Journal of Humanistic Mathematics

Volume 5 | Issue 2 July 2015

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Recommended Citation

Milos Savic, "On Similarities and Differences Between Proving and Problem Solving," *Journal of Humanistic Mathematics*, Volume 5 Issue 2 (July 2015), pages 60-89. DOI: 10.5642/jhummath.201502.06. Available at: https://scholarship.claremont.edu/jhm/vol5/iss2/6

Cover Page Footnote
I would like to thank the participants of the study who volunteered their time. I would also like to thank the reviewers of the manuscript for their suggestions. Finally, I thank my advisors, Drs. Annie and John Selden, whom made numerous contributions to the study.

On Similarities and Differences between Proving and Problem Solving

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Abstract

A link between proving and problem solving has been established in the literature [5, 21]. In this paper, I discuss similarities and differences between proving and problem solving using the *Multidimensional Problem-Solving Framework* created by Carlson and Bloom [2] with *Livescribe* pen data from a previous study [13]. I focus on two participants' proving processes: Dr. G, a topologist, and L, a mathematics graduate student. Many similarities between the framework and the proving processes of Dr. G and L were revealed, but there were also some differences. In addition, there were some distinct differences between the proving actions of the mathematician and that of the graduate student. This study suggests the feasibility of an expanded framework for the proving process that can encompass both the similarities and the differences found.

Keywords: *undergraduate mathematics education; problem-solving; proving; proof construction*

Proof and proving are central to advanced undergraduate and graduate mathematics courses. Yet there is often little systematic discussion in these courses on how proofs are constructed. Since proving and problem solving overlap [5, 21], one might look at the problem-solving literature in order to describe some aspects of the proof construction process. Here I use the *Multidimensional Problem-Solving Framework* created by Carlson and Bloom [2], coupled with a data collection technique [13] specifically aimed at collecting the real-time actions that a prover takes, to examine the proving actions of a topologist, Dr. G, and a mathematics graduate student, L. I discuss the adequacies and limitations of this framework for describing the observed proof construction processes. Using the framework, I also discuss the noticeable differences between Dr. G and L's proof construction actions. After that, I propose some educational strategies that might be useful for making explicit some specific actions taken during the proof construction process that do not appear in the final written proof. Finally, I suggest directions for future studies that can extend this research.

1. Background

Selden, McKee, and Selden [17] noted that proving "play[s] a significant role in both learning and teaching many tertiary mathematical topics, such as abstract algebra or real analysis" (page 128). In addition, professors teaching upper-division undergraduate mathematics courses often ask students to produce original (to them) proofs to assess understanding. When trying to do so, some naïve students might not know where to start or how to handle the proof construction process, thus they may not develop one or multiple proving "schemes" or how "students ascertain for themselves and persuade others of the truth of a mathematical observation" [7, page 243]. They may not grasp the complexity of carefully writing a mathematical proof. For example, students' proving might employ picture "proofs" or use empirical reasoning [4], which is an example of an *empirical proof scheme* [7]. While either approach may be a good start on proving a certain theorem, students that only use empirical proof schemes may need to understand and utilize more proof schemes. Researchers have concluded that both aspiring and current mathematicians seem to need flexibility in their proving styles in order to be successful in mathematics [9, 20].

One analytical tool to assist students in constructing a proof, designed by Selden and Selden [18], focuses on the problem-solving aspect; other aspects, according to the authors, can be enacted automatically in proving with sufficient practice. They describe two aspects of a written proof, the formal-rhetorical part and the problem-centered part. They write:

> *The formal-rhetorical part of a proof (what we have also referred as the proof framework) is the part of a proof that depends only on unpacking and using the logical structure of the statement of the theorem, associated definitions, and earlier results . . . The remaining part of a proof* [is] *the problem-centered part . . . that*

> *does depend on genuine problem solving, intuition, and a deeper understanding of the concepts involved* [16, pages 308–309].

The word "problem" as used in the problem-centered part is meant in the sense of Schoenfeld [15], who stated that a problem is a mathematical task for an individual if that person does not already know a method of solution for that task. Past mathematics education research has indicated connections between proving and problem solving. For example, Furinghetti and Morselli [5] stated that "proof is considered as a special case of problem solving" (page 71). Also, Weber [21] considered "proof from an alternative perspective, viewing proof construction as a problem-solving task" (page 351).

In the mathematics education literature, there are several other analytical tools concerning the proof construction process, including "proof schemes" (students' ways of "ascertain[ing] for themselves or persuad[ing] others of the truth of a mathematical observation" [7, page 243]), affect and behavioral schemas (i.e., habits of mind that further proof production) [3, 5, 17] and semantic or syntactic proof productions [22]; see [1] for a brief overview. Many of these tools do not specifically consider temporal order. I wish to use a problem-solving framework that utilizes not only how, but also when a participant acts in the proof construction process.

Mathematicians themselves first examined problem solving in mathematics. For example, Hadamard [6] investigated what other mathematicians did in their research, including how they approached a problem or proof in their own research. He cited Poincaré's problem-solving insight when stepping on a bus (for further details, see [11]). Pólya [12] also described many ways to go about problem solving that he summarized into four overarching steps: "(i) understanding the problem, (ii) developing a plan, (iii) carrying out the plan, and (iv) looking back." In the mathematics education literature, Schoenfeld [14], somewhat influenced by Pólya, described six processes when engaging in a problem-solving activity: "read, analyze, explore, plan, implement, and verify" (page 61). He then analyzed several students' attempts at problem solving using these six processes. Carlson and Bloom [2] utilized both Pólya's and Schoenfeld's ideas in creating their *Multidimensional Problem-Solving Framework*, which I describe in the next section.

2. The Multidimensional Problem-Solving Framework

The *Multidimensional Problem-Solving Framework* described by Carlson and Bloom [2] has four phases, each with the same four associated problem-solving attributes. The four phases are *orienting*, *planning*, *executing*, and *checking*. The four associated problem-solving attributes are *resources*, *heuristics*, *affect*, and *monitoring* (see Table 1 below for details).

Below I describe each phase, as well as the problem solving attributes associated with each phase. These phases and attributes described by Carlson and Bloom emerged during their analysis of the problem-solving processes of eight research mathematicians and four mathematics or mathematics education Ph.D. candidates. One of the problems posed in their study was

> *Problem 1: A square piece of paper ABCD is white on the front side and black on the back side and has an area of 3 in.2 Corner A is folded over to point A' which lies on the diagonal AC such that the total visible area is ? white and ? black. How far is A' from the fold line?* [2, page 71].

