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Existence of Continuous Solutions to a Semilinear Wave Equation

Ben Preskill
Harvey Mudd College

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Alfonso Castro, Advisor

Adolfo Rumbos, Reader

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Department of Mathematics

Abstract

We prove two results; first, we show that a boundary value problem for the semilinear wave equation with smooth, asymptotically linear nonlinearity and sinusoidal smooth forcing along a characteristic cannot have a continuous solution. Thereafter, we show that if the sinusoidal forcing is not isolated to a characteristic of the wave equation, then the problem has a continuous solution.

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To Alfonso Castro, Adolfo Rumbos, and the HMC Math Department. Without your guidance, I would not have made it this far.

Chapter 1

Introduction

1.1 The Wave Equation

The wave equation, particularly the *linear* wave equation, is a ubiquitous partial differential equation (PDE) that models the propagation, at least in theory, of a wave. In its most general n-dimensional form, the linear, inhomogeneous wave equation is written

$$u_{tt}(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) = p(\mathbf{x}, t), \tag{1.1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $p(\mathbf{x}, t)$ is a continuously differentiable function.

Here we will only be concerned with the case where $n = 1$. The linear, inhomogeneous wave equation is one dimension is written

$$u_{tt}(x, t) - u_{xx}(x, t) = p(x, t), \tag{1.2}$$

where $x, t \in \mathbb{R}$ and again $p(x, t)$ is a continuously differentiable function. In general, we abbreviate this to

$$\Box(u)(x, t) = p(x, t), \tag{1.3}$$

where

$$\Box(u) = u_t t - u_x x, \tag{1.4}$$

is the *D'Alembert operator*.

The linear wave equation arises in numerous physical applications. For example, if $p(x, t) \equiv 0$, it models the vibrations of a string, and the solutions are the harmonics of vibration. Similarly, solutions to the one-dimensional wave equation can be regarded as projections of solutions to the three-dimensional wave equation, as arise in electricity and magnetism, projected down on one dimension.

The linear wave equation can, at least on fairly simple subsets of $\mathbb{R}^2$, usually be solved nicely if stated as an initial value problem (see below). In general, linear PDEs are relatively tractable and may have solutions that can be found using analytical techniques. The equation studied here, a *semilinear* wave equation, is not so nice. Here semilinear implies that the *nonlinearity*, or portion of the equation that deviates from the linear wave equation, depends exclusively on the function $u(x,t)$ and not its derivatives. Consequently, the general one-dimensional semilinear wave equation is

$$\Box(u)(x,t) + g(u) = p(x,t), \tag{1.5}$$

where $g(t)$ is usually a differentiable, well-behaved function on $\mathbb{R}$. We commonly refer to $g(t)$ as the nonlinearity, since if $g(t) \equiv 0$, this reduces to the linear wave equation.

1.2 Boundary Value Problems

Although many differential equations, particularly those involving time, are specified at an arbitrary point (or line, in this case) defined by $t = 0$ in what is known as an *initial value problem*, this is not always practical or desirable. Indeed, if we consider a string held fixed at both ends and forced to vibrate, we need a way to specify that the displacement on either end is zero at all times. When considering a finite interval of time, we might even stipulate the initial and final configurations of the string, defining the values of the function on the entire boundary of a rectangle in $\mathbb{R}^2$.

The particular equation studied here is

$$\begin{cases} \Box(u) + g(u) = p(x,t) = p(x,t+2\pi) = p(x+2\pi,t) & x,t \in R \\ u(x,t) = u(x,t+2\pi) = u(x+2\pi,t) & x,t \in R, \end{cases} \tag{1.6}$$

where the region on which this is being solved is $\Omega = (0,2\pi) \times (0,2\pi)$. The boundary condition here is implicit in the statement that $u(x,t)$ is 2π-periodic in x and t, since its values on the boundary are constrained. Note that we also demand that $p(x,t)$ be 2π-periodic in both variables as well.

Beyond specifying a specific boundary condition, we need also specify the form of $g(t)$ if we are to gain any traction in proving results about the equation. In this problem, we stipulate that

$$g(t) = \tau t + h(t), \tag{1.7}$$

with

$$\tau \in (0,\infty) - \{k^2 - j^2 : k, j \in \{0\} \cup \mathbb{N}\}, \tag{1.8}$$

for reasons outside the scope of the introduction. Here $h(t)$ is a differentiable function, although other constraints will be imposed on $h(t)$.

Herein, we will investigate conditions on $g(t)$ and $p(x,t)$ for which there either exists or does not exist a continuous solution to Equation 1.6.

Chapter 2

Background

2.1 L^p and Hilbert Spaces

Before discussing the theory involved in the wave equation itself, it's necessary to quickly define some concepts from graduate analysis. Throughout this section, we let Ω denote a *measure space*. For readers not acquainted with measure theory, simply think of Ω as a subset of $\mathbb{R}^n$ for some n.

Definition 2.1. *$L^p(\Omega)$ denotes the space of measurable (i.e., integrable) real functions $f : \Omega \to \mathbb{R}$ such that*

$$\int_\Omega |f|^p \, dx < \infty. \tag{2.1}$$

By the linearity of integration, $L^p(\Omega)$ is a vector space with norm

$$\|f\|_p \doteq \left(\int_\Omega |f|^p \, dx \right)^{\frac{1}{p}} \tag{2.2}$$

for $p \geq 1$. Although we will not use the fact here, $L^p(\Omega)$ is in fact complete with respect to this norm and is thus a *Banach space*.

We will be particularly interested in $L^2(\Omega)$; not only is the space complete, but it also has an inner product that generates the norm given by

$$\langle f, g \rangle = \int_\Omega fg \, dx, \tag{2.3}$$

and so $L^2(\Omega)$ is a complete inner-product space, also known as a *Hilbert space*. Hilbert spaces have most of the nice properties of finite-dimensional vector spaces and can often be thought of as "taking the limit" of finite dimensional vector spaces with dimension n as $n \to \infty$.

