of the first mathematician's proof, I point out that the use of this theorem depends on the interval $[x^*, y^*]$ being connected. Absent this condition, no such r^* need exist.

The new problem is illustrated in Figure 3. Hypothetically, there could be many r_i where $f(r_i) = f(y)$. The function need not "cross the line" with the values of f to the immediate right of r_i being less than $f(y)$. The suggested solution is that this has to be the case for some r^*: otherwise, the values of f would never get below $f(y)$. Again, I draw attention to the lack of a mathematical argument justifying the claim.

On previous occasions, the professional prover had suggested that the supremum of some appropriately defined set might be important to the proof.

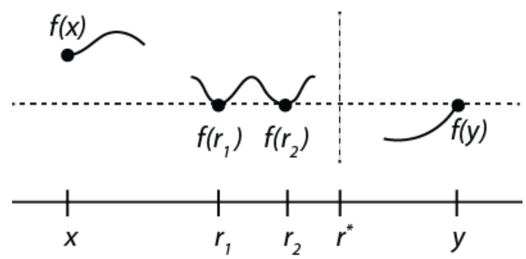


Figure 3: The Need to Find an Appropriate r^*

At this point, the professional made the suggestion definite in terms of Document 2: let $r^* = \sup\{r_i \in (x,y) \mid f(r_i) = f(y)\}$. A second intervention was to send me the proof-account in Appendix C. The proof shows that continuity is not needed: any pointwise increasing function on an open (or closed) interval is strictly increasing. I might have concluded that my present work was misdirected: Theorem 5 is not particularly interesting because the proof in Appendix C implies Theorem 5. Whether or not I fully understood this, the situation may reflect a difference in a professional's and a novice's orientation to the novice's work: I wanted to prove "my" theorem however much help I received.

Document 3

Theorem. If $f\colon (a,b) \longrightarrow \mathbf{R}$ is continuous on (a,b) and f has the local property LP `[that it is pointwise increasing on` (a,b)`]`, f is strictly increasing.

In order to prove the theorem, let $x < y$ be two points in (a,b). We want to show that $f(x) < f(y)$. The proof will be by contradiction where we assume that $f(x) \not< f(y)$. In fact, we can assume that $f(x)$ is strictly less `[corrected: strictly greater]` than $f(y)$: otherwise, there exists an x^+ in some interval $[x,\xi]$ of x `[delete "of x"]` such that $f(x) < f(x^+)$ and some y^- in some interval $(y-\rho, y]$ such that $f(y^-) < f(y)$ where $[x, x+\xi) \cap (y-\rho, y] = \emptyset$. The contradiction would then arise from the fact that $f(x^+) > f(y^-)$.

We now have a continuous function f on a compact interval $[x,y]$ such that $f(x) > f(y)$. By the Intermediate Value Theorem for continuous functions, there exists some point $r \in (x,y) \ni\colon f(r) = \frac{f(x)+f(y)}{2}$.

> Many such $r \in [x, y]$ may exist. The idea is to find the greatest $r \in [x, y] \ni$: $f(r) = \frac{f(x)+f(y)}{2}$ and to argue that LP `[the pointwise increasing property]` doesn't hold for this r.
>
> Earlier `[the professional prover]` had suggested that the supremeum `[supremum]` of an appropriately defined set might be involved in the original proof. Let $A = \{r_i \in [x, y] \mid f(r_i) = \frac{f(x)+f(y)}{2}\}$. It follows from the least upper bound property of real numbers that there exists an $r^* \in [x, y] \ni$: $r^* = \sup A$. Since $f(r^*) \neq f(y), r^* < y$. Moreover, by using the continuity of f or by using LP `[the pointwise increasing property]` and the Intermediate Value Theorem, $f(r^*) = \frac{f(x)+f(y)}{2}$. This, however, gives the contradiction: there exists a $\lambda > 0$ such that for all $p \in (r^*, r^* + \lambda), f(r^*) > f(p)$.