TABLE 1

Phase *Behavior*	**Resources**	**Heuristics**	**Affect**	**Monitoring**
Orienting *Sense making, Organizing, Constructing*	Mathematical concepts, facts and algorithms were accessed when attempting to make sense of the problem. The solver also scanned her/his knowledge.	The solver often drew pictures, labeled unknowns and classified the problem. (Solvers were sometimes observed saying, "this is an X kind of problem.")	Motivation to make sense of the problem was influenced by their strong curiosity and high interest. High confidence was consistently exhibited, as was strong mathematical integrity.	Self-talk and reflective behaviors helped to keep their minds engaged. The solvers were observed asking, "What does this mean?"; "How should I represent this?"; "What does this look like?"

Planning *Conjecturing, Imagining, Evaluating*	Conceptual knowledge and facts were accessed to construct conjectures and make informed decisions about strategies and approaches.	Specific computational heuristics and geometric relationships were accessed and considered when determining a solution approach.	Beliefs about the methods of mathematics and one's abilities influence the conjectures and decisions. Signs of intimacy, anxiety, and frustration were also displayed.	Solvers reflected on the effectiveness of their strategies and plans. They frequently asked themselves questions such as, "Will this take me to where I want to go?", "How efficient will Approach X be?"
Executing *Computing, Constructing*	Conceptual knowledge, facts, and algorithms were accessed when executing. Without conceptual knowledge, monitoring of constructions was misguided.	Fluency with a wide repertoire of heuristics, algorithms, and computational approaches were needed for the efficient execution of a solution.	Intimacy with the problem, integrity in constructions, frustration, joy, defense mechanisms and concern for aesthetic solutions emerged in the context of constructing and computing.	Conceptual understandings and numerical intuitions were employed to reflect on the sensibility of the solution progress and products when constructing solution statements.
Checking *Verifying, Decision Making*	Resources, including well-connected conceptual knowledge informed the solver as to the reasonableness or correctness of the solution attained.	Computational and algorithmic shortcuts were used to verify the correctness of the answers and to ascertain the reasonableness of the computations.	As with other phases, many affective behaviors were displayed. It is at the phase that frustration sometimes overwhelms the solver.	Reflections on the efficiency, correctness, and aesthetic quality of the solution provided useful feedback to the solver.

Table 1: Carlson and Bloom's *Multidimensional Problem-Solving Framework* [2, page 67].

2.1. Orienting

According to Carlson and Bloom [2][1], the *orienting* phase includes "the predominant behaviors of sense-making, organizing and constructing" (page 62). Examples of this phase included *defining unknowns*, *sketching a graph*, or *constructing a table*. They stated that an individual might execute these orienting actions with "intense cognitive engagement," ultimately understanding the nature of the problem. Use of *resources* in the orienting phase can include accessing mathematical concepts, facts, and algorithms. Use of *heuristics* in the orienting phase can include drawing pictures, labeling unknowns, and classifying the problem. *Affect* experienced during the orienting phase can include motivation to make sense of the problem, high confidence, and strong mathematical integrity. Finally, use of *monitoring* in the orienting phase can include self-talk and other reflective behaviors during sense making, such as asking, "What does this mean?"

2.2. Planning

Carlson and Bloom coded a *planning* phase in a transcript when a participant "appeared to contemplate various solution approaches by imaging the playing-out of each approach, while considering the use of various strategies and tools" (pages 62-63). In addition, they often observed a subcycle of (a) conjecturing a solution, (b) imagining what would happen using the conjectured solution, and (c) evaluating the validity of that solution during planning phases. In Carlson and Bloom's analysis, their participants could exhibit this subcycle either verbally or silently, but the entire planning phase occurred before the *executing* phase commenced. *Resources* used during the planning phase included conceptual knowledge and other facts needed to construct conjectures. *Heuristics* used, if visible to the researchers, included computations and geometric relationships. *Affect* exhibited by participants during the planning phase included beliefs about the methods or conjectures being employed and about their own abilities to solve the current problem. *Monitoring* exhibited by Carlson and Bloom's participants during the planning phase included self-reflection about the effectiveness of their current strategies.

[1] From here onwards all references to Carlson and Bloom are to [2].

2.3. Executing

Carlson and Bloom noted that the *executing* phase involved "mathematicians predominantly engaged in behaviors that involved making constructions and carrying out computations" (page 63). Specific examples included "writing logically connected mathematical statements," using concepts, facts, and procedures. *Resources* used were the same kind of concepts, facts, and procedures that had been used during the prior planning phase. *Heuristics* used during the execution of the solution included fluency with the algorithms and approaches employed. *Affect* exhibited in the executing process involved some emotional responses to the attempted solution, such as "frustration, joy, defense mechanisms, and aesthetics in the solution" (page 67). *Monitoring* involved the participants having some sensitivity to the progress of their solutions.

2.4. Checking

The *checking* phase was observed when the participants verified their solutions. These behaviors included "spoken reflections by the participants about the reasonableness of the solution and written computations ... contemplating whether to accept the result and move to the next phase of the solution, or reject the result and cycle back" [page 63]. *Resources* used during the checking phase involved "well-connected conceptual knowledge" for the "reasonableness" of their solutions. *Heuristics* used included knowledge of "conceptual and algorithmic shortcuts." *Affect* during the checking phase was similar to other affective behaviors, but frustration might overtake a participant if the solution seemed incorrect. *Monitoring* during this phase involved thinking about the "efficiency, correctness, and aesthetic quality of the solution" [page 63].

2.5. The cycle of problem solving

Carlson and Bloom stated: "it is important to note that the mathematicians rarely solved a problem by working through it in linear fashion. These experienced problem solvers typically cycled through the plan-execute-check cycle multiple times when attempting one problem" (page 63). They also noted that the cycle had an explicit *execution* phase, usually in writing, and formal *checking* that used computations and calculations that were also in writing. All cues exhibited by the participants and observed by the researchers in task-based interviews, whether written, verbal, or non-verbal, were used to distinguish between phases.

3. Research Questions

In this work I approach proof construction as a subset of problem solving, and as a consequence, I use the Carlson and Bloom framework to address the following research questions:

- What are some of the differences and similarities between proof construction and problem solving?
- What are some of the differences in proof construction actions between an intermediate prover (graduate student) and an expert prover (mathematician)?

