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Algebraic Reasoning in Elementary School Students

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Abstract

An exploratory study on instructional design for classroom activities that encourage algebraic reasoning at the elementary-school level. Assistance with the activities was provided as students needed further scaffolding, and multiple solutions were encouraged. An analysis of student responses to the activities is discussed.

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Acknowledgments

I would like to thank my advisors Professors Rachel Levy and Stacy Brown for all their help. Also thanks to the great people at Sycamore Elementary School and the members of the Claremont Unified School District who gave me the opportunity to work with such amazing learners.

For Jossue "Josh" Colato, a great friend who touched many hearts and had limitless potential.

Chapter 1

Introduction

Some of the earliest known practices of any type of algebraic reasoning can be found about four thousand years ago (Katz and Barton, 2007). At this point, mathematics was communicated almost entirely in spoken language, with little or no form of symbolization. Examples were generally offered as methodologies using specific quantities rather than variables. While there was much development of the type of problems that could be solved in this era, the presentation style remained fairly static.

Fast-forward to twelfth-century Europe, and we begin to see the early emergence of symbolic algebra and more sophisticated enumeration systems. However, in this era, answers less than zero and answers containing complex numbers were not accepted, so many problems were considered unsolvable.

Eventually, as a result of the work of François Viète (1540–1603) and Réne Descartes (1596–1650), a completely symbolic mathematical language has been established. From then on, math could be both reasoned symbolically as well as logically. Why is this important? Historically, it has taken thousands of years to get to a symbolic approach for solving mathematical problems. Some researchers believe this leap in representation to be related to why younger students may not automatically reach for algebraic methods to solve problems, and may have trouble when they first encounter algebra in school (Carraher et al., 2006).

For the better part of the twentieth century algebra has been thought of as the next level after arithmetic. As such, many curricula delay introducing students to algebraic concepts until later in their academic lives (middle or high school). The jump to using abstract symbolic notation and other forms of functional thinking was large enough to warrant a transitional

prealgebraic course (Carraher et al., 2006), but a growing body of research shows that students as early as the first grade can grasp these concepts (Blanton and Kaput, 2007). Furthermore, these studies indicate that concurrently introducing both arithmetic and algebra during the elementary school levels assists the necessary jump required to master algebra (Schliemann et al., 2007; Carraher et al., 2006).

Algebraic thinking is an umbrella term that encompasses many different types of thought. Most of these types of thinking can be divided into one of two forms: relational thinking and functional thinking (Blanton and Kaput, 2005).

Relational thinking, termed *generalized arithmetic*, deals with properties of numbers or expressions. These propeties includes relationships of parity, ideas of equality, and properties of inverses. For example, $\frac{2}{5}x = \frac{2x}{5}$.

Functional thinking involves understanding, utilizing, or synthesizing relationships in any of a number of forms, including symbolically, graphically, or as a table. For example, given $y = 2x + 1$, then if $x + 3$, $y = 7$.

In this study, I will be focusing primarily on functional thinking in fifth-grade students who have not been formally introduced to algebra. My expectation, based on previous studies, was that the students would demonstrate a good level of algebraic intuition. From personal experience, many students around the fifth-grade level can demonstrate relational thinking. In this study I also wanted to explore the degree or the level of functional thinking that students could engage in.

Chapter 2

Sycamore Elementary School

The school where I conducted my study was Sycamore Elementary School in the small college town of Claremont, California. Qualities that make Sycamore unique include

Multiaged Classrooms Unlike most traditional schools, where most of the students in a classroom are around the same age, classrooms at Sycamore will often have students spanning two to three grade levels. For example, all the students from this study come from classes having students from grades four through six. The classroom structure provides the possibility for multiaged social interactions in the entire population.

Individualized Learning Levels Because there are multiple levels in a class, students can join the group working at their level. This pace allows students who need to spend a longer on certain areas the time to catch up, and those who are excelling to move ahead or to help those in need.

Cooperative Group Learning Many of the activities that take place encourage working with others and sharing ideas openly.

Emphasis on Community Being a small town school, many students live in the surrounding neighborhood and walk or ride a bike to school. Also, many members of the community, including some from the local colleges, volunteer at the school. These visits give the students more focused attention as well as opportunities to view the material from a perspective other than that of the primary teacher.

I chose Sycamore as the site for this research because I had worked at the site as part of a service component for a mathematics for teachers course and continued volunteering after the course ended. I had thus established a relationship with some of the students and teachers at the school prior to the study. However, none of the students I had worked with in prior years were a part of this study. It is notable that during the year of the study a new curriculum was adopted, Everyday Mathematics, and a new principal, Amy Stanger, was hired.

Chapter 3

Methodology

A good portion of my research time was spent obtaining school, institutional review board (IRB) and school district approval for this study. During this time, I developed lesson plans, met with the principal, reviewed mathematics education literature, and developed forms to satisfy the requirements of the Claremont Colleges' IRB. These forms can be reviewed in Appendix A.

In the spring, two groups of six students were chosen from fifth-grade classes at Sycamore Elementary school by their teachers. One group met on Tuesdays and the other on Thursdays for six weeks. The groups participated in weekly activities lasting about 45 minutes. Each student was asked to keep all their work in a composition notebook, and was asked to write down their thoughts on a problem before discussing their answers. At the end of each lesson, I asked the students to take a moment to write down what they learned. All the sessions were videotaped with the participants' knowledge. This structure is influenced by the work of Elizabeth Warren (2006).

Originally, the purpose of having two cohorts was to allow the use of a different teaching style in each. One cohort would have me acting as an instructor, directly leading the students to their findings, while still attempting to present as many opportunities to learn as possible. For the other, I would use a more hands-off, Socratic, methodology. To determine the effectiveness of each, I planned to see how quickly students would solve the problems and how many different solution strategies they would generate. I would also see how well they could articulate their findings, and to what degree they were able to recollect techniques and reapply their knowledge.

My original plan proved to be more difficult then expected. After I gave

students a particular set of instructions, they were only able to push though the problem so far. I had to step in occasionally or progress would stall and I would feel as though I was wasting the students' and school's time. At other times the students would become less attentive. Thus I focused more on how far each student could go based on my scaffolding of the problems or based on their own revelations. Also, I noted the subtle differences in my own instructions between the groups to see if their understanding could have been influenced by a different phrasing or presentation I may have used between the two groups.

To keep student identities as anonymous as possible, students in group one (Tuesday), will be referred to using letters. Students in the second group (Thursday), will be referred to with numbers. I will describe each weekly activity and provide an overview and analysis of selected responses in Chapter 4.