One change in Document 3 is the use of $\frac{f(x)+f(y)}{2}$ rather than $f(y)$ to derive the contradiction. The advantage over using $f(y)$ is not clear; it may have arisen in developing the argument in Lemma A.1. Following the professional's suggestion, I consider the supremum of $A = \{r_i \in [x, y] \mid f(r_i) = \frac{f(x)+f(y)}{2}\}$.

The last paragraph in Document 3 glosses some of the technical work of the proof. For example, two alternatives—the continuity of f and the pointwise increasing property—are claimed to establish that $f(r^*) = \frac{f(x)+f(y)}{2}$. No elaboration is given. In the first case, I had asked the professional how such a claim might be substantiated, and the professional had given an argument for doing so. I assumed the mutual availability of the argument. On the other hand, the second alternative is not explicated either. It continues to use the central device of the developing proof: Let r^* be the least upper bound of $A = \{r_i \in [x, y] \mid f(r_i) = \frac{f(x)+f(y)}{2}\}$. If $f(r^*) \neq \frac{f(x)+f(y)}{2}$, either a point to the left of r^* or to the right of r^* can be constructed that contradicts the fact that $r^* = \sup A$. Thus, $f(r^*) = \frac{f(x)+f(y)}{2}$.

At this point, the argument becomes clouded. I simply assert "This, however, gives the contradiction: there exists a $\lambda > 0$ such that for all $p \in (r^*, r^* + \lambda), f(r^*) > f(p)$." The contradiction is never stated. The idea is that $\forall p \in (r^*, y), f(p) \neq \frac{f(x)+f(y)}{2}$ and, by the Intermediate Value Theorem, $f(r^*) > f(p)$. Thus, f can't be pointwise increasing at r^*. This is the realization of the plan of the proof in Document 1.

Document 4

Definition (PIP). A continuous function on an open interval (a, b),
$f\colon (a, b) \longrightarrow \mathbf{R}$ is said to have the Pointwise Increasing Property PIP if the following condition holds:

> $\forall x \in (a, b), \exists\, \delta_x > 0$ such that if $h \in (x - \delta_x, x)$, then $f(x - h) < f(x)$ and if $h \in (x, x + \delta_x)$, then $f(x) < f(x + h)$.

A function that has the pointwise increasing property is said to be "PIP."

Theorem. *A differential* `[differentiable]` *function such that* $\forall x \in (a, b), f'(x) > 0$ *is PIP.*

Proof. By the definition of the derivative, for any given $x \in (a, b)$, $\forall \epsilon > 0\ \exists\, \delta > 0 \ni\colon \forall h, 0 < |h| < \delta$,

$$\left| \frac{f(x + h) - f(x)}{h} - f'(x) \right| < \epsilon$$

By choosing $\epsilon < f'(x)$, it follows that, since $f'(x) > 0$,

$$\frac{f(x + h) - f(x)}{h} > 0$$

This implies, for any $h, \delta < h < 0$ such that $x - h \in (a, b)$, that $f(x - h) < f(x)$ `[correction: for any h with` $0 < -\mathtt{h} < \delta$, $\mathtt{f(x + h) < f(x)}$`]`, and for any $h, 0 < h < \delta$ such that $x+h \in (a, b)$, that $f(x) < f(x+h)$. Since (a, b) is an open interval, δ can always be chosen such that $(x - \delta, x + \delta) \subset (a, b)$. □

Theorem. *If* $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ *is differentiable and* $f'(x) > 0$ *for all* $x \in (a, b)$*, then* f *is strictly increasing.*

Proof. Let $x < y$. We wish to show that $f(x) < f(y)$. For an arbitrary $\lambda > 0, f|_{(x-\lambda, y+\lambda)} : (x - \lambda, y + \lambda) \longrightarrow \mathbf{R}$ is differentiable. Hence, by the preceding theorem, f is PIP on $(x - \lambda, y + \lambda)$ and, hence, on $[x, y]$. A contradiction results if $f(x) \leq f(y)$ `[correction:` $\mathtt{f(x) \geq f(y)}$`]`.

