12-31-2011

Review: On the Near Periodicity of Eigenvalues of Toeplitz Matrices

Stephan Ramon Garcia
Pomona College

Recommended Citation

MR2757527 (2012d:47084) Levitin, M., Sobolev, A.V., Sobolev, D., On the near periodicity of eigenvalues of Toeplitz matrices, Operator theory and its applications, 115125, Amer. Math. Soc., Transl. Ser. 2, 231, Amer. Math. Soc., Providence, RI, 2010. (Reviewer: Stephan R. Garcia)

MR2757527 (2012d:47084) 47B35 (15A18 15B05 47A75)
Levitin, Michael (4-RDNG-MS);
Sobolev, Alexander V. [Sobolev, Aleksandr Vladimirovich[2]] (4-LNDUC);
Sobolev, Daphne (4-OPENM)

On the near periodicity of eigenvalues of Toeplitz matrices.

Operator theory and its applications, 115–125, *Amer. Math. Soc. Transl. Ser.* 2, 231, *Amer. Math. Soc., Providence, RI*, 2010.

In order to study the spectrum of a given selfadjoint operator A on a Hilbert space H, one frequently considers the truncated operators $A_n = P_n A P_n$ where $\{P_n\}$ is a sequence of orthogonal projections onto certain finite-dimensional subspaces of H and such that P_n converges to I in the strong operator topology. This procedure is not necessarily straightforward since the approximating operators A_n may possess *spurious eigenvalues*, which do not, in the limit, correspond to spectral values of the original operator A. The presence of such spurious eigenvalues is sometimes called *spectral pollution* [see, e.g., E. B. Davies and M. Plum, IMA J. Numer. Anal. **24** (2004), no. 3, 417–438; MR2068830 (2005c:47027b)].

If $a\colon [-\pi,\pi) \to \mathbb{R}$ is a piecewise continuous function and $P_n\colon L^2(-\pi,\pi) \to L^2(-\pi,\pi)$ denotes the orthogonal projection onto the subspace spanned by the first n standard basis exponentials, let $T_n = T_n[a] = P_n A P_n$ denote the corresponding Toeplitz operator where A is the operator on $L^2(-\pi,\pi)$ of multiplication by the function a. If the range of the symbol a is disconnected, then it is known that the spurious eigenvalues fill in the gaps separating the components of the range of a. To be more specific, it can be deduced from the work of E. L. Basor [J. Math. Anal. Appl. **120** (1986), no. 1, 25–38; MR0861905 (88a:47027)] that an open interval I lying strictly inside the spectral gap contains $W \log n + O(1)$ eigenvalues of T_n where W is an explicitly computable constant depending on a and I.

In their paper, the authors consider the spectrum of the Toeplitz operator corresponding to the piecewise constant symbol

$$a(x) = \begin{cases} 0 & \text{if } x \in [-\pi, L), \\ 1 & \text{if } x \in [L, \pi), \end{cases}$$

where $L \in [0,\pi)$. Numerical work of M. R. Levitin and E. M. Shargorodskiĭ [IMA J. Numer. Anal. **24** (2004), no. 3, 393–416; MR2068829 (2005c:47027a)] indicates that if L is a rational multiple of π, then the spurious eigenvalues of $T_n[a]$ are in a certain sense "nearly periodic". Although an intuitive discussion of this phenomenon was carried out in [E. M. Shargorodskiĭ, in *Operator theory and its applications*, 173–180, Amer. Math. Soc. Transl. Ser. 2, 231, Amer. Math. Soc., Providence, RI, 2010; MR2757530 (2012d:47015)], a rigorous proof of this intriguing phenomenon has not yet been given.

The article in question takes a step in this direction by establishing the near periodicity of the eigenvalues for the squared Toeplitz operator $M_n = (T_n[a])^2$ with symbol

$$a(x) = \begin{cases} -1 & \text{if } x \in [-\pi, L), \\ 1 & \text{if } x \in [L, \pi). \end{cases}$$

To be more specific, the authors prove that if $pL = \pi q$ where $p \in \mathbb{N}$ and $q \in \mathbb{Z}$ are coprime, then, for each $\varepsilon \in (0,1)$ and each index j such that $\mu_j^{(n)} < 1 - \varepsilon$, there exists a corresponding $K > 0$ such that $n \geq K\omega\varepsilon^{-1}$ implies that the estimate

$$|\mu_j^{(n)} - \mu_j^{(n+\omega)}| \leq \frac{C\omega(1 + \log^2 n)}{\varepsilon n}$$

holds where the constant $C > 0$ is independent of p, q, n, j, ε and

$$\omega = \omega(p,q) = \begin{cases} 2 & \text{if } q = 0, \\ p & \text{if } p \text{ and } q \text{ are odd}, \\ 2p & \text{if either } p \text{ or } q \text{ is even}, \end{cases}$$

and $\mu_j^{(n)}$ denotes the jth eigenvalue of M_n, listed in ascending order.

{For the entire collection see MR2663905 (2012a:47001)}

Reviewed by *Stephan R. Garcia*