Chapter 4

Activities and Notes

This chapter contains a detailed description of the activities that the student groups completed. Following the general descriptions are the responses of the individual groups and examples of student work.

4.1 Lesson 1: Math Magic Tricks

The first activity was a set of mathematical magic tricks. I chose this activity mainly to gauge the students' understanding of algebraic relationships. I also thought it would pique their interest in the weekly groups. The magic tricks were sets of instructions that the students followed line-by-line starting with a number they chose.

The first algorithm was a simple one in which, regardless of what number a student initially picks, the output is five. The instructions were

> Pick a number, double it, add ten, divide by two, then subtract your original number.

I originally suggested picking a number from one through twenty to keep their calculations simple. When the students all tried their numbers and got five, I asked them why they thought they all got the same answer.

After the groups had presented their original thoughts on how the procedures might work, I proceeded to model the algorithm using different sized crayons as the manipulatives. Figure 4.1 shows responses to the various parts of this activity by Student D. Student D demonstrated his use of the model for the second algorithm I presented: pick a number, triple it, add nine, divide by three, subtract original number.

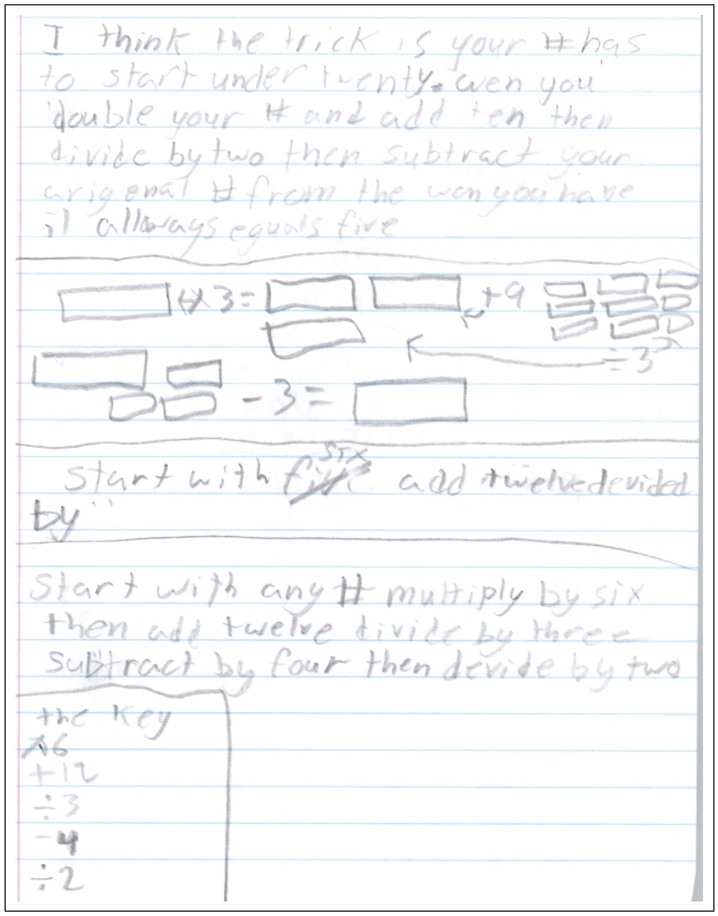

Figure 4.1: Work from Student D's notebook.

The manipulatives were useful, but they could have been more effective if there had been more than one type: different shapes and colors rather than simply two sizes of crayons. Because the manipulatives were distinguished by size and not necessarily by color, it was hard for some students to distinguish between what represented their variable and what represented their unit-number counters.

With the remaining time, I had the students try to develop their own math tricks and see if they could explain why their tricks worked.

4.1.1 Tuesday

After I showed them the first magic trick, I asked the students to model or try to explain why it worked. Perhaps because I did not give any examples on how to do so, most students could not come up with any explanation or model for the trick. Some responses included, "someone came up with a theory," by Student A; or simply a restatement of the steps, as shown in Figure 4.1 by Student D. A few of the students, because they were instructed to choose a number between one and twenty, thought that the limit was a necessary condition to have an output of five, but that idea was quickly overturned once they were asked to try larger numbers.

After I presented a way of modeling with the crayons, I gave the students a new problem and asked them to break the second problem into algebraic steps and analyze why it worked. Then they attempted to create their own math tricks. Many followed similar schemes to the problems I had previously presented. For example, Students A and C used quadrupling instead of doubling and tripling, whereas Student D, Figure 4.1, created one with an extra dividing step. Student B created her own trick with the form $a + b + c - b - c = a$.

4.1.2 Thursday

I again began by showing them the first trick and then giving them some time to explore why it worked. During their individual discussions, each group began to try to pick apart the steps to see what was going on. In one group, the discussion initially revolved around the parity of their new number after each step. At one point or another, both groups tried focusing on the range of values and the effects of doubling and adding ten. Eventually Student 1 realized that dividing by two was the key, in that it halved both the recently doubled original number and the added ten, which is where the five came from at the end. However, it seemed that the other group might have overheard this insight, which shifted their own conversation towards this idea almost immediately.

When creating their own algorithms, Student 1 used the fact that $n/\frac{n}{2} = 2$, forcing a two and allowing her free range on what happened thereafter. "Pick an even number and divide it by half that number." She added the restriction that n had to be even, although this method would work without it. Student 6 devised an unfinished algorithm that at best led to a range of possible values given a specific domain.

Overall, most students in both groups demonstrated some degree of

relational-thinking capability, mostly in terms of parity and additive or multiplicative inverses. As that was the goal, the results were a success.

4.2 Lesson 2: Popsicle Sticks and Squares

The next activity was one using an arrangement of Popsicle sticks in adjacent squares that share a side (see Figure 4.2). The problem is set by arranging three groups of sticks horizontally, one group making one square, one with two squares, and the last with three. I asked the students how many sticks were necessary to make each shape. Then I asked "How many sticks would be needed if I wanted four squares? What about six? One hundred?" I gave the students time to think about the problem to see what they would come up with.

I expected the students to notice the recursive additive property for the number of sticks quite readily. One of the things I hoped the students would be able to do was to represent the situation in a table with the number of squares in one column and the number or sticks in the other column. One thing I had to suggest to the students was to write the number of sticks as an unexecuted sum; see Figure 4.3. With or without the table, I wanted them to eventually be able to identify one of the many equivalent representations of the number of sticks in terms of the number of squares. Some examples include

$$\begin{aligned} S_n &= S_{n-1} + 3 \\ P_1 &= 3S + 1 \\ P_2 &= 3(S-1) + 4 \\ P_3 &= 4S - (S-1) \\ P_4 &= S + S + S + 1. \end{aligned}$$

From this, I observed where the discussion led.