4. Research Setting

One topologist, Dr. G, and one graduate student, L, were given a set of notes on semigroups and a *Livescribe* pen and paper, capable of capturing both audio and real-time writing using a small camera near the end of the ballpoint pen. These two were participants in a larger study of nine mathematicians and five graduate students [13], who were asked to answer two questions, provide seven examples, and prove thirteen theorems using the same notes. Use of this equipment was intended to allow the participants of the study to feel comfortable in order to capture naturalistic proof construction activities. This was done in response to a challenge posed by Liljedahl [10] to develop:

> "*A tracking method that allows for the accurate capture of the problem solving process while at the same time not restricting the participant's sincere engagement in the problemtrue problem solving requires that the participants be given lots of time and space to engage, rest, and reengage with the problem*" (page 203).

From the first use of the *Livescribe* equipment for proving or answering tasks in the notes until the last minute of equipment use, Dr. G logged five hours and 31 minutes, while L logged three days, 22 hours, and 11 minutes. These total times included all breaks, as the participants did not continuously attempt to answer the tasks in the notes. I focused the coding of the proof construction processes on Theorem 20: "*A commutative semigroup with no proper ideals is a group.*" I chose to concentrate on the proving of this theorem because most of the mathematicians and graduate students in the study did not seem significantly challenged by the earlier exercises, while this one turned out to be sufficiently challenging.

Theorem 20, which can be restated as "*If S is a commutative semigroup with no proper ideals, then S is a group*", is usually proved by showing that S has the two properties that distinguish a group from a semigroup: existence of an identity and inverses for every element of S. Since S is a semigroup, S is not an empty set; hence there exists an element $s \in S$. A common proof relies on showing that sS is an ideal, and since S has no proper ideals, $sS = S$. From this set equality, a prover can manipulate the resulting element equations. For example, $se = s$ for some $e \in S$. Hence, e acts like an identity for the element s, but a prover must still prove that e is the identity for all elements of S. Also, to prove inverses exist for all elements, one can see that given $e \in S$ is the identity element, then there is a $t \in S$ such that $st = e$, and so $s^{-1} = t$, and the proof is concluded.

From the first use of the *Livescribe* pen for their proof attempts of Theorem 20 until the last, Dr. G spent three hours and 17 minutes, while L spent just 41 minutes. Dr. G also attempted to prove Theorem 21 (*If K is a minimal ideal in a commutative semigroup S, then K is a group*) and answer Question 22 about isomorphisms of groups. I considered those portions as part of Dr. G's proving process of Theorem 20 due to the fact that moving on did not necessarily mean that Dr. G had given up or stopped subconsciously working on the proof.

I selected Dr. G's data because he spoke a significant amount of the time while proving, and he also encountered impasses when proving this theorem. I chose L's data because he was one of only two graduate students who attempted a proof of this theorem, and his was the only graduate student proof that could be analyzed using the Carlson and Bloom framework. The audio/video recordings were transcribed so that the audio and the actions on the *Livescribe* paper corresponded. All abbreviations written in the reported data are copied from participants' writing on the *Livescribe* paper. Once the proving sessions were transcribed, coding was done using the Carlson and Bloom framework. A sample of the transcription, with coding, can be seen in Table 2 below.

	Writing	Speaking	Attribute	Phase
(1)	Th 20: A comm semigp w/ no proper ideals is a gp. (1 minute pause)		Resources	Orienting
(2)	Hmm ... I'm taking a break, breakfast, etc. Back to this later. Must think on this.		Monitoring	Planning
BREAK 7:04 AM – 8:07 AM				Planning
(3)	Ok, I thought about this while on a cold walk in the fog.		Heuristics	Planning
(4)	Pf: Given $g \in S$, a semigp., consider the ideal g		Resources, Heuristics	Executing
(5)	(Then he stops and puts "comm." between "a" and "semigp.")		Monitoring	Checking
(6)	$gS = \{gs \mid s \in S\}$. Since S has no proper ideals, $gS = S$, so $\exists g^{-1} \in S \ni gg^{-1} =$		Resources, Heuristics	Executing
(7)	(32 second pause, then he strikes through the whole proof)		Monitoring	Checking
(8)	First need an identity, not given. (Then he goes back to the expression "$gg^{-1} =$" and writes a question mark with a circle around it.) Turn page.		Resources, Monitoring	Planning

Table 2: Sample coding of the proof construction process of Dr. G.

4.1. A Description of the Coding of the Sample

At 7:02 AM Dr. G started the proof construction process of Theorem 20; he had proved the theorems in the rest of the notes in the two hours prior to this first attempt. He wrote the theorem on paper (line 1), probably orienting himself to what he needed to prove. There was a one-minute pause, during which I infer that he was *orienting* himself to the theorem. Since he had proved Theorems 13-19 of the notes quite quickly, his decision to take a walk at 7:04 AM might have been because he had not quickly seen how to attempt this proof. I assume that during the walk he might have been *planning* how to prove Theorem 20, probably using the *conjecture-imagine-evaluate* subcycle mentioned above. At 8:07 AM, he started *executing* the idea that he had generated during the walk (line 3). He corrected his work by inserting "comm." to be precise, something that I coded as *checking* and *monitoring* (line 5). Then Dr. G went back to *executing* his idea, using an element, g, in the semigroup and multiplying it by the whole semigroup to create an ideal (line 6). There was a 32-second pause, and then he crossed out the entire proof that he had just written (line 7). This was coded as *checking*. In fact, at 8:09 AM, he wrote why he crossed out this (first) proof attempt:] he needed an identity, which had not been given (line 8). I coded this as *planning* (*resources* and *monitoring*), because Dr. G apparently used what he knew about groups to *check* this attempt and apparently recognized what he needed to have for a successful proof. Dr. G and L's full transcripts were coded in a similar manner using a decision process that inferred certain proof-construction situations. Samples of both transcripts are located in Appendices A and B.

4.2. Coding for reliability

For reliability, I asked two other colleagues to code three small excerpts of the transcripts using the Carlson and Bloom framework. There were certain segments that merited discussion, and we came to an agreement in all of those instances. Using their assistance, I then established a set of conventions to help refine the coding process. They were as follows:

A. Both participants had instances in their proof construction sessions that were pauses in their work. I defined a *pause* as a period of time (at least ten seconds) in the live data sessions during which the prover did not speak or write. I had asked the participants to prove the theorems at their own leisure with unlimited time, so I was not present to ask them contemporaneously about pauses in their proving. If a participant made corrections immediately after a pause, then I would code the pause as *checking*. If after a pause, the participant had an idea or could continue his progress, then I would code the pause as *planning* prior to the *executing* phase. I also coded participants' pauses based on what I thought a participant was accomplishing, using my own inferences about their proof construction process.

a. When a participant turned off the *Livescribe* pen and later turned it back on, I considered that a *break*. All breaks were considered *planning*. This is because almost immediately after a participant turned on the pen after a break, he or she had an idea to try, which is accounted for in the Carlson and Bloom framework as *executing*. For example, in Dr. G's transcript (Table 1), the break from 7:04 AM–8:07 AM was coded as *planning*.
b. Many pauses during the proof construction process were considered *planning*, because of the *executing* phase that occurred immediately afterwards.

