2009

Exploring Agreeability in Tree Societies

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May, 2009

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Abstract

Let $\mathcal{S}$ be a collection of convex sets in $\mathbb{R}^d$ with the property that any subcollection of $d-1$ sets has a nonempty intersection. Helly's Theorem states that $\bigcap_{s \in \mathcal{S}} \mathcal{S}$ is nonempty. In a forthcoming paper, Berg et al. (Forthcoming) interpret the one-dimensional version of Helly's Theorem in the context of voting in a society. They look at the effect that different intersection properties have on the proportion of a society that must agree on some point or issue. In general, we define a society as some underlying space X and a collection $\mathcal{S}$ of convex sets on the space. A society is (k, m)-agreeable if every m-element subset of $\mathcal{S}$ has a k-element subset with a nonempty intersection. The agreement number of a society is the size of the largest subset of $\mathcal{S}$ with a nonempty intersection.

In my work I focus on the case where X is a tree and the convex sets in $\mathcal{S}$ are subtrees. I have developed a reduction method that makes these tree societies more tractable. In particular, I have used this method to show that the agreement number of $(2, m)$-agreeable tree societies is at least $\frac{1}{3}|\mathcal{S}|$ and that the agreement number of $(k, k+1)$-agreeable tree societies is at least $|S| - 1$.

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Acknowledgments

I would like to thank Professor Francis Su for his patient guidance and encouragement both as my advisor for this thesis project and throughout my four years at Mudd and my second reader Professor Kimberly Tucker for all of her helpful comments. I would also like to thank the math department as a whole for providing a wonderful and supportive environment to learn and grow in for the last four years.

Chapter 1

Background

1.1 Helly's Theorem

The simplest version of the classical Helly's theorem states:

Helly's Theorem in 1-dimension. *Given n convex sets, that is intervals, in $\mathbb{R}$, if every pair has a nonempty intersection then all n intersect at a common point.*

Consider the example in Figure 1.1. Each of the colored lines represents an interval of $\mathbb{R}$, slightly displaced so they may be distinguished. Each pair of intervals has a nonempty intersection and we can see that there is a point contained in all of the intervals.

On the plane, the property that every pair of sets in a collection of convex sets has a nonempty intersection is no longer sufficient to guarantee that there will be a point contained in all of the sets. In Figure 1.2, we see three convex sets that intersect pairwise but still have an empty intersec-

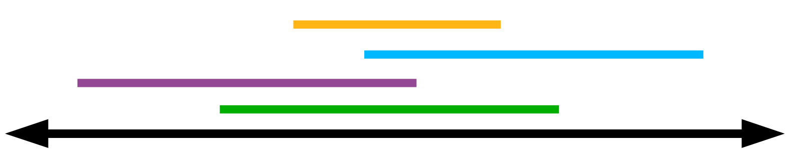

Figure 1.1: Four pairwise intersecting intervals.

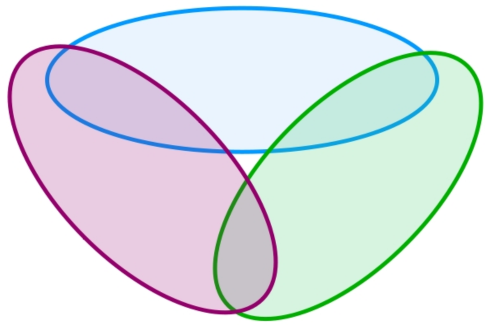

Figure 1.2: Three convex sets that intersect pairwise but have a nonempty intersection.

tion. However, if we change the hypothesis to require that every three sets in a collection of n convex sets has a nonempty intersection, we guarantee that there is a point in the intersection of the n sets. In general, the d-dimensional version of the classical Helly's theorem states:

Theorem 1.1 (Helly)**.** *Given n convex sets in $\mathbb{R}^d$ where $n > d$, if every $d+1$ of them has a nonempty intersection, then they all intersect at a common point.*

In moving to higher dimensions, it is necessary to change the intersection properties of the sets in the hypothesis to guarantee the same conclusion. It is also interesting to consider what results can be obtained when we alter the intersection properties in the hypothesis. This has been well explored by Berg et al. (Forthcoming) both in the linear case and in $\mathbb{R}^d$. In the linear case, Berg et al. (Forthcoming) also provide an interesting application to approval voting systems, which we describe below.

1.2 Agreeable Societies

If we consider $\mathbb{R}$ as the political spectrum and an interval (convex set) as the region of the political spectrum of which a particular voter approves , we can use a collection of intervals to model the political preferences of a society. For example, in Figure 1.3 we have taken our earlier linear example and labeled the line as a political spectrum. We can interpret the line and collection of four intervals as a society with four voters and see that there is a region of the political spectrum approved by all of the voters.

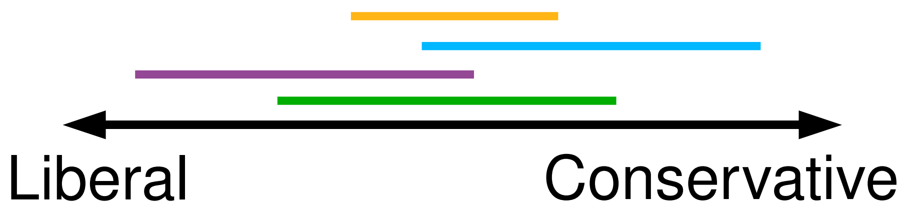

Figure 1.3: Interpreting the real line as a political spectrum.

If X is a linear space and $\mathcal{S}$ is a collection of convex sets of X, we call the ordered pair $(X, \mathcal{S})$ a *linear society*, view X as a political spectrum, and associate a voter v to each element $A_v \in \mathcal{S}$ called the voter's *approval set*. If a candidate's position falls within A_v we say that the voter v would approve that candidate. Let the *agreement number* $a(\mathcal{S})$ of a society $(X, \mathcal{S})$ be the size of the largest subset of $\mathcal{S}$ that has a nonempty mutual intersection. In a society satisfying the hypotheses of the classical Helly's Theorem, such as the society in Figure 1.3, we have $a(\mathcal{S}) = |\mathcal{S}|$.

Translating Helly's theorem in $\mathbb{R}$ using this context, we see that in a linear society $(\mathbb{R}, \mathcal{S})$ in which the convex sets corresponding to every pair of voters have a nonempty intersection, we have $a(\mathcal{S}) = |\mathcal{S}|$, that is, there is some point on the political spectrum of which all voters in the society approve. However, if we are having an election, it is not terribly useful to have a point on the spectrum approved by all voters if there is no candidate at that point for whom to vote.

To incorporate the idea of candidates, we can either assume that there is a candidate at every point on the line or we can place a node for every candidate on the line thus moving from $\mathbb{R}$ into a discrete line, that is, a possibly infinite (if there are infinitely many candidates) linear tree. If we move to this discrete scenario, then we require the subsets corresponding to the voters to be subtrees of the underlying linear tree. In Figure 1.4 we add five candidates to the linear society we were looking at earlier and convert the intervals associated with each of the four voters into subtrees.