Because we are looking at solutions to PDEs, we will usually need more than just a constraint on the integral of a function itself; we will need a constraint on its derivatives, too. It turns out that we actually only need for a function's first-order derivatives to be in $L^2(\Omega)$ for the wave equation, as will be seen later. Consequently, we introduce *Sobolev space*.

Definition 2.2. *The Sobolev spaces $H^k(\Omega)$ consist of functions $u \in L^2(\Omega)$ such that all partial derivatives of u of order $\leq k$ are also in $L^2(\Omega)$.*

We will only be interested in $H^1(\Omega)$ here. Letting Ω and $u(x,t)$ be as in Equation 1.6, we have

$$H^1(\Omega) = \{f \in L^2(\Omega) : f_x, f_t \in L^2(\Omega)\}. \tag{2.4}$$

It should be noted that $H^k(\Omega)$, for $k \in \mathbb{N}$, are Hilbert spaces as well.

Note that throughout this paper we will write $H = H^1(\Omega)$ for concision. Observe that $u \in H$ can be thought of as a function on $\mathbb{R}^2$ by extending the domain under the assumption of 2π-periodicity. Although u is not defined on the boundary of Ω, because this set has *measure zero*, this extension is unique as an equivalence class in $H^1(\mathbb{R}^2)$.

2.2 Weak Derivatives

Before discussing the wave operator, we need to define the notion of a *weak derivative*. It turns out that to discuss solutions of the wave equation, we don't need (or want, necessarily) continuity or differentiability, even of the first order.

Definition 2.3. *We say that a function f on Ω has weak derivative f' if for every $\phi \in C_0^\infty(\Omega)$, we have*

$$\int_\Omega \phi f' \, dx = -\int_\Omega \phi' f \, dx, \tag{2.5}$$

where $C_0^\infty(\Omega)$ is the set of infinitely differentiable functions on Ω that vanish on the boundary.

With this definition, we can speak about u_x and u_t, even if u isn't continuous.

2.3 Weak Solutions

In line with weak derivatives, there is a corresponding notion of a *weak solution* to a PDE. Qualitatively, a weak solution is derived by multiplying a PDE by a test function ϕ and then integrating, using integration by parts if possible. We can do so for the particular case of Equation 1.6. In contrast to the case with weak derivatives, it is sometimes useful to modify the class of *test functions* ϕ. In this case, because we only need ϕ to be differentiable to the first order and since we are working in H to begin with, we can take $\phi \in H$.

Multiplying both sides of the PDE by ϕ and integrating gives

$$\int_\Omega (u_{tt} - u_{xx} + g(u))\, \phi \, dx = \int_\Omega p\phi \, dx. \tag{2.6}$$

Integrating by parts on the left, subtracting the integral on the right, and multiplying by -1 leads to the following.

Definition 2.4. *We say $u \in H$ is a* weak solution *to Equation 1.6 if for all $\phi \in H$,*

$$\int_\Omega \{(u_t\phi_t - u_x\phi_x) - (g(u) - p)\, \phi\} \; dx = 0. \tag{2.7}$$

From this point forward, we will use the terms solution and weak solution interchangeably.

2.4 The Wave Operator

We can regard $\Box$, the D'Alembert operator defined in Equation 1.4, as a *linear operator* on H by the linearity of the derivative and by noting that

$$\Box(u) = v \tag{2.8}$$

is equivalent to

$$-\int_\Omega (u_t\phi_t - u_x\phi_x) \; dx = \int_\Omega \Box(u)\phi \, dx = \int_\Omega v\phi \, dx \quad \forall\, \phi \in C_0^\infty(\Omega), \tag{2.9}$$

so that $\Box$ is well-defined on H.

Like any linear operator, we can talk about the *null space* N of $\Box$ in H. In fact, because H is a Hilbert space, N is a topologically closed subspace of H. Although we do not prove it here, N is spanned by

$$\mathcal{B} = \{\alpha_{k,k}, \beta_{k,k}, \gamma_{k,k}, \delta_{k,k} : k \in \{0\} \cup \mathbb{N}\}, \tag{2.10}$$

where we write

$$\alpha_{k,j}(x,t) = \sin(kx)\cos(jt) \qquad \beta_{k,j}(x,t) = \sin(kx)\sin(jt) \tag{2.11}$$

$$\gamma_{k,j}(x,t) = \cos(kx)\cos(jt) \qquad \delta_{k,j}(x,t) = \cos(kx)\sin(jt) \tag{2.12}$$

2.5 Previous Results

The most important previous work on Equation 1.6 is Caicedo and Castro (2009). Therein it is shown that if h is not only differentiable on $\mathbb{R}$ but also has support in some interval $[0, D]$ for $D > 0$ such that

$$h(D/2) < -\tau \frac{D}{2}, \tag{2.13}$$

then the following result holds.

Theorem 2.1. *There exists $c_0 \geq 0$ such that if $|c| \geq c_0$ and $p(x,t) = c\sin(x,t)$, then Equation 1.6 has no continuous weak solution.*

Much of the proof from Caicedo and Castro (2009) is repeated or invoked to prove the results established herein. A related problem, and one that ultimately informed the research on this problem is

$$\begin{cases} \Box(u) + g(u) = p(x,t) = p(x,t+2\pi) & (x,t) \in (0,\pi) \times \mathbb{R} \\ u(0,t) = u(\pi,t) = 0 & t \in \mathbb{R} \\ u(x,t) = u(x,t+2\pi) & (x,t) \in [0,\pi] \times \mathbb{R} \end{cases} \tag{2.14}$$

A number of results have been established regarding Equation 2.14. In Brezis and Nirenberg (1978), the following theorem was proved.