4.2.1 Tuesday

I began by asking, "How many sticks for one track? Two? Three?" and so on.

Almost immediately, some students quickly assumed that the relationship would be four times the number of squares, so would answer 4, 8, 12, respectively. However, they quickly realized that these answers were not

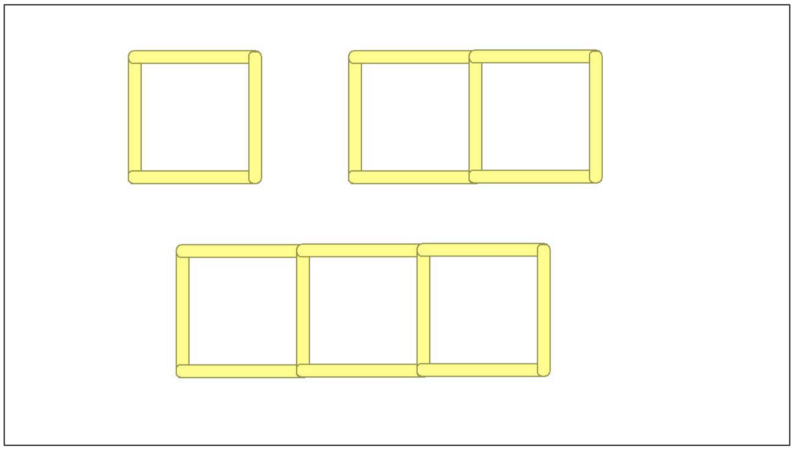

Figure 4.2: Initial arrangement of Popsicle sticks.

correct when another student directly counted. Student E quickly recognized the relationship as

$$\text{the number of sticks} = 1 + 3 \times \text{the number of squares},$$

showing great functional thinking. The other students did not hear his explanation, so the relationship was not given away, which allowed them to further explore the task for themselves.

It wasn't long until the students realized the recurrence of adding three to the previous number of sticks. Thus, one response to, "How many sticks are needed for one hundred squares?" was, "Don't you have to know what 99 is? 'Cause aren't you just adding three the whole time?" by Student C. My task at this point was to get the rest of the students to make the connection between repeated addition and multiplication, as Student E had done earlier.

My next instruction was to construct a table with one side having the number of squares and the other side the number of sticks. Also, I wanted them to write the number of sticks as a sum based on the previous result. Not everyone was experienced with this type of representation, but we still made some progress. After explicitly showing the repeated addition by means of a table, the link to multiplication was quickly realized by most students.

The next task involved tracks that had double the height. I asked a similar set of questions.

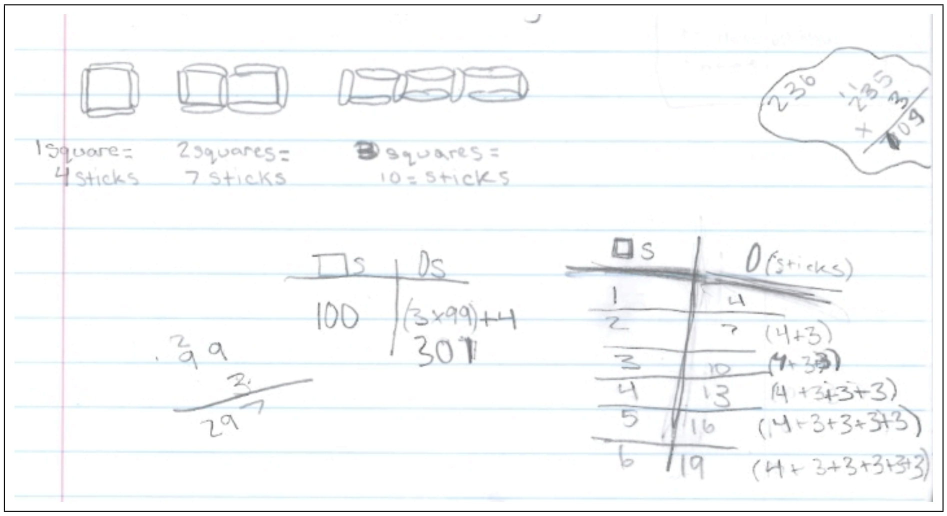

Figure 4.3: Student C's work on the Popsicle-stick problem, including her table with unexecuted sums.

There was some interesting talk about the best way to count the constructions in the beginning. One method that came to light was that, after we have two sticks in a line, for each length of track we add an E shape made of five sticks, so the formula that everyone determined was

$$2 + 5 \times \text{number of squares}.$$

As a final question, I asked, "Keeping this same construction, suppose we only had 52 sticks. How many tracks could we make?"

One student continued his table, using a form of repeated addition. He continued until he had 52 on the number of sticks side, thus determining his answer was ten. Many of the others divided 52 by five, getting ten with a remainder of two, to which Student C commented, "You need the two to close up the shape." Her comment was interesting because it also demonstrated some ability to manipulate a formula.

4.2.2 Thursday

Recalling some of the possible solutions for this problem I expected to see.

$$
\begin{aligned}
S_n &= S_{n-1} + 3 \\
P_1 &= 3S + 1 \\
P_2 &= 3(S-1) + 4 \\
P_3 &= 4S - (S-1) \\
P_4 &= S + S + S + 1
\end{aligned}
$$

Again, I began by asking, "How many sticks for one track? Two? Three?" and so on.

Student 5 quickly answered four for one track. For two tracks, Student 3 initially thought the answer was eight, but quickly said, "Wait, nope!" and Student 6 corrected with seven, as Student 3 pointed out, "Because they share the middle one." For three tracks, I asked for a volunteer to reconstruct the three track case; Student 5 did so by appending three more sticks to the two-track construction, answering ten sticks needed for three tracks. For four, five, and six tracks, I quickly got the correct answers without needing to construct them.

I asked, "What kind of patterns are you guys seeing?" Student 3 verbally, and Students 5 and 6 in writing, all pointed out the add-three recurrence.

Students 1, 3, and 4 all came up with strategies similar to Equation P_3 above. For example, Student 4 explains, "If you make three tracks like that," pointing to the three-track construction, "you would subtract two because these two are sharing," pointing specifically to the center sticks. "Umm, they are put together to make one."

