Humanistic Mathematics Network Journal

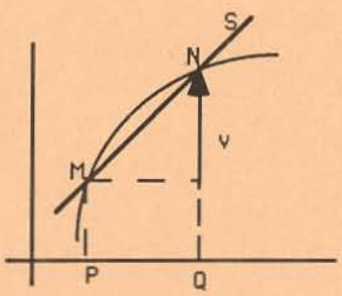
Issue 9 Article 13

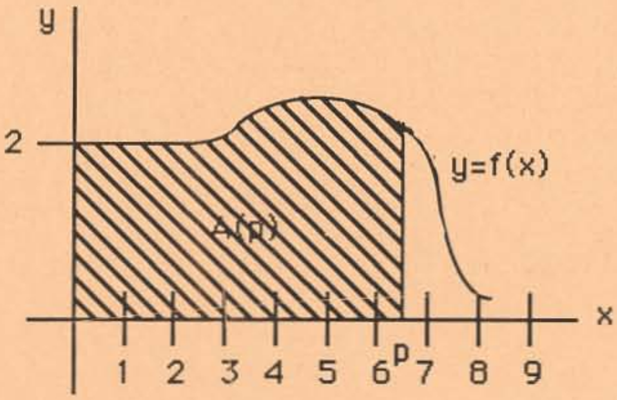
2-1-1994

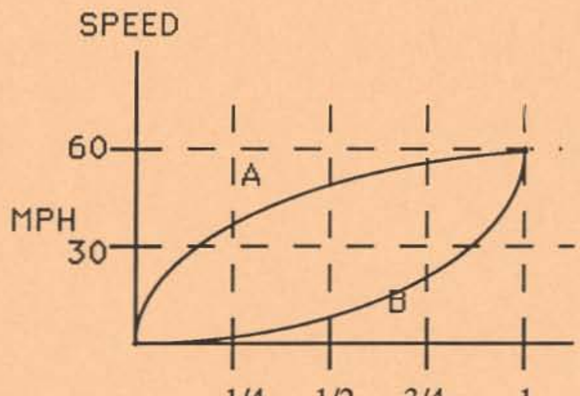
Complete Issue 9, 1994

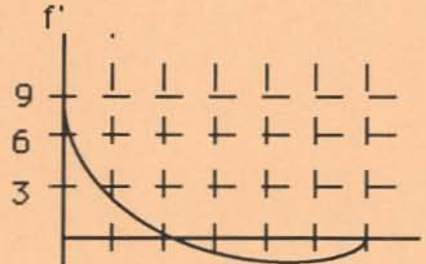
ISSN 1065-8297

Humanistic Mathematics Network
Journal #9
February 1994









INVITATION TO AUTHORS

Essays, book reviews, syllabi and letters are welcome. Two copies, double spaced, should be sent to Alvin White, HUM. MATH. NET., Harvey Mudd College, Claremont, CA 91711. If possible, avoid footnotes and put references and bibliography at the end using a consistent style. If you use a word processor please send a diskette in addition to the typed paper. *The Journal* is assembled using Microsoft Word 4.0 and PageMaker 4.0 on a Macintosh. It is possible, however, to convert from other word processing systems. Clean typed copy can be scanned (but not dot matrix). Your essay should have a title, you name and address, and a brief summary. Your telephone number (not for publication) would be helpful. Essays and communications may be transmitted by electronic mail to the editor at AWHITE@SIF.CLAREMONT.EDU.
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NOTE TO LIBRARIANS
The Humanistic Math Network Journal #8 and #9, ISSN# 1065-8297 are the successors to the Humanistic Math Network Newsletter, ISSN# 1047-627X.

COVER

These diagrams were used in G.S. Monk's article, "Students' Understanding of Functions in Calculus Courses," which appears on pages 21-27. His study was based on four multipart problems for beginning calculus students. Each problem had one or two questions that tested for a Pointwise understanding of a function and one or two questions that tested for an Across-Time understanding of the same function.

Supported by a Grant from the EXXON EDUCATION FOUNDATION

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Letter from the Editor

The humanistic mathematics movement which began as the personal vision of a few, has now become a major part of mathematical culture. What was viewed with skepticism is now accepted an expected.

Our journal attracts new readers and new authors. The thoughts and experiences that are shared advance the health and strength of mathematics and the mathematical community. The mix of themes in each issue enriches us all.

Stephanie Singer's lively report of a course for a general audience is an example of what can happen when math students are in a student-friendly environment. Professor Singer describes the excellent atmosphere "which I long to recreate in my more traditional classes." I think that she speaks for many of us.

E.G. Bernard also discusses language and writing mathematics. "All young children, except the most severely handicapped, possess a language,... (and) possess the ability to learn a system of symbolic knowledge."

Frederick Reiner considers the many resemblances between mathematics and the arts. "if we claim aesthetic values are inherent to our doing mathematics, can we specify what those values are or the role they play in our practice?" "Like the philosopher of art, we are not so interested in developing the concepts of beauty as we are in applying these to a particular type of activity."

M.K. Siu and C.I. Fung of Hong Kong write about the loss of self-esteem among math majors. "Students lack a global view of the subject and rarely appreciate or enjoy it from a cultural aspect, and too little emphasis is put on the subjects' own worth."

Y. Zheng of Nanjing University relates the philosophy of math, math education, and the philosophy of math education. Along the way, he remarks that the "new math" paid little attention to the "actual cognitive processes, of how humans think about mathematics."

The paper by G.S. Monk is reprinted from issue #2, 1988 when the newsletter was mailed to fewer than a hundred readers. Issue #9 is being sent to over a thousand, distributed over every continent.

The Exxon Education Foundation has supported the movement and journal since 1986, for which we are all grateful. Their support has also made possible the timely publication of the Math Association of America Notes Volume 32, Essays in Humanistic Mathematics. Many readers have contributed $25 or more, which have helped stretch the budget and to buy computer software and hardware for the desk top publishing. The voluntary, tax deductible contributions may be sent to Harvey Mudd College, c/o Alvin White, Humanistic Mathematics Network. The College sends each donor an acknowledgement and thanks. I want to add my special thanks to all.

Issue #10 should be mailed in May or June.

Alvin White

Letter to the Editor

The Einstein Effect. Many authors have commented on the annoying fact that most people believe that you have to be gifted, if not a genius, to succeed at mathematics. Consequently many people brag about being weak at mathematics, while very few admit that they cannot read or write and they certainly aren't proud of this fact. I wonder if this problem can be placed at the door of Albert Einstein. All my life the idea that you have to be a genius to understand mathematics and physics seems to have been tied to the fact that only geniuses could understand relativity and other work of Einstein. So my question is this: Did this popular view that only the gifted few can do mathematics precede Einstein? In other words, did the annoying phenomenon described above occur back in the year 1900? Or were "readin', writin' and 'rithmetic" equal in those days? I would be interested to know.

Ken Ross
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The Language of Mathematics: A Quantitative Course for a General Audience

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The purpose of this article is to convince anyone with the inclination to design a mathematics course for a general audience that it's worth the effort and to provide some concrete examples of successful activities and assignments.

The course I designed and taught this year, "The Language of Mathematics," was the most enjoyable class I have taught to date. The atmosphere was relaxed. The students plunged whole-heartedly into the in-class problem-solving in groups of three or four. They were forthcoming in the class discussions, willing to share emotional responses as well as insights from their own experiences and other studies. I attribute the excellent atmosphere (which I long to recreate in my more traditional classes) to the daily small group activities, the absence of a specified body of material to be mastered, and the fact that students were not graded individually on their problem-solving abilities.

The several goals of the course were: 1) to give students the opportunity to do some mathematics in a pressure-free, fun environment; 2) to teach students to be agressively critical of quantitative facts presented to them (e.g., by the media); and 3) to help students recognize and become conversant in the mathematical mode of language use. This third goal merits some explanation. There is a particular mode of language use essential to the discussion of mathematics (as well as other endeavors, such as the sciences and the law.) The primary feature is lack of ambiguity. While this mode of language use is as natural as breathing to mathematicians, it is extremely foreign to many people. We love theorems for their infallibility, but many other people believe that to every rule there is an exception. We feel free to use any word we like for any concept, as long as we define the word clearly, but most people learn most words from context and from experience. No wonder, then, that mathematics is viewed as a foreign language by many students—not only is the vocabulary unfamiliar, but even the process by which one learns the vocabulary is different! This problem has been well documented by Rin, who studied linguistic difficulties experienced by linear algebra students. The third goal of the course is to help students recognize that this other mode of using language exists and is useful.

The class met for one and a half hours twice each week. There were twenty students from Haverford College, almost evenly split among freshmen, sophomores, juniors and seniors. Fifteen were women. No student was a physical science or math major, and many described themselves as "not good at math." They were graded not on individual problem-solving ability

There is a particular mode of language use essential to the discussion of mathematics. The primary feature is lack of ambiguity. While this mode of language use is as natural as breathing to mathematicians, it is extremely foreign to many people.

but on their class participation and their written work, described in more detail below. The drawbacks of this grading system (e.g., that students could get away with failing to understand some concepts) were well worth the benefits for these students,as these unsolicited comments attest: "After so many bad math/science experiences, this was the first class of that sort which I enjoyed;" "Thank you so much for wiping away a stereotype and an insecurity and a paranoia concerning math—one that's been around for me since the very beginning when I

was told that I 'didn't think mathematically;'" and "It was great to look at math at a different angle and not to have all the pressure that I usually associate with math classes."

The first assignment of the semester (suggested by Victor Donnay of Bryn Mawr College) was to write about a good math memory—a memory of noticing a pattern. I described mowing a rectangular lawn and noticing the rectangle of uncut grass in the center switch from skinny to squarish and back again. Students will have no trouble finding delightful memories if you give them encouragement and an example.

To accustom students to doing mathematics in a relaxed environment, each class began with a group exercise, either a word problem, a definition exercise (e.g., find a verifiable definition for "vertical") or an estimation problem. This practice had the added advantage of focusing the students' attention on the class—waking and warming them up. (A tip: when supervising several groups, always have something up your sleeve for the groups who finish early.) Some problems came from John Harte's excellent book, *Consider a Spherical Cow: A Course in Environmental Problem Solving*. One could easily build a course around this book alone, which attacks serious problems with algebra, estimation and a sense of humor. Working through just the first fifteen pages provided much food for thought and discussion: orders of magnitude, using physical units, exponentials and logarithms. Another rich source of problems was the Shell Centre for Mathematical Education (University of Nottingham, NG7 2RD, England). Their examination modules for secondary schools have

Most important of all is to take problems from your own local environment. Ask students to estimate, e.g., how much money is deposited in the nearest soda machine each semester.

(among other things) thoughtful and well-prepared worksheets for group activities. I used the module called "The Language of Functions and Graphs." Most important of all is to take problems from your own local environment. Ask students to estimate, e.g., how much money is deposited in the nearest soda machine each semester. Have them make up their own estimation problems. You can also throw in some of the classic old puzzles about coins, flies, water jugs and people with twin children.

There were two assignments designed specifically to encourage aggressive criticism of quantitative facts presented in the media. The first concerned sneaky ads: *Find two advertisements in the press, on the radio or on television that couch not-so-impressive quantitative statements in a misleadingly impressive way. Turn in a copy of each advertisement (if you choose to do a radio or TV ad you must turn in a tape or an exact transcript) with a brief explanation of (1) the false conclusion the advertisers hope you will draw and (2) the true meaning of the statement.*

Students found ads ranging from the heinous (boldface "0% interest" with fine print detailing 21% interest to be refunded 1 year later in the form of store credit) to the silly ("100 free wigs to 100 women who send for our free catalog"). The second similarly solicited critiques of quantitative statements and graphs in newspaper or magazine articles.

The language issues entered the class explicitly in two ways. First of all, we spent considerable time on the concept I have come to call "verifiable definition." A verifiable definition of a word is a description of the meaning of the word giving a **category** of objects to which the word may apply and an unambiguous **criterion** for deciding which members of the category may be truthfully described by the word. For example: a person having two X chromosomes is called *female*. Another example: a *function* is a collection of pairs of numbers with the following property: if (a,b) and (a,c) are both in the collection, then b = c. The categories in these examples are "people" and "collections of pairs of numbers," respectively. This concept is a variation on Arons' "operational definition," described in *A Guide to Introductory Physics Teaching*, with more emphasis on lack of ambiguity than on the nature of the criterion. Students had plenty of practice finding verifiable definitions (e.g., for "area", "speed" and "to count"). We discussed the concept explicitly in class and finally students wrote essays on what verifiable definitions are,

when they are useful and when they are inappropriate. Second of all, we focused on mathematical texts as texts, i.e., as words written down by one human being in an attempt to communicate. Over a period of weeks, students read and compared four very different treatments of the topic of functions. Any sufficiently varied treatments would work; I used an old-fashioned book (Griffin), a product of recent reform (Hughes-Hallett et al.), a common cookbook approach (Shenk) and a text stressing rigor (Spivak). The first assignment of a series was to react to the texts: *First Reaction: skim all four texts. What do you notice? What do you like? Dislike? How do they make you feel? How does the writing differ from other kinds of writing? Which has the best pictures? The best typeface? Criticize! Annotate! Compare and contrast! Submit a reaction in the medium of your choice (e.g., words, music, drawing). Be creative!*

This first assignment compelled examination of the texts without requiring comprehension. The second assignment, designed to teach students a method for approaching a difficult text, was assigned repeatedly, with reinforcement from classroom work and discussions on functions:
Detailed Reaction: Choose one of the four texts. For each of the first four new ideas you read, either (1) give an explanation in your own words and an illustrative example or (2) describe what it is you don't understand.

The final assignment allowed students to synthesize and me to evaluate what they had learned: *Compare and Contrast: read the definition of "function" in each of the four texts. How do the definitions differ? Are all of the texts describing the same idea? Discuss the relative merits of each definition.* Judging from student feedback, a final, unifying class discussion of this series of assignments would have been helpful.

Other major, graded written work included an essay on proofs, truth and validity, a proof that the square root of 3 is irrational and an explanation of where an analogous attempt to prove the square root of 4 would break down (after discussion of a proof that the square root of 2 is irrational) and a few write-ups of problems. There were numerous minor, ungraded writing assignments, mostly reactions to various readings and write-ups of problems discussed in class.