Furthermore, we know that any given group of people is highly unlikely to all agree on anything, so it is interesting to ask what happens if, for example, we know that some pair amongst every three people in a society agree even though any given pair might not. The following theorem gives a bound on the agreement number of a linear society in which every

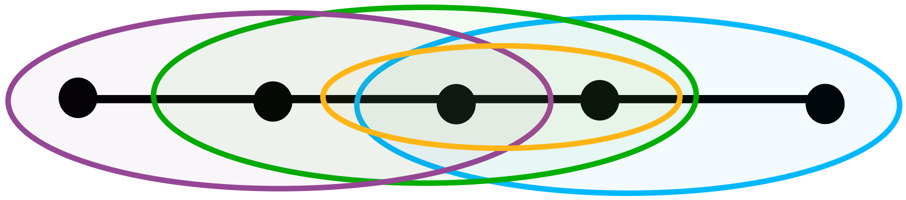

Figure 1.4: A discrete linear society.

collection of m voters has some subset of k voters that all approve of some candidate.

Theorem 1.2 (Berg et al. (Forthcoming)). *Let* $2 \leq k \leq m$. *If* $(X, \mathcal{S})$ *is a* (k, m)*-agreeable linear society, then*

$$a(\mathcal{S}) \geq \lceil (|\mathcal{S}| - \rho) / q \rceil$$

where $m - 1 = (k-1)q + \rho$, $\rho \leq k - 2$.

In addition, Berg et al. (Forthcoming) provides a construction showing that there are linear societies for which this lower bound is attained.

We define a *circular society* $(X, \mathcal{S})$ analogously, where we now require X to either be a continuous circle or a cycle graph. The following recent result mirrors the above results for linear societies:

Theorem 1.3 (Hardin (Forthcoming)).

1. *If* $(X, \mathcal{S})$ *is a* (k, m)*-agreeable circular society, then*

   $$\frac{a(\mathcal{S})}{|\mathcal{S}|} > \frac{k-1}{m}$$

   equivalently, $a(\mathcal{S}) \geq \lfloor \frac{k-1}{m} |\mathcal{S}| \rfloor + 1$.

2. *For every* $N \geq m \geq k \geq 1$, *there is a* (k, m)*-agreeable circular society* $(X, \mathcal{S})$ *with* $|\mathcal{S}| = N$ *and* $a(\mathcal{S}) = \lfloor \frac{k-1}{m} |\mathcal{S}| \rfloor + 1$.

Hardin's work is discussed further in Section 1.4.

1.3 Graphs and the Helly Property

Helly's theorem has also given rise to the definition of a *Helly family* of sets. A collection of sets $\mathcal{S}$ form a *Helly family* if every pairwise intersecting subset of $\mathcal{S}$ has a common intersection. In the graph theory literature, a certain class of graphs with a designated type of subgraph is said to have the *Helly property* if the collection of all subgraphs of the designated type form a Helly family. In recent years, a great deal of work has been done identifying and characterizing classes of graphs with a specific type of subgraph that have the Helly property. The simplest and most well known such class is trees with their subtrees.

Theorem 1.4 (Helly Property for Trees)**.** *If $G_1, \ldots, G_k$ are pairwise-intersecting subtrees of a tree G, then G has a vertex that belongs to all of $G_1, \ldots G_k$.*

A second class that has been well studied are the *clique-Helly* graphs, those graphs for which the collection of maximal cliques form a Helly family. Lin and Szwarcfiter (2007) provide a characterization of clique-Helly graphs that leads to a polynomial-time algorithm for recognizing them. In addition to clique-Helly graphs, the current literature has much to say about *biclique-Helly graphs* — those graphs in which the maximal bicliques form a Helly family, *neighborhood Helly graphs* — where the neighborhoods of the graph form a Helly family, and such increasingly complicated classes as *self-clique Helly circular-arc graphs*.

1.4 Agreeability on Graphs

We noted in Section 1.1 that when the classical Helly's theorem is generalized in $\mathbb{R}^d$, it is the intersection property that is changed in the hypothesis, not the type of subset. Generalizations of Helly on continuous domains, such as Berg et al. (Forthcoming) have tended to involve exploring sets with different intersection properties. It thus seems odd that the generalizations of Helly's theorem in the graph theory literature have focused almost exclusively on determining classes of graphs that satisfy the Helly property rather than starting with an intersection property and exploring what can be said about families that satisfy it.

That said, Hardin (Forthcoming) uses discrete representations, specifically cycle graphs, to study circular societies. His work provides a method for turning any circular society into a discrete circular society, where the underlying space is a cycle graph and each subset is a connected subgraph.

He further shows that every discrete circular society is equivalent to a discrete circular society in which none of the subsets is completely contained in another subset and the collection of endpoints of the subsets alternates between right and left endpoints as one proceeds around the cycle. By showing that all circular societies are equivalent to discrete circular societies with certain structural properties, Hardin is able to make the graphs he is working with much more manageable.

Hardin (Forthcoming) also points out that it is unknown whether or how Theorem 1.2 generalizes to trees with collections of subtrees. My work has been exploring this question, using Hardin's technique of converting arbitrary discrete societies into discrete societies with some amount of known structure.

1.5 Terminology and Notation

Let T be a tree. Then a subtree s is a connected subgraph of T.

When considering the intersection of a collection of subgraphs of some graph G, we say that the intersection of the subgraphs is nonempty if the subgraphs have at least one common vertex.

Let $G = (V, E)$ be any graph and $v \in V$. Then $G - v$ is the subgraph of G induced by all vertices of G except v.

If T is a tree and $\mathcal{S}$ is a collection of subtrees of T and $s \in \mathcal{S}$, then $\mathcal{S} - s$ is the result of removing one copy of s from $\mathcal{S}$. Unless we specifically state otherwise, we allow any collection $\mathcal{S}$ of subtrees to contain multiple, distinguishable copies of the same subtree. We will sometimes refer to subsets of $\mathcal{S}$, but if $\mathcal{S}$ has multiple copies of some subtree then the subsets are also allowed to contain multiple copies of that subtree.

A *tree society* $(T, \mathcal{S})$ is a tree T along with a collection $\mathcal{S}$ of subtrees. We say that $(T, \mathcal{S})$ is (k, m)*-agreeable* if every m-element subset of $\mathcal{S}$ has a k-element subset with a nonempty intersection. The *agreement number*, $a(\mathcal{S})$, of a tree society $(T, \mathcal{S})$ is the size of the largest subset of $\mathcal{S}$ that has a nonempty intersection.

If $(T, \mathcal{S})$ is a tree society and v is a vertex of T , we let

$$\mathcal{S}_v = \{t \in \mathcal{S} | v \in V(t)\}\,;$$

that is, the set of subtrees in $\mathcal{S}$ that contain the vertex v.

1.6 Our Main Goal

Given a (k, m)-agreeable linear society, Theorem 1.2 provides a sharp lower bound on the agreement number for the society. Hardin (Forthcoming) has provided a sharp lower bound for the agreement number of (k, m)-agreeable circular societies. Our goal has been to provide a lower bound on the agreement number for (k, m) agreeable tree societies.

While we have not found a sharp lower bound for (k, m)-agreeable tree societies in general, we provide such bounds for $(k, k+1)$-agreeable and $(2, m)$-agreeable tree societies in Theorem 3.2 and Theorem 3.3 respectively. In addition, in Chapter 2 we develop a notion of equivalence of tree societies and show that every tree society can be reduced to a tree society with particular structural properties.