Theorem 2.2. *If g is monotone and asymptotically linear (i.e. $\lim_{|t|\to\infty} g(t)/t$ exists and is a constant), then Equation 2.14 has a weak solution in $L^2([0,\pi] \times [0,2\pi])$.*

It was also shown in Brezis and Nirenberg (1978) that if there exists $\epsilon > 0$ such that $g'(z) \geq \epsilon > 0$ for all $z \in \mathbb{R}$, then such a solution is smooth (class C^∞) if p is smooth. The significance of this result is that it relies heavily on the fact that g is monotone; as the result in Caicedo and Castro (2009) implies for Equation 1.6, this does not generalize to nonmonotone g.

In fact, Equation 2.14 has been studied for nonmonotone g in Willem (1981) and Hofer (1982), where it was proved that Equation 2.14 has a solution for p in a dense set of $L^2([0,\pi] \times [0,2\pi])$. Thus, although there are "a lot" of functions p such that solutions exist, it might be the case that there is a dense set of functions p such that no solution exists as well.

These results help inform explorations of the related problem Equation 1.6.

Chapter 3

Implementation

3.1 Preliminaries

As before, we let $\Omega = (0, 2\pi) \times (0, 2\pi)$, $H = H^1(\Omega)$, and N be the null space of $\Box$, the D'Alembert operator, subject to the periodic boundary condition. Because H is an inner-product space with respect to both the H^1 and L^2 inner-products and N is a closed subspace, we can define a closed subspace Y of H by $Y = N^\perp$, where orthogonality is taken with respect to the L^2 inner-product, and we then have

$$H = Y \oplus N \tag{3.1}$$

so that every element $u \in H$ may be written uniquely as $u = y + v$ where $\langle y, v \rangle_{L^2} = 0$. We may then modify the formulation of the weak solution to Equation 1.6 in Equation 2.7, substituting $u = y + v$ and $\phi = \hat{y} + \hat{v}$.

We can then say that $u = y + v$ is a weak solution of Equation 1.6 if

$$\int_\Omega \{(y_t \hat{y}_t - y_x \hat{y}_x) - (g(u) - p)(\hat{y} + \hat{v})\}\, dx\, dt = 0, \tag{3.2}$$

where we have used the fact that $\Box(u) = \Box(y) + \Box(v) = \Box(y)$ because $v \in N$.

3.2 Regularity (Nonexistence)

Let $u = y + v$ be a weak solution to Equation 1.6. Here we define

$$\alpha_1(x, t) = \sin(x + t), \quad \alpha_2(x, t) = \sin(x - t), \tag{3.3}$$

although we will only use α_1 for now.

Let a be the projection of v onto the linear subspace spanned by α_1. Then we can orthogonally decompose v such that $v = a\alpha_1 + w$, with $\langle \alpha_1, w \rangle_{L^2} = 0$. That is,

$$\int_\Omega \alpha_1 w \, dx \, dt = 0. \tag{3.4}$$

We can then further decompose w by defining

$$\bar{v} = \frac{1}{4\pi} \int_\Omega v \, dx \, dt = \frac{1}{4\pi} \int_\Omega w \, dx \, dt, \tag{3.5}$$

because α_1 has zero integral over Ω. Note that Equation 3.5 is just the average of v over Ω. We then write $w = \bar{v} + z$ where $z = w - \bar{v}$. However, because $z \in N$ (since $v \in N$) and

$$\int_\Omega z \, dx \, dt = 0, \tag{3.6}$$

we can write

$$z(x,t) = z_1(x+t) + z_2(x-t) \tag{3.7}$$

with

$$\int_\Omega z_1(x+t) \, dx \, dt = \int_\Omega z_2(x-t) \, dt \, dt = 0 \tag{3.8}$$

We can now state and prove the following.

Lemma 3.1. *With the above definitions,* $\|z_i\|_\infty \leq 3\|h\|_\infty / \tau$ *and* $|\bar{v}| \leq \|h\|_\infty / \tau$.

Proof. Because we assume $u = y + v$ is a weak solution, we know Equation 3.2 holds for $\hat{y} = 0$ and $\hat{v} = \alpha_1$. Substituting these values in, we have

$$\begin{aligned}
\int_\Omega -(g(u) - p)\alpha_1 \, dx \, dt &= \int_\Omega (c\alpha_1 - \tau u - h(u))\, \alpha_1 \, dx \, dt && (3.9)\\
&= \int_\Omega (c\alpha_1 - \tau y - \tau v - h(u))\, \alpha_1 \, dx \, dt && (3.10)\\
&= \int_\Omega (c\alpha_1 - \tau y - \tau a\alpha_1 - \tau w - h(u))\, \alpha_1 \, dx \, dt && (3.11)\\
&= \int_\Omega \left\{ (c - \tau a)\, \alpha_1^2 - h(u)\alpha_1 \right\} \, dx \, dt && (3.12)\\
&= 0, && (3.13)
\end{aligned}$$

where we have used the L^2-orthogonality of y and α_1 and w and α_1. Equation 3.9 reduces to

$$(c - \tau a)\|\alpha_1\|_2^2 = \int_\Omega h(u)\alpha_1 \, dx \, dt, \tag{3.14}$$

and so we have

$$|\tau a - c| \|\alpha_1\|_2^2 \leq \|h\|_2 \|\alpha_1\|_2 \tag{3.15}$$

by Hölder's inequality. Clearly $\|\alpha_1\|_2 = \sqrt{2}\pi$ by a simple calculation, and because $m(\Omega) = 4\pi^2$, we know $\|h\|_2 \leq 2\pi\|h\|_\infty$. Combining these, we are left with

