

1-1-2012

## Review: The Dirichlet Space: A Survey

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### Recommended Citation

MR2782728 (2012b:30055) Arcozzi, N., Rochberg, R., Sawyer, E.T., Wick, B.D., The Dirichlet space: a survey, *New York J. Math.* 17A (2011), 45–86. (Reviewer: Stephan R. Garcia)

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**MR2782728 (2012b:30055)** 30C85 (30E05 31C25 46E22)

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**The Dirichlet space: a survey. (English summary)**

*New York J. Math.* **17A** (2011), 45–86.

This paper is a thorough survey of many recent and historical function-theoretic results about the classical Dirichlet space  $\mathcal{D}$  on the open unit disk  $\mathbb{D}$ . As the authors themselves point out, generalizations to other domains or to several variables and connections to operator theory are not discussed. Nevertheless, there is still a wealth of information covered in this survey.

The authors begin with the basic definitions, including several alternate characterizations of  $\mathcal{D}$  due to R. Rochberg and Z. J. Wu [Illinois J. Math. **37** (1993), no. 1, 101–122; [MR1193132 \(93j:30039\)](#)] and B. Bøe [Proc. Amer. Math. Soc. **131** (2003), no. 1, 235–241; [MR1929043 \(2003g:46024\)](#)]. An in-depth study of Carleson measures for  $\mathcal{D}$  is undertaken in Section 3. In particular, the original characterization of Carleson measures for  $\mathcal{D}$  due to D. A. Stegenga [Illinois J. Math. **24** (1980), no. 1, 113–139; [MR0550655 \(81a:30027\)](#)] and several more recent approaches of E. Tchoundja [Ark. Mat. **46** (2008), no. 2, 377–406; [MR2430733 \(2009g:32012\)](#)] and the first three authors [Rev. Mat. Iberoamericana **18** (2002), no. 2, 443–510; [MR1949836 \(2003j:30080\)](#)] are treated. A detailed exposition of the *tree model* of the unit disk and its application to the Dirichlet space are explored in Section 4. In particular, the authors study Carleson measures, capacities, and testing conditions from this viewpoint. A brief discussion of the complete Nevanlinna-Pick property is conducted in Section 5 (a more complete treatment can be found in the book [J. Agler and J. E. McCarthy, *Pick interpolation and Hilbert function spaces*, Grad. Stud. Math., 44, Amer. Math. Soc., Providence, RI, 2002; [MR1882259 \(2003b:47001\)](#)]), and Section 6 studies the multiplier space  $\mathcal{M}(\mathcal{D})$  and other spaces which are intrinsic to  $\mathcal{D}$  theory. In particular, the weakly factored space  $\mathcal{D} \odot \mathcal{D}$ , the  $\bar{\partial}$ -equation in the Dirichlet space, and the corona theorem for  $\mathcal{D}$  are considered. After a detailed discussion on interpolating sequences for  $\mathcal{D}$  and its multiplier space  $\mathcal{M}(\mathcal{D})$ , the paper concludes with several open problems.

This paper will no doubt become a standard reference on the subject and also the starting point for many graduate students.

Reviewed by *Stephan R. Garcia*

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*Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.*