Chapter 2

A Reduction Method

In this chapter we begin by discussing what it means for two tree societies to be equivalent. We then show that every (k,m)-agreeable tree society can be reduced to an equivalent (k,m)-agreeable tree society with certain useful structural properties. This reduction process and the accompanying notion of irreducible tree societies play a vital role in the work we do to determine lower bounds on the agreement number in Chapter 3.

2.1 Equivalent Tree Societies

Given a (k,m)-agreeable tree society $(T,\mathcal{S})$, we seek a lower bound for the agreement number of the society. Both our initial conditions on $\mathcal{S}$ and our question are purely in terms of intersection properties of the subtrees in $\mathcal{S}$. Thus it makes sense to consider two tree societies to be equivalent if their subtrees intersect in the same ways.

More formally, we say that two tree societies $(T,\mathcal{S})$ and $(T',\mathcal{S}')$ are *equivalent* if there is a bijection $\varphi : \mathcal{S} \to \mathcal{S}'$ that preserves the intersection properties of the subtrees, that is, if $R \subseteq \mathcal{S}$, then $\bigcap_{r\in R} r$ is empty if and only if $\bigcap_{r\in R} \varphi(r)$ is empty. Figure 2.1 shows a pair of equivalent tree societies.

2.2 Irreducible Tree Societies

In this section we define the notion of irreducible tree societies and explore the structural properties of these societies. In Theorem 2.3 we show that every tree society is equivalent to an irreducible tree society.

Let $(T,\mathcal{S})$ be a tree society and suppose l is a leaf of T and v is the vertex

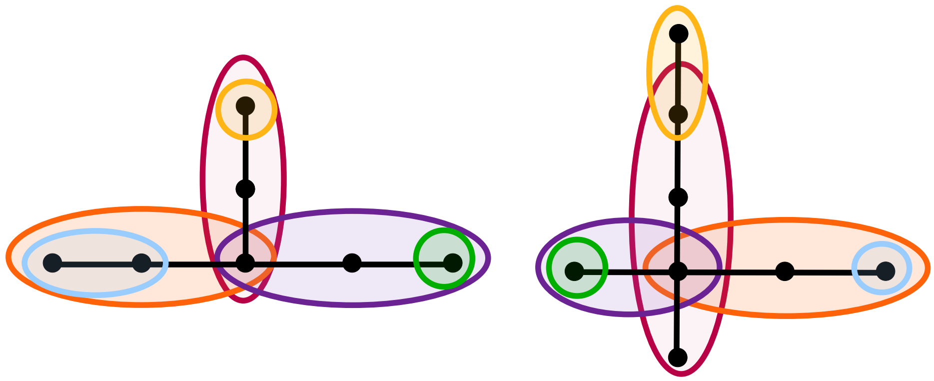

Figure 2.1: These two tree societies are equivalent. The bijection is given by pairing the subtrees of the same color.

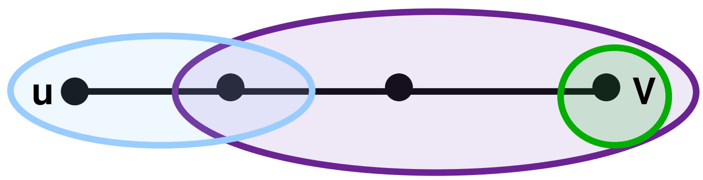

Figure 2.2: The leaf u is reducible, but the leaf v is not.

adjacent to l. If $\mathcal{S}_l \subseteq \mathcal{S}_v$, then every subtree containing l also contains its neighbor v and we call l a *reducible leaf*. In Figure 2.2 we provide an example of a tree society that has both a reducible leaf and a nonreducible leaf.

We define an *irreducible tree society* $(T, \mathcal{S})$ as a tree society with no reducible leaves. If the underlying tree of a society is a single vertex then the society is irreducible. The following theorem provides a useful characterization of irreducible tree societies:

Theorem 2.1. *A tree society $(T, \mathcal{S})$ is irreducible if and only if every leaf of T is covered by at least one single-vertex subtree from $\mathcal{S}$.*

Proof. Suppose $(T, \mathcal{S})$ is irreducible. By definition, if a leaf is not reducible, then there is some subtree in $\mathcal{S}$ that contains that leaf but not the vertex adjacent to it. Because every leaf of T is not reducible, we can find a single-vertex subtree in $\mathcal{S}$ that covers each leaf.

If a leaf l is covered by a single-vertex subtree t, then t does not contain the vertex adjacent to l. Thus l is not reducible. If every leaf of T is covered by at least one single-vertex subtree from $\mathcal{S}$, it follows that none of the leaves are reducible. Thus $(T, \mathcal{S})$ is irreducible. □

The following structural result for irreducible tree societies follows directly from this characterization:

Lemma 2.1. *In an irreducible (k, m)-agreeable tree society $(T, \mathcal{S})$, the underlying tree has at most $m - k + 1$ leaves.*

Proof. $(T, \mathcal{S})$ is irreducible, so we can find a single-vertex subtree in $\mathcal{S}$ that covers each of the leaves of T. Let $L \subseteq \mathcal{S}$ be a minimum set of single-vertex subtrees that together cover all of the leaves of T. Note that $|L|$ is the number of leaves in T and that the elements of L are pairwise disjoint. Let $R \subseteq \mathcal{S} - L$ be a set of $m - |L|$ subtrees and consider the set $R \cup L$. $R \cup L$ is a collection of m elements of $\mathcal{S}$, so it must have a k-element subset that shares a common vertex. Because the elements of L are disjoint, only one of these k elements can come from L, so there must be at least $k - 1$ elements in R. Therefore $m - |L| \geq k - 1$, which directly implies that $|L| \leq m - k + 1$. Thus we see that T must have at most $m - k + 1$ leaves. □

Corollary 2.1. *If $(T, \mathcal{S})$ is an irreducible $(n, n+1)$-agreeable tree society then T has at most two leaves and is thus either a single vertex or a path.*

These structural properties of irreducible tree societies make them much easier to work with than arbitrary tree societies.

2.3 A Reduction Process

In this section we prove that every tree society is equivalent to an irreducible tree society. This will allow us to prove theorems about general tree societies while only working directly with irreducible tree societies.

We begin by considering the effect of removing a single reducible leaf from a tree society.

Theorem 2.2. *If $(T, \mathcal{S})$ is a tree society and l is a reducible leaf, then $(T', \mathcal{S}')$ is equivalent to $(T, \mathcal{S})$ where $T' = T - l$ and $\mathcal{S}' = \{t - l : t \in \mathcal{S}\}$.*

Proof. Let v be the unique neighbor of the leaf l. It is clear that each $t - l$ is a nonempty subtree of T', so $(T', \mathcal{S}')$ is a tree society. Define a map $\varphi : \mathcal{S} \to \mathcal{S}'$ by $\varphi(t) = t - l$. The construction of $\mathcal{S}'$ ensures that φ is a bijection. Furthermore, as every subtree in $\mathcal{S}$ that contains l also contains v, the image of any collection of subtrees will have a nonempty intersection if and only if the original set had a nonempty intersection. Thus $(T, \mathcal{S})$ and $(T', \mathcal{S}')$ are equivalent tree societies. $\square$

In the next theorem we extend this method for removing a single reducible leaf to an algorithm that converts any tree society into an equivalent irreducible tree society.

Theorem 2.3. *Every finite tree society $(T, \mathcal{S})$ is equivalent to an irreducible tree society.*

Proof. The following algorithm takes any finite tree society and produces an equivalent irreducible tree society.