> To derive the contradiction, we can assume without loss of generality that $f(x) > f(y)$. Otherwise, PIP insures that there exists an x_+ in some interval $[x, \xi)$ such that $f(x) < f(x_+)$ and some y_- in some interval $(y - \rho, y]$ such that $f(y_-) < f(y)$. The numbers ξ and ρ can be chosen such that $[x, x + \xi) \cap (y - \rho, y] = \emptyset$. The contradiction would then arise from the fact that $f(x_+) > f(y_-)$.
>
> Since f is continuous on $[x, y]$, there exists some $r \in (x, y) \ni$: $f(r) = \frac{f(x)+f(y)}{2}$ by the Intermediate Value Theorem. Let $r^* = \inf\{r_i \mid r_i \in (x, y) \text{ and } r_i = f(r) = \frac{f(x)+f(y)}{2}\}$. Then $r^* \in [x, y]$. Moreover, by continuity, as $r_i \longrightarrow r^*, f(r_i) \longrightarrow f(r^*)$. Since, by definition, for all $i, r_i = \frac{f(x)+f(y)}{2}$, it follows that $f(r^*) = \frac{f(x)+f(y)}{2} \neq f(x)$. Thus, $r^* \in (x, y)$.
>
> By Pointwise Increasing Property PIP, there exists some $\xi > 0$ such that, for all $p \in (r^* - \xi, r^*), f(p) < f(r^*)$. Let $q \in (r^* - \xi, r^*)$. Then $f(x) > f(q)$ and, again by the Intermediate Value Theorem, there exists some $r_k < r^*$ such that $r_k = f(r) = \frac{f(x)+f(y)}{2}$ This contradicts the definition of r^*. $\square$

Document 4 is a mess. On the positive side, a more descriptive name for the “local property” has been found, and the definition has been disentangled from the statement of the theorem and from the proof that uses it. The chief aim of the document seems to be to return to the original theorem involving differentiability (Theorem 4), structure a course of proving to lead to this end, and produce a proof that does not use the Mean Value Theorem. A better structure would begin with the definition of a pointwise increasing function, prove Theorem 5, and then claim Theorem 4 as a corollary.

Earlier the professional had commented that, in regard to Document 3 and in contrast with the proof in Appendix C, I had derived the contradiction from the values of the function (that the function would not be pointwise increasing at r^*) rather than from the assumed existence of the supremum. I make this change in Document 4. Further, mimicking the argument in Appendix C, I use the infimum rather than the supremum, deriving the contradiction on the left rather than on the right. Although expressed poorly, the idea is that, since f is pointwise increasing at r^*, there will be a point q to the left of r^* such that $f(q) < \frac{f(x)+f(y)}{2}$. By the Intermediate Value Theorem, there then exists a point $r' \in (x, q)$ such that $f(r') = \frac{f(x)+f(y)}{2}$.

This contradicts the fact that $r^* = \inf\{r_i \in (x, y) \mid f(r_i) = \frac{f(x)+f(y)}{2}\}$. The set $\{r_i \in (x, y) \mid f(r_i) = \frac{f(x)+f(y)}{2}\}$ must be the empty set, and $f(x) \not\geq f(y)$.

Appendix A gives a later version of my much-assisted proof-account.

One thing these documents suggest is that proof-accounts and the proofs that they describe are shaped over time. Although we have considered the work of a novice prover, nothing indicates that the same is not true for the proofs of professional mathematicians. Nor do we know the extent and character of collaborative and collegially-assisted proving. What we can see is that the activity of discovering proofs and writing proof-accounts concerns, and is animated by, the technical, mathematical details of developing arguments. The pursuit of mathematical clarity lies nowhere else.