$$|\tau a - c| \leq \sqrt{2}\|h\|_\infty. \tag{3.16}$$

Let b be a positive odd integer. Now because any function of $(x+t)$ or $(x-t)$ is in N, we see that $z_1^b(x+t)$ and $z_2^b(x-t)$ are in N. Consequently, we can set $\hat{v}(x,t) = z_1^b(x+t)$ in Equation 1.6 while keeping $\hat{y} = 0$, and we have, noting again that y is L^2 orthogonal to all elements of N,

$$\int_\Omega -(g(u) - p) z_1^b(x+t)\, dx\, dt \tag{3.17}$$

$$= \int_\Omega (c\alpha_1 - \tau a \alpha_1 - \tau w - h(u))\, z_1^b(x+t)\, dx\, dt \tag{3.18}$$

$$= \int_\Omega \left(c\alpha_1 - \tau a\alpha_1 - \tau\bar{v} - \tau\left(z_1^b(x+t) + z_2^b(x-t)\right) - h(u)\right) z_1^b(x+t)\, dx\, dt \tag{3.19}$$

$$= \int_\Omega \{(c - \tau a)\,\alpha_1 - \tau(z_1(x+t) + z_2(x-t)) - h(u)\}\, z_1^b(x+t)\, dx\, dt \tag{3.20}$$

where we have used the fact that $\tau\bar{v}$ is a constant, and

$$\int_\Omega z_1^b(x+t)\, dx\, dt = 0 \tag{3.21}$$

because b is odd. Because functions of $(x+t)$ and $(x-t)$ are orthogonal, this simplifies to

$$\tau \|z_1\|_{b+1}^{b+1} = -\int_\Omega \{h(u(x,t)) - (c - \tau a)\alpha_1(x,t)\}\, z_1^b(x,t)\, dx\, dt \tag{3.22}$$

$$\leq 3\|h\|_\infty |\Omega|^{\frac{1}{b+1}} \left(\int_\Omega |z_1(x,t)|^{b+1}\, dx\, dt\right)^{\frac{b}{b+1}}. \tag{3.23}$$

And so we have that

$$\tau \|z_1\|_\infty \leq 4\|h\|_\infty |\Omega|^{\frac{1}{b+1}}. \tag{3.24}$$

Letting $b \to \infty$ and using the fact that $\|z_1\|_\infty = \lim_{b\to\infty} \|z_1\|_{b+1}$, we have

$$\tau \|z_1\|_\infty \leq 4\|h\|_\infty. \tag{3.25}$$

This argument holds similarly for z_2, so we also have

$$\tau\|z_2\|_\infty \leq 4\|h\|_\infty. \tag{3.26}$$

We now have

$$4\pi^2\tau|\bar{v}| = \tau\left|\int_\Omega w(x,t)\,dx\,dt\right| = \left|\int_\Omega h(u(x,t))\,dx\,dt\right| \leq 4\pi^2\|h\|_\infty, \tag{3.27}$$

and we are done. □

Lemma 3.2. *There exists $K > 0$, independent of c, such that if $u = y + v \in Y \oplus N$ is a weak solution to Equation 1.6, then $|y(x,t)| \leq K\|h\|_\infty$ for all $(x,t) \in \Omega$ and $\|y\|_H \leq K$.*

Proof. Because $\{\alpha_{k,j}, \beta_{k,j}, \gamma_{k,j}, \delta_{k,j} : k, j \in \{0\} \cup \mathbb{N}\}$ is a basis for H and $y \in N^\perp$, we can write

$$y = \sum_{k \neq j}\left(a_{kj}\alpha_{k,j} + b_{kj}\beta_{k,j} + c_{kj}\gamma_{k,j} + d_{kj}\delta_{k,j}\right). \tag{3.28}$$

Furthermore, writing $u = y + v$, we can look at the projection of $h(u)$ onto the subspace Y as

$$P_Y(h(y+v)) = \sum_{k \neq j}\left(A_{kj}\alpha_{k,j} + B_{kj}\beta_{k,j} + C_{kj}\gamma_{k,j} + D_{kj}\delta_{k,j}\right). \tag{3.29}$$

Because P_Y is a projection, we have

$$\|P_Y(h(v+y))\|_2 \leq \|h(y+v)\|_2 \leq 2\pi\|h\|_\infty. \tag{3.30}$$

Now, because we assume $p(x,t) \in N$, we can project both sides of Equation 1.6 onto Y and get

$$\Box(y) + \tau y + \tau P_Y(h(u)) = 0. \tag{3.31}$$

Substituting in for y and $P_Y(h(u))$, this equation becomes

$$\sum_{k \neq j}\left\{\left(\tau + k^2 - j^2\right)\left(a_{kj}\alpha_{k,j} + b_{kj}\beta_{k,j} + c_{kj}\gamma_{k,j} + d_{kj}\delta_{k,j}\right)\right. \tag{3.32}$$

$$\left. + \tau\left(A_{kj}\alpha_{k,j} + B_{kj}\beta_{k,j} + C_{kj}\gamma_{k,j} + D_{kj}\delta_{k,j}\right)\right\} = 0, \tag{3.33}$$

and reducing by components gives

$$a_{kj} = \frac{A_{kj}}{k^2 - j^2 + \tau}, \tag{3.34}$$

and similarly for b_{kj}, c_{kj}, and d_{kj}. Parseval then gives

$$|y(x,t)| = \left| \sum_{k \neq j} \left(a_{kj}\alpha_{k,j} + b_{kj}\beta_{k,j} + c_{kj}\gamma_{k,j} + d_{kj}\delta_{k,j} \right) \right| \tag{3.35}$$

$$\leq \left(\sum_{k \neq j} \left(A_{kj}^2 + B_{kj}^2 + C_{kj}^2 + D_{kj}^2 \right) \right)^{1/2} \left(\sum_{k \neq j} \frac{1}{(k^2 - j^2 + \tau)^2} \right)^{1/2} \tag{3.36}$$