Input: A tree society $(T, \mathcal{S})$

Initialization: Let L be the set of reducible leaves of $(T, \mathcal{S})$ and set $T' = T$, $\mathcal{S}' = \mathcal{S}$.

Iteration: If L is empty, we are done and $(T', \mathcal{S}')$ is an irreducible tree society equivalent to $(T, \mathcal{S})$. Otherwise, choose $l \in L$. Set $T' = T' - l$, $\mathcal{S}' = \{t - l : t \in \mathcal{S}'\}$. Let L be the set of reducible leaves of $(T', \mathcal{S}')$ and iterate.

At every iteration we produce a tree society $(T', \mathcal{S}')$ that is equivalent to $(T, \mathcal{S})$. Because each iteration decreases the number of vertices in T', T is finite, and every society whose underlying tree is a single vertex is irreducible, the algorithm must terminate. $\square$

See Figure 2.3 for an example of the application of this algorithm.

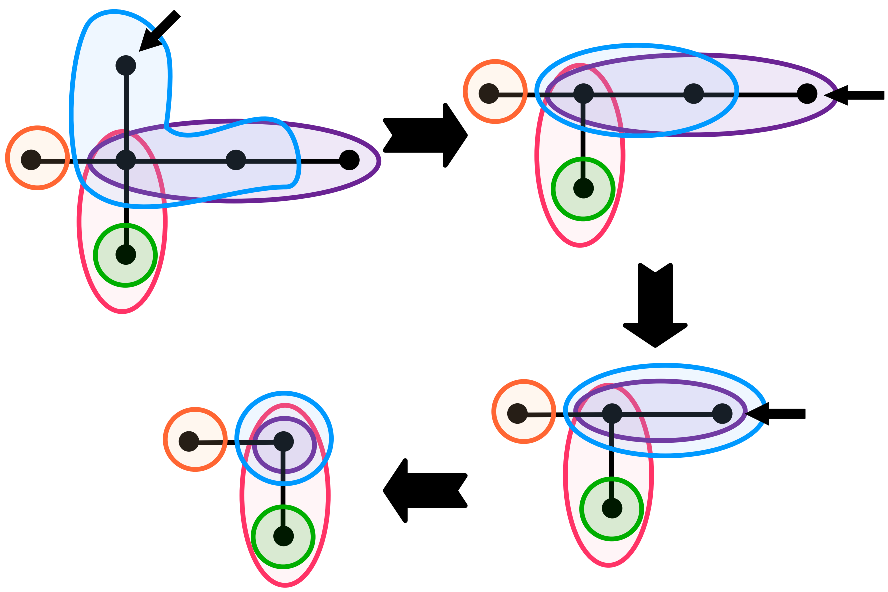

Figure 2.3: This figure illustrates the application of the algorithm in Theorem 2.3 to a specific tree society. At each step the next vertex to be removed is marked with an arrow.

Chapter 3

(k,m)-Theorems for Tree Societies

In this chapter we explore lower bounds on the agreement number of (k,m)-agreeable tree societies. In particular, in Theorem 3.2 and Theorem 3.3 we provide sharp bounds for the agreement number of $(n,n+1)$- and $(2,m)$-agreeable tree societies respectively. Table 3.1 summarizes our bounds for (k,m)-agreeable tree societies when $m \leq 6$.

3.1 Definitions

Recall from Section 1.5 that we define a *tree society* as an ordered pair $(T,\mathcal{S})$ where T is a tree and $\mathcal{S}$ is a collection of nonempty subtrees of T. We allow $\mathcal{S}$ to contain multiple copies of a particular subtree, but we assume that we can distinguish between these copies.

A tree society $(T,\mathcal{S})$ is (k,m)*-agreeable* if every m-element subset of $\mathcal{S}$ has a k- element subset with a nonempty intersection. We define the *agreement number*, $a(\mathcal{S})$, of a tree society $(T,\mathcal{S})$ to be the size of the largest subset of $\mathcal{S}$ that has a nonempty intersection.

If v is a vertex of T, we let

$$\mathcal{S}_v = \{t \in \mathcal{S} | v \in V(t)\};$$

that is, the set of subtrees in $\mathcal{S}$ that contain the vertex v.

3.2 $(n, n+1)$-Agreeable Tree Societies

In this section we provide a bound for $(n, n+1)$-agreeable tree societies. We begin by examining the case where $n = 2$ and then consider the $n > 2$ case.

Theorem 3.1. *In a $(2,3)$-agreeable tree society $(T, \mathcal{S})$ there is some vertex in at least $\frac{1}{2}$ of the subtrees in $\mathcal{S}$ and this result is sharp.*

Proof. Let $(T', \mathcal{S}')$ be an irreducible tree society equivalent to $(T, \mathcal{S})$ and let $\varphi : \mathcal{S} \to \mathcal{S}'$ be bijective and intersection-preserving. We know by Corollary 2.1 that T' has at most two leaves.

Case 1: If T' has fewer than two leaves, it must be a single vertex, v. In this case, $\bigcap_{s' \in \mathcal{S}'} s' = \bigcap_{s \in \mathcal{S}} \varphi(s) = v$ and, because φ is intersection-preserving, we see that there is some vertex of T that is contained in every $s \in \mathcal{S}$.

Case 2: T' has two leaves, u and v. Then because (T', S') is reduced, Theorem 2.1 implies that there exist find single-vertex subtrees x and y in S' covering u and v, respectively. Note that x and y are disjoint.

Because $(T', \mathcal{S}')$ is $(2,3)$-agreeable, the set $\{x, y, s\}$ must have some pair with a nonempty intersection. It follows that that every $s \in \mathcal{S}' - \{x, y\}$ must contain at least one of u and v. Let

$$A' = \{s \in \mathcal{S}' : s \cap x \neq \emptyset\} \text{ and } B' = \{s \in \mathcal{S}' : s \cap y \neq \emptyset\}.$$

Then A' and B' are not necessarily disjoint, but $A' \cup B' = \mathcal{S}'$. Thus $|A'| + |B'| \geq |\mathcal{S}'|$, so without loss of generality $|A'|$ must be at least $\left\lceil \frac{|\mathcal{S}'|}{2} \right\rceil$. Because A' has a nonempty intersection in $\mathcal{S}'$, $\varphi^{-1}(A') = A$ has a nonempty intersection, so there is some vertex in at least $|A| = |A'|$ subtrees in $(T, \mathcal{S})$.

Thus, in either case, there is some vertex in at least half of the subtrees of $\mathcal{S}$. The tree society shown in Figure 3.1 shows that this minimum is attained. $\square$

When $n > 2$, we have the following much stronger result:

Theorem 3.2. *Let $n > 2$. In every $(n, n+1)$-agreeable tree society $(T, \mathcal{S})$, there is some vertex v in at least $|\mathcal{S}| - 1$ of the subtrees in $\mathcal{S}$.*

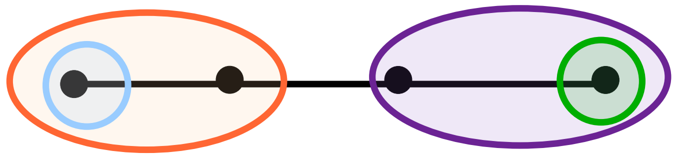

Figure 3.1: $(2, 3)$-agreeable tree society with minimal agreement number.