In lieu of an exam, students wrote a final 3-5 page paper. There were three choices: to investigate an endeavor that, like mathematics, has developed a language of its own, to relate ideas learned in the class to topics of their own choosing, or to write up several mathematical proofs (e.g., from exercises in Herstein and Kaplansky). Students wrote on a variety of interesting subjects. One paper discussed the stylized language of African talking drums. Another proposed an experiment to determine if students of science and students of the humanities perform linguistic tasks (such as giving directions or describing objects) in measurably different ways.

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One student chose to write about his personal experience in the class; anyone doubting the worth of such a class should consider this excerpt:

> Not only have I been given a chance to understand math, but I was encouraged to question and explore it. Aspects of math which had before seemed inconceivable or illogical began to make sense. Numbers which I had always assumed were infallible were now open to question and approachable. Imaginary numbers and non-Euclidean geometries, I found, were not esoteric inventions of distant brilliant mathematicians, but actually logical systems which could be explained concretely with a pencil, ruler, and a piece of paper, or maybe with an orange. For me, any number with over three digits was large and inconceivable, but through practical estimations and critical reading of quantitative information I have learned the discrepancies beween large numbers. I have also become aware of how easily they can be manipulated to fool people like me who when they see more than

three digits do not even take the time to read the number. The realization that numbers can be used deceptively and that they can reveal essential information has been very beneficial to me. I am more comfortable, inquisitive, and alert when handling quantitative information for my other classes, not to mention in my daily encounters with numbers. I can even feel comfortable reading my bank statements.

Anyone wanting further information should not hesitate to contact me.

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Writing Mathematics

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All young children, except the most severely handicapped, possess a language, and can repeat stories sequentially and in detail after hearing them a surprisingly few times. People retain these stories for their whole lives. Virtually all young children, then, possess the ability to learn a system of symbolic knowledge. Can techniques from language experiences assist in the learning and assessment of mathematics?

In order to attempt an answer to this question, I visited several mathematics' classes and told the students that I was going to give them an unusual mathematics assignment today. First, I'll give you a non-mathematical example of the assignment and

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let you try it. We'll discuss and take up aspects of this assignment. Second, I'll give you the mathematical assignment. Third, I'll return in a few days, take up the assignment, explain to you what I'm trying to accomplish, and solicit your views.

I'm going to put two non-mathematical expressions on the board. When you see two expressions, I want you to tell me the content (the story) that you associate with these two expressions.

Here are the two expressions: glass slipper and pumpkin.

Now, tell me the content (the story) that you associate with these two expressions. (Teachers may have to use other examples depending upon the cultural backgrounds of their students.)

Most students said these expressions refer to the story of Cinderella. When asked to commence telling the story, most students began "Once upon a time", and were able to tell the story. I pointed out, that although they had likely not heard this story for a long time, they had an amazing amount of correct detail in their stories and the content of their stories was well sequenced.

Now for the mathematics assignment. This time I will put a title on the board, and you write the story. (The classes I visited had recently completed their study of fractions, and I had the students' names and scores on their fraction tests.)

<u>The differences between adding and multiplying fractions.</u>

Remember the Writing Process in your English classes. List the appropriate content, that is, what you know about the title, sequence the content, and write out a <u>full</u> account of the story. Give a lot of detail. Use only words.

When I read the stories, many students employed introductory and concluding sentences. For instance, "There are a few differences between the addition and multiplication of fractions. ... These are the differences between the addition and multiplication of fractions."

In a few cases, students incorporated their content into a traditional type of story. For instance, "Once upon a time there was a boy named Bob who didn't know how to do fractions. His friend helped him after school one day. ... Bob finally understood how to do fractions, and passed his fraction test." In another version, the conclusion stated, "and the boy not only passed his next

fraction test, years later he became a Math teacher." And in another case, "Adding and Multiplying were talking to each other one day. Adding asked Multiplying to tell how he is different from Adding regarding fractions. ... Well, although we both deal with numbers, we are different in many ways."

I had prepared a marking scheme prior to administering these assignments. I awarded marks for the title being present, for an introductory sentence with a bonus for a creative introduction, for a conclusion, and for spelling, grammar, and writing style. In marking the content of the addition of fractions, I was looking for some discussion of finding a common denominator, of adding the numerators, and placing the sum of the numerators over the common denominator. In marking the multiplication of fractions, I was looking for a description of multiplying the numerators together, and placing this product over the product of the denominators. I also awarded marks for a reference to reducing answers to their lowest terms, of dividing in the multiplication of fractions to reduce the size of the numbers, a reference to the order of operations, and a discussion of dealing with mixed numbers.

As many as forty percent of the students' marks increased on this assignment over their marks on traditional fraction tests, and many substantially, for instance: 37 to 65, 23 to 45, 63 to 80, and 72 to 100. Some of the students received marks similar to their marks on the traditional fraction tests. As many as forty percent of the students' marks decreased, for instance: 71 to 60, 60 to 35, 53 to 10, and 63 to 0. Most of the students whose marks decreased were New Canadians; they were unable to describe or write about mathematical operations albeit some of them scored higher on the traditional types of tests.

I believe that exercises of this type offer opportunities for many students to exhibit knowledge not requested in traditional mathematics tests. It is important in all subject areas that we employ a variety of types of questions, and a variety of types of evaluation instruments in order to meet the various talents of each of our students, and to allow students to be successful in some area of each course. For instance, in visual arts courses students are not evaluated in only one medium. Various media are explored, for instance, sketching, watercolours, oil, charcoal, clay, etc. as well as art history, and the greater the variety, the more likely we will match an aspect of the course with a talent of a student. Consequently, in mathematics, we should employ the traditional types of testing as well as assignments of the type I am suggesting in this paper in order to meet the variety of talents that students possess.

The business-world and post-secondary schools want graduates who can communicate clearly and creatively. They appear to want young people who can describe as well as apply what they know. If this is true, then a student who can apply theory and articulate it clearly should score higher than someone who can only perform the calculations properly. If writing mathematics is an important aspect in understanding and reporting mathematical concepts, then I hope this paper may point to a way of commencing this activity in our classrooms.

When I returned to the classes in order to take up the assignments, to explain what I was attempting to do, and to solicit the students' views, I discussed how very young children, except the most severely handicapped, have learned a language and stories. In fact, it is amazing how a young child can repeat a story, such as, Cinderella, after hearing it a surprisingly few times. You have

It is important in all subject areas that we employ a variety of types of questions, and a variety of types of evaluation instruments in order to meet the various talents of each of our students, and to allow students to be successful in some area of each course.

heard the story of fractions many more times than you heard the story of Cinderella, and you have heard about fractions more recently than Cinderella. You should be able to describe operations involving fractions. Some students said, but there was no pressure or expectation to learn Cinderella at our mothers' knees, and at the time, Cinderella was interesting to us. Fractions (or mathematics in general) is not interesting to many of us. These students have pointed out the very basis for commencing this type of exercise (writing mathematics) at a much earlier stage in the

educational process. I explained to them that I was seeking additional methods for assessing students in mathematics. Some students thought that seeking additional assessment methods in mathematics was a good idea. Others, of course, did not like this idea. Some students recalled that they had been asked in the past to write about how

Topics in Mathematics need to be presented in an overview rather than in isolated pieces. By learning (or seeing) the whole story, association and sequence are present.

they felt after writing a math test, and some said that they had been asked to write poetry concerning their experiences in mathematics.

In order to implement "Writing Mathematics" I believe we need some applications in learning and assessing mathematics which arise from learning language and stories.

- In learning languages and stories, we give countless examples of how words are used. In Mathematics, we need to give countless examples of how numbers are used. Further, we need to give examples of applications, for instance, from Science, Technology, Computer Studies, Sports, Geography, Business, and so on.

- I suggest that topics in Mathematics need to be presented in an overview rather than in isolated pieces. Language and stories are learned holistically. We did not learn Cinderella by hearing bits of the story now and then, and out of sequence. By learning (or seeing) the whole story, association and sequence are present. Association and sequence are two aspects of what is meant by the material having "meaning", of not being like nonsense syllables. There is a lot of research on the amount of time and effort required to learn nonsense syllables as opposed to learning meaningful material. Nonsense syllables are learned more slowly (research suggests that four times the effort is required), and are quickly forgotten. Does Mathematics come across as nonsense syllables to some children?

  In mathematics, we can go back later and deal with difficult points in isolation, rather than teaching the whole topic in a series of isolated points. I suggest that we need to present the "whole fraction story": what are fractions, examples of what fractions look like, how fractions are used, pictures of pieces of pie, the parts of a fraction named and labelled, the four operations as they apply to fractions noting the similarities and differences among the operations, examples of the four operations, and so on. As well as association and sequence being present in this methodology, students would have the whole story of fractions in their notes in one place for study purposes.

- Part of the testing in Mathematics, then, should include the natural questioning which is used in evaluating language and stories as well as the traditional means of testing.

- Assignments and assessment in Mathematics need to involve great variety,and not just textbook-like questions. For instance, in introductory Geometry, assignments and assessment need to include problem-solving, research, writing, conceptual knowledge, aesthetics, and so on. These attributes can be met through learning the shapes, learning the parts of each shape, labelling, learning perimeter, area, and volume, applying the formulae to perimeter, area, and volume, creating mobiles of shapes, researching these shapes as they appear in nature, writing the story of, say, Area, the use of computers in generating these shapes, estimation of widths of rivers and heights of trees, and so on.

- The common elements of "problem-solving" need to be discussed noting that there are often more ways than one to solve problems. The learning style of each student may mean that each student prefers one way over another.

- One very important method of learning in the Humanities involves a student asking questions prior to commencing a topic, and then having his own questions answered during his readings. We remember the

answers to our own questions better than we remember the answers to someone else's (the teacher's) questions. For instance, if we were about to study the American Civil War, we might ask what are the dates of the war, who were the leaders of both sides, what were the two sides, how did the war start, what were the causes, what was the first battle, what were the major battles, what was the last battle, who were the generals, who won, reasons for the outcome, what was going on in the world at the time of this war, does anyone really win a war, what are the effects of the war immediately after, and today, are there any feelings about the war still present in America, and so on.

As a person reads about a topic, new questions will arise. As a person has more experience asking his own questions, he will get better at asking questions prior to commencing a topic, and thus, will get better at remembering significant details about a topic.

In the case of introducing fractions: what are fractions, what does the word fraction mean, can you add, subtract, multiply and divide fractions, what are the rules for these operations with fractions, what are some applications of fractions, how are fractions used, what new vocabulary arises from the study of fractions, is there a relationship between fractions and decimals, and fractions and percent, and so on.

If there is a germ of an idea in this paper, I suspect each reader will be able to add many other applications. I would be very glad to hear from you concerning new applications that you can make to this idea.

Mathematics and the Arts: Taking Their Resemblances Seriously

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[Though] mathematics is often excluded from the arts....such exclusion does not justify denial or neglect of the respects in which [it] resembles the arts we do recognize.

—Francis Sparshott
[S3, pp. 145-146]

The above comment from a poet and philosopher expresses an attitude with which many mathematicians would probably feel comfortable. From at least the time of Aristotle, writers of various philosophical stripes have paid tribute to the aesthetic appeal of mathematical studies, and today there exists a considerable body of introspective literature on the affinities between mathematics and the arts. Certainly the readers of a journal devoted to humanistic mathematics should sympathize with any reaffirmation of their subject's place among creative human activities. Indeed, to some Sparshott's remark is surely overcautious: Isn't mathematics already an art?

For example, if we claim aesthetic values are inherent to our doing mathematics, can we in any way specify what those values are or the role they play in our practice?

I believe, however, that we should treat Sparshott's comment more as a suggestion than as another (and actually rather weak) tribute to the aesthetic nature of our subject. By virtue of its not assuming mathematics to be an art it encourages us to speculate on the issue. The results of such speculation may well be fresh insights into the nature of mathematics, its role among human endeavors, and even the manner in which mathematics education should proceed.

To see this, consider the extent to which the dominant mathematical philosophy of formalism concentrates on the rigorous, analytic structure of theories and results as opposed to the complex, creative and often downright hazy methods by which these results are obtained. In the eyes of formalism this is not merely a matter of emphasis—the latter notions are intentionally brushed aside as irrelelvent to the question "What is mathematics?" (see, e.g., [H1]). Can such an approach possibly be consistent with the belief that mathematics fundamentally engages our aesthetic interests, or that it proceeds in manners akin to those of art? If we truly feel these things about mathematics, why do we cleave to a philosophy and to practices that deny them?

Thus what I see Sparshott suggesting is that we take our generally unexamined beliefs on this issue and examine them seriously. For example, if we claim aesthetic values are inherent to our doing mathematics, can we in any way specify what those values are or the role they play in our practice? If we claim mathematics to be an art by virtue of entailing properties X, Y and Z, are we prepared to counter the charges that with regard to art itself, X is irrelevent, Y insuffucient or Z actually antithetical? Do we know what art is any more than we know what mathematics is? To answer questions like these—even just to ask them intelligently—requires a depth of analysis which few from either the aesthetics/art or mathematics sides of the issue have attempted.

While I make no claim of attaining such a depth in what follows, I do hope I can indicate something

of what is likely to be involved in the attempt, and where some of the benefits to be obtained by the effort may lie.

Aesthetics, Art and Mathematics

In keeping with a practice all too common among writers on this issue, I have thus far been using two different terms—art and aesthetics—in a dangerously synonymous way. It is time for a distinction. By aesthetics I mean the particular type of inquiry that Scruton [S2, p. 15] characterizes as the "philosophic study of beauty and taste." Similarly, by "aesthetic" I refer either to an attribute such an inquiry would study (e.g., aesthetic distance) or to some system that embraces these notions in a particular way (e.g., a culture or era with an aesthetic different from our own).

Art, on the other hand, is a trickier notion with which to come to grips. Too complex to take as a primitive term, it is one of those concepts—like mathematics itself?—for which we seem to have enough of a sense to use with impunity and still be understood—at least among people with backgrounds similar to our own. Yet a careful delineation of the concept always seems beyond our grasp. Perhaps we might regard this as good enough, but nagging questions of considerable import nevertheless keep arising: Is such-and-such a worthy enough work of art to merit our attention? Does its maker merit our support? Should art itself be publically supported? Should it be taught in schools? Can it be taught at all? As might be expected, the closer we get to the "edges" of the concept—to education, to ethics, to the boundaries of "non-art"—the tougher the questions get. Hence arises the philosophy of art.

Like the philosopher of art, we are not so interested in developing the concepts of beauty and taste per se as we are in applying these to a particular—though very broad—type of activity.

