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Review: The Spectrum of Some Compressions of Unilateral Shifts

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A Banach space E of analytic functions on $\mathbb{D}$ is called star-shaped if $\lim_{r\to 1^-}\|f^{[r]}-r\|=0$ where $f^{[r]}(z)=f(rz)$ for $r\in[0,1]$. Let $Sf=zf$ and $Tf=(f-f(0))/z$ denote the forward and backward shifts, respectively, acting on some fixed E. Finally, if M is a z-invariant subspace of E, let S_M denote the operator defined by $S_M(f+M)=Sf+M$ for $f\in E$.

The main result of the article is the following.

Theorem 4.3. Let E be a star-shaped Banach space of analytic functions on $\mathbb{D}$ satisfying

$$\sum_{n\geq 0}\frac{\log^+\|S^n\|+\log^+\|T^n\|}{1+n^2}<+\infty$$

and let M be a z-invariant subspace of E such that $\dim(M/zM)=1$. Let $\zeta\in\mathbb{T}$ and $r>0$; if there exists $f\in M$ having an analytic extension to $\mathbb{D}\cup D(\zeta,r)$ and such that $f(\zeta)\neq 0$, then $\zeta\notin\operatorname{Spec}(S_M)$.

For $\alpha\in(\frac{1}{2},1)$, consider the weighted Hardy space $H^2_{\sigma_\alpha}(\mathbb{D})$ corresponding to the weight sequence $\sigma_\alpha(n)=e^{-n^\alpha}$. Although no classification of z-invariant subspaces M of $H^2_{\sigma_\alpha}(\mathbb{D})$ satisfying $\operatorname{Spec}(S_M)=\{1\}$ has been obtained, partial results and examples have been obtained by A. Atzmon [in *Proceedings of the Ashkelon Workshop on Complex Function Theory (1996)*, 37–52, Bar-Ilan Univ., Ramat Gan, 1997; MR1476702 (98k:30036)] and A. A. Borichev, H. Hedenmalm and A. L. Vol′berg [J. Funct. Anal. **207** (2004), no. 1, 111–160; MR2027638 (2005d:30052)]. In [*Selected problems of weighted approximation and spectral analysis* (Russian), Izdat. "Nauka" Leningrad. Otdel., Leningrad, 1974; MR0467269 (57 #7133a)], N. K. Nikol′skiĭ proved that the functions

$$\psi_{\alpha,c}(z)=\exp\left(\sum_{k=0}^{\lfloor\frac{\alpha}{1-\alpha}\rfloor}\frac{b_k}{(1-z)^{\alpha/(1-\alpha)-k}}-\frac{1}{(1-z)^c}\right)$$

for $c\in(0,1)$ generate nontrivial z-invariant subspaces $M_{\alpha,c}$ of $H^2_{\sigma_\alpha}(\mathbb{D})$. It follows immediately from Theorem 4.3 that $\operatorname{Spec}(S_{M_{\alpha,c}})=\{1\}$.

Reviewed by *Stephan R. Garcia*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.