Proof. Let $(T', \mathcal{S}')$ be irreducible and equivalent to $(T, \mathcal{S})$ and let $\varphi : \mathcal{S} \to \mathcal{S}'$ be bijective and intersection-preserving. We know that T' has at most two leaves by Corollary 2.1.

Case 1: If T' has fewer than two leaves, it must be a single vertex, v. In this case, $\bigcap_{s' \in \mathcal{S}'} s' = \bigcap_{s \in \mathcal{S}} \varphi(s) = v$ and as φ is intersection-preserving we see that there is some vertex of T that is contained in every $s \in \mathcal{S}$.

Case 2: T' has two leaves, u and v. Then, because (T', S') is reduced, Theorem 2.1 implies that there exist single-vertex subtrees x and y in S' covering u and v respectively.

Suppose, for the sake of contradiction, that we could find $a, b \in \mathcal{S}' - \{x, y\}$ such that $u \in a, u \notin b$ and $v \in b, v \notin a$. Note that this construction ensures that x and b are disjoint and that y and a are disjoint. Let $R \subseteq \mathcal{S}'$ be any collection of $n+1$ subtrees such that $\{x, y, a, b\} \subseteq R$. Any n-element subset of R must contain all but one of the subtrees in R and thus must contain either the pair x and b or the pair y and a. In both cases, this n-element subset contains some disjoint pair of elements and thus must have an empty intersection. This violates the intersection property given in the hypothesis, so no such pair a and b can exist.

Let $R \subset \mathcal{S}' - \{x, y\}$ be an arbitrary collection of $n-1$ subtrees. Because $R \cup \{x, y\}$ must have an n-element subset with a nonempty intersection, R itself must have a nonempty intersection. Furthermore, either u or v must be in $\bigcup_{r \in R} r$. As this is true for any such R, we see that every element of $\mathcal{S}'$ must contain at least one of u and v.

We have shown that every subtree in S' must contain at least one of u and v and that there is no pair of subtrees $a, b \in S' - \{x, y\}$ such that a contains u but not v, and b contains v but not u. It follows that either every subtree in $\mathcal{S}' - \{x,y\}$ contains the leaf u or every subtree in $\mathcal{S}' - \{x,y\}$ contains the leaf v. Without loss of generality, assume that every subtree in $S' - \{x,y\}$ contains u. Then the only subtree in $\mathcal{S}'$ that does not contain u is y, that is, u is in the intersection of the subtrees in $S' - \{y\}$. Because φ is a bijective, intersection-preserving map, $S - \varphi^{-1}(y)$ must also have a nonempty intersection.

Thus there is some vertex of T in at least $|\mathcal{S}| - 1$ of the subtrees in $\mathcal{S}$.

□

3.3 $(2,m)$-Agreeable Tree Societies

In this section we prove that every $(2,m)$-agreeable tree society $(T,\mathcal{S})$ has some vertex contained in $\frac{1}{m-1}|\mathcal{S}|$ of the subtrees in $\mathcal{S}$ and that this lower bound is attainable. First, however, we examine the specific case where $m = 4$.

Theorem 3.3. *If $(T,\mathcal{S})$ is a $(2,4)$-agreeable tree society then there is some vertex of T contained in at least $\frac{1}{3}|\mathcal{S}|$ subtrees in $\mathcal{S}$.*

Proof. Let $(T,\mathcal{S})$ be a $(2,4)$-agreeable tree society and let $(T',\mathcal{S}')$ be the equivalent reduced society guaranteed by Theorem 2.3. Because it is reduced, we know by Lemma 2.1 that T' has at most $m-k+1 = 4-2+1 = 3$ leaves. We consider three cases:

Case 1: If T' has one leaf, then it is a single-vertex tree. Thus every subtree in $\mathcal{S}'$ must contain that single vertex and so in the original society $(T,\mathcal{S})$ there must have been some vertex contained in every subtree in $\mathcal{S}$.

Case 2: Suppose T' has two leaves, u and v. Define the collections

$$A = \{s \in \mathcal{S}'; u \in s\} \qquad B = \{s \in \mathcal{S}'; v \in s\}$$

and let $C = \mathcal{S}' - (A \cup B)$ be the complement of $A \cup B$ in S'. Note that this construction ensures that every subtree in A contains the vertex u, that every subtree in B contains the vertex v and that every subtree of $\mathcal{S}$ appears in at least one of A, B, and C. As $(T',\mathcal{S}')$ is a reduced society, we can find $a \in A$ and $b \in B$ such that a is a single-vertex subtree containing u and b is a single-vertex subtree containing v. If $c,d \in C$,then the fact that $(T',\mathcal{S}')$ is $(2,4)$-agreeable tells us that some pair of a, b, c, d must have a nonempty intersection. By construction, a and b do not have a common intersection and do not intersect with any subtree in C, so it must be the case that c and d have some vertex in common. Thus we see that the subtrees in C are pairwise intersecting. Therefore there is some vertex of T' contained in every subtree in C.

By construction, every subtree in $\mathcal{S}'$ is contained in at least one of A, B and C, so we know by the pigeonhole principle that one of the three contains $\frac{1}{3}|\mathcal{S}'|$ subtrees. We have shown above that these $\frac{1}{3}|\mathcal{S}'|$ subtrees must have some vertex in common. Thus when the reduced tree society has two leaves there is some vertex of the original tree contained in at least $\frac{1}{3}|\mathcal{S}|$ of the subtrees in $\mathcal{S}$.

Case 3: Suppose T' has three leaves, call them u, v, and w. Because $(T', \mathcal{S}')$ is reduced, we can find single-vertex subtrees $a, b, c \in \mathcal{S}'$ covering u, v, and w, respectively. Any subtree $s \in \mathcal{S}' - \{a, b, c\}$ must have a nonempty intersection with one of a, b, and c and thus must contain at least one of u, v, w. Because every subtree in $\mathcal{S}$ must contain at least one of u, v, or w, the pigeonhole principle tells us that one of them must be contained in at least $\frac{1}{3}|\mathcal{S}'|$ of the subtrees in $\mathcal{S}'$. Thus when the reduced tree has three leaves there must be some vertex in the original tree T contained in at least $\frac{1}{3}|\mathcal{S}|$ subtrees in $\mathcal{S}$.

In all three cases we see that there is some vertex of T contained in at least $\frac{1}{3}|\mathcal{S}|$ subtrees in $\mathcal{S}$, so this is true for all $(2, 4)$-agreeable tree societies. □

We note that in the two-leaf case the reduced tree society is actually linear, so we could have simply appealed to Theorem 1.2. The above argument is included to provide a more concrete example of the inductive process we use to prove the following theorem.

Theorem 3.4. *If $(T, \mathcal{S})$ is a $(2, m)$-agreeable tree society then there is some vertex of T contained in at least $\frac{1}{m-1}|\mathcal{S}|$ subtrees in $\mathcal{S}$. Furthermore, this bound is sharp.*

Proof. When $m = 2$, this is simply the Helly property for trees and we cover the $m = 3$ case in Theorem 3.1.

Assume for the sake of induction that for $m < n$, any $(2, m)$-agreeable tree society $(T, \mathcal{S})$ has some vertex contained in at least $\frac{1}{m-1}|\mathcal{S}|$ subtrees of $\mathcal{S}$.