Although historical precedents for confusion are ample, it is almost certainly mistaken to equate this latter field with aesthetics [see S4, pp. 15-16]. The philosopher of art may well be concerned with questions of ontology (What is a work of art? A physical object? A mental state?), epistemology (Is there such a thing as artistic knowledge? What is it knowledge of? Is it somehow verifiable?) or ethics (Can a work of art be morally neutral? Does its carrying any particular moral value affect its status as art?). These are concerns whose connection with aesthetics proper is not entirely clear. Conversely, the aesthetician may well be interested in our responses to nature or to dreams, or to the incidents of our daily lives. One may argue that art is broad enough to embrace these notions as well, but this is certainly something to be demonstrated and not assumed.

So where does this put us with regard to mathematics? Ideally, we might wish to establish a relationship between aesthetics and the philosophy of mathematics analogous to that between aesthetics and the philosophy of art. Problems immediately arise, however. On the one hand, many of the concepts of aesthetics have developed (even if controversially so) with specific reference to art, and applying these as they now stand to mathematics may be unwarranted. But even more troublesome difficulties arise from the side of the philosophy of mathematics itself, which in this century has so estranged itself from aesthetic concerns that prospects for an immediate dialogue seem dim. The two disciplines apparently lack a common language.

It is here that we might appeal to the philosophy of art. Like the philosopher of art, we are not so interested in developing the concepts of beauty and taste per se as we are in applying these to a particular—though very broad—type of activity. Similarly, we should be willing to admit that the most important aspects of our subject may well lie outside the realm of aesthetics. Finally, we can note from the start that several important concepts in the philosophy of art (e.g., representation, form, the distinction between pure and applied activities) bear prima facie resemblance with concepts in mathematics. Possibly this is no more than a superficial coincidence, but again this is a point requiring demonstration. (I suspect that in many instances the coincidence is not superficial.) In keeping with the sense of Sparshott's suggestion, we should compare mathematics and art before we even consider equating them. We may even find that such an equation may be the least illuminating of the insights we gain.

Essentialist Theories

Traditional attempts to formulate philosophies of art have largely focused on the question of definition—of finding those qualities which uniquely constitute the essence of art. As DeWitt Parker stated in an influential article from 1939:

> The assumption underlying every philosophy of art is the existence of some common nature present in all the arts....There is some...set of marks which, if it applies to any work, applies to all works of art, and to nothing else....[This] constitutes the definition of art. [P2, p. 61]

Note that this essentialist approach—that of reducing art to necessary and sufficient conditions—promises a result not unlike those formulas for the identity of mathematics (e.g., "Mathematics is logic", or "Mathematics is the study of formal systems") that have arisen in our own field. The resemblance is deeper than this, however. The very attempts to specify these qualities in each subject have themselves paralleled each other in remarkable and significant manners.

Although any attempt to classify diverse theories runs the risk of oversimplification, I shall deal here with three general forms of essentialist arguments. The first, realism, involves identifying this essence in some relationship between a constructed object (or "artifact") and a world exterior both to that object and its maker. By contrast, expressionism concentrates on relationships between the artifact and a world interior to either its maker, audience or both. Finally, formalism identifies the essence of art strictly in terms of qualities contained within the artifact itself—e.g., the lines, colors or shapes in a painting, the arrangement of tones in a musical composition, etc.

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I do not wish to imply that all essentialist theories fit snugly into these categories. Hybrid theories combining elements from each of these also exist, as do theories that strike off in different, often quite sophisticated and radical, directions (see, e.g., [T1] & [D3]). For introductory purposes, however, this scheme (which parallels that of [H5]) should be adequate, and will also serve to highlight the parallels between traditional philosophies of art and mathematics. Let us briefly consider each in turn.

Realism

For centuries realistic views had constituted the dominant theories of art, and although these attitudes have generally fallen from favor, both artists and audiences alike continue to be influenced by their appeal. While a broad spectrum of realistic views are possible, these theories in general identify art as a system of knowledge with a status claim not entirely unlike that of science. Most explicitly, Leonardo declared that, "Truly painting is a science, the true born child of nature" and some three centuries later the landscapist Constable reaffirmed: "Painting is a science, and should be pursued as an inquiry into the laws of Nature." By these lights (which can be clearly discerned in the works of Aristotle), the artist reveals reality to us through abstract representations of its various aspects—aspects incapable of such revelation by any means but art. In some variations of the theory, these revelations are actually of a "higher" plane of reality than mere physical existence. For example, Piet Mondrian viewed his skeletal arrangements of lines and solid colors as presenting images of "true reality and true life," a position more akin to neo-Platonism than the inductive realism of Aristotle. These views are linked, however, by a conception of art as essentially a means of discovering truth.

Have attitudes such as these ever been displayed toward our mathematical "artifacts," and are they still likely to exert an influence on us today? To both questions the answers are definitely yes. For centuries Euclidean geometry did not stand as one of a multitude of formal geometric systems but was regarded as representing the Real geometry of Real Space, based on inductively self-evident truths abstracted from reality and developed through the unassailable method of deduction. (Writing before Euclid, Aristotle accurately observed some of the

consequences of rejecting the assertion that the sum of the angles in any triangle was two right angles. He abandoned these conclusions because of their "obvious" inability to square with the evidence of experience. Spherical geometry, of course, had already developed a long history, but this was Real geometry on a Real surface—not a model of a non-Euclidean system. See [T3].) True, the insistence upon logical deduction may appear outwardly to be difficult to reconcile with art, but I believe this perception largely arises from simultaneous tendencies to downplay the role of logical development in art and to overplay its role in mathematics. The logical "flaws" in Euclid's geometry or Newton's calculus were recognized as profoundly troublesome, but these problems were as little compared to the practical insights and successes the systems as a whole presented (see [K1]). The knowledge gleaned from mathematics was looked upon as genuine knowledge nonetheless, and throughout this period there was little doubt that this knowledge was that of reality.

The rise of non-Euclidean geometry required abandonment of this geometry-as-the-science-of-real-space conception, but by no means did it spell the end of realist conceptions of mathematics. By one view, different geometries or systems of mathematics could be conceived as separate entities of a Platonic world of forms, with the mathematician regarded as a sort of suprascientist who discovers the properties of these entities. And

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although this position ignores several not inconsiderable aspects of Plato's thought on mathematics, the belief that any well-framed mathematical question has a definite answer independent of our existence characterizes the position known as mathematical Platonism. Phrased in this manner, the position becomes as much the creed of a mathematical theology as it does the basis of a descriptive philosophy. As the former it seems to be of practical value to individual mathematicians [see, e.g., T2]; as a critical theory it seems as suspect as Mondrian's supposed ability to envision "true reality." Did art experience its own "non-Euclidean revolution?" In a sense it did, even if its occurence cannot be as precisely specified as that of mathematics. (Although the late-18th and early-19th centuries figure prominently in the revolutions in both fields. See [A1].) The process in this case involved—at least in part—the realization that representation itself was neither a necessary nor sufficient condition for art. Its insufficiency (long recognized) can be ascertained simply by considering common occurences of representation that do not qualify as art: courtroom records, plastic anatomical models, or snapshots (the history of photography's status as an art provides an interesting trial case of realistic theories). But even more telling is the apparent case for unnecessity. Not only do non-representational artworks exist, but entire branches of art appear to be non-representational by nature. Debussy's La Mer may be taken to represent the sea, but does not this conclusion follow primarily by virtue of the work's title alone? Without this title, would the work still be perceived as resembling the sea more than it does anything else? Would we even consider the work in representational terms? Would we still consider it a work of art? (For this example see [H5, p. 708].)

Emile Zola—one of the late 19th century's most prominent exponents of realism through his ideal of the naturalistic "experimental novel"—once characterized art as life seen through a "temperment." But even from before the time of Zola it had become increasingly apparent that realistic theories had been in error in not placing more emphasis on the role of this "temperment" itself. In the philosophies of expressionism this role would lie at the very heart of the matter.

Expressionism

For an example of an extreme expressionist view, consider the theory developed in the first half of this century by the philosophers Benedetto Croce [C2] and R. G. Collingwood [C1]. (These are actually separate theories, but close enough in both spirit and detail to be regarded as one for our purposes.) In this theory art is taken as being identical with expression, the revealing of an

internally constructed mental state. Even more simply, art is realized imaginative activity, and in fact stands as the most basic form of human language. Although this is a particularly broad pronouncement, it is accompanied by explicit statements of what art is not. First (and possibly most difficultly) it is not any set of existing physical or symbolic artifacts such as paintings, texts or sculptures. These are simply the necessary embodiments of the imaginative constructions that Collingwood calls the "work[s] of art proper" [C1, p. 37]. Thus while we may be accustomed to referring to the Mona Lisa itself as a work of art, the work of art proper lies in the imaginative constructions that resulted in Leonardo's painting it and in our interpreting it. (Along these lines Collingwood approvingly quotes Coleridge's "we know a man for a poet by the fact that he makes us poets" [C1, p. 118].) Similarly, art should not be confused with any particular kinds of media. These, rather, are simply the necessary channels through which particular expressions flow. (Again from Collingwood: "Every gesture that each one of us makes is a work of art" [C1, p. 285]) Finally, art should not be identified with any activity engaged in for a specific purpose or outcome. Art may be used to represent reality, entertain us or persuade us, but it does these things en passant. Indeed, if any specific purpose guides the production of a work we are dealing not with art at all, but with craft or techne—what Collingwood calls "art falsely so-called."

This last may seem to place art in a paradoxical position. On the one hand it is equivalent to the most basic human attribute (communication) and on the other it is divorced from purposive human activities. This is resolved through the nature of the internal constructions of emotions, intuitions and experiences that result in successful expressions. Collingwood explains that to attain these the artist must "experience what all experience" and comments that "in ivory towers art languished" [C1, p. 119]. Similarly, Croce disposes of the romantic notion of the artist as necessarily being a specimen of genius (a view which also haunts mathematics) by noting that "inspiration is not something from heaven, but is in the essence of humanity itself," and that "the man of genius who poses as that...finds his punishment in becoming somewhat ridiculous" [C2, p. 16]. In this theory artists are born and not made, but we are all born artists. Differences are of degree, not of kind.

It may seem unlikely that there could be an overall view of mathematics corresponding to an expressionist theory of art—particularly a theory as extreme as that of Croce and Collingwood. It is somewhat startling to find, therefore, that it is not difficult to find particular passages in these writer's works which are virtually identical to ones found in the work of such figures as Poincaré, Brouwer and later members of the constructivist school.

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Consider Brouwer's intuitionism [B]. To Brouwer it is above all mistaken to confuse mathematics with its final symbolic forms. These, rather, are merely the manifestations of the internal constructive processes that constitute the proper identity of mathematics. Other mathematicians may use their interpretation of these symbols as a basis for their own constructions, but their interpretation is itself a personal construction. (Sparshott notes that both he and H. S. M. Coxeter know the Pythagorean theorem, but that what "he knows in knowing it is something I cannot even imagine" [S4, p. 33].) To Brouwer, logical rules do not guide the process of mathematical construction; rather, a pragmatic logic is effectively created along the way. Similarly, Collingwood notes that the artist does not create rules so much as construct them. These viewpoints do not leave judgements in either activity purely subjective, however. In mathematics, such judgements are constrained by the requirement of constructive existence; in art by the commonality of actual human experience. Even on the point where the theories apparently diverge most severely—the conception of language—the difficulty reduces to a difference in terminology. Croce and Collingwood conclude that art is language, whereas Brouwer virtually declares that mathematics is anything but language. But to Collingwood, language is any expression of internal constructions; to Brouwer language is primarily symbolic notation (which Collingwood refers to as "language falsely so-called" and Croce

as "the mere grammatical").

Though these theories may assume different names in different fields—expressionism in art, intuitionism or constructivism in mathematics—all share a focus upon the freedom and constraints of the creative process as essential to the activity at hand. In art, expressionism has been particularly criticized for its apparent disregard for the artistic product as an entity onto itself—the theory's notion of artistic existence appears skewed [H4]. Similarly, mathematical constructivism establishes a criterion of existence often viewed as unnecessarily severe ([H2]). In response to these charges in both areas, philosophies of formalism—already present in each field—have arisen as alternatives.

Formalism

Whether in mathematics or in art, formalism in its purest state is a philosophy of detachment. Artworks and mathematical systems are viewed as independent phenomena standing apart from their creators, their audiences and any external world. Meanings or values can be considered only with reference to elements contained within the work itself. And although ideal formalist criticism may be difficult to achieve in practice, in that ideal works stand apart even from the history of their very construction. As Werhane puts it, a work exists virtually "in spite of its creator...and its audience" [W4, p. 99].

I should make at least one distinction among formal theories, however—particularly with respect to art. In the pure state described above—a position sometimes referred to as absolutism—the meaning of a work is entirely separated from any quality not within the work itself. Thus to recognize a work of art is to recognize such qualities as symmetry, internal dynamics or the relationships of color and shape; to create art is to impose these qualities on an artifact. However, in another brand of formalism it is important that one look at both the formal structure of a work as well as any meanings that may be expressed by that structure. Art effectively becomes defined as symbolic form. But unless one wishes to regard virtually everything from everyday language through, say, mathematics as art, a more precise description of what constitutes artistic forms is needed. Such theories—as for example those of Suzanne K. Langer [L] or Nelson Goodman [G2]—evolve largely into "meta-theories" of syntactic and semantic rules, and thus to a large extent diverge from pure formalism. Simultaneously, however, they also seem to diverge from the actual experience of creating and appreciating art. Writing with regard to Langer's theory, Scruton comments that her "analysis gives no procedure for interpretation, nothing that would give application to the claim that in understanding a work of art we understand it as a symbol" [S2, p. 22].

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In mathematics, pure formalism—comparable to aesthetic absolutism—is that view which takes mathematics to be strictly the study of formal systems (see [H1]). Where the formalist in art seeks internal structure, the mathematical formalist seeks consistency. Where the former detaches meaning or expression from form, the latter constructs "uninterpreted" systems.

I feel compelled here to mention that Hilbert (the "father of formalism") was not a formalist in this sense, as his introduction to Geometry and the Imagination [H3] makes abundantly clear. The metamathematical strategy of Hilbert's Program required him to treat mathematical systems as uninterpreted for the sake of a consistency proof—hardly the same as believing mathematics itself to be fundamentally uninterpretable or meaningless. Curry, one of the leading exponents of mathematical formalism, has himself stated that "it is unfortunate that many persons identify formalism with what should be called Hilbertism" [C3, p. 156]. Indeed, it would be interesting to determine the degree to which the aims and methods of metamathematics in general parallel those of, say, Goodman's strategy in Languages of Art [G2].