$$\leq 2\pi \|h\|_\infty \left(\sum_{k \neq j} \frac{1}{(k^2 - j^2 + \tau)^2} \right)^{1/2} \tag{3.37}$$

$$\equiv K_1 \|h\|_\infty, \tag{3.38}$$

where K_1 is simply a constant. Note that the above series does in fact converge. We also have that

$$\|y\|_H^2 \leq 2 \sum_{k \neq j} \frac{(k^2 + j^2)(A_{kj}^2 + B_{kj}^2 + C_{kj}^2)}{(k^2 - j^2 + \tau)^2} \tag{3.39}$$

$$\leq K_2 \|h(u)\|_2^2 \tag{3.40}$$

$$\leq 4\pi^2 K_2 \|h\|_\infty^2, \tag{3.41}$$

where K_2 is another constant. We will let $K = \max\{K_1, 2\pi\sqrt{K_2}\}$, and the lemma follows. □

Let $D > 0$ be as in Equation 2.13. Now we have

$$|u(x,t)| = |a \sin(x+t) + \bar{v} + z(x,t) + y(x,t)| \tag{3.42}$$

$$\geq \left[(|c| - 2\|h\|_\infty) |\sin(x+t)| - 7\|h\|_\infty \right] / \tau. \tag{3.43}$$

Thus

$$h(u(x,t)) = 0 \text{ if } |\sin(x+t) \geq \frac{\tau D + 7\|h\|_\infty}{|c| - 2\|h\|_\infty}. \tag{3.44}$$

And this implies that there exist positive constants c_0 and m such that $|c| \geq c_0$ implies

$$m(\{(x,t) \in \Omega : h(u(x,t)) \neq 0\}) \leq \frac{m}{c}. \tag{3.45}$$

And Equation 3.45 furnishes a bound on $\|h(u)\|_2$. In particular,

$$\|h(u)\|_2 \leq m^{1/2} \|h\|_\infty c^{-1/2}, \tag{3.46}$$

for $|c| \geq c_0$. Using this inequality in Equation 3.35, we have

$$|y(x,t)| \leq K\|h\|_\infty c^{-1/2}, \tag{3.47}$$

and also for $|c| \geq c_0$. We also need a bound on $|\bar{v}|$. But note that

$$\tau|\bar{v}| = \left|\int_\Omega h(u(x,t))\,dx\,dt\right| \tag{3.48}$$

$$\leq \|h\|_\infty m(\{(x,t) \in \Omega : h(u(x,t)) \neq 0\}) \tag{3.49}$$

$$\leq \frac{m\|h\|_\infty}{c}. \tag{3.50}$$

And by adapted reasoning, we also have

$$|\tau a - c| \leq \frac{m\|h\|_\infty}{c}. \tag{3.51}$$

For $0 \leq r \leq s \leq 2\pi$, define $\chi_{[r,s]}$ to be the 2π-periodic extension of the characteristic function of $[r,s]$. Furthermore, define

$$\phi(x,t) = \chi_{[r,s]}(x-t). \tag{3.52}$$

Because $\phi \in N$, the mean value theorem for integrals furnishes

$$0 = \int_\Omega \phi((a\tau - c)\alpha_1 + \tau(z_1 + z_2) + \bar{v} + h(u))\,dx\,dt \tag{3.53}$$

$$= 2\pi(s-r)\tau z_2(s_2) + \int_\Omega \phi h(u)\,dx\,dt + 2\pi\bar{v}(s-r), \tag{3.54}$$

where $s_2 \in (r,s)$. Clearly we have

$$\left|\int_\Omega \phi h(u)\,dx\,dt\right| \leq \|h\|_\infty (r-s)\frac{m}{c}, \tag{3.55}$$

and so

$$|z_2(r)| \leq \frac{M\|h\|_\infty}{c}, \tag{3.56}$$

where M is a constant independent of c. Now, letting

$$\psi(x,t) = \chi_{[r,s]}(x+t), \tag{3.57}$$

we still have $\psi \in N$, and so multiplying Equation 1.6 by ψ and integrating yields

$$0 = \int_\Omega \psi((a\tau - c)\alpha_1 + \tau(z_1 + z_2) + \bar{v} + h(u))\,dx\,dt \tag{3.58}$$

$$= 2\pi(s-r)((a\tau - c)\alpha_1(0,s_3) + \tau z_1(s_1)) + \tau\bar{v}2\pi(s-r) \tag{3.59}$$

$$+ \int_\Omega \phi(h(u) - h(a\alpha + \bar{z}_1))\,dt\,dt + \int_\Omega \psi h(a\alpha + \bar{z}_1)\,dx\,dt, \tag{3.60}$$

and here $s_1, s_3 \in (r, s)$. Let $s \to r$ and we have

$$0 = 2\pi((a\tau - c)\alpha_1(0, r) + \tau z_1(r) + h((a\alpha + \bar{z}_1)(0, r)) + \bar{v}) \tag{3.61}$$
$$+ \int_0^{2\pi} \phi(h'(u(x, r - x))(\bar{v} + z_2(x, r - x) + y(x, r - x))\, dx. \tag{3.62}$$

Combining all results together, we see that

$$\tau z_1(r) + h(a\alpha(0, r) + z_1(r)) = O(c^{-1/2}). \tag{3.63}$$

3.3 Existence

Recalling that we can write $u(x, t) = y(x, t) + \bar{v} + v_1(x + t) + v_2(x - t)$, with $\bar{v} \in \mathbb{R}$ and $y \in Y$, to prove the existence of a solution, we show that there exists a quadruplet $(y, \bar{v}, v_1, v_2)$ such that Equation 1.6 is satisfied. In particular, existence is shown if we can prove that the projection of Equation 1.6 onto each of N and Y are satisfied. This is what we will do.