Let $(T, \mathcal{S})$ be a $(2, n)$-agreeable tree society and let $(T', \mathcal{S}')$ be an equivalent $(2, n)$-agreeable irreducible society. Because T' is irreducible, Lemma 2.1

tells us that it has $l \leq n - k + 1 = n - 2 + 1 = n - 1$ leaves. For $i = 1, \ldots, l$, let these leaves be denoted v_i and define

$$A_i = \{s \in \mathcal{S}'; v_i \in s\} \qquad B = \mathcal{S}' - \bigcup_{i=1}^{l} A_i.$$

Because the society is reduced, we know that each A_i contains a single-vertex subtree a_i covering v_i. Let L be the set of these a_i. Note that $|L| = l$ and that the subtrees in L are all pairwise disjoint. Let $C \subseteq B$ be any collection of $n - l$ subtrees. By construction, C and L are disjoint collections of subtrees, so $|C \cup L| = (n - l) + l = n$.

$(T', \mathcal{S}')$ is $(2, n)$-agreeable, so there must be some pair of subtrees in $C \cup L$ with a nonempty intersection. Each $a_i \in L$ contains a single leaf of T' and the subtrees in C necessarily contain none of the leaves of T', so any two subtrees of $C \cup L$ with a nonempty intersection must both come from C. Thus we see that (T', B) is a $(2, n - l)$-agreeable tree society. By the inductive hypothesis we can find some vertex u contained in $\frac{1}{n-l-1}|B|$ of the subtrees in B.

The subtrees in each A_i have the vertex v_i in common, so if $|A_i| \geq \frac{1}{n-1}|\mathcal{S}|$ for any i we would be done. Suppose $|A_i| < \frac{1}{n-1}|\mathcal{S}|$ for all i. Now, $\mathcal{S} = \cup\{A_1, \ldots A_l, B\}$, so

$$|\mathcal{S}| \leq |B| + \sum_{i=1}^{l} |A_i| < |B| + \frac{l}{n-1}|S|,$$

which implies that

$$\frac{n-1-l}{n-1}|\mathcal{S}| < |B|.$$

Thus u is contained in at least

$$\frac{1}{n-l-1}|B| > \frac{1}{n-1}|\mathcal{S}|$$

subtrees of B.

Hence, there is some vertex of T' contained in at least $\frac{1}{n-1}|\mathcal{S}|$ of the subtrees in $\mathcal{S}'$ and the equivalent statement holds for the original unreduced tree society $(T, \mathcal{S})$.

By the principle of induction it follows that in every $(2, m)$-agreeable tree society $(T, \mathcal{S})$ there is some vertex of T contained in at least $\frac{1}{m-1}|\mathcal{S}|$ subtrees in $\mathcal{S}$.

In Figure 3.2, we provide an example of a $(2, 4)$-agreeable tree society in which every vertex is contained in at most $\frac{1}{3}$ of the subtrees. It is easy

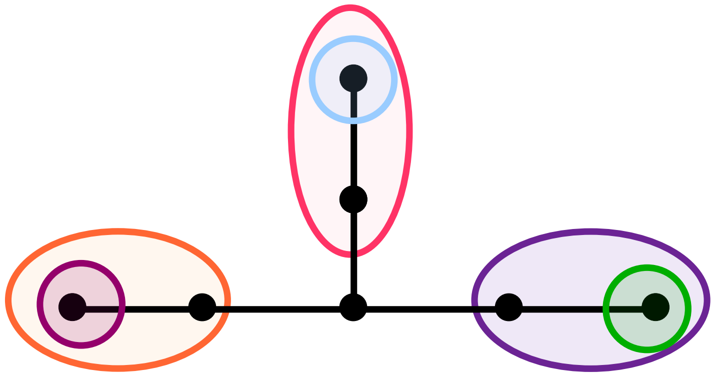

Figure 3.2: $(2,4)$-agreeable tree society with minimal agreement number.

to modify this example to obtain a $(2,m)$-agreeable tree society in which every vertex is contained in at least $\frac{1}{m-1}$ of the subtrees.

$\square$

3.4 More Small Cases

In this section I discuss a few small cases not covered by the preceding theorems. In particular, I will provide sharp bounds on $(3,5)$- and $(4,6)$-agreeable tree societies and give the best bounds I have found for $(3,6)$-agreeable tree societies.

Theorem 3.5. *If $(T,\mathcal{S})$ is a $(3,5)$-agreeable tree society, then its agreement number is at least $\frac{|\mathcal{S}|}{2}$. Furthermore, if an equivalent reduced tree society has three leaves, then $a(\mathcal{S}) \geq |\mathcal{S}| - 3$.*

Proof. Let $(T,\mathcal{S})$ be a finite $(3,5)$-agreeable tree society. By Theorem 2.3 we can find an equivalent reduced society $(T',\mathcal{S}')$, and, by Lemma 2.1, we know that T' has at most $5-3+1=3$ leaves.

Case 1: If T' has only a single leaf, then every subtree in $\mathcal{S}'$ contains that leaf and we have $a(n) = |\mathcal{S}'|$. If T' has two leaves, then $(T',\mathcal{S}')$ is linear, so we may apply Theorem 1.2 to obtain the sharp bound $a(\mathcal{S}') \geq \frac{|\mathcal{S}|}{2}$.

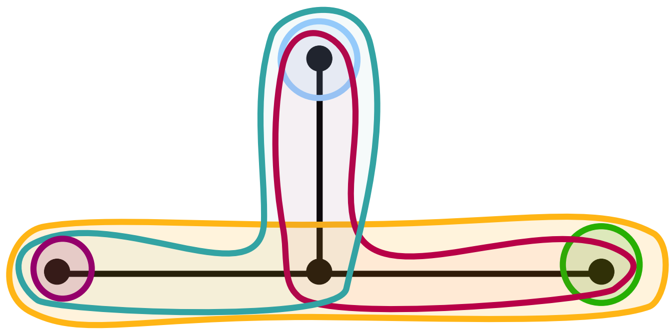

Figure 3.3: A $(3,5)$-agreeable tree society with $|\mathcal{S}| = 6$ and $a(\mathcal{S}) = |\mathcal{S}| - 3 = 3$.

Case 2: Suppose T' has three leaves, call them a, b, and c. Then, according to Theorem 2.1, we can find single-vertex subtrees $t_a, t_b, t_b \in \mathcal{S}'$ covering a, b, and c respectively. Let $R = \mathcal{S}' - \{t_a, t_b, t_c\}$ and let $u, v \in R$. Because $(T', \mathcal{S}')$ is $(3,5)$-agreeable and $\{t_a, t_b, t_c, u, v\}$ is a five-element subset of $\mathcal{S}'$, some three elements of this set must have a nonempty intersection. As t_a, t_b, and t_c are single-vertex subtrees covering different vertices, they are pairwise disjoint, so the three-element subset can contain only one of them and thus must contain both u and v. In particular, u and v have a nonempty intersection. Hence we see that any two arbitrary elements of R have a nonempty intersection and that (T', R) must be $(2,2)$-agreeable. By Theorem 1.4, there must be some vertex of T' contained in every subtree in R. This gives us the bound $a(\mathcal{S}') \geq |R| = |\mathcal{S}'| - 3$. Figure 3.3 shows a society in which this bound is attained. Furthermore, $|\mathcal{S}'| > 6$ implies that $|S'|/2 > 3$ and hence $|S| - 3 > |S|/2$, and $|\mathcal{S}| = 5$ necessarily gives $a(\mathcal{S}) \geq 3$, so we see that $a(\mathcal{S}')$ satisfies the weaker bound $a(\mathcal{S}') > |S|/2$ as well.