Both Students 5 and 6 used strategies modeled by Equation P_2. As Student 5 explains, "To make one you use four, and then after that you just add three each time."

I then asked each of them to determine how many Popsicle sticks were necessary to make one hundred tracks. Everyone was able to get the correct answer of 301 with their individual methods.

I wrote the two methods that were used on the board, and included two other methods that they did not come up with, Equations P_1 and P_4. I then asked them to try to explain if and why the various methods gave the same answer, regardless of the inputs.

In general, most explanations consisted of taking particular values for the number of squares and verifying that the different formulas gave the same answers.

Student 6, when comparing Equations P_2 through P_4, explained, "Because, when you're adding three times S minus one, when you're adding the four you're adding back the three and just adding one, and when you do it that way, you're just adding one because you're not taking anything away."

4.2.3 Afterthoughts

An interesting divergence occurred between the two groups in terms of trajectory. In one group, they got a bit more practice with creating connections between inputs and outputs, having done two examples instead of one. In the other group, the students demonstrated a model of how to write out a function using symbolic notation. This divergence actually created an opportunity to test out each group's ability to model a problem, which was continued with the next session.

4.3 Lesson 3: Table-Seating Problems

For this activity, the students were asked to to determine a relationship between the number of trapezoidal tables in a particular arrangement to the number of people who could comfortably sit around such a table arrangement. For this activity, to help the students physically represent the situation, I had them use trapezoidal pattern blocks for the tables and unit cubes to represent the people.

4.3.1 Tuesday

Again, I began by asking the students for a few initial cases, and then asked them directly what pattern they recognized. Initially, there was some argument about whether the pattern was plus four or plus three. As Student D explains, "You're basically adding four [chairs], but taking away one from the other [tables]."

When all the students were in agreement what the recursive pattern was, I next asked them about the case of one-hundred tables. Student C went first, stating that the answer would be "five plus ninety-nine threes," which got her to the correct answer of 302. I proceeded to write down $5 + 3(T - 1)$ on the board, although I used a trapezoid instead of T. Next, Student E offered his counting method of counting the number of tables at the end first, followed by three times the number of tables in the middle, and so I wrote $4 + 4 + 3(T - 2)$.

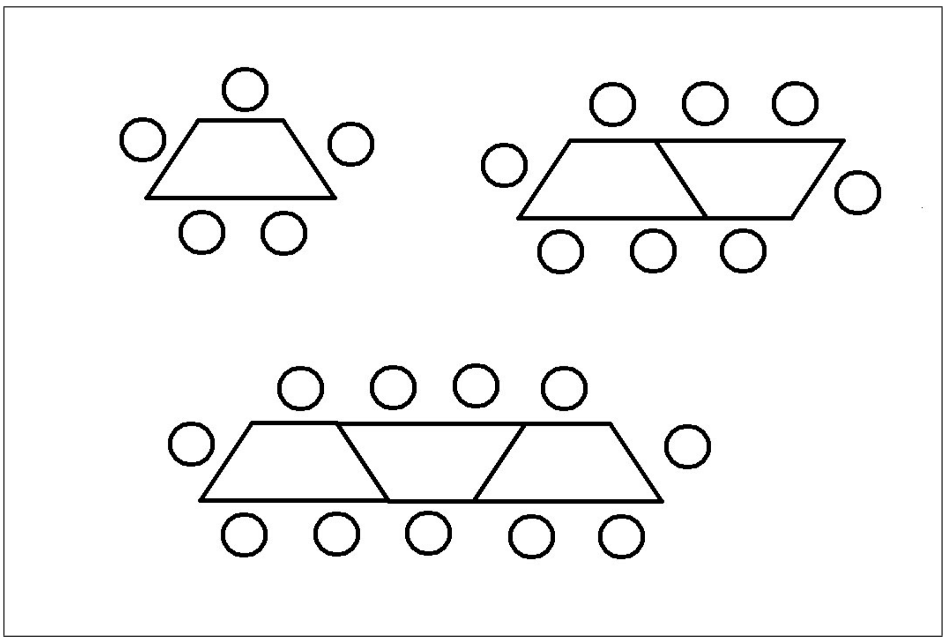

Figure 4.4: First three arrangements of table-seating problem.

Most students had some idea of getting at least one closed form of the problem, so for the rest of the hour, Professor Levy and I led them through other possible ways of counting the tables. For example, Professor Levy led them through the $3 \times T + 2$ by "fixing the answer." This strategy involves assuming that the formula uses multiplication, say, $3 \times T$, and adds to it how much it is off from the actual problem, in this case two.

4.3.2 Thursday

Student 1 was the first to explicitly state how similar this activity was to the Popsicle-stick problem, but was a bit confused as to why. When asked how many people can sit around one hundred tables, there was a tentative answer of three hundred, to which I asked them, "Why?" The first explanation was one similar to Student D's revelation, which led to the form $3 \times T + 2$. Student 3 tried to explain a different method that took into account all the chairs that would be lost once the tables were stuck together; in other words $5 \times T - 2 \times (T - 1)$. She seemed to realize that her solution simplified to the previous method, and adopted the simpler method thereafter.

I spent time trying to connect the different methods by trying to point out some simplification methods, without making much headway in terms of the students' understanding. Because we had some time left, I changed to using hexagonal tables and followed the same instructional steps used before.

4.3.3 Afterthoughts

This activity was very similar to the previous one, and it seemed that more students moved towards a functional relationship more readily then in the previous task. At this point, it seemed most students were getting a hold of generating closed-form formulas for linear relationships. It would be some time before we returned to linear problems, but my hope was that they would be able to retain these ideas until then.