B. Speaking was never considered *executing*. Any phase coded as *executing* occurred within the participants' written work, and was only coded this way when it furthered (either correctly or incorrectly) their attempted proof.

C. Any crossing out or elimination of any part of the *executing* phase was considered *checking*. An example of this, which occurred on line 7 in Dr. G's transcript, can be found in Figure 1.

D. Many problem-solving attributes were difficult to code, since, unlike Carlson and Bloom in their study, I was not present when the participant was proving. *Affect*, in particular, was difficult to gauge. I attempted to associate the attribute that best matches what I heard from the audio recordings and saw on the *Livescribe* paper.

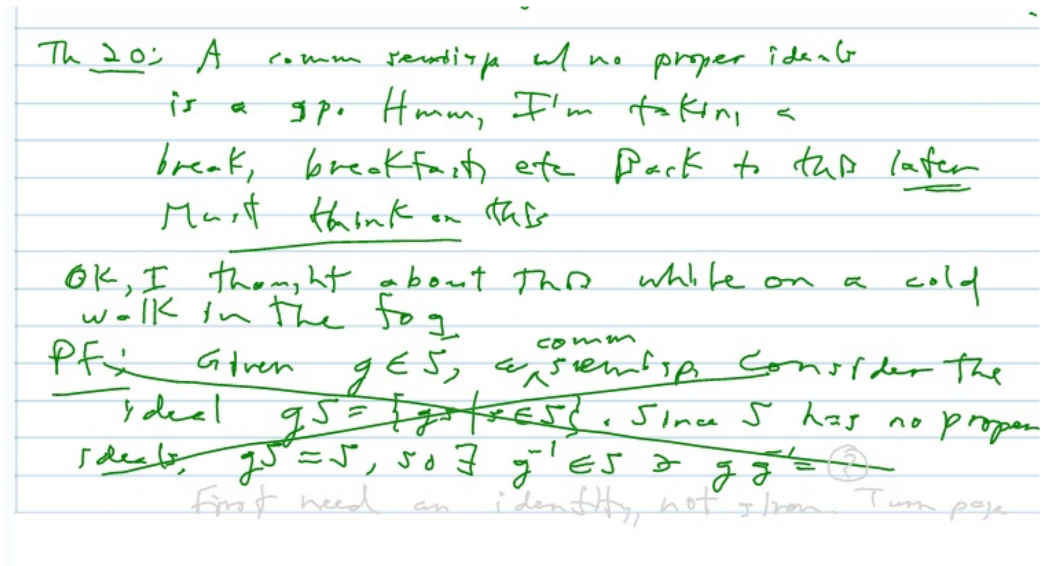


Figure 1: Dr. G's crossed-out work on Theorem 20.

5. Results

The *Multidimensional Problem-Solving Framework* aligned well with much of what the two participants (Dr. G and L) did during the proof construction process. Using the phases (*Orienting*, *Planning*, *Executing*, and *Checking*) and the problem solving attributes (*Resources*, *Heuristics*, *Affect*, and *Monitoring*), I coded both transcripts that pertained to Theorem 20 and analyzed the situations that agreed and those that disagreed with those in Carlson and Bloom's framework.

5.1. Instances of agreement with Carlson and Bloom's framework

For most portions of the transcripts, Carlson and Bloom's *Multidimensional Problem-Solving Framework* could be used to code and describe the proof construction process. There were multiple situations in both transcripts that involved both the *planning* subcycle (*conjecturing*, *imagining*, *evaluating*) and the larger cycle of *planning*, *executing*, and *checking*. I illustrate these below.

5.1.1. Planning subcycle

In his spoken discussion of Theorem 21 ("*If K is a minimal ideal of a commutative semigroup S, then K is a group*"), Dr. G demonstrated a *planning* subcycle (described in Section 2.2 and seen in Table 3 below):

	Writing	**Speaking**	**Subcycle**	**Phase**
(26)		If it has a zero element, then that will be a minimal ideal. Does that make it a group?	Conjecturing	Planning
(27a)		(Silence for 13 seconds)	Imagining	Planning
(27b)		Well no,	Evaluating	Planning
(28)		what about the non-negative integers?	Conjecturing	Planning

Table 3: An example of the Conjecturing-Imagining-Evaluating subcycle.

5.1.2. Example of a full cycle of planning-executing-checking

In his proof of Theorem 20, L demonstrated the full *planning-executing-checking* cycle, as seen in Table 4 (lines 8-11) below. L did not speak during his entire proof construction process, so I conjectured the phases using only his written work.

	Writing	**Attribute**	**Phase**
(7)	First we want to show S has an identity 1.	Monitoring	Executing
(8)	(pauses one minute and 5 seconds)		Planning
(9)	If possible.	Monitoring	Executing
(10)	Suppose S has no identity. Then for every $a \in S$, $ab \neq a$ for all	Resources	Executing
(11)	(pauses for 25 sec, then lines out "Then for every $a \in S$, $ab \neq a$ for all")	Monitoring	Checking/ Planning
(12)	Let $a \in S$. Let $A = \{ab : b \in S, ab \neq a\}$.	Resources	Executing

Table 4: An example of the Planning-Executing-Checking cycle.

5.2. Instances of differences with Carlson and Bloom's framework

5.2.1. Cycling back to orienting

At one point, Dr. G was trying to create a counterexample for Theorem 21 ("*If S is a semigroup with a minimal ideal K, then K is a group*") and in fact had to *reorient* himself to an understanding of the definition of a minimal ideal. This can be seen in Table 5 below.

	Writing	**Speaking**	**Attribute**	**Phase**
(30)		I mean, isn't the ideal generated by zero just zero? It's a minimal ideal, methinks.	Resources	Planning
(31)		Let's just go back and check this.	Monitoring	Planning
(32)		(ruffles paper, then silence for 13 seconds) Where's the definition of minimal again? I can't find it. Yeah doesn't properly contain any other ideal. Sure, and your ideals are all non-empty by requirement. So, ok let me write this down.	Resources	Orienting

Table 5: An example of reorienting.

5.2.2. Planning-executing-checking cycle is non-linear

L had certain instances in his proof construction process where there was both *planning* and *checking*. Also, the phases were non-linear, meaning that *executing* did not always follow from *planning*, and *checking* did not always follow from *executing*. This discrepancy with the Carlson and Bloom framework is displayed in Table 6 below.

