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Review: Common Cyclic Vectors for Unitary Operators

Stephan Ramon Garcia
Pomona College

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Ross, William T. (1-RICH-CM); **Wogen, Warren R.** (1-NC)

Common cyclic vectors for unitary operators. (English summary)

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The authors continue their study of common cyclic vectors [Indiana Univ. Math. J. **53** (2004), no. 6, 1537–1550; MR2106335 (2005h:47041)] and determine whether or not certain natural classes of unitary multiplication operators have common cyclic vectors. To be more specific, let μ be a finite Borel measure on the unit circle $\mathbb{T}$ such that $\operatorname{supp}\mu = \mathbb{T}$ and let $H(\mathbb{T})$ denote the set of all homeomorphisms of $\mathbb{T}$. If μ is discrete, then the class $\{(M_\psi, L^2(\mu)) : \psi \in H(\mathbb{T})\}$ has a common cyclic vector. On the other hand, the authors show that if μ is continuous, then this class does not have a common cyclic vector (for μ continuous the authors showed in [op. cit.] that the larger class of multiplication operators on $L^2(\mu)$ whose symbols are univalent μ-almost everywhere does not have a common cyclic vector).

The restrictions that the authors place upon μ are natural, in light of several known results. If $K = \operatorname{supp}\mu \neq \mathbb{T}$, then a classical theorem of Lavrent′ev [J. B. Conway, *The theory of subnormal operators*, Amer. Math. Soc., Providence, RI, 1991; MR1112128 (92h:47026) (p. 232)] quickly implies that any $f \in L^2(\mu)$ such that $|f| > 0$ μ-a.e. is cyclic for M_ψ for any homeomorphism ψ of K. Thus it suffices to consider $\mu = \mu_{\rm c} + \mu_{\rm d}$ where $\mu_{\rm c}$ is continuous and $\mu_{\rm d}$ is discrete. Related work by R. V. Sibilev on Denjoy-Wolff series handles the discrete case [Algebra i Analiz **7** (1995), no. 1, 170–199; MR1334156 (96j:30006)]. The decomposition

$$\operatorname{Lat}((M_z, L^2(\mu))) = \operatorname{Lat}((M_z, L^2(\mu_{\rm c}))) \oplus \operatorname{Lat}((M_z, L^2(\mu_{\rm d})))$$

implies that it suffices to consider continuous measures whose support is $\mathbb{T}$. In fact, the authors reduce their common cyclic vector problem to the special case when μ is the normalized Lebesgue measure m on $\mathbb{T}$. In this setting, they obtain:

Theorem 1.2: Let B denote the homeomorphisms h of $[0, 2\pi]$ such that h is absolutely continuous with $h' \neq 0$ almost everywhere, and let B_1 denote the $h \in B$ such that h^{-1} is Lipschitz in some interval I_h.

(i) The class $\{(M_{e^{ih}}, L^2(m)) \colon h \in B\}$ does not have a common cyclic vector.

(ii) The class $\{(M_{e^{ih}}, L^2(m)) \colon h \in B_1\}$ has a common cyclic vector and in fact, $\varphi \in L^2(m)$ is a common cyclic vector for this class if and only if $|\varphi| > 0$ almost everywhere and $\log|\varphi|$ is not integrable on any arc of $\mathbb{T}$.

The paper concludes with a number of open questions.

Reviewed by *Stephan R. Garcia*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.