Now we note that by the definition of equivalent tree societies, $|\mathcal{S}| = |\mathcal{S}'|$ and any bound on $a(\mathcal{S}')$ is also a bound on $a(\mathcal{S}')$. Thus the above arguments show that $a(\mathcal{S}) \geq |\mathcal{S}|/2$ in all cases and that when T' has three leaves we have the tighter bound $a(\mathcal{S}) \geq |\mathcal{S}| - 3$.

□

Theorem 3.6. *If $(T, \mathcal{S})$ is a $(4,6)$-agreeable tree society, then its agreement number is at least $|\mathcal{S}| - 2$.*

Proof. Let $(T, \mathcal{S})$ be a finite $(4,6)$-agreeable tree society. By Theorem 2.3 we can find an equivalent reduced society $(T', \mathcal{S}')$ and, by Lemma 2.1, we know that T' has at most $6 - 4 + 1 = 3$ leaves.

Case 1: If T' has only a single leaf, then every subtree in $\mathcal{S}'$ contains that leaf and we have $a(n) = |\mathcal{S}'|$. If T' has two leaves, then $(T', \mathcal{S}')$ is linear so we may apply Theorem 1.2 to obtain the sharp bound $a(\mathcal{S}') \geq |\mathcal{S}| - 2$.

Case 2: Suppose T' has three leaves, call them a, b, and c. Then according to Theorem 2.1 we can find single-vertex subtrees $t_a, t_b, t_b \in \mathcal{S}'$ covering a, b and c respectively. Let $R = \mathcal{S}' - \{t_a, t_b, t_c\}$ and let $u, v, w \in R$ be distinct. (T', S') is $(4,6)$-agreeable, so there must be some four element subset of $\{t_a, t_b, t_c, u, v, w\}$ with a nonempty intersection. We know that the t_i are disjoint, so this four-element set must contain $\{u, v, w\}$. Thus we see that every three-element subset of R has a nonempty intersection and it follows that there is some vertex contained in every subtree in R.

Now, let $V = \cap\{t \in R\}$, that is, the set of vertices in every subtree in R. If either a or b is in this set then adding the appropriate single-vertex subtree back into R gives us a collection of $|R| + 1 = (|\mathcal{S}'| - 3) + 1 = |\mathcal{S}'| - 2$ subtrees with a nonempty intersection and we are done. Otherwise, we can find some $t \in R$ such that t does not cover a. If t covers b, then we can also find a $u \in R$ that does not cover b, otherwise, we let u be any element of $R - \{t\}$. Let $v \in R - \{t, u\}$ be arbitrary. Now consider $\{t_a, t_b, t_c, t, u, v\}$. Because it is a six-element subset of $\mathcal{S}'$, it must have a four-element subset with a nonempty intersection. By construction, t_a and t_b are both disjoint from at least three of the other subsets, so they cannot be part of the four-element subset. Hence it must be the case that $\cap\{t_c, t, u, v\}$ is nonempty. This is true for every $v \in \mathbb{R}$, so we see that there is some vertex in $\cap(\{t \in R\} \cup \{t_c\})$. Thus we have shown that there is a vertex contained in at least $|R| + 1 = (|\mathcal{S}'| - 3) + 1 = |\mathcal{S}'| - 2$ subtrees of $\mathcal{S}'$.

Now, we note that by the definition of equivalent tree societies, $|\mathcal{S}| = |\mathcal{S}'|$ and that any bound on $a(\mathcal{S}')$ is also a bound on $a(\mathcal{S}')$. Thus the above arguments show that $a(\mathcal{S}) \geq |\mathcal{S}| - 2$ regardless of the number of leaves in the reduced society.

□

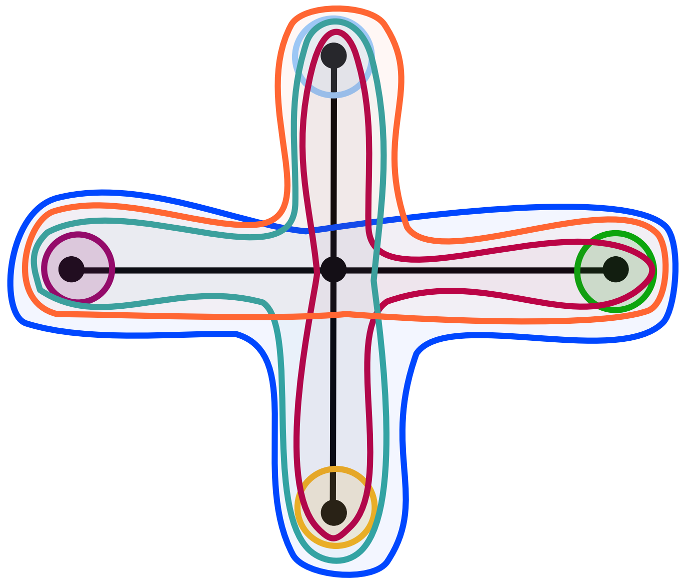

Figure 3.4: A $(3,6)$-agreeable reduced tree society with four leaves and agreement number $|\mathcal{S}| - 4$.

The case where $(T, \mathcal{S})$ is a $(3,6)$-agreeable society has proven to be much more difficult to understand, particularly when an equivalent reduced society has three leaves.

Example 3.1. *An exploration of* $(3,6)$*-agreeable tree societies*

Lemma 2.1 tells us that in an equivalent reduced society $(T', \mathcal{S}')$ there can be at most $6 - 3 + 1 = 4$ leaves. As usual, if the reduced society has a single leaf, then we know $a(\mathcal{S}) = |\mathcal{S}|$. When T' has two leaves, the reduced society is linear and Theorem 1.2 tells us that

$$a(\mathcal{S}) \geq \left\lceil \frac{|\mathcal{S}| - 1}{2} \right\rceil$$

and that this bound is sharp.

If T' has four leaves, let $t_a, t_b, t_c, t_d \in \mathcal{S}'$ be single-vertex subtrees covering the leaves. Let u, v be any two other distinct elements of $\mathcal{S}'$ and consider $\{t_a, t_b, t_c, t_d, u, v\}$. We know that some three-element subset of this six-element set must have a nonempty intersection, but the t_i are, by construction, all disjoint. Hence we see that the three-element subset must contain both u and v. As this is true for any choice of u and v, the elements of $\mathcal{S}' - \{t_a, t_b, t_c, t_d, u, v\}$ are pairwise intersecting and thus the whole collection has a nonempty intersection. This gives us the bound $a(\mathcal{S}) = a(\mathcal{S}') \geq |\mathcal{S}'| - 4$. Figure 3.4 gives an example of a society which attains this bound. I suspect that the bound may improve to $a(\mathcal{S}) \geq |\mathcal{S}'| - 3$ when $\mathcal{S}$ is sufficiently large, but have not managed to prove it. Regardless of whether this potential stronger bound holds, the bound we do have is stronger than the bound on linear societies.