5.3. Attributes associated with phases

During the coding, attributes were associated with each phase. For example, in Table 6, on line 27, L was in the *checking* phase, and the attribute associated with that phase was *monitoring*. This was due to his involved thinking about the "efficiency, correctness, and aesthetic quality of the solution" [2, page 63], since he returned to a certain point in the proof and wrote a correction (b'' instead of a) to ensure the aesthetic quality of the proof.

	Writing	**Attribute**	**Phase**
(27)	(writes over a in (25, last line) to be b'', then 20 sec pause)	Monitoring	Checking
(28)	This ~~contra~~ gives me the contradiction that S has no proper ideal. So a has an identity.	Resources	Executing
(29)	(pauses for four minutes and five seconds)		Planning
(30)	If $a \notin S$ has the property that $ab'' \neq b''a \neq b''$ for all $b'' \in S$	Resources	Executing
(31)	(crosses out $a \notin S$ (30) and writes above $a \in S$)	Monitoring	Checking

Table 6: An example of not doing the final checking.

I have collected and counted all instances of the attributes enacted by the two participants during their Proof of Theorem 20, and have listed them in Table 7 below.

Participant	Dr. G	L
Resources	43	17
Heuristics	9	2
Affect	17	0
Monitoring	22	13

Table 7: Attributes enacted by the participants.

6. Discussion

6.1. Successes and limitations of the coding

The four phases of the Carlson and Bloom framework were generally relevant to the proof construction processes exhibited by Dr. G and L. At first glance, the two participants were always in one of the phases (*Orienting*, *Planning*, *Executing*, and *Checking*) during their entire proof construction sessions for Theorem 20. This suggests that the four phases of problem solving are very important for the proof construction process. This further suggests that an expansion of Carlson and Bloom's framework could potentially provide the mathematics education community a proof construction-process framework, complete with additional problem-solving attributes that a prover experiences.

Though the Carlson and Bloom framework offers much promise, there is clearly room for productive improvement. In particular there is reason to suggest new phases in the process. Some additional problem-solving phases may include *incubation* (taking a break) and instances of multiple phases (*checking* and *planning*) occurring during a pause in the midst of proving or during a break from proving. An *incubation* phase may or may not involve conscious work on the proof. As Pólya [12] stated: "The fact is that a problem, after prolonged absence, may return to consciousness essentially clarified, much nearer to its solution than it was when it was dropped out of consciousness" (page 198). A student or mathematician does not do mathematics continuously, so an enhanced proof-construction process framework should attempt to reflect the subconscious work that may or may not ever be revealed. Furthermore, there may be a need for more flexibility between phases in a proof-construction framework. As noted in Table 6, L did not cycle as the Carlson and Bloom problem-solving framework indicates.

Overall Carlson and Bloom's framework describes the process of problem solving well. They provide ample examples from their study to support their framework. When confronted with a problem like those that Carlson and Bloom posed (see Problem 1 above), mathematicians can rather easily and quickly get conversant with the constraints (*orienting*) and then go about solving the problem (*planning-executing-checking*). However, in my study [13], the mathematicians were given a theorem to prove (Theorem 20), and had to go about *orienting* themselves for quite some time. Some mathematicians (e.g., Dr. G) *executed* their ideas (e.g., about modifying the hypotheses) early to see where that might lead, but then had to look at the theorem again to analyze in detail why the hypotheses needed modifying. This could have been the result of either not believing that the theorems in the notes were true or re-orienting to develop more intuition. In fact, if one questions whether a statement (in this case, a theorem) could be true or false, one must next decide whether to prove or find a counterexample. This is somewhat different than mathematical problem solving, where, unless a problem is posed as a true-or-false question, some textbook problems can often implicitly be assumed to have a solution.

Finally, Carlson and Bloom audiotaped the mathematicians in their study while they were solving the problems, and were also in the room to take notes and answer questions. In my study, however, participants were given a set of notes with unlimited time and not much direction, except to do what they normally would do when proving theorems and considering examples or questions. As a result, I was unable to observe their non-verbal actions, something that I conjecture provided Carlson and Bloom considerable help with their coding. This was a limitation of my study. On the contrary, I was able to capture incubation periods and insight, which were not accounted for in the Carlson and Bloom framework. This influenced my coding. Breaks are a crucial part of creativity and problem-solving for mathematicians [13], yet were not considered in Carlson and Bloom's problem-solving framework.

6.2. Observed differences between the mathematician and the graduate student

When analyzing all participants [13] in the data collection (nine mathematicians and five graduate students), I found that coding the proof attempts on Theorem 20 would give the best comparison of how expert provers and intermediate provers attempt a proof. Six of the nine mathematicians in my larger study experienced impasses when attempting a proof of Theorem 20, but only two out of five graduate students even attempted a proof. Constructing a proof of Theorem 20 was not trivial for any of my participants, and this provided a nice comparison between the attempts of Dr. G and L. Notice that Dr. G analyzed situations dealing with the theorem, such as "Why should the nonexistence of proper ideals force existence of an identity?" Dr. G often questioned the constraints of the hypotheses of the theorem. He went a step further and even thought that he might be able to construct a counterexample. According to the coding, L *oriented* himself at the beginning of the proof construction period for Theorem 20, and did not question the truth of the theorem, nor the constraints given. My conjecture is that the mathematician (Dr. G) has had substantial experience both with conjecturing his own theorems and adjusting the precise wording of those theorems after attempting unsuccessfully to prove them. He must have had to reorient himself rather often when engaging in mathematical research.

Another observable difference between Dr. G and L was how each handled the *checking* phase. Most *checking* phases of L were incorporated with the *planning* phase, where after multiple pauses during the proving of Theorem 20, he both crossed out a certain amount of his previous proof attempt and immediately proceeded to move to the next *executing* phase. There was not as much observable mixture of phases in Dr. G's proof construction process.

This may have been due to the amount of speaking that Dr. G did, which helped separate the *planning* and *checking* phases. But the more important aspect of Dr. G's *checking* phases were that he tried to make sense of his failed proof construction attempts and gain insights from them. For example, in Table 2, Dr. G stated, "First need an identity, not given (line 8)." Here he noticed that his previous proof construction attempt required an identity element, so he had to adjust his next proof construction attempt to accommodate that requirement. During L's proof construction attempts, he would make minor adjustments after each attempt but maintained the same main idea (supposing there is no identity) during most of the 41 minutes spent on that proof. *Checking* of the structure of the proof might have helped L gain more information from his proof construction attempts.