In both mathematics and art formalism has become the most established theoretic view of this century. (In both fields, in fact, it has independently acquired the descriptive phrase "Modernism.") In mathematics education, its influence—under the pervasive guidance of Bourbaki and the New Math—has been nearly universal. However, as Davis and Hersh ([D4, pp. 343-344]) note, cracks have begun to show in the formalist wall. The underlying theme of both [D4] and [D5] is in essence an assault on mathematical formalism, as are the selections in [T4]. Similarly, many of the essays in Arthur Danto's recent collection [D2] constitute strong cases against aesthetic formalism and its separation from human experience and meaning.

Non-Essentialism

In an often anthologized paper, "The Role of Theory in Aesthetics" [W1] the philosopher Morris Weitz argued that the entire essentialist approach to the philosophy of art is fundamentally misguided (see also [W3] for an elaboration). To Weitz, art is an example of what he calls an inherently undefineable open concept. He develops this more fully by referring to what he calls the perreniel flexibility and debatability of art. Art is perennially flexible in that it is always capable of expanding its scope to embrace previously unconsidered artifacts, styles, media, etc. It is perenially debatable in that not only are the criteria by which new works are judged subject to continual criticism and reevaluation, but even established works continue to fall under such scrutiny. Oedipus Rex may be established as a masterpiece of tragedy, but the matter of exactly which of its qualities grant it that status continues to be contentious.

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Weitz bases his discussion in Wittgenstein's notion of family resemblance [W5, pp. 31-33]. To use Wittgenstein's example, if we are confronted with the question "What is a game?" we shall find ourselves unable to provide a satisfactory set of necessary and sufficient conditions. Not all games have rules, not all involve competition, not all are purely recreational, etc. What we can do is point out resemblances between those entitities which we have chosen to call games. If a new candidate for gamehood presents itself, we base our decision whether to expand the concept to embrace it on the resemblances we perceive between the new activity and those previously admitted. Weitz notes that with regard to art we could always choose to "close the concept" by legislating strict standards of admittance, but to do so would, as he puts it, "foreclose on creativity"—a seemingly self-defeating decision.

This last approach—that of choosing to close the concept—is the option foundational philosophies of mathematics seem to offer. But when mathematics is regarded primarily as an activity (i.e., historically, culturally and in actual practice) this option as well seems to be a particularly self-defeating decision. Viewed this way, in fact, mathematics increasingly presents itself as a flexible and debatable open concept. As an example, consider the attitudes mathematicians have historically held regarding the existence of infinitesimals. Does the long history of this issue between the times of Eudoxus and modern non-standard analysis suggest anything but the perennial flexibility and debatability of this concept? What of "dimension," "continuity," or "real number?" Perhaps most tellingly, consider the notion of "proof." Is this a closed concept, or have our criteria of what constitutes a valid mathematical argument themselves been (and continue to be) flexible and debatable? Consider the positions available with regard to the use of computers in proof. Tymockzo [T4, pp. 243-266] asks whether if we regard the 4-color theorem as having been proved, must we not also admit that its proof is of an entirely different nature from any that have come before it?

Although Weitz considers essentalist arguments to be misguided in purpose, he does not consider the particular issues they raise to be irrelevant. These theories provide, in fact, "a series of invaluable...directions for attending to art," and concludes that:

> If we take aesthetic theories literally...they all fail; but if we reconstrue them, in terms of function and point...as recommendations to concentrate on certain criteria [they are] far from worthless. [They] teach us what to look at in art. ([W1], p. 156)

Similarly, if we demand of our philosophies of mathematics a secure foundation capable of supporting the entire structure of the subject we may well be fundamentally misguided. However, foundational philosophies have been successful in raising relevant issues and thereby provoking us to fresh insights. In both art and mathematics, non-essentialism moves us toward a pluralistic view of our endeavors, one that recognizes that there is not a single, absolute perspective but many—possibly not even mutually consistent—viewpoints. By no means, however, should pluralism be confused with a kind of bland eclecticism or—even worse—blind subjectivism. On the contrary, the holding of firm individual views is probably necessary to create the productive tensions that drive our activities as a whole. One can be generally non-pluralist with regard to one's own philosophy and generally pluralist with regard to mathematics or art as a whole. (For a discussion of this stance with regard to art see Danto's "Learning to Live with Pluralism" in [D2]).

The recognition of both art and mathematics as open concepts leaves behind it a curious casuality. The claim that mathematics is an art seems to have lost its meaning. But in its place has arisen the prospect of a much more active agenda—that of seeking family resemblances between the two fields. That any resemblances we find are indeed familial ones I do not find objectionable—the underlying conceptions of the essentialist theories in each field justifies their membership in a common philosophical "family." In the remainder of this article I would like to address just a few of the areas where some useful resemblances of this kind might be sought.

Family Resemblances between Art and Mathematics

Mathematics and Techne

One of the central points in the Croce-Collingwood theory is the distinction between art and techne, or craft. Recall that in this theory the former is equivalent to expression, while the latter is identified with activities engaged in to attain a specific result. Even without wholly accepting this theory, the desire for such a distinction seems desirable. Techne is composed of specific, trainable skills; art not only marshals these skills, but transcends them in some not clearly defineable manner. Mixing paints or mastering scales is techne; The Madonna of the Rocks or an original jazz improvisation is art. Propaganda as propaganda is techne; propaganda in the form of Eisenstein's Potemkin somehow becomes art.

A similar distinction seems valid in mathematics, and could well be crucial in education. Finding the greatest common divisor of two numbers through following some recipe is techne; deriving a process along the lines of Euclid's algorithm, convincing oneself and others that it works, or developing a geometric interpretation of it (in short, understanding it) is mathematics. Even a brief perusal of most secondary level textbooks or examinations reveals a tremendous emphasis on techne. But to what degree are "skills" nonetheless necessary, and which particular ones? Can and should these be approached in a more "mathematical" manner? If mathematics proper is something that cannot be directly taught, how can it be productively developed? Approaching these issues seems to require a sharper disitnction between "math proper" and "math falsely so-called" than now exists. (Consider: To what degree is formal proof a matter of techne? Is much of the "mathematics proper" completed by the time we reach this stage?)

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A final open note on this subject: The math/techne distinction appears to cut across that of pure/applied mathematics, and in certain respects may prove to be more a fruitful one. Consider architecture. Is this pure or applied art? Is it art or

techne? Which question gives us a more useful perspective in contrasting Frank Lloyd Wright's Fallingwater with tract housing? Now what of, say, the theory of relativity, or the activity of mathematical modelling?

Style

Among the most perplexing questions concerning the nature and history of art are those involving the issue of style. Why have different individual artists, or the artists of different cultures, nations and ages represented the world in such different ways? How is it even possible for them to have done so, and then for others to be able to recognize and appreciate their works? How does an approach to these issues regarding representation translate into the perception of style among non-representational works? Could art have evolved in only one possible way? What is the significance of style? (The first questions are broached by Gombrich [G1]; the last by Goodman [G3])

These topics have not been entirely alien to mathematics, even if they have not been approached in terms of the overriding concept of style. Comparisons of Greek and modern mathematics have often been approached with regard to what are effectively stylistic considerations, and from at least the time of Poincaré writers have contrasted individual geometric "styles" with arithmetic or analytic ones. However, style has not played the unifying role in historical or philosophical studies of mathematical thinking that it has in art. Possibly this is due to an attitude which views mathematical progress in a deterministic, unitary manner—a great chain of mathematical being, so to speak.

There have been indications of change. Educators speak of the role of "cognitive styles" and "multiple representations" in the learning of the subject, and Otte has written of a "complementary" relationship between arithmetic and geometry which is as suggestive of corresponding artistic styles or media as it is of modern physics [O]. Furthermore, the emerging field of ethnomathematics ([D1] & [A2]) seems to offer a particularly fresh approach to the nature of mathematical styles developed outside traditonal academic settings. Possibly all these workers may have something to gain from the methods and results of art history and psychology with respect to style.

One further note along these lines: In [D4, p. 318], Davis and Hersh point to the relative permanence of mathematical results as opposed to scientific ones. This suggests a further stylistic affinity between mathematics and the arts. Just as Picasso did not relegate Rembrandt to the attic, modern geometry does not invalidate Euclid. (Note the current state of Aristotelian physics, however.) Still, just as we can no longer look upon Rembrandt with anything but modern eyes, we must reinterpret Euclid in light of our current position. The development of mathematics has certainly been allied closely with that of science; however, the nature of the subject's history itself may bear more similarity with that of art. (This point is suggested by Thomas Kuhn [K2, p. 345].) A conception of the history of mathematics in terms of stylistic change might provide a better handle on this phenomenon than one which views the development of mathematics as analagous to that of science.

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Ethics

If the philosophy of art often seems a rather dry and erudite affair, there is at least one set of issues over which it becomes both heated and public—that of censorship and the moral obligations of the artist. Are there general topics from which the artist, or particular works from which the public, should be restricted access? Similarly, is the artist obliged to deal with any particular topics or situations? If each of the opposed paths of strict moralism and unrestricted aestheticism appear undesireable, what positions are left open to us? These have been among the most discussed issues in the philosophy of art, and can be traced from such figures as Confucious and Heraclitus through

current commentators on the role of the National Endowment for the Arts.

Issues of a similar nature have arisen more frequently in mathematics in recent years. (Davis and Hersh [D4, pp. 87-89] contrast the extreme ethical positions of "Mathematical Maoism" and a form of mathematical aestheticism derived—perhaps inaccurately—from Hardy.) In general, however, mathematics (and to a slightly less extent, science) has been disappointingly slow in embracing such topics. As more mathematicians and educators enter into such discussions, it may well be to their advantage to acquaint themselves with the manners in which the corresponding issues have been approached with respect to art. It might be worthwhile to point out here one particular area in which the functioning of each of mathematics, art and science have grown tremendously similar: All to a large degree depend primarily for support on governmental or industrial support ("patronage"). The time is ripe for a genuine dialogue between these areas on this issue alone.

Aesthetic Issues

Oddly enough, purely aesthetic considerations have made few appearances in this discussion. If we were to adopt a strictly Platonic view of either art or mathematics we might account for this simply through invoking Keats: Beauty is truth, truth beauty. Though a simplification, this formula nonetheless captures much of the attitude that allowed realistic views of art and mathematics to coexist for centuries with formal conceptions of beauty (as outlined in Plato's Philebus). To a modern day formalist, on the other hand, aesthetic considerations may well be deemed irrelevant, either relegated to the "emotive" qualities of the positivists, or actually defined as the meeting of certain formal criteria. Neither of these positions, however, is particularly helpful in ascertaining the role or nature of aesthetic judgements in actually creating art or mathematics.

This is a topic on which the mathematical literature is virtually mute. Poinçare [P4, p. 392] wrote of aesthetics as serving the role of the "delicate sieve" through which successful mathematical "combinations" passed on their way to consciousness, but his suggestive metaphor has remained little more than that for some nine decades. (Though some interesting development is attained in [P1].) This is not to imply that the artistic literature has itself resolved the problems posed by the creative imagination, but it has approached them directly and arrived at apparently useful distinctions and insights. This same literature may provide a fertile starting ground for mathematical investigations. For an introduction to these issues from philosophical and psychological persepctives, see [S1] and [P3], respectively.

I would like to mention briefly one aesthetic value which does seem to be pervasive in the mathematical literature—that of elegance. As aesthetic values go, this seems to be rather peculiar

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to mathematical studies. (Indeed, the word actually carries a somewhat negative connotation in, say, painting or music.) Far from being able to separate the positive or negative effects this concept has had on development, mathematical writers have not even been particularly clear on just what elegance is or how it stands in relation to such other concepts as clarity, conviction, understanding or significance. It is, however, an aesthetic value (or at least an apparent aesthetic value) with some degree of currency in the mathematical community, and as such may provide an entry point for approaching the nature of mathematical aesthetics in general. My own suspicion is that an overly formal conception of elegance—say one strictly in terms of efficiency—could hinder development as much as it motivates it, whereas a pluralistic view of various attitudes could be genuinely productive. I would greatly appreciate hearing others' views on mathematical elegance—particularly examples of what they find to be elegant and some (however imprecise) description of what leads them to say so. (Questions: Does your conception of elegance primarily involve results or procedures? Does visualization have anything to do with it? Are there arguments which you find convincing but still not satisfying?)

Conclusion

Perhaps the most appropriate way to close this article is to refer again to the epigraph with which it opened. Superficially, mathematics may appear as the most austere of human endeavors: rigorous, analytic and precise. More deeply, it reveals itself as one of the most mysterious, aspiring to aspects of experience most commonly associated with philosophy or religion (both of which, incidentally, Hegel concluded art eventually becomes). To understand mathematics at this deeper level—to grasp it as a human activity which itself humanizes—requires a perspective that embraces other humanist endeavors. Art—in its richness, its mysteries and its humanism—offers an abundant wealth of experience and insight from which mathematics may have much to learn of itself.

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Students' Understanding of Functions in Calculus Courses

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reprinted from the Humanistic Mathematics Network Newsletter #2, March, 1988.

1. Introduction

I have long been interested in the relationship between students' understanding of basic concepts in mathematics and their performance on material that is supposed to be built up out of these concepts. But such an interest immediately raises the question of what concepts underlie any particular body of mathematical material and how one might describe student understanding of these concepts as something separate from their mastery of the material.

To a mathematics teacher, the central concepts of a beginning calculus course are limit, derivative, and integral. But these concepts do not underlie the subject: they are to be developed in the course of studying it. Most teachers would agree that, while the course is in progress, students' understanding of these concepts could not stand much of a test. The concept that the subject is built out of, the one that lies behind such notions as limit, derivative, and integral is that of function. The question this paper addresses is: What do students understand of the concept of function while they are in the process of mastering the material of a beginning calculus course?

Boiled down to its simplest form, a function is a correspondence between two sets or between two variables. At first, one usually describes functions in terms of every day examples using tables, algebraic formulas, graphs, and various artificial rules. This idea is so simple that we have difficulty imagining that our students are not already familiar with it. And, indeed, if a function is given by a table, or we use its graph or formula as if it were only a table, reading particular numerical values of the independent variable as corresponding to particular numerical values of the dependent variable, then this *IS* a very simple concept—one that most of our students acquired in high school. I call such use or such a view of this concept a "Pointwise understanding" of functions. I have always found it interesting that when authors of calculus and precalculus texts give their obligatory introductions of the concept of function, it is a Pointwise view of the concept they are trying to get across. But this is not actually the way the concept of function is used in calculus.