3.3.1 Equation in The Kernel

Let $u = v + y \in N \oplus Y$ be a weak solution to Equation 1.6. We can write $v(x, t) = \bar{v} + v_1(x + t) + v_2(x - t)$ with $\bar{v} \in \mathbb{R}$, v_1, v_2 2π-periodic functions with

$$\int_0^{2\pi} v_1(s) ds = \int_0^{2\pi} v_2(s) ds = 0. \tag{3.64}$$

For $0 \le r \le s \le 2\pi$, let $\chi_{[r,s]}$ be the 2π-periodic function such that $\chi_{[r,s]}(t) = 1$ if $t \in [r, s]$, and $\chi_{[r,s]}(t) = 0$ if $t \in [0, 2\pi] - [r, s]$. Let $\phi(x, t) = \chi_{[r,s]}(x - t)$, and $A = \{(x, t); x \in [0, 2\pi], t \in [x - s, x - r]\}$. Using that $\phi \in N$ and the mean value theorem for integrals we have

$$\begin{aligned}
\int_A p(x, t) dx dt &= \int_\Omega \phi(x, t) p(x, t) dt \\
&= \int_\Omega \phi(x, t)(\tau v(x, t) + h(u(x, t))) dx dt \\
&= \int_A (\tau \bar{v} + v_1(x + t) + v_2(x - t) + h(u(x, t)) dx dt \\
&= 2\pi\tau(s - r)(\bar{v} + \tau v_2(s_2)) + \int_A h(u) dx dt,
\end{aligned} \tag{3.65}$$

where $s_2 \in (r, s)$. Dividing by $s - r$ and taking limit as $s \to r$ we have

$$\int_0^{2\pi} p(x, x - r) dx = 2\pi\tau(\bar{v} + v_2(r)) + \int_0^{2\pi} h(u(x, x - r)) dx. \tag{3.66}$$

Letting $\psi(x,t) = \chi_{[r,s]}(x+t)$ and arguing as in Equations 3.64-3.66 we have

$$\int_0^{2\pi} p(x, r-x)dx = 2\pi\tau(\bar{v} + v_1(r)) + \int_0^{2\pi} h(u(x, r-x))dx. \tag{3.67}$$

We know that any weak solution u must satisfy Equations 3.66 and 3.67. We now prove that, for each y and $\bar{v}$, there exist v_1 and v_2 such that these equations are satisfied.

We restrict to the case $p(x,t) = c(\sin(x+t) + \sin(x-t))$, so we have

$$\begin{aligned} 2\pi c \sin(r) &= 2\pi\tau(\bar{v} + v_1(r)) + \int_0^{2\pi} h(u(x, r-x))dx, \\ 2\pi c \sin(r) &= 2\pi\tau(\bar{v} + v_2(r)) + \int_0^{2\pi} h(u(x, x-r))dx. \end{aligned} \tag{3.68}$$

Let $z_i(r) = (c/\tau)\sin(r) - v_i(r) \equiv C\sin(r) - v_i(r)$, $i = 1, 2$. Hence

$$\begin{aligned} z_1(r) &= \bar{v} + \frac{1}{2\pi\tau}\int_0^{2\pi} h(y(x, r-x) + \bar{v} + C\sin(r) \\ &\quad + C\sin(2x-r) - z_1(r) - z_2(2x-r))dx, \\ z_2(r) &= \bar{v} + \frac{1}{2\pi\tau}\int_0^{2\pi} h(y(x, x-r) + \bar{v} + C\sin(2x-r) \\ &\quad + C\sin(r) - z_1(2x-r) - z_2(r))dx. \end{aligned} \tag{3.69}$$

Lemma 3.3. *There exists $c_0 > 0$ and a continuous map $T : Y \times \mathbb{R} \to C(0, 2\pi) \times C(0, 2\pi)$ such that, for $|c| \geq c_0$,*

$$T(y, \bar{v}) = (v_1(y, \bar{v}), v_2(y, \bar{v})), \tag{3.70}$$

satisfies the system in Equation 3.69.

For z_1, z_2 2π-periodic functions with $||z_i||_\infty \leq K$ for some constant K, we define $N_1(z_1, z_2)(r)$ as the right hand side of the first equation in Equation 3.69 and $N_2(z_1, z_2)(r)$ as the right hand side of the second equation in Equation 3.69. Also we write

$$N(z_1, z_2)(r) = (N_1(z_1, z_2)(r), N_2(z_1, z_2)(r)). \tag{3.71}$$

Let $z_1, z_2, \hat{z}_1, \hat{z}_2 \in L^2(\mathbb{R}^2/(2\pi\mathbb{Z} \times 2\pi\mathbb{Z}))$, and

$$W(x,t) = y(x,t) + \bar{v} + C\sin(x+t) + C\sin(x-t) \tag{3.72}$$

By the mean value theorem,

$$\begin{aligned} h(u(x,r-x)) - h(\hat{u}(x,r-x)) &= h(W(x,r-x) - z_1(r) - z_2(2x-r)) \\ &\quad - h(W(x,r-x) - \hat{z}_1(r) - \hat{z}_2(2x-r)) \\ &= h'(\sigma(x,r-x))\,([\hat{z}_1(r) - z_1(r)] \\ &\quad + [\hat{z}_2(2x-r) - z_2(2x-r)])\,, \end{aligned} \tag{3.73}$$

where $\sigma(x,t)$ is in the segment $[W(x,r-x) - z_1(r) - z_2(2x-r), W(x,r-x) - \hat{z}_1(r) - \hat{z}_2(2x-r)] \cup [W(x,r-x) - \hat{z}_1(r) - \hat{z}_2(2x-r), W(x,r-x) - z_1(r) - z_2(2x-r)]$. From the definition of N_1 and Equation 3.73, we have

$$\begin{aligned} &|N_1(z_1,z_2)(r) - N_1(\hat{z}_1,\hat{z}_2)(r)| \leq \\ &\int_0^{2\pi} |h'(\sigma(x,r-x))|(\|z_1 - \hat{z}_1\|_\infty + \|z_2 - \hat{z}_2\|_\infty)dx. \end{aligned} \tag{3.74}$$