4.4 Lesson 4: Handshake Problem

The students were asked to determine the relationship between the number of people and the number of handshakes necessary so that every one of them would shake each other's hand exactly one time. Seeing that the

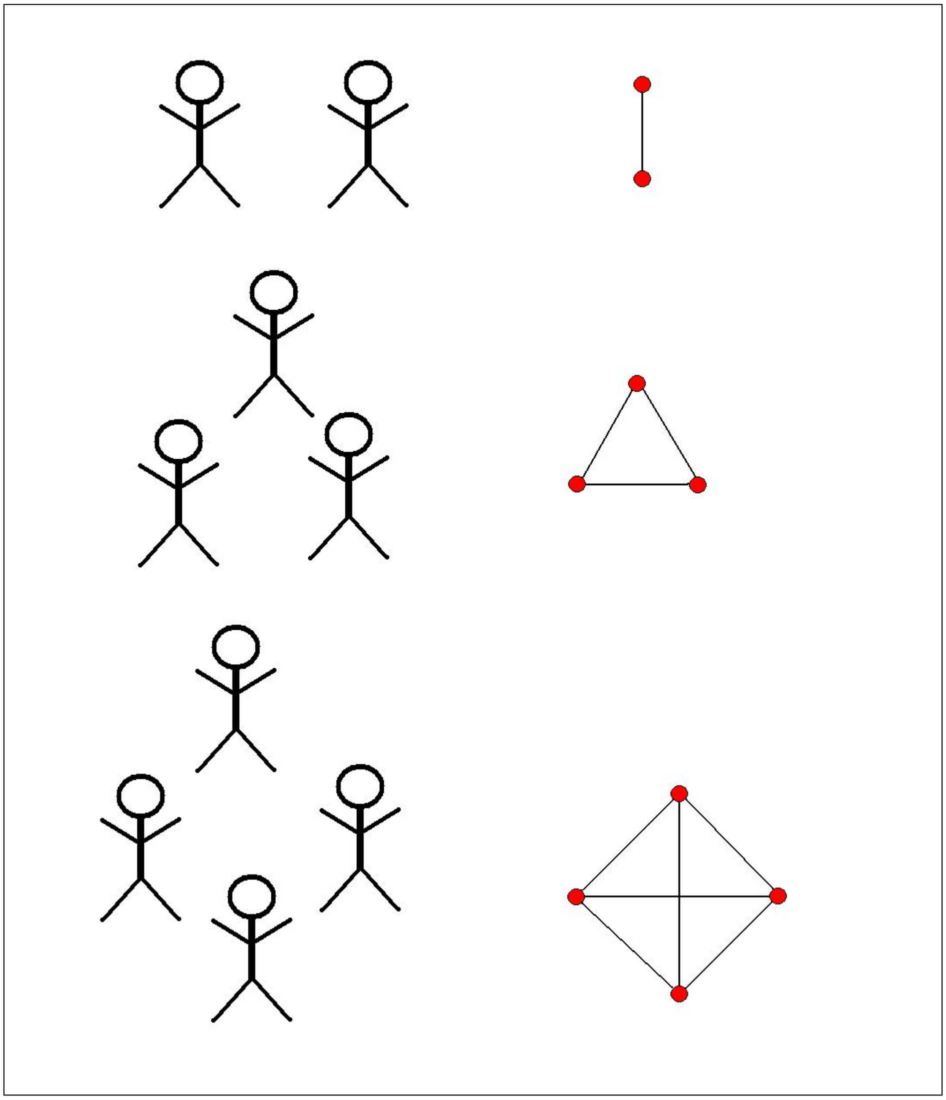

Figure 4.5: A possible way of representing the handshake problem.

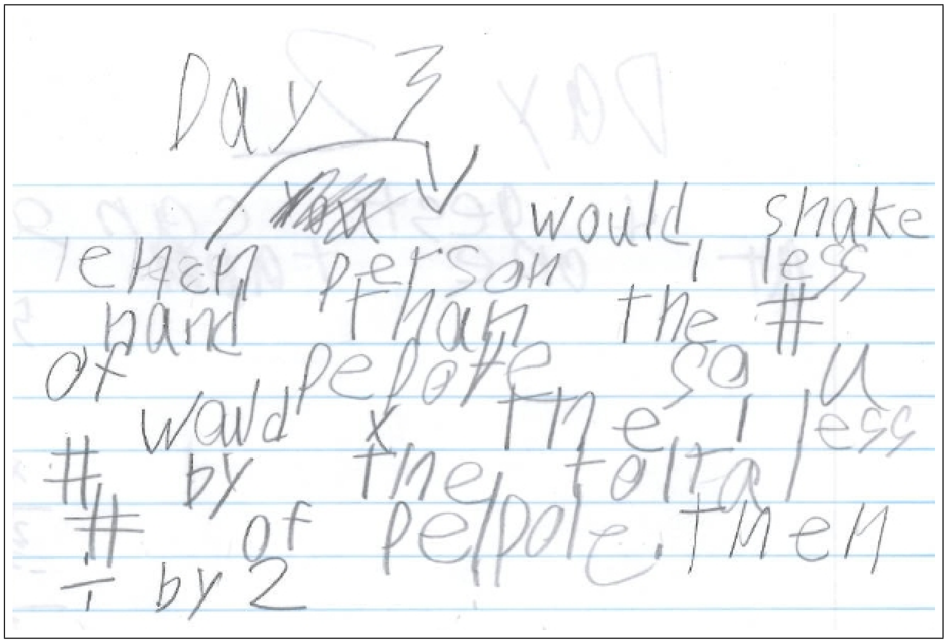

Figure 4.6: Excerpt of student's work showing a closed formula.

function is quadratic, $h = \frac{p(p-1)}{2}$, the students were expected to identify the recurrence relationship, but not necessarily the closed-form expression right away. I could then lead the students to different trajectories, such as connections to triangle numbers or connections between their own varied responses.

4.4.1 Tuesday

As usual, I began with asking "How many handshakes for two people? Three people? Four people?"

For two people, everyone quickly answered one. For three people, there was a quick guess of two handshakes by some, three by others. After suggesting for them to simulate the situation, an consensuses of three came quickly. When I asked for four people, there was an initial guess of four. Then the answers of five and six were jumping around. Again, I had the students simulate the situation, arriving at six. Just to have all cases covered, I also asked the situation for one person, which everyone quickly answered zero.

Then I asked, "What is the pattern you are seeing?"

Some initial guesses included, "Is it just plus three every time?" and, "There is no pattern." However, Student D excitedly exclaimed, "It's one, plus two, plus three, plus 4. . . !" Some more experimentation helped everyone arrive at the same conclusion. When asked why this pattern was the case, Student 5, after some false starts, answered, "Because there is already five people here, and one more person, and those five people have to shake his hand one time. . . ."

I next had them work to find some formula to describe the situation, with no real progress. So instead I led them through a discussion of the triangle numbers, $T_n = \frac{n(n+1)}{2}$, which has a similar solution to the handshake problem.

4.4.2 Thursday

To start off with the general case, after describing the handshake problem, I asked, "How many handshakes for a given number of people?"

The first question the students wanted answered was exactly how many people were shaking hands. I refused to give them a definite answer. Most quickly settled on six people, presumably because that was the size of the group. An initial guess was the number times itself, but then Student 5 suggested the answer would be the number of people times itself minus one, a response somewhat closer to a valid solution. The Student 5's reasoning was, "They don't shake their own hand." Most others quickly anchored onto solutions similar to this one.