Finally, a significant difference between the graduate student and the mathematician was the use of the *monitoring* attribute. Dr. G exhibited the attribute 22 times during the proof construction process, while L displayed it only 13 times. This may be due to L not talking during proving, but Dr. G did write certain *monitoring* questions, such as, "Why should the nonexistence of proper ideals force existence of an identity?" (Appendix A, line 9). The number of *monitoring* attributes found was not as significant as the quality of *monitoring*. Dr. G's above question is an example of a larger mathematical understanding, and would be considered, in my opinion, an insightful mathematical question. L's *monitoring* was of a more specific nature, usually involving a correction of his work. For example, in Table 6, lines 27 and 31, L corrected some notation. This difference is convincing enough to suggest that insightful mathematical *monitoring* may not be emphasized explicitly enough in undergraduate/graduate proof-based courses. Significant questions such as "Can the hypothesis be weakened and the conclusion still remain true?" might lead students to alter their proof construction process or think about theorem-stating carefully. *Monitoring* may also be a byproduct of persistence, which has been shown to be an influential factor for graduate students who successfully complete their Ph.D.s in mathematics [8].

7. Future Research

Instructors' use of *Livescribe* pens along with Carlson and Bloom's framework when teaching might assist students in considering their actions during the proof construction process. For example, a professor or a graduate assistant could create a proof using a *Livescribe* pen. This creation process could demonstrate many of the phases and attributes shown by Dr. G. The professor could then make a real-time "movie" of his or her proof construction process (edited for time considerations). Then the assignment for the students would be to code the movie using the framework. The purpose would be to make students aware of certain phases and attributes, thus hoping to make the students mindful of those phases and attributes in the future. In particular, one could determine whether some students recognize, or fail to recognize, the *checking* phase or the *monitoring* attribute in their own problem solving or proving. An example of this phenomenon occurred with L's work in Table 6, when he finished his proof without any *checking*. Making phases of the proving process explicit might help undergraduate students make the transition to proof-based classes more quickly, thus shifting the focus of those initial upper-division proof-based courses, like abstract algebra or real analysis, towards their principal purpose of illuminating content.

Additionally, an enhanced data collection technique that could capture more phases and attributes would help in developing an enhanced proof construction-process framework. My study gathered written data in real time with synchronized audio, but there was no collection of gestures, including when the participants viewed the notes to *orient* themselves to the statement of a theorem or to gather ideas during a *planning* phase. In Carlson and Bloom's study, the participants were in an interview room for a more-or-less fixed time working continuously on the problems posed. Because their participants had no time for a break or other distracting activity, their data collection technique might have influenced their participants' creativity. A combination of the two data collection techniques (*Livescribe* pen and videoed interview sessions) would be much more informative, but would take more time and resources than I had.

I propose that there are three phases when constructing a proof: *planning*, *executing*, and *checking*, with *orienting* included in *planning*. This framework would be more of a Venn diagram (see Figure 2), with all three phases pairwise-overlapping each other, but there would never be a situation where *planning*, *checking* and *executing* would occur at the same time. *Planning* and *checking* were in many pauses that L had (e.g., Table 4, line 11), and *executing* and *checking* were prevalent in the mind of Dr. G (or else he could not have seen his small error in Table 2, lines 4-6). Since you must have something *executed* in order to *check* an attempt, there has to be an initial proving attempt from the *planning* phase to the *executing* phase.

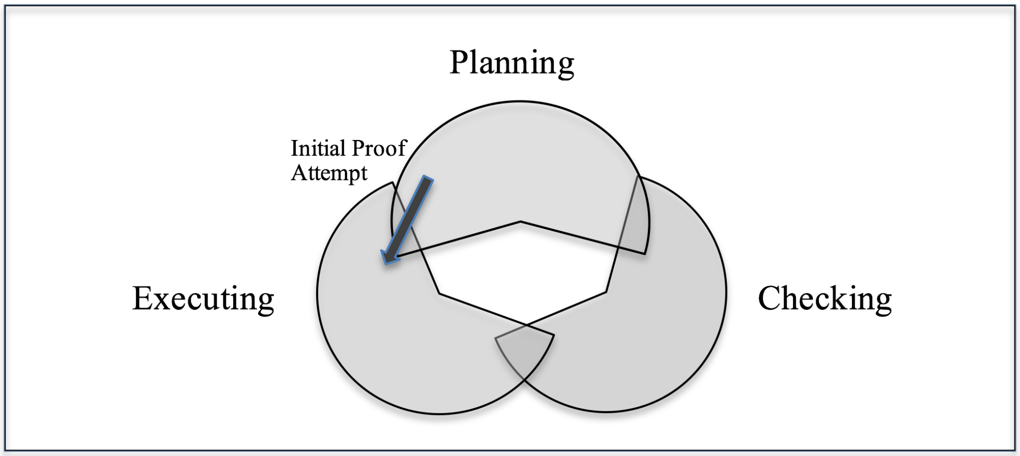


Figure 2: Proposed proof production framework.

I conjecture it might be the only directional phase shift in the framework. Finally, I claim that, while separated in the coding, L *planned* and *executed* his work simultaneously (Table 4, lines 7-9). The Venn diagram could take care of these "grey areas." An enhanced proof construction framework might allow mathematics education researchers to analyze their proof and proving data in a new way.

There might also be refinements of the *checking* and *planning* phases in a future enhanced proof construction-process framework. There were instances of local planning or proceeding on only a small part of a proof, and global planning or approaching a proof with a certain proving attempt [19]. An example of local planning was exhibited when Dr. G went through the planning subcycle in Table 3 and asked himself whether a semigroup that he had created was a group. An example of global planning was when L considered whether to prove Theorem 20 directly or by contradiction in Table 4 (lines 7-10). Checking could also be split into local checking (e.g., finding minor errors) and global checking (i.e., seeing if a proof attempt is sound). For example, local checking occurred in the middle of Dr. G's first proof attempt when he stopped executing to write "comm." between "a" and "semigroup" (Table 1, line 5). Global checking happened about a minute later, where he crossed out his entire proof attempt (seen in both Table 2 line 7 and Figure 1).

When using such a framework in the classroom, a teacher could isolate the phases (*Orienting*, *Planning*, *Executing*, and *Checking*) or the problem-solving attributes (*Resources*, *Heuristics*, *Affect*, and *Monitoring*) that need to be worked on, and focus instruction on that particular phase/attribute. Likewise, discussing the phases and attributes of a proof construction framework with students might assist them with their proof meta-cognition, which could assist them in their subsequent proof courses. This envisioned enhanced proof construction-process framework would be focused more on students' maneuvers through the phases and attributes, and less on correct or incorrect proofs. Notice that Dr. G had some incorrect statements, but in the *checking* phase and the *monitoring* attribute he identified what was missing and what was needed to advance his proof.