Because

$$m\{x \in [0,2\pi]; |W(x,r-x) + z_1(r) + z_2(2x-r)| \leq D\} \leq O(c^{-1/2})\}, \tag{3.75}$$

$$\begin{aligned} &|N_1(z_1,z_2)(r) - N_1(\hat{z}_1,\hat{z}_2)(r)| \leq \\ &\leq O(c^{-1/2})(\|z_1 - \hat{z}_1\|_\infty + \|z_2 - \hat{z}_2\|_\infty). \end{aligned} \tag{3.76}$$

Similarly,

$$\begin{aligned} &|N_2(z_1,z_2)(r) - N_2(\hat{z}_1,\hat{z}_2)(r)| \leq \\ &\leq O(c^{-1/2})(\|z_1 - \hat{z}_1\|_\infty + \|z_2 - \hat{z}_2\|_\infty). \end{aligned} \tag{3.77}$$

From Equations 3.76 and 3.77 we see that N defines a contraction. Hence for each $(y,\bar{v})$ there is a unique pair $(z_1,z_2) \in C(0,2\pi)$ that satisfies Equation 3.68. Moreover $(z_1(y,\bar{v}), z_1(y,\bar{v})$ depends continuously on $(y,\bar{v})$.

Lemma 3.4. *If v_1, v_2 satisfy Equation 3.68 then*

$$\tau v + Q(h(y+v)) = Q(p), \tag{3.78}$$

where Q denotes the projection of $L^2(\Omega)$ onto N.

Proof. Because any function $\eta \in N$ can be written as $\eta(x,y) = \eta_1(x+y) + \eta_2(x-y)$, we will show that for $\eta = \phi,\ \psi$ as defined above,

$$\int_\Omega \eta\,(\tau v + Q(h(y+v)) - Q(p))\,dxdt = 0. \tag{3.79}$$

Because linearity implies that if the above equation holds, then it holds for simple functions as well, dominated convergence implies that it in fact holds for all $\eta \in N$. We have

$$
\begin{aligned}
\int_\Omega \phi Q(p) dxdt &= \int_\Omega \phi p dxdt \\
&= \int_0^{2\pi} \int_0^{2\pi} \chi_{[r,s]}(x-t) p(x,t) dtdx \\
&= \int_0^{2\pi} \chi_{[r,s]}(v) \left(\int_0^{v} p(u, u-v) du \right) dv \\
&+ \int_{2\pi}^{4\pi} \chi_{[r,s]}(v) \left(\int_{v-2\pi}^{2\pi} p(u, u-v) du \right) dv \\
&= \int_0^{2\pi} \chi_{[r,s]}(v) \left(\int_0^{2\pi} p(u, u-v) du \right) dv \qquad (3.80) \\
&= \int_r^{s} \int_0^{2\pi} p(x, x-t) dxdt \\
&= \int_r^{s} \left[2\pi\tau(\bar{v} + v_2(t)) + \int_0^{2\pi} h(u(x, x-t)) dx \right] dt \\
&= \int_r^{s} \int_0^{2\pi} \left[\tau(\bar{v} + v_2(t)) + h(u(x, x-t)) \right] dxdt \\
&= \int_\Omega \phi \left[\tau v + Q(h(y+v)) \right] dxdt,
\end{aligned}
$$

where we have used a change of variables $u = x$ and $v = x - t$ as well as Equation 3.66. We obtain a similar result using ψ. $\square$

3.3.2 Equation In The Range

Lemma 3.5. *If v_1, v_2 satisfy Equation 3.68 then there exist $y \in Y, \bar{v} \in R$ such that*

$$\square y + \tau y + P(h(y+v)) = P(p), \qquad (3.81)$$

where P denotes the projection from $L^2(\Omega)$ to Y.

Proof. Fix $\bar{v} \in \mathbb{R}$. Define an operator $T : Y \to Y$ such that

$$T(y) = -(\square + \tau I)^{-1} P(h(y + \bar{v} + v_1(\bar{v}, y) + v_2(\bar{v}, y))). \qquad (3.82)$$

We will show that T is compact and maps $B_R(0)$ into itself for some $R > 0$. Schauder's fixed point theorem will then imply that T has a fixed point y satisfying the above equation.

Because $-(\Box + \tau I)^{-1}$ is compact and because P is continuous, to show that T is compact, we need only show that $S : Y \to L^2$ given by

$$S(y) = h(y + \bar{v} + v_1(\bar{v}, y) + v_2(\bar{v}, y)), \tag{3.83}$$

is continuous. To see this, note that for $y, y' \in L^2(\Omega)$, we have,

$$\begin{aligned} ||S(y) - S(y')||_{L^2} = &\int_\Omega [h(y + \bar{v} + v_1(\bar{v}, y) + v_2(\bar{v}, y)) \\ &-h(y' + \bar{v} + v_1(\bar{v}, y') + v_2(\bar{v}, y'))]^2 \, dxdt \\ &= \int_\Omega \left[h'(\xi(\bar{v}, y)(x,t)) \left([y - y'] \right.\right. \\ &+ \left.\left. [v_1(\bar{v}, y) - v_2(\bar{v}, y')] + [v_2(\bar{v}, y) - v_2(\bar{v}, y')]\right)\right]^2 dxdt \\ &\leq M \left(||y - y'||_{L^2} + ||v_2(\bar{v}, y) - v_2(\bar{v}, y')||_{L^2} \right. \\ &+ \left. ||v_2(\bar{v}, y) - v_2(\bar{v}, y')||_{L^2}\right), \end{aligned} \tag{3.84}$$

where M is a bound on h'. Because the maps $(\bar{v}, y) \mapsto v_i(\bar{v}, y)$ are continuous in $L^2(\Omega)$ we see that S is continuous.