Next I asked them if they could model or demonstrate the situation.

Student 3 originally modeled this situation by having hexagons represent people, and stacking squares on top representing the handshakes, implying a entirely multiplicative solution. As Student 1 described it, "Everybody shakes five hands because they wouldn't shake themselves."

I simplified the problem by asking them to solve the handshake problem with only three people. Student 3 quickly answered "Fifteen! No wait. . . ." Soon, most participants, including Student 3, came up with six. In other words; they still believed their original method was working. To clarify for them, I asked, "If I shook your hand, and you shook mine, how many total handshakes are there?" everyone who replied answered one, so they seemed to understand the problem vaguely, but they were not yet taking this same answer into account in their calculations for the case with three people.

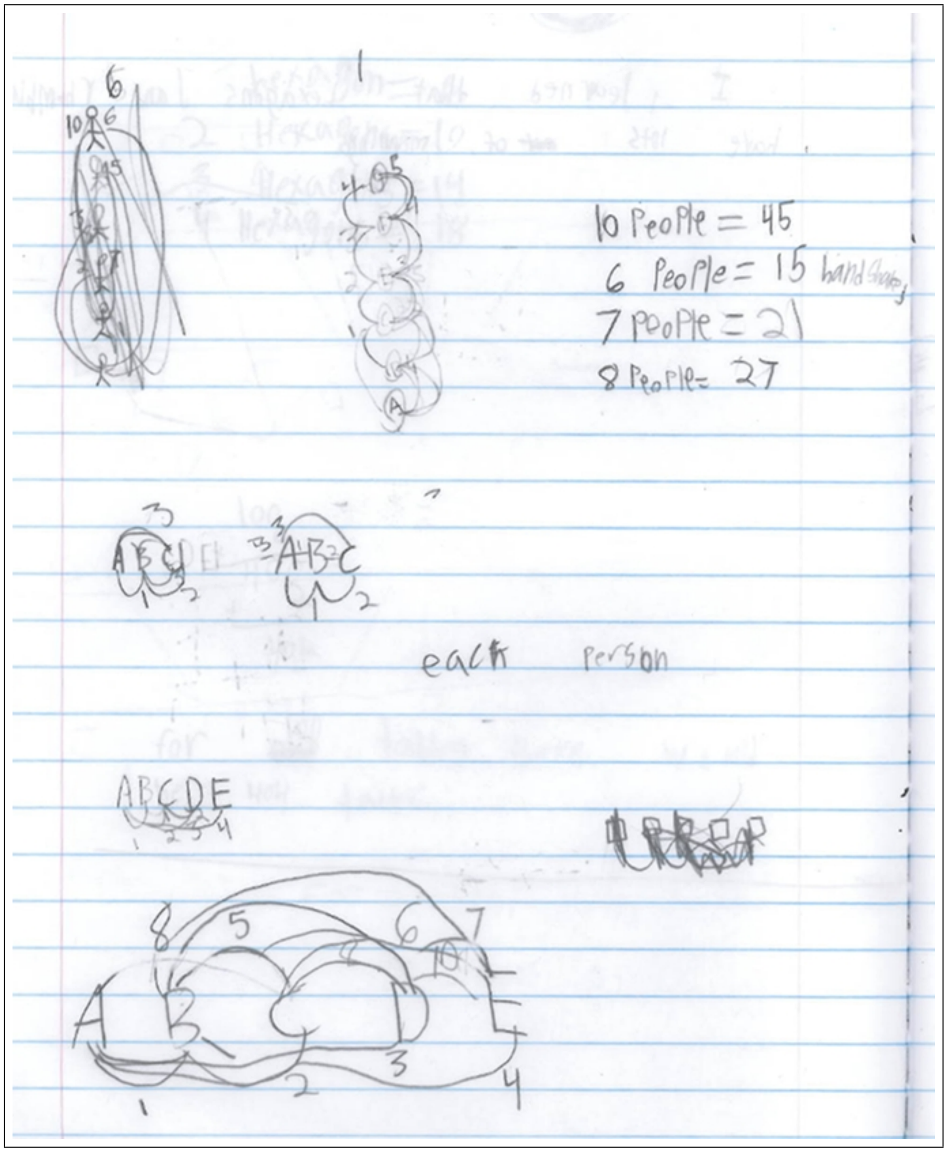

Figure 4.7: Example of Student 3′s counting method for the handshake problem.

Student 2 was one of the first to get a different answer: "I got three." Looking at his notebook, I noticed that his method was assigning each person a letter, then connecting each person to each other with a line once, keeping track of the connections.

I next had some volunteers demonstrate the five-person handshake problem. After some trouble in both bookkeeping and double handshaking, the group eventually came up with ten handshakes, the correct value. Student 5 asked, "Why is ten the answer?" still unsatisfied that his method was not working. Here, I did something I probably should not have, practically giving him the solution; namely that he always had double the actual answer due to double counting, and all he had to do was divide his answer by two. From this point on, Student 5 would cling to this method, ignoring the recursive element of the problem entirely.

After some experimentation, many of the students (as demonstrated through a variety of different methods) realized that they were adding the integers from one to the total number of people less one.

I asked the students to write their strategies up on the board.

Student 5 wrote formulas for both the generic situation and the specific case of five people. Almost everyone else had some variation of adding up the numbers one through four. An interesting note was that Student 4 was the only one to add up the numbers in different orders.

4.5 Lesson 5: Regions on the Plane

The object is to determine the amount of regions created by a certain number of intersecting lines. The lines must all intersect each other and no more the two lines can intersect at any point. I began by showing them the first four cases, including the case with zero lines, and let them explore from there.

I expected that the students would discover the recursive pattern, namely that the number of regions for N lines is N plus the number of regions for $N - 1$ lines, or something to that effect. A more advanced response would have been the students being able to recognize a connection to the handshake problem or triangle numbers, maybe even suggesting that the closed form would be one plus the closed form of the triangle number, $N(N+1)/2+1$.