5. Disciplining Provings' Work

The central interest in this paper is in how practically rigorous proofs are discovered, not in terms of the transcendent truths they are seen to reveal, but as witnessable objects that allow such truths to be exhibited. The awkwardness of my work, rather than being detrimental to such an inquiry, gives insight into how such objects are produced. The ordinary, familiar ways this is done seem to make up the disciplinary character of current mathematical practice: mathematicians are not only aware of the lived (as opposed to the formal) practices of proving, they constantly monitor and insist on them. Those practices are, in fact, integral to recognizing proofs as the practically rigorous proofs of ordinary mathematical practice.

Consider, first, my difficulties in producing coherent notation. On each occasion when notation is introduced, I am looking around for anything that I can find. Document 1 begins by using x as a real variable and a and b as fixed points on the real line, with λ as a distinguished point and ξ as a parameter. The end of the document returns to a Greek γ and δ with a Roman z between them. In Documents 1 and 2, I use x^* and y^*; in Document 3, I switch to x^+ and y^- as mnemonics for a point to the right and left respectively. In Document 4, I write $\delta < h < 0$ instead of $0 < -h < \delta$ even confusing myself, leading me to the erroneous conclusion that $f(x - h) < f(x)$. The notation x^+ and y^- is now changed to x_+ and y_-. After defining a set $\{r_i\}$, I use r_k to name a point less than any of the r_i.

The ability to generate consistent notation in the course of proving is different from, but related to, the use of notational conventions. The worldly reference of "conventional notation" is not to textbook or journal conventions, but to the situations where provers are proving theorems for and among each other. The skills of generating and using notation are cultivated, practiced, and monitored in those collegial settings.

Next, I had difficulty maintaining a consistent level of detail. Document 3 introduces unnecessarily $\frac{f(x)+f(y)}{2}$ to replace $f(y)$. I construct the set $A = \{r_i \in [x, y] \mid f(r_i) = \frac{f(x)+f(y)}{2}\}$ and argue, given the assumption that $f(x) > f(y)$, that $r^* = \sup A$ exists. Yet I then claim, without any elaboration, that $r^* < y$. Next, I write that $f(r^*) = \frac{f(x)+f(y)}{2}$ follows either from the continuity of f or the fact that f is pointwise increasing. No details are given. I conclude by simply claiming "[t]his, however, gives the contradiction: there exists a $\lambda > 0$ such that for all $p \in (r^*, r^* + \lambda), f(r^*) > f(p)$." Document 4 ends by asserting that the existence of a point r_k "contradicts the definition of r^*" without clarifying what this means—that the set of r_i's must be empty and, therefore, that $f(x) \ngtr f(y)$.

In terms of an eventual proof, the balance is wrong. Rather than exhibiting the identifying detail of the proof, the proof-account has a jigsaw puzzle effect where the pieces are offered, but the way they fit together is left to the reader. Instead of working through the pairing of text and lived-work, I settled for glosses, leaving to others, and particularly to the professional prover, the details of their association.

Throughout the documents, I am trying to extract the definition of a pointwise increasing function from the subsequent proof. I am also seeking an appropriate way of naming the identified property and referring to it. For the most part, my argument has an accountable, "followable" structure. The basic argument is set out: to pick arbitrary $x, y \in (a, b), x < y$ and to show that $f(x) < f(y)$ by assuming the contrary and showing that this leads to a contradiction. I eliminate the need to consider $f(x) = f(y)$, define an appropriate set, and show that the existence of a least upper bound gives rise to a contradiction.

In contrast with the structuring of the central proof, the larger structure of the argument goes completely awry in Document 4. First I prove that a differentiable function with positive derivative is pointwise increasing;

then I prove that a differentiable function with positive derivative is strictly increasing. The proof that a continuous, pointwise increasing function is monotonically increasing is hidden away, and I fail to mention that differentiability implies continuity. It is unclear what I want to prove and why it should have any interest. I seem to have ignored the work that I had already done and want to return to the question of whether the Mean Value Theorem is needed for the original textbook proof.