Now recall that $-(\Box + \tau I)^{-1}$ is bounded on Y, with operator norm N. As such, we have that

$$||T(y)||_Y \leq N||S(y)||_{L^2} \tag{3.85}$$

But

$$||S(y)||_{L^2} = \int_\Omega h(y + \bar{v} + v_1(\bar{v}, y) + v_2(\bar{v}, y))^2 dxdt \leq 4\pi^2 M^2 \tag{3.86}$$

where M is a bound on h. Thus setting $R = 4\pi^2 M^2 N$, we have that

$$||T(y)||_Y \leq R, \tag{3.87}$$

and we are done. $\Box$

Chapter 4

Results

4.1 Proof of Main Regularity Result

We now proceed to prove the following theorem.

Theorem 4.1. *There exists $c_0 \geq 0$ such that if $|c| > c_0$, h is defined as in Caicedo and Castro (2009), and $p(x,t) = c\sin(x+t)$, then Equation 1.6 has no continuous (weak) solution.*

Proof. This is the result originally presented in Caicedo and Castro (2009). Assume that there exists a continuous solution to Equation 1.6. We will produce a contradiction. Without loss of generality, consider the case where $c > 0$. For sufficiently large c, we have that

$$a\alpha_1\left(0, \frac{\pi}{2}\right) + z_1\left(\frac{\pi}{2}\right) > D, \tag{4.1}$$

and

$$a\alpha_1\left(0, \frac{3\pi}{2}\right) + z_1\left(\frac{3\pi}{2}\right) < 0, \tag{4.2}$$

and so the intermediate value theorem furnishes t_1, t_2 with $\pi/2 < t_1 < t_2 < 3\pi/2$ and so that

$$a\alpha_1(0, t_1) + z_1(t_1) = D/2, \tag{4.3}$$

and

$$a\alpha_1(0, t_2) + z_1(t_2) = 0. \tag{4.4}$$

But from Equation 3.63, we see that

$$\tau z_1(t_1) = -h(D/2) + O(c^{-1/2}). \tag{4.5}$$

It follows that

$$a\alpha_1(0,t_1) = \frac{D}{2} - z_1(t_1) = \frac{D}{2} + \frac{h(\frac{D}{2})}{\tau} + O(c^{-1/2}) < 0, \tag{4.6}$$

from the above conclusions. However, Equation 3.63 implies that

$$\tau z_1(t_2) = -h(0) + O(c^{-1/2}), \tag{4.7}$$

which implies that

$$\begin{aligned} a\alpha_1(0,t_2) &= -z_1(t_2) && (4.8)\\ &= O(c^{-1/2}) && (4.9)\\ &> O(c^{-1/2}) + \frac{1}{2}\left(\frac{D}{2} + \frac{h(\frac{D}{2})}{\tau}\right) && (4.10)\\ &> a\alpha_1(0,t_1). && (4.11) \end{aligned}$$

But because $t \to \alpha_1(0,t)$ is a decreasing function of t on $[\pi/2, 3\pi/2]$, since $\alpha_1(0,t) = \sin(t)$, we have a contradiction. □

4.2 Proof of Novel Regularity Result

Theorem 4.2. *Let h be as in Caicedo and Castro (2009) but instead of requiring compact support, assume that $h(t)t^\alpha \leq M_0$ for some constants M_0, $0 < \alpha < 1$ and all $t \in \mathbb{R}$. Then there exists $c_0 \geq 0$ such that if $|c| > c_0$, h and $p(x,t) = c\sin(x+t)$, then Equation 1.6 has no continuous (weak) solution.*

Proof. The full proof is mostly a repetition of the arguments used to prove the Caicedo and Castro (2009) result. Minor changes are needed to establish that

$$m\{(x,t) \in \Omega : h(u(x,t)) > c^{-1/2}\} \leq \frac{m}{c}, \tag{4.12}$$

which then replaces the corresponding equation in the proof. The rest of the result is fundamentally identical. □

4.3 Proof of Existence Result

Theorem 4.3. *There exists c_0 such that for $|c| \geq c_0$ and $p(x,t) = c(\sin(x+t) + \sin(x-t))$, Equation 1.6 has a continuous (weak) solution.*

Proof. This is just a combination of previous lemmas. First, Lemmas 3.3 and 3.4 imply that for each $(y, \bar{v})$, there exist v_1 and v_2, depending continuously on y and $\bar{v}$, that satisfy Equation 3.68. Moreover, if $u = y + \bar{v} + v_1 + v_2$, then u satisfies the projection of Equation 1.6 in the kernel N.

Next, Lemma 3.5 implies that for $|c| \geq c_0$, there exist $y \in Y$ and $\bar{v} \in \mathbb{R}$ so that the projection of Equation 1.6 in the range Y is satisfied. Because $H = Y \oplus N$, this completes the proof. $\square$

Chapter 5

Conclusion

5.1 Concluding Remarks

We have successfully shown that Equation 1.6 has no continuous solution for $p(x,t) = c\sin(x+t)$ with c sufficiently large. Moreover, we have extended the result to account for asymptotically linear $g(t)$. Finally, we have shown that in the alternative case that $p(x,t) = c(\sin(x+t) + \sin(x-t))$, Equation 1.6 does in fact possess a continuous solution. Future work will hopefully examine the case of general forcing $p \in N$, where $p(x,t) = q_1(x+t) + q_2(x-t)$. Under fairly relaxed conditions, we speculate that such forcing will generate continuous solutions to Equation 1.6.

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