A look at almost any page of a calculus book shows that the crucial question asked about functions is: How does change in one variable lead to change in others? How is the behavior of the output variables influenced by variation in the input variable? I call an ability to answer such questions an "Across-Time understanding" of the concept of function. The definition of the tangent line to a graph or of its slope would be utterly meaningless to someone who could only look at the graph or the function at a few specific points at a time. The essence of the definition is a tendency of the behavior of secant lines or difference quotients as the incremental change in the independent variable is decreased.

This paper is based on a study of student responses to two types of questions on final examinations in calculus classes, one of which requires only Pointwise understanding and the other of which requires Across-Time understanding. The results of this study show that these two kinds of understanding are clearly distinguishable. But, more crucially for teachers, they show that, at least in simple situations, students have a confident and secure Pointwise understanding of functions, but even at the end of the first or second quarter of calculus—when we tend to assume they have already acquired this ability—they are still struggling to see functions in an Across-Time manner.

2. Description of the Study

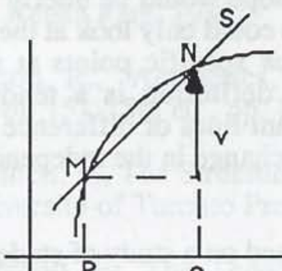
This study is based on four multipart problems for beginning calculus students. Each problem has one or two questions that test for a Pointwise understanding of a function and one or two questions that test for an Across-Time

understanding of the same function. A number of faculty members at the University of Washington each agreed to include one of these problems as a regular part of an examination in a calculus class with the results to be counted toward the exam grade. Each of the questions was used once on a final exam in a first-quarter calculus class (Math 124). One was also used on a late mid-quarter exam in the first quarter calculus class and two were used on final exams in second-quarter calculus classes (Math 125). In all, 628 students were involved in the study.

In order to facilitate analysis and discussion, these problems are presented below in a highly compressed form. They were expanded into a form more readily comprehensible to students for the exams. In most cases, they were written as multiple choice questions.

The Problems

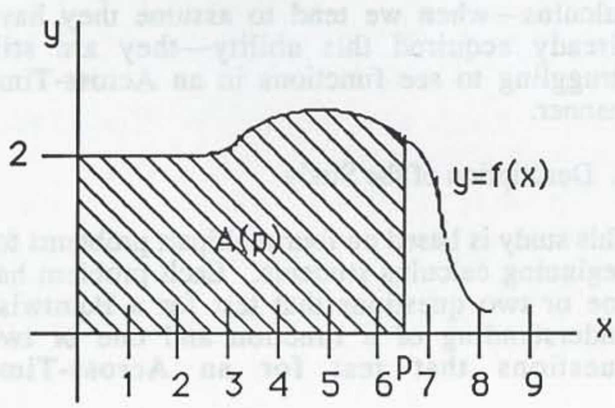
SLIDING SECANT



POINTWISE: Find the slope of the secant and value of v when M and N have coordinates (1.6) and (4.12).

ACROSS-TIME: The point Q moves toward P. Does the slope of S increase or decrease? Does the value of v increase or decrease?

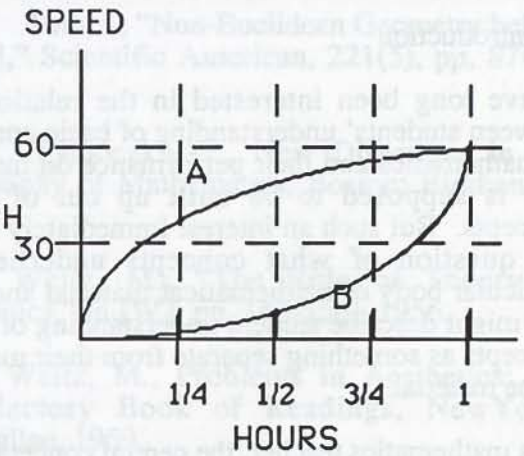
AREA UNDER THE GRAPH



POINTWISE: Determine the values of A(1) and A(3).

ACROSS-TIME: The point p moves from 4.5 to 6.0. Does the area A(p) increase or decrease?

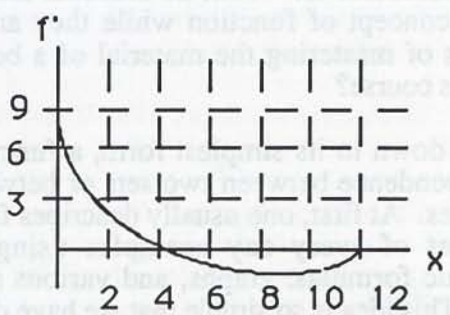
TWO SPEED GRAPHS



POINTWISE: Which car is going farther at time 3/4: Is Car A going faster at time 1/4 or 3/4?

ACROSS-TIME: Tell whether or not the cars are closest together at time t=1. In the time from 1/2 to 1 hour, do the two cars get further apart or closer together?

SECOND DERIVATIVE TEST



The graph shown is of the derivative, f' (x).

POINTWISE: Determine the slop of the tangent line to f(x) at x=2. Give a value of x for which the the tangent line to f(x) is horizontal.

ACROSS-TIME: Describe the shape of the graph of the function f(x) for the interval 1 to 4.

Each student's answers to the various questions

were then coded and recorded, so that combinations of rightness and wrongness, or combinations of incorrect answers could be studied. A summary of the statistical results of this study is given in Table 1. Each grid indicates the results of one class's response to the two kinds of questions on one problem. Thus, for instance in the first grid of "Sliding Secant", we see that of the 116 students who answered the question, 53% got both the Pointwise and Across-Time Questions correct, and 34% got the Pointwise Question correct while they got the across-Time Question incorrect. We also see that, in all, 87% got the Pointwise Question correct. There are two grids for each of the problems Sliding Secant, Area Under the Graph, and Two Speed Graphs because each of these problems was given in two different classes.

Sliding Secant

		Across-Time		
	N=166	Right	Wrong	T
P	Right	53%	34%	87%
T	Wrong	4%	9%	13%
W	T	57%	43%	

Sliding Secant

		Across-Time		
		Right	Wrong	T
P	Right	42%	40%	82%
T	Wrong	10%	10%	20%
W	T	52%	50%	

Area Under the Graph

		Across-Time		
	N= 89	Right	Wrong	T
P	Right	63%	19%	82%
T	Wrong	0%	18%	18%
W	T	63%	37%	

Area Under the Graph

		Across-Time		
		Right	Wrong	T
P	Right	60%	24%	84%
T	Wrong	4%	11%	15%
W	T	64%	35%	

Two Speed Graphs

		Across-Time		
		Right	Wrong	T
P	Right	40%	50%	90%
T	Wrong	0%	10%	10%
W	T	40%	60%	

Two Speed Graphs

		Across-Time		
		Right	Wrong	T
P	Right	49%	35%	84%
T	Wrong	0%	16%	16%
W	T	49%	51%	

Second Derivative Test

		Across-Time		
		Right	Wrong	T
P	Right	44%	8%	52%
T	Wrong	22%	28%	50%
W	T	66%	36%	

Table 1 —Summary of Test Results

3. Interpretation of the Test Results

The most striking observation to be made about the data shown in Table 1 is that the percentages of completely correct solutions to these problems are so low. Overall, only 50% of the solutions to these problems were completely correct. Yet most mathematics instructors would agree that these are extremely simple, if not naive, problems, which test the kinds of understanding we tend to assume our students have as they work on the early material in calculus. The problems are so easy that

Pointwise understanding of graphs is prerequisite to Across-Time understanding, but the jump from the one to the other is a considerable one for students.

many argue against the results by saying that the students were tricked or that they were probably not paying complete attention to them. But these problems were on exams which counted toward the students' grades in very competitive classes. These students were strongly motivated to carefully read and think about these questions. Moreover, each problem had a very simply Pointwise question, placed there, at least in part, to settle the student into the problem and to check from his or her answer that the quantity being asked about was clearly distinguished. The fact that such a high percentage did get the Pointwise questions correct (79% on all problems) indicates that the students were paying sufficient attention for us to make inferences about their thought processes.

In relation to the overall percentages of correct answers, it is probably worth noting that of the students who were assigned final grades in these classes, 83% received a grade of 1.6 or higher. Since all but a handful of students who take a final or very late mid-quarter exam receive a grade, it seems safe to assume that these data do not include a large pool of students who are lost in the material, as, say, an early mid-quarter exam might. Presumably, most of the students who took these tests felt reasonably comfortable with their ability to perform in the course.

To draw more refined inferences from these data we must separate the last problem, "The Second Derivative Test" from the others. It presents a different picture of students' understanding which can only be described in relation to the picture indicated by the other data—the pairs of results from the first three problems.

3.1 The First Three Problems

Common sense would indicate that at some stage students learn to read a graph one point at a time—in a Pointwise manner—and once they have done so, it is only a small jump to reading a graph at many points. From that position, it is again only a small step to reading a graph at infinitely many points, or with a continuously changing variable, i.e., reading graphs in an Across-Time manner. Indeed a study of precalculus texts indicates that authors and many teachers assume that this is how things happen when students are taught functions. However, since, in the first three problems in this study, 85% of the students got the Pointwise questions correct while only 53% of the students got the Across-Time questions correct, it does NOT seem to be the case that an Across-Time understanding comes easily and automatically after a Pointwise understanding has been developed.

A study of the first six grids in Table 1 illustrates this point forcefully. It shows that reading graphs in a Pointwise manner is a necessary, but far from sufficient condition to reading graphs in an Across-Time manner. In particular this data says:

a) Students who get the Pointwise questions wrong are unlikely to get the Across-Time questions right. (Overall, 15% who got the Pointwise questions wrong got the Across-Time questions right.)

b) Students who get the Pointwise questions right have variable chances of getting the Across-Time questions right—depending on the question, the class, etc. (Overall, 59% of the students who got the Pointwise questions correct also got the Across-Time questions correct.)

c) Students who get the Across-Time questions correct are extremely likely to get the Pointwise questions correct. (Overall, 96% who got the Across-Time question right got the Pointwise question right.)

Another way of putting this is that Pointwise understanding of graphs is prerequisite to Across-Time understanding, but the jump from the one to the other is a considerable one for students.

That the difference between these two kinds of questions is a qualitative one can be seen by analyzing the answers given to these questions by the population of those students who got the Pointwise questions correct. For these students we can be reasonably confident that they can read the graph in a rudimentary fashion and that they have a basic comprehension of the set-up of the problem. What one sees in these answers is that, when pressed, the students do inconsistent things; in each wrong answer there is a self-contradiction.

> Sliding secant: 43% of this population got the Across-Time question wrong. The most common error was one in which the secant line is regarded as moving, with its slope increasing, while the vertical distance v is regarded as fixed. In fact, if one views this diagram as made up of a system of inter-connected "moving parts", (as the course material requires) then all of the incorrect answers are self-contradictory. They either give one part as moving and another as fixed, or they give two parts moving, but in inconsistent ways. It must be that these students do not see this diagram as such a system, that such a dynamic, Across-Time use of a graph is quite alien to them.
>
> Area Under the Graph: 25% of this population got the Across-Time question wrong. These students knew well enought how to compute area from the two dimensions, height and base, for the two particular values p=1 and p=3 given, but then, when they were asked about the behavior of A(p) as p goes from 4.5 to 6.0, they responded as if they thought that A(p) was to be found by simply looking at the height of the given graph. This seems to be a case of a student transforming a question into one that is easier when the given question cannot be answered.
>
> Two Speed Graphs: 50% of this population got one of the Across-Time questions wrong. Most of these, (34% of this population) gave answers to both Across-Time questions as if the graphs shown were of position vs time—while answering the Pointwise questions as if the graphs were of speed vs time. The remaining students in this population were conflicted. They answered one Across-Time question as if the graphs were of speed vs time and the other as if the graphs were of position vs time. In both cases, we see the students shifting their interpretation of the quantity represented by the vertical axis when confronted with a need to use the information in the graphs in a novel way.

It would seem to be the case that students who give contradictory answers have lost their hold on the situation described in the problem—that they are, in some way, overloaded. To understand why these Across-Time questions have this effect would require a separate study that would at least include an analysis of the mental processes of students who do get the correct answers to these problems. My own speculation is that in order to do the Sliding Secant and Area Under the Graph problems correctly, the student would have to evoke in his or her own mind a version of the given diagram that can be made to move and then draw conclusions from these mental experiments. In order to do the Two Speed Graphs problem correctly the student would have to evoke a mental model of two cars, read the information from the graphs that one car is going farther than the other for the entire period, and draw conclusions from this fact, while specifically disattending from the striking pictorial qualities of the graphs. There is very little in the experience of first-quarter calculus students that would prepare them to do these things, and so it does not surprise me that the students do so badly.

It would seem to be the case that students who give contradictory answers have lost their hold on the situation described in the problem—that they are, in some way, overloaded.

3.2 Interpretation of Results of "Second Derivative Test"

The statistical results of "Second Derivative Test" are very different from the results of all of the first three problems. On this problem students did

better on the Across-Time question than on the two Pointwise questions (66% vs 52%). Furthermore, in contrast to the results we saw before:

a) 22% of the students who got the Across-Time question wrong got the Pointwise questions right.

b) 66% of the students who get the Across-Time question correct got the Pointwise questions correct.

c) 85% of the students who got the Pointwise question correct also got the Across-Time question correct.

Thus, extending the kind of analysis used to confirm that Pointwise understanding of graphs is prerequisite to Across-Time understanding on the first three problems, we arrive at the conclusion that an Across-Time understanding of this problem is prerequisite to a Pointwise understanding. But this makes no sense at all, because any understanding of how to draw inferences from the shape of the graph of a derivative, would have to include an ability to make pointwise readings of this graph.

But, in fact, it can be seen that of the 108 students who took the test, 24 were able to use the graph of f '(x) to select the correct shape of f(x) over the interval [1,4], but at the same time, could not read from the graph the value of f '(2) or the value of x for which f '(x) = 0. These students are getting correct answers, but not from a base of understanding. For this class, this problem is not a good test of understanding, since the students seem to be responding with fragments of partially digested course material. If anything, this problem underscores the distinction to be made between understanding and the ability to produce correct answers to selected questions.

4. Implications of this Study

The problems used in this study were specifically chosen for their proximity to the standard calculus curriculum, so that the results of each problem bear upon assumptions made by calculus instructors as they teach particular topics in the subject.