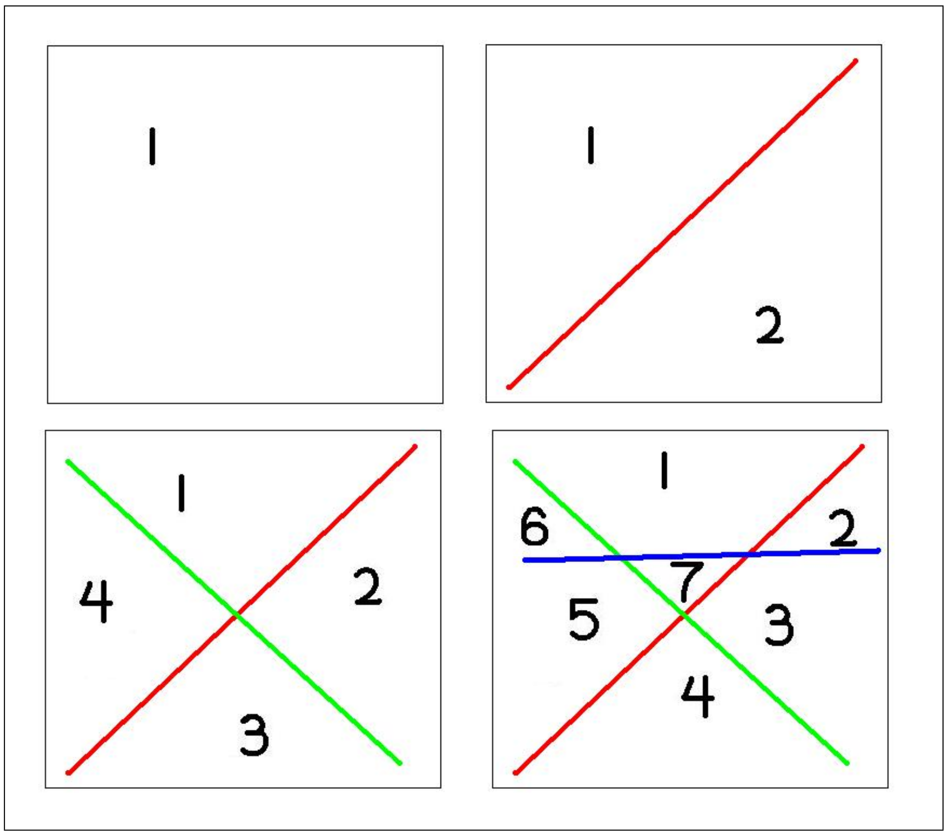

Figure 4.8: First several iterations of the region-on-the-plane problem.

4.5.1 Tuesday

After describing the problem and showing them how the cases of one, two, and three lines appeared, I let the students explore on their own.

Student C guessed at the pattern by the fourth case, not wanting to check any other situation. I asked her directly to compare the number of regions to the number of handshakes. She responded, "They're different numbers." Then she flipped pages back and forth and said, "They're one less," referring to the number of handshakes. I then asked her, "If you had a formula for the handshake problem, what would you add to make it work for the new situation?" to which she responded, "Add one to each one." She appeared a bit frustrated so I moved on to the other students.

Many discrepancies between the students' theories arose from the difficulty with constructing a valid depiction of the regions; either because some of the lines ended up being parallel or making intersections too close to one another. Because of these differences, many of the other students had reservations about suggesting a pattern. For example, Student D thought the pattern began by adding one and two for the first two lines, respectively, and three for the rest. Later, he got the general pattern, but was off by one on the three-line case, so the number of regions he got continued to be off by one for four lines and on.

To try to get everyone on the same page, I decided to create my own table for everyone to see. I put up a table with the correct number of regions based on lines up to the case of five lines. I then had everyone compare these values to those found during our exploration of the handshake problem and triangle numbers.

4.5.2 Thursday

This group had many of the same troubles that the first group had. I tried not to coach them as much as the first, but still ended up having to provide the table with the correct answers. There were no significant differences between this group and the previous one. We ran out of time before we were able to make the comparison to the handshake problem.

4.6 Lesson 6: Name that Rule

This was an activity all the students were familiar with. It involved showing them a flashcard containing a table of values and challenging the students to determine what was "the rule" to get from the values on the left

X	Y
1	9
2	11
3	13
4	15
5	17
10	27
15	37
X	?

Figure 4.9: Example of a name that rule card. The rule is $y = 2x + 7$.

to the values on the right. The rules began with only addition or multiplication of a single constant, sometimes the generic constant a. The last few cards contained linear functions with nonzero constant terms, much like the activities the students had worked on before. In essence, this activity was an abstraction of what the students had been previously working on.

4.6.1 Tuesday

I did not remind everyone to write down their responses before the discussion, so many students shared their initial thoughts or guesses, which probably influenced their responses. Everyone was easily able to determine the rules for additive rules. There were slightly more trouble from multiplication-only rules, but still no real difficulties. I also made the suggestion at this point to phrase their answers as, "Y equals something," which they were all able to do. When the constant was just the general a itself, the students did well.

When the rule was of the form $y = ax + b$, we hit a bit of a snag. The students initially tried to find patterns within a quantity. For example, when the rule was $y = 2x + 7$, the students first commented that the rule was, "Plus eight, plus nine, plus ten...," but with a bit of struggling managed to come up with a proper answer (although some peeked inadvertently.) As the problems included increasingly larger numbers and subtraction, the students were lost until I suggested that they consider the patterns they saw within quantities before and compare the rules for across the patterns, similar to "fixing the answer" introduced earlier in the table-seating problem. After this suggestion, they were able to solve all of the remaining problems quite easily.

4.6.2 Thursday

I made sure that everyone wrote down their responses before I would allow them to answer aloud. However, I didn't insist on the students stating their answers in the form, "Y equals something," so many of the answers were simply, "Times 3," or "Plus 2". There was some confusion when adding (or multiplying) by the general constant a. While Student 5 wrote "Plus a," many of the other students either wrote nothing or wrote "plus 0." It took a little while for me to notice this misconception. When I asked them for their answers, Student 5 was the only one to respond. Even after continued discussion, there was still some confusion, settling that there were multiple

correct answers, one of them being zero. I was unsatisfied with this result, but still had to move on.

This group had the same problems as Tuesday's group, but after some struggling, they had a response for the card, $y = 2x + 7$. After suggesting the "fix the answer" approach, they, too, were able to breeze through the rest of the responses.

Chapter 5

Teacher Debriefing

At the end of my study, I met with the principal and the participating students' teachers so I could discuss my findings. I focused my discussion on the students' various abilities when it came to generating relationships between and among quantities, as well as some conceptual difficulties that they had with symbolization. But the most valuable information exchanged was what I learned from the teachers.

One of the first questions I asked was why they had chosen these students. Their general response was that most of these students were more than capable of quickly finishing any assignment in their new fifth-grade curriculum. However, the students were not quite ready for the sixth-grade material. As such they all felt that the students would get a good amount of engagement from my proposed activities.