Suppose T' has three leaves a, b, c and let t_a, t_b, t_c be the single-vertex covers guaranteed by Theorem 2.1.

Case 1: Suppose we can find some $t \in \mathcal{S}'$ that does not cover any of a, b, c. Then choosing any two other distinct subtrees u, v in $\mathcal{S}'$, we obtain a six-element set $\{t_a, t_b, t_c, t, u, v\}$ where t_a, t_b, t_c and t are pairwise disjoint. Then we can use t in the same way we used t_d in the four-leaf case to obtain the bound $a(\mathcal{S}) = a(\mathcal{S}') \geq |\mathcal{S}'| - 4$.

Case 2: If there is no t satisfying the condition in Case 1, we know that every subtree in $\mathcal{S}'$ covers at least one of the leaves. Now, we note that every three-leaf tree has exactly one vertex of degree 3. Call this vertex d and define $\mathcal{A} = \{t \in \mathcal{S}' | a \in t, d \notin t\}$ and $\mathcal{B}$ and $\mathcal{C}$ analogously. Note that these sets are nonempty as each contains the relevant t_i. These sets are disjoint, so if we could find two elements in each set that would give us a six-element subset of $\mathcal{S}'$ with no three-element subset with a nonempty intersection. Thus at least one of them must contain only one element. Without loss of generality, we may say this is $\mathcal{C}$, that is, we have t_c covering c, and every other subtree in $\mathcal{S}'$ that covers c must also cover d. Now, removing t_c from $\mathcal{S}'$ does not change the intersection properties of the remaining subtrees, so $(T', S' - \{t_c\})$ is also a $(3, 6)$-agreeable society. Furthermore, we know that every subtree that covers c also covers d, so this society reduces to a linear society and we can apply Theorem 1.2 to obtain the bound

$$a(\mathcal{S}' - \{t_c\}) \geq \left\lceil \frac{|S' - \{t_c\}| - 1}{2} \right\rceil = \left\lceil \frac{|S'| - 2}{2} \right\rceil .$$

As a subset of $\mathcal{S}'$ satisfies this bound, $\mathcal{S}'$ itself must as well and we have

$$a(\mathcal{S}') \geq \left\lceil \frac{|S'| - 2}{2} \right\rceil .$$

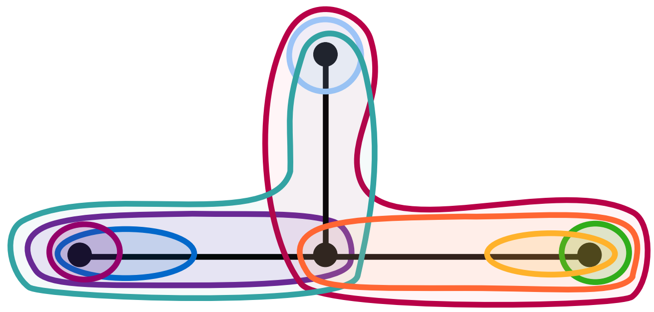

Figure 3.5: A $(3,6)$-agreeable reduced tree society with three leaves and $a(\mathcal{S}) = \left\lceil \frac{|\mathcal{S}|-1}{2} \right\rceil$.

> This gets us very close to the linear bound, and when $|\mathcal{S}'|$ is odd it even agrees with the linear bound, but is not quite there. The fact that $S' - \{t_c\}$ still satisfies $(3,6)$-agreeability when t_c is added back in seems like it ought to put a stronger bound on the agreement number of $\mathcal{S}'$ than $\mathcal{S}' - \{t_c\}$, but I have been unable to prove this. However, Figure 3.5 gives an example of a three-leaf $(3,6)$-agreeable reduced tree society in which the lower bound on linear societies is obtained. Thus we at least know that we will not be able to find a stronger bound than the one we have for the linear case.

The following table summarizes our results for (k, m)-agreeable tree societies $(T, \mathcal{S})$ where $k \leq m \leq 6$ and l is the number of leaves in an equivalent reduced society. When $l = 1$ the agreement number is clearly $|\mathcal{S}|$ regardless of k and m, so I have left that case out of the table.

k	m	l	Bound on $a(\mathcal{S})$	Reference
2	3	2	$\frac{\lvert\mathcal{S}\rvert}{2}$	Theorem 3.2
2	4	$2, 3$	$\frac{\lvert\mathcal{S}\rvert}{3}$	Theorem 3.4
3	4	2	$\lvert\mathcal{S}\rvert - 1$	Theorem 3.2
3	5	2	$\frac{\lvert\mathcal{S}\rvert}{2}$	Theorem 3.5
3	5	3	$\lvert\mathcal{S}\rvert - 3$	Theorem 3.5
4	5	2	$\lvert\mathcal{S}\rvert - 1$	Theorem 3.2
2	5	$2, 3, 4$	$\frac{\lvert\mathcal{S}\rvert}{4}$	Theorem 3.4
2	6	$2, 3, 4, 5$	$\frac{\lvert\mathcal{S}\rvert}{5}$	Theorem 3.4
5	6	2	$\lvert\mathcal{S}\rvert - 1$	Theorem 3.2
4	6	2	$\lvert\mathcal{S}\rvert - 2$	Theorem 3.6

Table 3.1: Summary of Small Cases.

Chapter 4

Future Work and Discussion

4.1 Comparing Linear Societies and Tree Societies

In Theorem 3.2 and Theorem 3.3, we provide lower bounds for the agreement numbers of $(n, n+1)$-agreeable tree societies and $(2, m)$-agreeable tree societies, respectively. Both of these results agree with the results for $(n, n+1)$-agreeable linear societies and $(2, m)$-agreeable linear societies given by Berg et al. (Forthcoming). In the $(n, n+1)$ case this is expected, as we can use Corollary 2.1 to reduce any $(n, n+1)$-agreeable tree society to a society with at most two leaves – a linear society. Though it uses the fact that the society is essentially linear, our proof is of a very different nature than the proof presented by Berg et al. (Forthcoming). For the $(2, m)$ case this correspondence is somewhat more surprising, as our current reduction process does not suggest that all $(2, m)$-agreeable tree societies can be reduced to linear societies.

For both $(3, 5)$- and $(4, 6)$-agreeable tree societies, we have shown that having more leaves can only give us stronger bounds. In the case of $(3, 6)$-agreeable societies, we have shown that the bounds on the agreement number in a reduced society with four leaves are at least as strong as those on a linear society. For the three-leaf case, we have been unable to prove that there cannot exist a society that does not satisfy the bounds on the linear case, but I have also been unable to find a counter example. The most immediate goal is thus to better understand this case.

The statement of Theorem 1.2, that in a (k, m)-agreeable linear society $(X, \mathcal{S})$, $m - 1 = (k-1)q + \rho$, $\rho \le k - 2$ implies that $a(\mathcal{S}) \ge \lceil (|\mathcal{S}| - \rho) / q \rceil$, looks like the sort of statement that ought to be able to be shown via a pigeonhole argument. It seems plausible that such an argument could then be extended from linear societies to tree societies.

4.2 Other Possible Directions

Before discovering that these questions were not yet understood on trees, I spent some time playing with similar questions on the cube graph. It would be interesting to explore the analogous questions on the hypercubes or possibly other classes of graphs more fully. This brings with it the added challenge of determining what subgraphs are appropriate to consider, as it is less clear how we should define a convex subset of an arbitrary graph.

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