It is difficult to imagine how one could present the notion of a tangent line and its slope as approximated by slopes of secant lines without using some version of the diagram in the Sliding Secant problem. Likewise, it is hard to conceive of a discussion of the Fundamental Theorem of Calculus that is not illustrated by something like the diagram used in the Area Under the Graph problem. The results of this study indicate that these diagrams do NOT carry the meaning for our students that we assume they do. The students can read them in a Pointwise manner, but large numbers of them cannot read them in an Across-Time manner, as the subject demands they do. Of course, it could be argued that an outcome of 50% to 65% of the students getting these problems correct is not all that bad, but the rejoinder is to point out how naive these questions are in relation to those that arise when the slope of the tangent or the area function are actually used in a calculus class. For instance, in the Sliding Secant problem,

A student who is faced with graph sketching or optimization problems who has no such understanding is forced to memorize a series of arcane rules and procedures, which will only move him or her further from the possibility of comprehension of this subject.

one could ask whether or not the slope of the secant line increases without bound, and in the Area Under the Graph problem, one could ask about the behavior of the function ΔA (for a fixed increment of p) as the variable p increases. It seems clear that the results would have been so much worse that a legitimate issue would have arisen as to the fairness of such problems on a final exam.

One of the key issues in the first quarter of calculus is that the students come to understand how the behavior of the derivative f '(x) of a function gives us information about the behavior of the function f(x) itself. A student who is faced with graph sketching or optimization problems who has no such understanding is forced to memorize a series of arcane rules and procedures, which will only move him or her further from the possibility of comprehension of this subject. The Two Speed Graph and Second Derivative problems indicate that large numbers of students have little or no basis for arriving at such an understanding, because under the pressure of making Across-Time

reading of f '(x) and then Across-Time inferences about f(x), they confound these two quantities, thinking part of the time that they are being given the graph of one and part of the time that they are being given the graph of the other.

The implications of this study are not restricted to the particular aspects of the calculus curriculum that the problems refer to. It would not be difficult to write problems related to other crucial topics in calculus that would show just as clearly that students can use functions in a Pointwise manner, but not in an Across-Time manner as the subject demands. For instance, we could begin with almost any related rate problem, show the students the corresponding diagram, and have them indicate specific corresponding numerical values of the variables. What they would not be able to do is tell how change in one of these variables causes change in another, either in the form of how constant increments in one lead to a pattern of increments in the other, or qualitatively, in terms of related rates of change.

Across-Time understanding of functions is critical to an understanding of calculus. This study indicates that most students come to a calculus course neither equipped with it nor on the verge of acquiring it. Specific instruction toward Across-Time understanding is clearly indicated, but this raises genuine questions as to the type of instruction that would be effective; what sorts of activities should the students carry out to be able to draw Across-Time conclusions about a function? I have written, and have used for several years, material that takes one approach to this problem. It is based on a variety of graphs and diagrams at the same level of complexity as the above problems. As with the Two Speed Graphs problem, much of the material requires that students translate between a graph and a concrete context. The material is in the form of instructions and questions that ask students to distinguish and interpret the various kinds of information contained in the graphs and to draw further conclusions from this information. Watching students struggle with this material has convinced me even more that the difficulties students have with Across-Time understanding are real and that one of our tasks as calculus and precalculus instructors is to directly address these difficulties and help our students overcome them.

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Mathematics for Math Majors: Loss of Its Self-esteem

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Many may think that mathematics for math majors presents no serious problem since math majors are supposed to form a highly motivated and capable group who can study mathematics almost on their own. In reality our teaching experience tells us otherwise. We see a wide spectrum of students who are labelled as math majors, at least so labelled for administrative reason because they take a certain number of units of courses in mathematics including such and such compulsory courses in the programme. At one end of the spectrum we are gratified to find a few very motivated students with high calibre and strong background in mathematics. At the other end we are depressed to find a small group of unmotivated and listless students whose sole aim is to obtain a university degree with (less than?) minimal effort. They are not necessarily weak, though they usually end up weak in mathematics. In between, we find a rather large group of students, some quite motivated and some not as motivated, some adequately prepared and some less prepared, but all reasonably willing to put in the effort. It is mainly this middle group that we wish to address here, but certainly what we say applies to the first group as well. There is little we can do about the remaining group, whose attitude is moulded by priority and value judgement formed within a society of which they form a part. But if we do not do our job well, many in the middle group may be driven to the end of the spectrum. (In a small way we can still hope to influence the socio-political scenario of the time!) A student in the middle group may start as one who likes mathematics, at least not one who dislikes or is apprehensive of mathematics, but in the course of time may feel more and more confused, alienated and frustrated. When he was in secondary school he could compute and solve specific problems assigned by the teacher, and derive delight and satisfaction from doing that. Now, he is immersed in concepts which seems neither related to what he has learnt nor suggestive of where he is heading. Faced with this fragmentary picture he may resort to learning by rote. But that usually ends up in disaster. Even if he is lucky and passes the test which happens to ask for knowledge reproducible from rote learning, he is disillusioned. For that does not seem like what an intellectually exciting subject should be. In either case he no longer feels the same affinity to mathematics as he had felt before becoming a math major! (In this article "he", "his" are used in a generic sense. It is tiresome to write "he or she", "his or her" each time instead.) What goes wrong?

Mathematics is too vast and too old (yet forever new) a subject so that when school children learn of achievements of modern sciences in the present century, their mathematics lessons basically cover what had been done up to the sixteenth and seventeenth centuries.

In a poster at ICME-5 of 1984, one of us wrote: "Although mathematics is universally recognized as a most basic, important, useful and encompassing discipline, it is also the least understood, the most misunderstood and the most neglected subject by the public. ... Mathematics is too vast and too old (yet forever new) a subject so that when school children learn of achievements of modern sciences in the present century, their mathematics lessons basically cover what had been done up to the sixteenth and seventeenth centuries. Even at

universities, most students study mathematics that was done up to the beginning of the nineteenth century; only a few math majors may go beyond that. Mathematics gradually acquires a language of its own, which can sound quite obscure to one without that training. It must also be admitted that mathematics demands abstract thinking so that one must put in the requisite amount of time and effort to really understand it. .... Thus, in schools, mathematics teaching tends to emphasize the technical content with the advantage that a reasonable amount of knowledge can be transmitted in the time allotted so that students can learn their language and skill in a reasonably short time. However, in so doing, the cultural aspect is bound to be neglected. Students may be totally unaware that mathematics has its life, that it has a past as well as a future, that it is not just a mess of neatly packed but lifeless formulae and theorems." (M.K. Siu, History of Mathematics Teachers, Mathematical Tall Timbers, Mu Alpha Theta, No. 11 (March 1985), French translation in Bull. de l'Association des Prof. de Math., no. 354 (1985), 309319.) He also wrote in the same article: "What do we mean by a learned teacher? A learned teacher should possess the following qualities: (i) ability, (ii) knowledge, (iii) wisdom. They differ from one another but are closely related, each complementing the other two. An eighteenth century Chinese scholar, Yuan Mei, once said (but in a literary context), 'Knowledge is like the bow, ability like the arrow; but it is wisdom which directs the arrow to the bull's eye.' .... A well-balanced mathematics curriculum should address three aims (i) training of the mind, (ii) transmission of technical knowledge, (iii) awareness of the cultural aspect. If the reader is willing to bear with a looser usage of vaguer terms, we shall characterize the three aims as: (i) ability, (ii) knowledge, (iii) wisdom, which brings us back to our starting point." This viewpoint was further elaborated: "in schools we tend to emphasize the technical content of mathematics and teach it as a skill and a tool. By so doing we attend to 'knowledge' or perhaps even some 'ability', but unfortunately not 'wisdom' as well. ... This may seem effective, at least it seems so from examination performance. In Hong Kong, the passing percentage and average score in mathematics in the Secondary School Leaving Certificate Examination are usually higher than that in other subjects. The recent report of the 2nd International Evaluation of Educational Achievement reveals that Hong Kong students score pretty high marks in mathematics among some twenty countries/districts. But does this really signify success in our mathematics education? Are we complacent about this? I am doubtful of that. I even feel confused, because among the students I have taught only a few show interest in mathematics or perform well, the remaining majority, if not downright abhorrent of mathematics, are indifferent to mathematics. Is it because, in demanding technical ability, we sacrifice other aspects and pay a price for those qualities which cannot be assessed by short-term standard tests? ..... From a broad viewpoint of education, this is a big failure. A.N. Whitehead once said in *The Aims of Education* (1929): 'Culture is activity of thought, and receptiveness to beauty and human feeling. Scraps of information have nothing to do with it. A merely well-informed man is the most useless bore on God's earth. What we should aim at producing is men who possess both culture and expert knowledge in some special direction." (M.K. Siu, Who needs history of mathematics (in Chinese), Shuxue Tongbao (Mathematics Bulletin), 4 (1987), 42-44.)

In short, students lack a global view of the subject of mathematics and rarely appreciate or enjoy it from a cultural aspect, and too little emphasis is put on the subject's own worth. This is rather at variance with the view expressed by great mathematicians since antiquity. Let us just quote that of a great mathematician in our century, Hermann Weyl, who said: "We do not claim for mathematics the prerogative of a Queen of Science, there are other fields which are of the same or even higher importance in education. But mathematics sets the standard of objective truth for all

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intellectual endeavors; science and technology bear witness to its practical usefulness. Besides language and music, it is one of the primary manifestations of the free creative power of the human mind, and it is the universal organ for world-understanding through theoretical construction. Mathematics must therefore remain an essential element of the knowledge and abilities which we have to teach, of the culture we have to transmit, to the next generation" The unsatisfactory situation of mathematics teaching in schools seems to persist at the university level, which is further aggravated by the fact that the subject material becomes more abstract and diversified at that level. Students tend to have a feeling that they are on a "sight-seeing" tour of museum exhibits, only they are not particularly interested as museum-goers! Throughout the undergraduate years for the majority of students, if they do form a good opinion of mathematics, it would be that mathematics is a useful tool. But then, for those who do not need to use that tool that much, what worth is there left of the subject? We share the education philosophy that views "the critical and the evaluative function of education as its central contribution to the intellectual and spiritual development of the student" (Abe Shenitzer, "Some thoughts on the teaching of mathematics," Math. Intelligencer, 8 (1986), 21-24). Abe Shenitzer goes on to say: "Uninterpreted or underinterpreted technical material .... destroys mind and soul, and teaches cynicism and contempt for teacher and subject alike." (A recent article of Abe Shenitzer, titled "An unorthodox "test" which appears in vol. 99 (January, 1992) of Amer. Math. Monthly will give us much food for thought.)

Some may think that as long as students acquire the technical content, it is rather secondary whether students see the cultural value of mathematics or not. We disagree. Looking at the history of our subject we believe that the development of mathematics education and even of mathematics itself is, to a large extent, dictated by the general prevalent "Mathematics Anschauung" of the community at the time and the place. (As an illustration, see a poster display at this Congress: M.K. Siu, Mathematics Education in Ancient China: What Lesson Do We Learn From It?) Emphasizing only the usefulness of mathematics at the expense of its cultural aspect leads to a loss of self-esteem for the subject (and for those math majors who study the subject). As a result, the supply of good teachers and good researchers is dwindling. Furthermore this problem has a snowballing effect, for quite a number of math majors are to become teachers later, in schools and universities. If they lack a general outlook of the subject and fail to realize how the subject is woven into the full tapestry of a general education, then they will produce crops of students who will be exposed to such an outlook and in turn some of these students will become teachers, and so on.

Our teaching aim in mathematics is to let students acquire good study habits and the ability to stand on their own, to teach them to think both logically and heuristically, to give them the room to think, to let them appreciate the beauty and import of mathematics, and more generally to instil in them a regard for learning. This seems far more important than to make sure they know a hundred theorems. (But if our aim is achieved, they will know those hundred theorems as a by-product!) Actually, the same objective has been stressed in other reports, for instance Standard 6 of developing perspectives in "A Call For Change: Recommendations For the Mathematical Preparation of Teachers of Mathematics MAA Report", Mathematical Association of America (editor, J.R.C. Leitzel), 1991 and the five goals in "Curriculum and Evaluation Standards for School Mathematics", National Council of Teachers of Mathematics, 1988. What we wish to add is that these goals are equally important, if not more, for those math majors who go on to become research mathematicians.

In short, students lack a global view of the subject of mathematics and rarely appreciate or enjoy it from a cultural aspect, and too little emphasis is put on the subject's own worth.

What can be done? We believe attention to a general attitude is more important than the subject-content itself, which is generally agreed upon except for details. We believe the curriculum for math majors, for both prospective research mathematicians or mathematics teacher (these two should not be exclusive of each other) should be the same; for as Klein put it in his 1872 Inaugural

Lecture at Erlanger on mathematics education (not to be confused with his more famous "Erlanger Program" which was on geometry and which was written before his appointment; see D.E. Rowe, Felix Klein's Erlanger Antrittsrede, Hist. Math., 12 (1984), 123-141): "We want the future teacher to stand above his subject, that he have a conception of the present state of knowledge in his field, and that he generally be capable of following its further development. ... Although hardly one in ten will later for himself take up scientific research, anyone who completes such a study even once takes with him an altogether different type of certainty in judgement and liveliness of conception." We call this a "mathematical taste", which corresponds to the item "wisdom" mentioned before. In this respect, the "interface" between school mathematics and university mathematics should be paid attention to in all undergraduate courses whenever possible, in the spirit of Klein's "elementary mathematics from an advanced standpoint". Although skills in mathematics are to be acquired, understanding and a global view should be of primary importance. In certain instances such awareness cannot be tested easily, but likewise many time-honoured aims in education cannot be tested either; they show up in the growth of personality. We should encourage the doing, the reading, the writing and the talking of mathematics among students. We should pay attention to maintaining a "continuity" in the courses, i.e. what was learnt in a previous course may pop up in subsequent courses, as part of the content, as examples, or even just as remarks in passing. We should incorporate a "sense of history" into all courses to let students appreciate the long process of evolution and of the formation of key ideas and concepts. (We need not stress further the usefulness of mathematics, not because it is not but because enough has been said about that. Ironically, its usefulness, in a sense, attributes to its neglect as an intellectual discipline!) To regain the self-esteem of mathematics (and mathematics majors) we must convince the audience that mathematics is an intellectually rewarding subject which plays a central role in human culture.