8. Conclusion

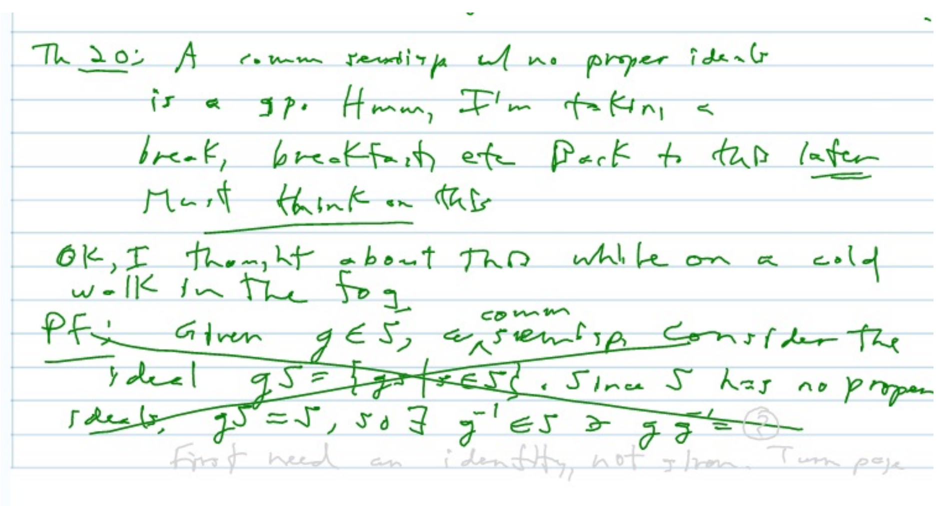
The Carlson and Bloom *Multidimensional Problem-Solving Framework* did help describe much of the proof construction of the professor's (Dr. G) and the graduate student's (L) proving processes. Indeed, I claim that there is a large overlap between proving and problem solving. However, there were also certain portions that did not match well with Carlson and Bloom's framework due to instances of subconscious, conscious, and silent, or non-linear work. Also, the differences between Dr. G and L that were highlighted by the coding included how each handled *checking* or re-evaluating previous work. Dr. G seemed to use the checking phase much more to his advantage than L, who went through several proving paths without analyzing or utilizing the incorrectness of his previous attempts. One could potentially use problem-solving or proof-constructing frameworks in the classroom to highlight areas for improvement in a student's proving process. I conjecture that focusing on the proof-construction process could have a lasting effect on the correctness of a student's proofs in their future.

Acknowledgments

I would like to thank the participants of the study who volunteered their time. I would also like to thank the reviewers of the manuscript for their suggestions. Finally, I thank my advisors, Drs. Annie and John Selden, who made numerous contributions to the study.

A. Excerpt of Dr. G's Proving Process of Theorem 20

	Writing	Speaking	Attribute	Phase
(1)	Th 20: A comm semigp w/ no proper ideals is a gp. (1 minute pause)		Resources	Orienting
(2)	Hmm ... I'm taking a break, breakfast, etc. Back to this later. Must think on this.		Monitoring	Planning
BREAK 7:04 AM – 8:07 AM				Planning
(3)	Ok, I thought about this while on a cold walk in the fog.		Heuristics	Planning
(4)	Pf: Given $g \in S$, a semigp., consider the ideal g		Resources, Heuristics	Executing
(5)	(Then he stops and puts "comm." between "a" and "semigp.")		Monitoring	Checking
(6)	$gS = \{gs \mid s \in S\}$. Since S has no proper ideals, $gS = S$, so $\exists g^{-1} \in S \ni gg^{-1} =$		Resources, Heuristics	Executing
(7)	(32 second pause, then he strikes through the whole proof)		Monitoring	Checking