At first, such observations seem to describe the obvious failings of a novice prover. Viewed differently, they point to general features of mathematical practice. Even so, they seem to involve minor, unimportant, trivial details, distracting attention from the substantive concerns of mathematical proofs and the work of proving. I was struggling with work that, for professionals, is unremarkable, ordinary, practical, and utterly mundane. Yet, like the ordinary techniques of the chemistry laboratory, that work is everywhere where provers are engaged in proving, not abstractly, but in the intimate mathematical details of what they are doing. It provides the continually produced background for discovering and describing naturally accountable proofs.

Although the problems have been gathered under the rubrics "notational coherence," "consistency of detail," and "intrinsic structure," such difficulties in a course of proving are not seen, in the first instance, as standing for general problems, but as self-exhibiting, technical problems in their own detail. When seen, they are specifically noticed, even in their apparent inconsequentiality: the third paragraph of Document 3 has the statement "[w]e now have a continuous function f on a compact interval $[x, y]$ such that $f(x) > f(y)$." This is not exactly what I meant or what should have been said: given the ability to construct points x^+ and y^-, the possibility that $f(x) = f(y)$ need not be considered. All that needed to be said is that $f(x) > f(y)$ can be assumed without loss of generality.

This may seem like carping: "everybody" understands what was intended. At the same time, "everybody" sees the "mistake." In terms of the formula

the material text / the lived-work of proving

a slight distortion has been introduced in the left-hand side. In this case, the error and mispairing are easily recognized and easily repaired.

When professional work conditions are considered, my various assorted practices could only cause major problems. Professional theorem provers are not proving just one theorem, but proving theorems full time. For them, the ability to produce coherent argumentation is a continually practiced skill, clarifying discovery work and providing for its recognizability. That skill is part of the ever-present produced background for discovering and describing congregationally recognizable proofs for and among other theorem provers.

In part, my work on the proof begins to elucidate the *sine qua non* of disciplinary mathematics: the uninteresting "routine grounds" [4] of discovering and producing the congregationally recognizable, practically rigorous proofs of ordinary mathematical practice. At the same time, it also begins to reveal something else: the continual management of those practices and the collective and self supervision of them. Everywhere where proving is going on, provers are managing and supervising the ordinary, routine ways in which proving is done, not in general, but as it is done here and now, in and as the technical details of particular courses of proving. That management and supervision is not personal but anonymous, belonging to the witnessable conditions and circumstances of proving theorems.

The disciplinarity of proving seems so ingrained, so natural, and so ordinary that no one seems to have commented on it. It provides the grounds for discussions of the nature and truth of mathematics yet such discussions immediately distance and disassociate themselves from it. Nevertheless, it has pride of place in any claim that a previously unproved and resistant theorem has now been proved: one marvels at the brilliance of intellect but demands the witnessable shaping of disciplinary practice.

6. Discipline and Truth

The truth of a theorem does not insure that the theorem will be seen as true; strings of symbols and words do not constitute a proof, but need to be recognized as such. How do mathematicians, as everyday practice, come to recognize mathematical arguments as practically rigorous proofs? An anthropological, ethnographic, participant observer approach offers one way of investigating this situation: anthropological because it treats mathematicians as belonging to a culture of theorem provers and seeks to understand the ways of that culture and how it works; ethnographic because it attempts

to describe what members of the culture, in real time, actually do and uses documentary materials to further such inquiries; a participant observer approach because only by standing within and trying to participate in that culture can we begin to see what is relevant for its members, and how they orient to and engage in the work identifying of their membership.

Some years ago a colleague[6] pointed out that triathletes, as an integral part of their daily training, use electronic devices to measure their heart rate and caloric expenditure, their pace, speed, and cadence when running, to identify their swimming strokes, and to count the strokes and calculate the distance per stroke when bicycling and swimming. They keep detailed records of the data; they keep detailed schedules and records of their exercise and dietary routines. They are engaged in massive regimes of monitoring, habituating, and regulating their bodies.