Philosophy of Mathematics, Mathematics Education and Philosophy of Mathematics Education

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As a philosopher of mathematics, I have been thinking about, or rather, worried about the following question: is there any important relationship between the philosophy of mathematics and actual mathematical activities (including mathematical research, teaching and learning)? Or, does the philosophy of mathematics have any important influence on actual mathematical activities? I think the answer is 'yes'; and I have also tried to do some things in this direction by working in the field of methodology of mathematics (cf. Y. Zheng, 1985, 1991a, 1991b). But it is only a personal opinion and has limited influence, so when I came to the USA as a visiting scholar in 1991, this problem was still deeply rooted in my mind. However, what I have learnt in the field of mathematics education in the USA is really a great pleasure for me, as it does show clearly that there is a close relationship between the philosophy of mathematics, mathematics education and mathematics as well: it is modern research in the philosophy of mathematics which offers the necessary ideological foundation for the new reform movement of mathematics education in the USA, and then, in this way, the philosophy of mathematics can exert a great influence on the future of mathematics.

While the advancement of human society, is the most important external impetus, it is the theoretical studies of mathematics education during the past decade which have laid the necessary foundation for the new reform movement.

The first and second parts of this paper will use the modern development of mathematics education in the USA as a background to make an analysis of the great influence of the philosophy of mathematics on mathematics education. The third section discusses the problem of how to develop the subject 'philosophy of mathematics education', which in fact can be regarded as an impetus from mathematics education to the further development of philosophy of mathematics and philosophy in general.

1. New Developments of Mathematics Education

Mathematics education in the USA is now undergoing a new reform movement, to which Everybody Counts, published by the National Research Council in 1990, gives the following description:

"Over the next two decades, the nation's schools, colleges, and universities will undergo major transitions in mathematics programs—transitions that will involve fundamental changes in curricular content, in modes of instruction, in teacher education, in professional development, in methods of assessment, and in public attitudes." (p. 87)

While the advancement of human society, i.e. the transition from the industrial age to the information age, is the most important external impetus, it is the theoretical studies of mathematics education during the past decade which have laid the necessary foundation for the new reform movement. In this section, we make a brief survey of the new theoretical studies. They are chiefly: the emphasis on problem solving, the psychology of mathematical learning, and the social-cultural approach to mathematics education.

(1). The emphasis on problem solving

'Problem Solving' was the main slogan for mathematics education during the eighties. "Problem solving must be the focus of school mathematics" (NCTM, 1980, p. 2); and by

'problem solving', it means 'to use a variety of mathematical knowledge and methods effectively to solve nonroutine problems, including both actual problems and those originated from mathematics itself'.

Putting forward the idea of focusing on problem solving is a giant step for mathematics education, because the idea represents a great shift in the conception of mathematics education, i.e., the idea itself is a direct negation of the traditional conception of mathematics education, especially, the teaching method based on 'transmission of information' and the trend of 'separating learning from application'. To explicate, the key points of 'focusing on problem solving' are as follows: First, students should learn mathematics by the activities of solving problems. That is, "'knowing' mathematics is 'doing' mathematics ....instruction should persistently emphasize 'doing' rather than 'knowing that'." (NCTM, 1989, p. 7) Second, by solving problems, especially those having actual meaning, students can learn to value mathematics, and become more confident in their own mathematical ability. Third, the final aim of mathematics education should be to improve students' ability of problem solving, especially help them learn to think mathematically.

Generally speaking, the idea that problem solving must be the focus of school mathematics is now widely accepted; and as this idea is directly opposite to the traditional conception of mathematics education, it is said that 'solving nonroutine problems is the central theme of the current reform movement in school mathematics.' (T. Romberg, 1991, p. 9)

(2). The emphasis on the psychology of mathematical learning

The study of the psychology of mathematical learning is itself a result of the further development of psychology: it has been beyond the level of general study and penetrated into special fields. Furthermore, where modern studies of the psychology of mathematics learning are concerned, we should pay more attention to the cognitive science approach to mathematics education and "the constructivist view of mathematics learning".

To explicate, the basic position of cognitive psychology is that the study of psychology should not (as behaviorists suggest) be limited to 'visible behavior' but penetrate into the inner information processing of the mind, including the storage, retrieval, representation, development of knowledge and so on. Also,the so-called 'constructivist view' can be regarded as a main conclusion of cognitive psychology: as far as mathematics learning is concerned, it asserts that the learning of mathematics is not a passive reception but a process of construction based on previous experience and knowledge.

If the idea "focusing on problem-solving" is a direct negation of the traditional concept of mathematics education, then the cognitive studies of mathematics learning, especially the constructivist view of mathematics learning, have offered further arguments for this fundamental transition from the microscopic view. And just for

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this reason, the constructivist view on mathematics learning has recently attracted great attentions in the field of mathematics education. For example, as R. Davis, C. Maher and N. Noddings say in Constructivist Views on the Teaching and Learning of Mathematics:

"The idea of "constructivism"—hardly mentioned a few years ago—nowadays attracts a lot of attention in the world of mathematics education. A great many people now think and write about it, and the people who do so do not agree with one another ...Still, beneath the theoretical argumentation, there is a substantial agreement about the nature of learners, the nature of mathematics, and the appropriate form of pedagogy." (R. Davis, C. Maher and N. Noddings 1991, p. 187)

(3). The social-cultural approach to mathematics education

The first implication of the social-cultural study of mathematics is that we should take as a background the whole culture of human society in the study of

mathematics education. This is to say, mathematics education should represent clearly the features of the time. In fact, it is exactly the most important feature of the new reform movement of mathematics education in the USA: it is the transition from the industrial society to the information society which offers the most important impetus to the movement, and the final aim of the movement is to create the kind of mathematics education that not only meets the need of the time but also uses fully the new technology; in a word, we should create the mathematics education of the information age.

Secondly, the social-cultural approach to mathematics education has also made clear the social nature of mathematics learning and teaching. Although the construction of mathematics knowledge should be carried out relatively independently by all individuals, such activities are carried on in some 'social environment', and must include the processes of expressing, communicating, comparing, criticizing, improving and so on, so that it is in fact a 'social construction'. Besides, the social nature of mathematics teaching can be seen clearly by the fact

First, every mathematics teacher is (consciously or unconsciously) doing his work under the influence of some conception of mathematics and mathematics education, and the latter are in fact manifestations of the social nature of mathematics education.

that the role played by teachers is just the 'intermediate' between the whole system of education and the objects of education. In other words, teachers' duty is to carry out the overall aim of mathematics education in facing the concrete students and the concrete situation of teaching in general.

Finally, one more important implication of the social-cultural approach to mathematics education is the importance of conception both to mathematics teaching and learning: First, every mathematics teacher is (consciously or unconsciously) doing his work under the influence of some conception of mathematics and mathematics education, and the latter are in fact manifestations of the social nature of mathematics education. Secondly, as far as students are concerned, the importance of conception lies in the fact that mathematics learning is a process in which not only mathematical knowledge is constructed but also some conception, belief and attitude of mathematics are formed, and the latter in turn will exert great influence on the learners' further study of mathematics and even for their whole life (as a part of their whole ideology). For example, it is just by such consideration that Curriculum and Evaluation Standards for School Mathematics, which is one of the most important documents shaping the new reform movement, lists 'learning to value mathematics' and 'becoming confident of one's own ability' as the first two goals for mathematics education.(pp. 5-6)

Obviously, if the psychology of mathematics learning is the study on the microscopic level, then the social-cultural study belongs to the macroscopic level; and just as J. Kilpatrick points out in his A History of Research in Mathematics Education, 'Researchers were taking the social and cultural dimensions of mathematics education more seriously.' (1992, p. 30)

2. From the philosophy of mathematics to mathematics education

Research in the above three directions as a whole represents a new conception of mathematics education, whose kernel is new ideas about the questions 'what is mathematics' and 'what it means to know mathematics'. At just these points, we can see clearly the important influence exerted by philosophy of mathematics on mathematics education.

To explicate, philosophy of mathematics had for a long time been under the tradition of 'foundational studies'. The common position for all the main schools in the study of mathematics foundations, i.e. logicism, intuitionism and formalism, took mathematics as a body of mathematical knowledge, and it was hoped that, by the logical analysis of the inner structures of mathematical knowledge, they could lay a firm foundation for mathematics so that the problem of the soundness of mathematics could be solved forever.

The researchers of the above three schools had produced many important results. As far as their final aims were concerned, however, they all failed, and as time passed, a big deficiency of the foundational studies has become clear, i.e, it deviates terribly from actual mathematical activities. So, after the period of the 'the golden age' (about 1890-1940), the study of the philosophy of mathematics stagnated.

In the sixties, mainly under the influence of the philosophy of science, some new phenomena appeared in the field of philosophy of mathematics, which in turn represented a transition of the basic positions. The new position was that mathematics should be mainly regarded as creative activities of human beings rather than a specific body of fixed mathematical knowledge. Thus, in comparison with the traditional view of mathematics, the new conception—which may be called 'the human view of mathematics'—contains the following changes:

First, the new view emphasizes the development of mathematics: as creative activities of human beings, mathematics is not something static and fossilized but has been changing all the time and will keep on changing in the future. Particularly, as daily mathematical activities are concerned, they are necessarily complicated processes including conjectures, errors and tests.

Second, the development of mathematics is not only a process of accumulation but also includes qualitative changes. That is, there are also revolutions in mathematics.

Third, the human view of mathematics also confirms that mathematics consists of meaningful activities, so that it should not be identified as the mechanical manipulation of meaningless symbols.

As the human view of mathematics represents a big transition of basic ideas, it has also opened new directions for the study of the philosophy of mathematics.

For example, there is firstly the social-cultural approach to mathematics. To be concrete, mathematicians in modern society are all working in some social environment, and therefore are, in fact, members of 'mathematical communities'. In fact, the working aim for most mathematicians is to get mathematical statements which are representable by the language uniformly accepted by the community, and are resolutions to those problems uniformly regarded as important or significant by the community, and are based on arguments or methods uniformly accepted by the community. (cf. P. Kitcher, 1984) In fact, such a prescriptive role of the mathematical community based on individual mathematicians is just the concrete manifestation of what might be called 'mathematical culture' (in the level of graduate school and mathematics research).

Furthermore, as mathematical researches are social activities, we can therefore study the impetus and laws for the development of mathematics from a higher level. This is to say, we can transcend an individual's work and take the whole human society as a background for the study of the historical development of mathematics. (cf. R. Wilder, 1981) Obviously, such studies denote that philosophy of mathematics has extended from daily mathematical activities to macroscopic studies.

Also, from the microscopic view, mathematical activities are all mental processes. In particular, the creation of all mathematical concepts is a process of construction. To be concrete, mathematical entities are not objects existing in the empirical world but creations of abstraction. Furthermore, in strict research, no matter whether the entities concerned have or do not have empirical backgrounds, we cannot rely on intuition but on deduction from the corresponding definitions. Therefore, the process of mathematical abstraction is, in fact, an activity of

as creative activities of human beings, mathematics is not something static and fossilized but has been changing all the time and will keep on changing in the future.

construction. That is, mathematical entities are constructed by the corresponding definitions (including explicit and implicit definitions), and only by those processes of "logical construction" can the corresponding mathematical entities be transferred from 'the inner creations of the mind' to 'the outer independent existence'. (cf Y. Zheng, 1991b.) Furthermore, because mathematical entities are not objects in the empirical world, the study of mathematical entities must include a

process of 're-creation' (in comparison with 'the primary creation'). That is, people must actually construct the corresponding mathematical entities in the mind, so that what had been 'objectified' with the aid of language can be transferred back into 'inner elements of the mind'.

Putting together the above discussion of the modern developments of mathematics education and of the philosophy of mathematics, we can see clearly that it is modern research in the philosophy of mathematics which has offered the important ideological foundation for the new reform movement of mathematics education in the USA. For example, the emphasis on problem solving is obviously a necessary consequence of the human view of mathematics. In fact, an important starting

the distinct feature of the 'new math' was that little attention was paid to the actual cognitive processes, of how human beings think about mathematics.

point of the new reform movement of mathematics education is just the recognition that school mathematics under the old tradition is not 'real mathematics', and the idea of 'focusing on problem solving' in turn is to put students in the same situation as mathematicians. T. Romberg says on this point:

'For over two thousand years, mathematics has been viewed as a body of infallible truth far removed from the affairs and values of humanity. These views are being challenged by a growing number of philosophers of mathematics....Such a dynamic view of mathematics has powerful educational consequences. The aims of teaching mathematics need to include the empowerment of learners to create their own mathematical knowledge; ....When mathematics is seen in this way, it needs to be studied in living contexts that are meaningful and relevant to the learners, including their languages, cultures, and everyday lives, as well as their schoolbased experiences.' (T. Romberg, 1992, p. 751)

Secondly, although 'constructivist' is a new terminology in the world of mathematics education, it is quite familiar to philosophers of mathematics. Therefore, the 'rise' of the constructivist view on mathematical learning and teaching can be regarded as an extension or transition from philosophy of mathematics to mathematics education. What should also be noted is that, mathematics educators have found important illuminations for instruction from modern studies of the philosophy of mathematics and philosophy of science in general. For example, based on the discussion about scientific revolutions, especially about the transition of 'paradigm' in philosophy of science, some mathematics educators suggest that forming 'conceptual conflict' is a requisite and efficient way for promoting students' mathematical thoughts, especially for the correction of their wrong ideas.

Finally, the social cultural approach to mathematics education obviously corresponds directly to the social cultural studies of mathematics. For example, in correspondence with the concept of 'mathematical community', mathematics educators have introduced the concept of 'mathematics education community', which consists of mathematics teachers, mathematics education researchers, directors for mathematics teacher's training, supervisors for mathematics curriculum, makers of policies for mathematics education, designers of mathematics examinations and so on, and the main feature of a mathematics education community is also that all its members share (consciously or unconsciously) somewhat the same conception of mathematics education.