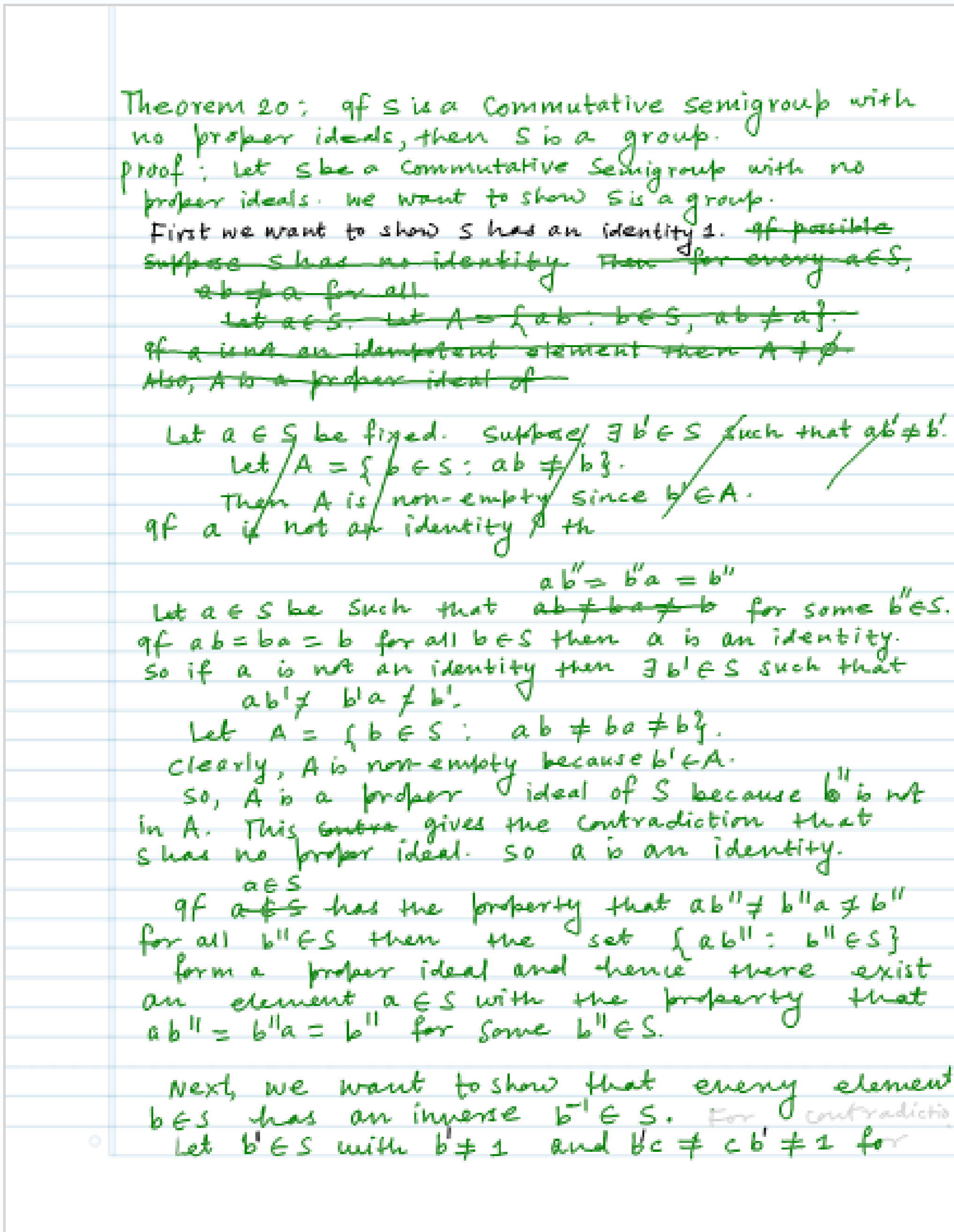


	Writing	**Speaking**	**Attribute**	**Phase**
(8)	First need an identity, not given. (Then he goes back to the expression "$gg^{-1} =$" and writes a question mark with a circle around it.) Turn page.		Resources, Monitoring	Planning
BREAK 8:09 AM – 9:44 AM				Planning
(9)	Later. I'm suspicious that this is true. Why should the nonexistence of proper ideals force existence of an identity?		Monitoring	Planning
(10)	But I don't know many examples, so I don't see a counterexample. (Silence for a minute, followed by ruffled papers, then silence)		Resources, Heuristics	Organizing
(11)	I'll do some talking.	Ok, so, I don't have to be quiet for anyone sleeping anymore, got a little music in the background. And I'm just going to talk a little bit. So I'm tossing around this idea of whether a semigroup with no proper ideals has to have an identity, in which case I could prove it's a group,	Resources	Planning
(12)		but I don't see why it would have to have an identity.	Monitoring	Planning

	Writing	**Speaking**	**Attribute**	**Phase**
(13)		Umm, I was trying to think of you pick an element in the semigroup and then the ideal that it generates has to be the whole thing.	Resources	Planning
(14)		But couldn't it translate ... couldn't it multiply to each element to give an element other than itself, so that neither would be the identity ever.	Resources	Planning
(15)		Why can't that happen?	Monitoring	Planning
(16)	(writes over the word "talking") (writes over the word "talking" again)	So it's sort of like a is a translation, but then if you ... you think you would get sub translation ideals of certain translations, except if you don't the semigroup would be very small, like only one element.	Resources, Heuristics	Planning

B. Excerpt of L's Proving Process of Theorem 20

	Writing	**Attribute**	**Phase**
(1)	Theorem 20: If S is a commutative semigroup with no proper ideal, then S is a group. Proof:	Resources	Orienting
(2)	(Turns page to his proofs of Theorem 18, Theorem 19, then to page with his proofs of Theorem 14, 15, and 16, then to page with proof of Theorem 17, then back to proof)	Resources, Heuristics	Orienting
(3)	(pauses one minute and 30 seconds)		Planning
(4)	Let S be a commutative semigroup with no proper ideals.	Resources	Executing
(5)	We want to show S is a group.	Monitoring	Executing
(6)	(pauses one minute and 27 seconds)		Planning
(7)	First we want to show S has an identity 1.	Monitoring	Executing
(8)	(pauses one minute and 5 seconds)		Planning
(9)	If possible.	Monitoring	Executing
(10)	Suppose S has no identity. Then for every $a \in S$, $ab \neq a$ for all	Resources	Executing
(11)	(pauses for 25 sec, then lines out "Then for every $a \in S$, $ab \neq a$ for all")	Monitoring	Checking
(12)	Let $a \in S$. Let $A = \{ab : b \in S, ab \neq a\}$.	Resources	Executing
(13)	(pauses one minute and 25 seconds)		Planning/ Checking
(14)	If a is not an idempotent element then $A \neq \emptyset$. Also, A is a proper ideal of	Resources	Executing
(15)	(pauses for three minutes and five seconds, then lines out "Let $a \in S$. Let $A = \{ab : b \in S, ab \neq a\}$. If a is not an idempotent element then $A \neq \emptyset$. Also, A is a proper ideal of,")	Monitoring	Checking
(16)	(pauses 35 seconds, then lines out "If possible. Suppose S has no identity," then pauses for 10 seconds)	Monitoring	Planning



Theorem 20: If S is a commutative semigroup with no proper ideals, then S is a group.

proof: Let S be a commutative semigroup with no proper ideals. We want to show S is a group.

First we want to show S has an identity 1. ~~If possible suppose S has no identity. Then for every $a \in S$, $ab \neq a$ for all~~

~~Let $a \in S$. Let $A = \{ab : b \in S, ab \neq a\}$.~~

~~If a is not an idempotent element then $A \neq \emptyset$.~~

~~Also, A is a proper ideal of~~

~~Let $a \in S$ be fixed. Suppose $\exists b' \in S$ such that $ab' \neq b'$.~~

~~Let $A = \{b \in S : ab \neq b\}$.~~

~~Then A is non-empty since $b' \in A$.~~

~~If a is not an identity $\emptyset$ th~~

Let $a \in S$ be such that ~~$ab \neq ba \neq b$~~ $ab'' = b''a = b''$ for some $b'' \in S$.

If $ab = ba = b$ for all $b \in S$ then a is an identity.

So if a is not an identity then $\exists b' \in S$ such that $ab' \neq b'a \neq b'$.

Let $A = \{b \in S : ab \neq ba \neq b\}$.

Clearly, A is non-empty because $b' \in A$.

So, A is a proper ideal of S because b'' is not in A. This ~~contra~~ gives the contradiction that S has no proper ideal. So a is an identity.

If ~~$a \in S$~~ $a \in S$ has the property that $ab'' \neq b''a \neq b''$ for all $b'' \in S$ then the set $\{ab'' : b'' \in S\}$ form a proper ideal and hence there exist an element $a \in S$ with the property that $ab'' = b''a = b''$ for some $b'' \in S$.

Next, we want to show that every element $b \in S$ has an inverse $b^{-1} \in S$. For contradictio

Let $b' \in S$ with $b' \neq 1$ and $b'c \neq cb' \neq 1$ for

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