The unrelenting disciplining of mathematical practice seems to have attracted almost no interest, not because it is hidden, but because of its ordinariness, utter familiarity, and omnipresence, and because it is concerned with and lodged within the minuscule, technical details of proving theorems. Yet the continually produced "steady state" of professional practice provides the background against which the adequacy of mathematical argumentation is witnessed. The infelicities and clumsiness of my attempt to prove a theorem help exhibit how closely mathematicians attend to the details of such work: unavoidably and inextricably, theorem provers watch how proving is done and, in watching, monitor and supervise those practices, insisting on the recognized, witnessable proprieties of how proving is done here (and now).

Mathematicians know the work of proving theorems not as abstract, general knowledge, as a matter of reflection, or as the result of scientific investigation, but as unavoidable participants in that work, practically skilled through their participation in it. It may be here, in the ordinariness and mundanity of disciplinary practice, that the amazing achievement of mathematical practice can best be appreciated—not in the transcendental truths of mathematical proofs, but as the produced, common witnessability of such truths.

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Appendix A. A Proof Using Continuity

Definition. A function $f\colon D \subset \mathbf{R} \longrightarrow \mathbf{R}$ is said to be *pointwise strictly increasing* (or *p.s.i.*) at $x \in D$ if $\exists\, \delta_x > 0 \ni: \forall p \in (x - \delta_x, x) \cap D$ and $\forall q \in (x, x + \delta_x) \cap D, f(p) < f(x) < f(q)$.

Lemma A.1. Let $f\colon [a, b] \longrightarrow \mathbf{R}$ be continuous on $[a, b]$ and pointwise strictly increasing on (a, b). Then f is pointwise strictly increasing on $[a, b]$.

Proof. If f is not p.s.i. at b, then for all $\delta > 0$, there exists an $x \in (b-\delta, b)$ such that $f(x) \geq f(b)$. Let s be one such x. By the p.s.i. property of f on (a, b), we can assume that $f(s)$ is strictly greater than $f(b)$; by the Intermediate Value Theorem, there exists some point $p_1 \in (s, b)$ such that $p_1 = \frac{f(s)+f(b)}{2}$. Now let $A = \{p_i \in (s, b] \mid p_i = \frac{f(s)+f(b)}{2}\}$, and let $p^* = \sup A$. By the continuity of f, it follows that $f(p^*) = \frac{f(s)+f(b)}{2} > f(b)$. Hence, $p^* < b$. Since f is p.s.i. at p^*, a similar argument would allow the construction of a point $p', p^* < p' < b$, such that $p' = \frac{f(s)+f(b)}{2}$. Thus, $p^* \neq \sup A$. The set A must be the empty set and, hence, f must be p.s.i. at b.

The argument that f is p.s.i. at a is similar. □

Theorem A.2. If $f\colon [a, b] \longrightarrow \mathbf{R}$ is continuous on $[a, b]$ and pointwise strictly increasing on (a, b), then f is a strictly increasing function.

Proof. By Lemma A.1, f is p.s.i. on $[a, b]$. Let $x, y \in [a, b]$ with $x < y$. We want to show that $f(x) < f(y)$. Assume to the contrary that $f(x) \geq f(y)$. By the p.s.i. property of f at x and y, there exists a $\delta_x > 0$ and a $p \in (x, x + \delta_x)$ such that $f(p) > f(x) \geq f(y)$, and there exists a $\delta_y > 0$ and a $q \in (y - \delta_y, y)$ such that $f(q) < f(y)$, where $(x, x + \delta_x) \cap (y - \delta_y, y) = \emptyset$.

By the Intermediate Value Theorem, $\exists\, r_1 \in (p, q)$ such that $f(r_1) = f(y)$. By the p.s.i. property of f, there exists some point $p_1 > r_1$ such that $f(p_1) > f(y) > f(q)$. Again by the Intermediate Value Theorem, $\exists\, r_2 \in (p_1, q)$ such that $f(r_2) = f(y)$. Now let $S = \{r_i \in (x, y) \mid f(r_i) = f(y)\}$.