In addition to the above discussion, what should be noted is that we can analyze the relationship between the philosophy of mathematics and mathematics education in a more general sense. For example, the traditional conception of mathematics education reflects to a great extent the 'absolute view of mathematics' (we should also see the influence of mechanism here). Besides, it is the foundational study mentioned above that offers the necessary ideological foundation for the 'new math movement', which was seen throughout all the western countries during the sixties. In fact, the distinct feature of the 'new math' was the emphasis on the logical structures of mathematical knowledge and little attention was paid to the actual cognitive processes, of how human beings think about mathematics. We can see here very clearly the influence of the 'foundationists'. The French mathematician R. Thom, while commenting on 'new math', clearly raised the following question:

'"Modern" Mathematics: an Educational and Philosophical Error?' And as an answer, he says, for example,

'Set theory....is the essential litany intoned by those who advocate the so-called modern mathematics. Some affirm that the use of set theory permits the entire renovation of mathematics teaching and that, thanks to this change, the average student will be able to achieve mastery of the curriculum. Needless to say, this is pure illusion ...Everything considered, the excessive optimism bred by the use of set theory symbol has its roots in a philosophical error.' (Thom, 1971, p. 75)

To summarize, we should definitely confirm that there is an important relationship between the philosophy of mathematics and mathematics education; and then, via mathematics education, philosophy of mathematics will exert great influence on the future of mathematics.

3. Towards a philosophy of mathematics education

The above discussion shows clearly the important relationship between the philosophy of mathematics and mathematics education; however, at the same time we should not identify the philosophy of mathematics with the theoretical foundation of mathematics education. In other words, mathematics education should have its own relatively independent theoretical foundation.

In fact, every subject has its own history during which it has formed its special field, problems and theories. With this view, we can see clearly the differences between the philosophy of mathematics and the theoretical foundation of mathematics education:

On the one hand, the philosophy of mathematics, as philosophical analysis of mathematics, has its special problems. In fact, in comparison with those problems mentioned above, the ontology and epistemology of mathematics are of a more basic nature. The ontology of mathematics can be described as: do mathematical entities have an independent existence? If the answer is 'yes', then what kind of existence is it; if the answer is 'no', what is the meaning of mathematics? On the other hand, the epistemology of mathematics is focusing on whether mathematical statements are a priori or empirical. The fact that the ontology and epistemology of mathematics have occupied important positions in the philosophy of mathematics is a natural result of the speciality of mathematics, especially its abstractness (the speciality of mathematics lies not only in the contents of mathematical abstraction, but also in its degree and method. cf. Y. Zheng, 1991). And just for this reason, although there have been some new directions in the field of the philosophy of mathematics since the sixties, any systematic theory in the philosophy of mathematics still has to give definite answers (or detailed analysis) to the ontology and epistemology of mathematics. In fact, just as P. Benacerraf points out in his paper Mathematical Truth, the difficulty in the study of philosophy of mathematics just lies in 'the dilemma of the ontology and epistemology of mathematics'. This is to say, those theories which are satisfactory in the ontology always have serious deficits in the epistemology; while the others which are satisfactory in epistemology always have deficits in the ontology. However, all these discussions do not seem to have important implications for mathematics education.(P. Benacerraf, 1983)

On the other hand, the theoretical foundation of mathematics education obviously should include the following contents:

(1) **The View of Mathematics.** This is the answer to the question 'what is mathematics'. It should include not only analysis about the relationship between objective mathematical knowledge and the creative activities of the human, but also an explication of the subject (and nature) of mathematics. For example, according to the modern view, mathematics should be defined as 'the science of patterns' (cf. L. Steen, 1988 and Y. Zheng, 1991), and this definition seems to be a confirmation of the duality of mathematics, i.e., it is both descriptive and prescriptive.

(2) **The Analysis of the Nature of Mathematical Learning.** Differing from the study of epistemology in the philosophy of mathematics, the final aim of the analysis of mathematical learning is not to get a definite conclusion about the a priori and empirical nature of mathematical statements but rather to study the actual information process of the mind and explicate its implications for mathematics education. Therefore, the key question here is

whether mathematics learning is a process of 'transmission of information' centering on teachers or an activity of discovery (re-creation) by students. Besides, from the social-cultural view, there is also the question whether mathematics education is an isolated activity or an organic part of the whole cultural system of the human.

(3) The Aim of Mathematics Education. As a conscious activity of humans, mathematics education has its definite aim which should reflect the features of the time, i.e., it should meet the needs of the time and reflect the advance of science and technology. Particularly, we should analyze carefully the great influence on mathematics education exerted by the transition from the industrial age to the information age and the rapid development of computer technology. For example, as the information age is in some sense 'the age of mathematizing', the development of the society has made a higher standard for every student an historical necessity for mathematics education (cf. NRC, 1990, 1991). In addition, the rapid development of computer technology has not only offered efficient tools but also opened a new prospect for mathematics education. For example, with the help of computers, people can really be freed from the influence of the traditional conception of mathematics education that emphasizes very routine skills, and then concentrate on the promotion of the students' ability in problem-solving.

By the above discussion, we can now see clearly that there are both some important relationship and differences between the philosophy of mathematics and the theoretical foundation of mathematics education. What is more, it is obvious that we should also differentiate naive conceptions of mathematics education from systematic theories. Therefore, there is a deep need to introduce the concept 'philosophy of mathematics education'. To be explicit, philosophy of mathematics education consists mainly of the following contents: the view of mathematics, the analysis of the nature of mathematics learning (and teaching), and the aim of mathematics education; and as a whole it forms the theoretical foundation for mathematics education.

To make things clearer, we are going to make a brief introduction and comment on the most popular view of mathematics education in China. According to this view, the theory of mathematics education mainly consists of the following three parts: the theory of mathematics curriculum, the theory of mathematics teaching, and the theory of mathematics learning. 'The theoretical foundations of the theory of mathematics education include the following subjects: dialectical materialism (philosophy), mathematics, education, psychology, logic, and computer science.' (Cao Cai-han, 1989, p. 9) So, the basic framework of this theory of mathematics education is as follows:

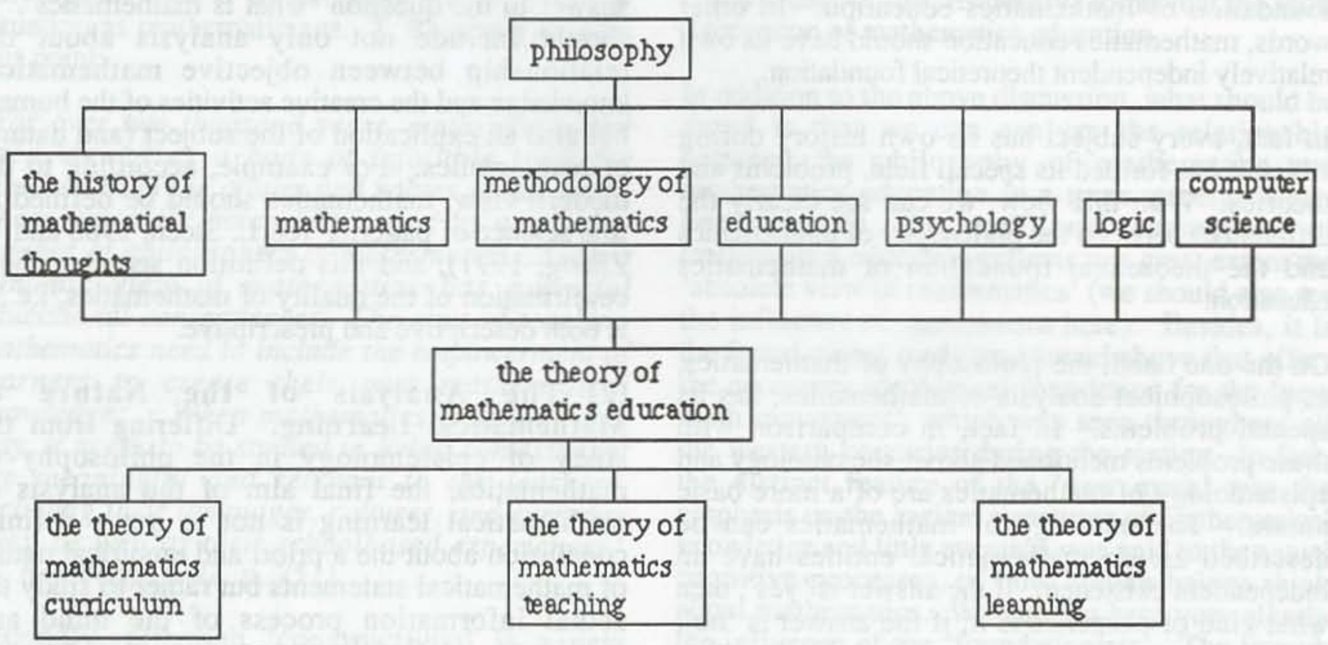

Figure 1

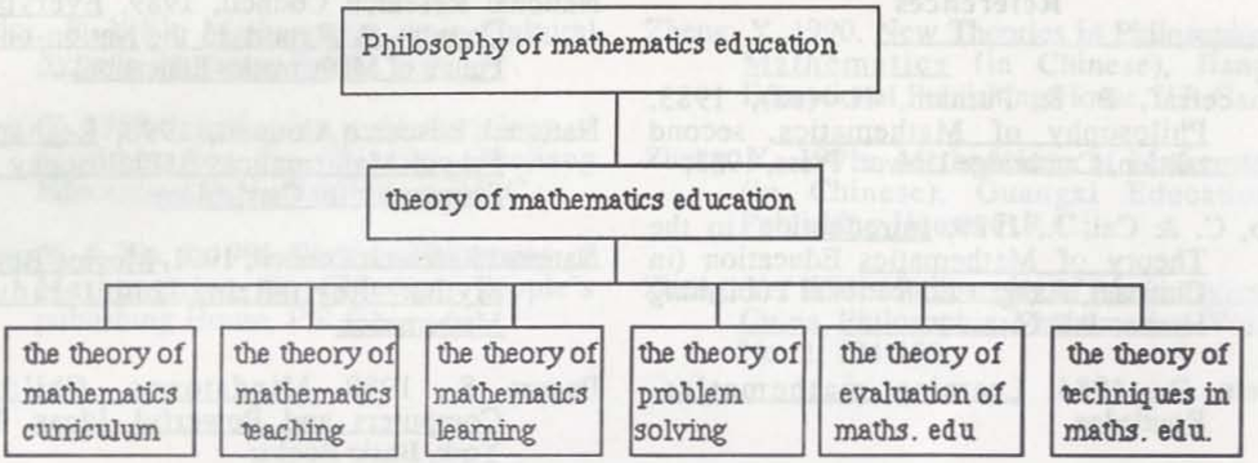

Figure 2

However, we now know clearly that the analysis of the theoretical foundations of mathematics education should not be limited to listing all the relevant subjects; instead, we should set up its own theoretical foundation, i.e., the philosophy of mathematics education. Therefore, we are, in fact, introducing the following new theoretical framework for mathematics education (in which we have also made some improvement and extension of the contents of the theory of mathematics education; however, it goes beyond the topic of this paper), as shown above.

Finally, what should be emphasized is that, although there is already some preliminary work in this direction (c.f., P. Ernest, 1990), philosophy of mathematics education is still a new field waiting for further studies. We can see by the above discussion that the founding of a systematic theory of philosophy of mathematics education needs cooperation between philosophers and educators. Actually, the most important thing is to introspect one's own concept of mathematics education, so as to transfer from the old, backward conception to the advanced and scientific conception of mathematics education. In fact, just as Everybody Counts, which is another important document for the new reform of mathematics education in the USA, points out, the following transitions 'will dominate the process of change during the remainder of this century':

Transition 1: The focus of school mathematics is shifting from a dualism mission—minimal mathematics for the majority, advanced mathematics for a few—to a singular focus on a significant common core of mathematics for all students.

Transition 2: The teaching of mathematics is shifting from an authoritarian model based on "transmission of knowledge" to a student centered practice featuring "stimulation of learning."

Transition 4: The teaching of Mathematics is shifting from preoccupation with inculcating routine skills to developing broad based mathematical power. (p. 81-82)

These transitions, of course, can not be carried out spontaneously in practice; just the opposite, 'naiveness' in philosophy (one frequent form of 'naiveness' is the ignoring of philosophy) always leads people to become slaves of some 'modern', but at the same time 'bad' philosophy. For example, what is called 'the radical constructivist view', which seems to be a 'modern fashion' in the world of mathematics education in the USA, is, in fact, a revision of intuitionism in the philosophy of mathematics. And as intuitionism necessarily leads to mathematical mysticism' and 'mathematical solipsism' by its denial of the representability and objectivity of mathematics, this philosophical view has already been widely criticized. Obviously, it shows more clearly the importance of the study of philosophy of mathematics education; and it in turn can also be regarded as an impetus for mathematics education to the further development of the philosophy of mathematics and philosophy in general.

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Poems by Lee Goldstein

Plighted Symbolism

Through the credential nonverbality
Are theorems of that that is of the not benamed abstraction,
And where the verbality, thence the symbolism seems may be to willing,
While the masters of the symbolism
Know that unwillingness and yet can act them;
For the locus of what is not benamed
Bodes mathematical,
And such constructivism is believed
In this very alterity.

1993

The Imagination

The infiniteth inbeing of desire expressed objectively,
For instance, 'the set of all sets which do not include themselves',
Implies an ineluctable phenomenon
That precludes mental escape,
Unless there is admitted the glamourous search
Of the not at the object,
But of a living, instead, past the paradoxes implicit in desired (or undesired) objects
Where truthful objectedness arises, identically,
Only upon a nonce imagination of the "things ideal."

1993

Space Venture

Edward E. Chipman
830 N. Shore Dr. NE
Palm Shrs., 319
St. Petersburg, FL 33701

Into the vast mystery of the universe
Man does probe with man-made star,
Thrusting into heavenly course
Toward some goal unknown, afar.
Mighty is the venturing deed,
And the whole world stands in awe;
Yet he moves by spirit's need,
Still he follows heaven's law.

Need of striving in his soul,
Moving outward toward the dark;
Striving toward a beckoning goal
Gendered by the spirit's spark.
Restless from the earth he moves,
Searching out the heavenly seas.
Among far planets now he roves;
More than flesh he seeks to please.

This his nature, given of God:
To search out the unknown star,
Discontent to lag and plod
Where the sated pleasures are.
His a nature heaven-endowed,
Made for searching and far sight;
Knowing to himself is owed,
And to God, the spirit's flight.

Into the vast mystery of the universe
Man does probe with man-made star,
Thrusting into heavenly course
Toward some goal unknown, afar.
Fray he, as he serves heaven's law
In implicit destiny,
Never shall he lose the awe,
Wonder of the majesty

That endows him with the power
To thrust forth to farther space.
Pray on earth he may explore
With the same intensity
Brotherhood, and friendship's dower,
Love's unplumbed immensity:
And as he moves from more to more,
Gain God's peace, and inner grace

7-10-69

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