Next, I wanted to know what sort of background the students had in topic related to algebra. From my understanding, the students all had some experience solving problems with one unknown; for example, $3x + 9 = 2x - 5$. These problems were generally introduced as a literal balance problem in the new curriculum. The teachers went on to explain that while balancing both sides of a balance was understandable to the students, they had trouble transitioning to the symbolic manipulations of an equation.

Another aspect related to relational thinking that we discussed was the students' exposure to scientific notation. Specifically, many of the students were having difficulties intuitively understanding $10^0 = 1$. One possible way of alleviating this difficulty, as suggested by my advisor, is explaining that a number raised to the zeroth power being one is a convention that most mathematicians had adopted, and introducing the students to the indeterminate quantity 0^0. Because the students were introduced to negative

exponents and multiplying by fractions, my suggestion was to try teaching additive exponential rules; that is, $10^0 = 10^1 \times 10^{-1} = 1$, but the teachers thought this approach might go over their heads.

Chapter 6

Conclusions

As expected, students were usually able to identify recursive patterns with far less difficulty than functional relationships. However, given the right hints or revelations, finding a functional relationship was always within their grasp.

A common misconception was the difference between unknowns and variables. These terms are usually used interchangeably by both instructors and students, and both are often represented by a letter. However, there is a key difference between these concepts. An *unknown* refers to something with a value (or set of values) that have yet to be identified. For example, in the problem, "Solve for x in $x + 2 = 5$," x is an unknown. *Variables* can take on any of a range of values, such as both x and y in $y = 2x + 3$. Often, this distinction is not made, which causes confusion when students are tasked with finding a symbolic representation of the relationship between two variables when many previous activities before hand had treated letters as unknowns.

One way to address these misconceptions is to develop a student's understanding of unknowns and variables before developing their functional thinking. For example, there has been success in developing the idea of a variable through the exploration of number lines (Carraher et al., 2006).

Appendix A

Permission Forms for the Study

Before I was able to conduct this study, I had to obtain parental consent for each student. To do so, I created the following forms, all of which where approved by the Claremont Colleges' Institutional Review Board prior to the start of my investigation. Within the forms, I included a brief description of how my study would be structures, plus the reasons for having the study video recorded. Some of the forms contain some errors, but these were the final versions I sent out to the parents, so I have reproduced them as distributed for historical accuracy.

Permission Form: *Early Algebraic Reasoning of Elementary School Students*

You are being asked to allow the participation of your child in a research project conducted by Ivan Hernandez at Harvey Mudd College as part of his senior thesis in mathematics education, under the direction of Rachel Levy, a mathematics professor at Harvey Mudd College and a former Sycamore parent. Your child was selected because he/she was recommended by his/her classroom teacher.

Please return this form to your child's teacher by _________________________.

What will your child do? Your child will participate in a weekly mathematics program during which the students will engage in advanced problem solving activities. These enrichment activities align with Sycamore's new mathematics curriculum, *Everyday Mathematics*. The 45 minute sessions will be conducted by Ivan Hernandez and video recorded. A staff member or parent volunteer from Sycamore Elementary will always be present during the meetings as a silent observer. The study will take place during normal school hours in place of their usual math lesson over a six (6) week period.

PURPOSE: The purpose of this studying is to explore various teaching methods and examine the student's early algebraic thinking.

VOLUNTARY PARTICIPATION: Please understand that your consent and your child's participation are completely voluntary. At any time, you may, or your child may choose to withdraw from the study. This will not affect any relationship between you, Sycamore, or the Claremont Colleges.

VIDEO RECODINGS: During the study, your child will be video recorded. This is to ensure that the investigator does not miss any action that may provide some insight to the study. The raw video footage will only be available to the investigator and his advisors.
Selected excerpts from the recordings may be used during a professional presentation. Those that are chosen will be available for your review and approval before the presentation.

CONFIDENTIALITY: All consent forms and raw data will be secured and accessible only by Hernandez and his advisors. Pseudonyms will be used in all written reports and in presentations. All video footage not selected for professional presentations will be destroyed following the end of my research.

RISK/BENEFITS OF PARTICIPATION: Risks related to participation are minimal. The study is conducted at the students' regular campus with children they will interact with on a usual basis. Activities will resemble normal classroom activities. Benefits include exposure to interesting mathematical topics in a small group setting. No monetary compensation will be provided fro participation.

Figure A.1: Parental consent form.

Assent Form for Early Algebraic Reasoning of Elementary School Students

My name is Ivan Hernandez, and I go to Harvey Mudd College. I am asking for your participation in my study because I am really interested in how you learn.

PURPOSE: In this study, I want to learn how you think about mathematics. I will give you some lessons in mathematics that you may not have seen before and videotape how you solve the problems in a group. This study will take place during normal school hours.

RISKS & BENEFITS: Since this will be similar to your normal classrooms, there should not be anything dangerous.

I am hoping the activities will be challenging but very fun. Hopefully you will learn something new!

PARTICIPATION: You have been recommended by your teacher. I will also ask your parents for permission. However, you do not have to participate if you do not want to. No one will be upset if you say no, and you can always change your mind and leave the study at any time if you wish to stop.

VIDEO RECORDING: I will have a video camera recording the lessons. This is to make sure I hear and see what everyone has said to help me with my study. I may use a video clip in a professional presentation. Your parent/guardian will have a chance to see the clips and decide whether or not I can show them.

If you want to take part of this study, sign below.

Student's Name ____

Student's Signature______________________________________ Date _____________

Signature of Researcher _________________________________ Date _____________

Figure A.2: Student assent form.

Video Release Form:

Early Algebraic Reasoning of Elementary School Students

During the study, your child will be video recorded. This is to ensure that the investigator, Ivan Hernandez, will not lose any actions that may provide insights to how the students are interpreting the assignment. The raw video footage will only be available to the investigator and his advisors.

With your permission, some material may be selected to be shown during professional presentations at the end of the study. Prior to the presentation, the selected footage will be available for your approval.

All footage that is not to be used for presentations will be deleted/destroyed at the conclusion of the thesis.

I voluntarily release the videotaping and use of __________________________ image
Printed Name of Participating Child
for the use during this study. I understand that I will be allowed to view and approve any material that will be shown in professional presentations.

Printed Name of Parent or Guardian ________________________________

Signature of Parent or Guardian ____________________________________

Date __________________

Signature of Researcher ___________________________________

Figure A.3: Video release form.

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