Given the assumption that $f(x) \geq f(y), S \neq \emptyset$, so S has a least upper bound $r^* \in [x, y]$. Since $f(r_i) = f(y)$ for all i, by the continuity of f, as $r_i \to r^*, f(r_i) \to f(r^*) = f(y)$, with $r^* < y$ (since f is p.s.i. at y). By the same construction as before, $\exists\, r' \in (r^*, y)$ such that $f(r') = f(y)$. It follows that r^* cannot be the least upper bound of S. The set S must be empty and, hence, $f(x) \ngeq f(y)$. □

Appendix B. The Compactness Argument

Theorem B.1. If $f\colon [a, b] \longrightarrow \mathbf{R}$ is pointwise strictly increasing on $[a, b]$, then f is a strictly increasing on $[a, b]$.

Proof. Let $x, y \in [a, b]$ with $x < y$. At each point $p \in (x, y)$, let $B_{\delta_p}(p) = (p - \delta_p, p + \delta_p) \subset (x, y)$, where f is pointwise strictly increasing at p on $B_{\delta_p}(p)$. At $p = x$, let $B_{\delta_x}(x) = (x - \delta_x, x + \delta_x)$, where $[x, x + \delta_x) \subset [x, y)$ and f is pointwise strictly increasing at x on $B_{\delta_x}(x) \cap [x, y]$. Similarly at $p = y$, let $B_{\delta_y}(y) = (y - \delta_y, y + \delta_y)$, where $(y - \delta_y, y] \subset (x, y]$ and f is pointwise strictly increasing at y on $B_{\delta_y}(y) \cap [x, y]$. Then the collection of intervals $B_{\delta_p}(p)$ for all $p \in [x, y]$ is an open cover of $[x, y]$. By compactness, there exists a finite subcover $\{B_{\delta_{p_i}}(p_i) |\ i = 1, \ldots, m\}$ for some $m \in \mathbf{N}$. Remove any nested intervals from that subcover, and order the remaining intervals so that $p_i < p_{i+1}$. Then the set $\{B_{\delta_{p_i}}(p_i) |\ i = 1, \ldots, n\}$ with $2 \leq n \leq m$ is also a finite cover of $[x, y]$.

We can assume that there exists an $x_i \in B_{p_i}(p_i) \cap B_{p_{i+1}}(p_{i+1})$. If this were not the case, then for some $k = i + 2, \ldots, n, B_{p_i}(p_i) \cap B_{p_k}(p_k) \neq \emptyset$ because the $B_{p_i}(p_i)$ cover $[x, y]$ and $[x, y]$ is a connected set. But, then, $\delta_{p_{i+1}} < \delta_{p_k}$ and $B_{p_{i+1}}(p_{i+1}) \subset B_{p_k}(p_k)$, a nested interval. The x_i can always be chosen so that $p_i < x_i < p_{i+1}$ because $p_i < p_{i+1}$.

Because f is pointwise strictly increasing at each p_i, the theorem then follows: $f(x) = f(p_1) < f(x_1) < f(p_2) < f(x_2) < \cdots < f(x_n) < f(p_n) = f(y)$. □

Appendix C. The Professional's Proof Sketch

$F(x)$ is defined on $D = [0, 1]$ satisfying: for every x there is an $\epsilon > 0$ such that $F(x - a) < F(x) < F(x + a)$ for all $0 < a < \epsilon$, when defined.

Claim. *$F(x)$ is strictly increasing.*

Proof. Suppose that $F(A) \geq F(B)$ for some $A < B$.

Let S denote $\{x \mid x > A \text{ and } F(A) \geq F(x)\}$. Let $M = \text{infimum } S$.

By the "$F(x) < F(x + a)$" condition, $M > A$. By the "$F(x - a) < F(x)$" condition, $M < B$.

If $F(M) \leq F(A)$, then by the "$F(x-a) < F(x)$" condition, there are $y < M$ such that $F(y) \leq F(A)$, contradicting that M is a lower bound for S.

If $F(M) > F(A)$, then by the "$F(x+a) > F(x)$" condition, $F(x) > F(A)$ for all x in some interval $[M, M + \epsilon)$, and so M is not the greatest lower bound of S. $\square$