Radial Solutions to Semipositone Dirichlet Problems

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Abstract

We study the problem

$$\begin{cases} \Delta u + \lambda f(u) = 0, \\ u = 0 \text{ when } |x| = 1. \end{cases} \tag{1}$$

when $f(0) \leq 0$, f increases monotonically, and $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$. Specifically, we provide a nonexistence result for a particular choice of f, a nonexistence result for large λ, and an existence result for the case $f(u) = e^u - 1$.

We make use of Pohozaev identities, energy arguments, and bifurcation from a simple eigenvalue.

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Chapter 1

Historical Notes

We are interested in positive solutions u to a Dirichlet problem on the unit disc in $\mathbb{R}^n$. We study the problem

$$\begin{cases} \Delta u + \lambda f(u) = 0, |x| < 1, \\ u = 0 \text{ when } |x| = 1. \end{cases} \tag{1.1}$$

We say that a solution to (1.1) is radial if, for each $r \geq 0$, u is constant on the surface of the ball of radius r centered at the origin. In Gidas et al. (1979), it is proved that if u is a positive solution to (1.1), then u is radial. So it suffices to study radial solutions $u(r)$ of (1.1). This allows us to convert (1.1) to an ordinary differential equation using hyper-spherical coordinates. Assuming u is radial, we have the formula

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial x_i}\right)^2 + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x_i^2},$$

since $\frac{\partial u}{\partial \phi_i} = 0$ for each of the angular variables ϕ_i. Thus

$$\begin{aligned} \Delta u &= \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \\ &= \sum_{i=1}^{n} \frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial x_i}\right)^2 + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x_i^2} \\ &= \frac{\partial^2 u}{\partial r^2}\sum_{i=1}^{n}\left(\frac{\partial r}{\partial x_i}\right)^2 + \frac{\partial u}{\partial r}\sum_{i=1}^{n}\frac{\partial^2 r}{\partial x_i^2} \\ &= \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r}. \end{aligned}$$

Since the new differential equation has no dependence on the angular variables ϕ_i, the problem becomes an ordinary differential equation with boundary condtion $u(1) = 0$; concretely, we have

$$\begin{cases} (r^{n-1}u')' + \lambda r^{n-1} f(u) = 0, \\ u(1) = u'(0) = 0. \end{cases} \tag{1.2}$$

In Castro and Shivaji (1989), Castro and Shivaji extended the result of Gidas, Ni, and Nirenberg to nonnegative solutions to (1.1), by establishing in particular that a nonnegative solution is in fact a positive solution, arguing that nonnegative solutions may not have zeros on the interior of the region.

A solution to (1.2) is an ordered pair (u, λ) which satisfies the above equations for a given function f. This function f is called the *nonlinearity* of the Dirichlet problem.

Joseph and Lundgren Joseph and Lundgren (1970) studied (1.2) for $f(u) = (1 + \alpha u)^\beta$ with α, β real numbers and $f(u) = e^u$. They proved the following existence and uniqueness results. When $f(u) = e^u$, there exists $\lambda_* > 0$ such that (1.1) has

1. no solutions when $\lambda > \lambda_*$ $(n \geq 1)$
2. one solution when $\lambda = \lambda_*$ $(n \geq 1)$
3. two solutions when $0 < \lambda < \lambda_*$ $(n = 1, 2)$,
4. an infinite number of solutions when $\lambda = 2 (n = 3)$,
5. a finite but large number of solutions when $|\lambda - 2| \neq 0$ is small $(n = 3)$,
6. an infinite number of solutions when $\lambda = 2(n - 2)(n < 10)$
7. a finite but large number of solutions when $|\lambda - 2(n - 2)| \neq 0$ is small $(n < 10)$, and
8. one solution for each $\lambda < 2(n - 2)$ $(n \geq 10)$.

Similar properties are proved for when $f(u) = (1 + \alpha u)^\beta$.

In Jacobsen and Schmitt (2002), the problem (1.1) was considered for a larger class of partial differential operators. They provided existence and multiplicity results for these problems. They also motivated the study of

(1.1) as a limiting case of the problem

$$\begin{cases} v_t = \Delta v + \lambda(1-\varepsilon v)^m e^v(1+\varepsilon v), x \in \Omega, \\ v = 0, x \in \partial\Omega. \end{cases} \tag{1.3}$$

Per Jacobsen and Schmitt (2002), (1.3) is the *solid fuel ignition model* which arises in combustion theory. Nontrivial solutions to (1.1) are uthe steady-state solutions to (1.3). A detailed exposition of the existence and uniqueness results for (1.1) established in Joseph and Lundgren (1970) may be found in Bebernes and Eberly (1989), entitled *Mathematical Problems from Combustion Theory.*

Cohen and Keller (1967) defined a *positone* Dirichlet problem to be one where the nonlinearity $f(u)$ is positive-valued, continuous, and monotonically increasing. For example, the problem

$$\begin{cases} \Delta u + \lambda e^u = 0, |x| < 1, \\ u = 0 \text{ when } |r| = 1. \end{cases} \tag{1.4}$$

is positone. We are particularly interested in the case where $f(u)$ is continuous and monotonically increasing, but not necessarily positive at 0. For example, we study the case where $f(u) = e^u - 1 - \varepsilon$ for $\varepsilon > 0$. Such a Dirichlet problem is called *semipositone*. We restrict our investigation to $f(u)$ with the property $\lim_{u\to\infty} f(u)/u = \infty$. Such a function is called *superlinear*.

Chapter 2

A Nonexistence Result

Joseph and Lundgren studied problem (1.1) for the nonlinearity $f(u) = (1+\alpha)^{\beta}$. We are interested in an example of a similar semipositone problem which in fact has no solutions. We study the nonlinearity $f(u) = |-1 + \alpha u|^{\beta-1}(-1+\alpha u)$, $\alpha, \beta > 0$, in the supercritical case $\left(\beta > \frac{N+2}{N-2}\right)$. This function f is monotonically increasing, yet satisfies $f(0) = -1 < 0$. Thus the problem

$$\begin{cases} (r^{n-1}u')' + \lambda r^{n-1}|-1+\alpha u|^{\beta-1}(-1+\alpha u) = 0 \\ u(1) = u'(0) = 0 \end{cases} \tag{2.1}$$

is semipositone. Under the change of variables $v = -1 + \alpha u$, $x = r$, (2,.1) becomes

$$\begin{cases} (x^{n-1}v')' + \alpha\lambda x^{n-1}|v|^{\beta-1}v = 0 \\ v(1) = -1, v'(0) = 0 \end{cases} \tag{2.2}$$

This form is advantageous since for a function $v(x)$ satisfying the differential equation

$$(x^{N-1}v')' + \hat{\tau}x^{N-1}v^{\beta} = 0, \tag{2.3}$$

we may apply a convenient form of Pohozaev's identity.

Lemma 1 (Pohozaev's Identity): *If $v(x)$ satisfies,*

$$(x^{N-1}v')' + \hat{\tau}x^{N-1}v^{\beta} = 0,$$

then $v(x)$ satisfies the integral identity

$$x^N\left(\frac{(v'(x))^2}{2} + \frac{\hat{\tau}v^{\beta+1}}{\beta+1}\right) + \frac{N-2}{2}x^{N-1}vv' = \hat{\tau}\int_0^x s^{N-1}\left(\frac{N}{\beta+1} - \frac{N-2}{2}\right)v^{\beta+1}\,dt.$$

Proof. We first multiply the differential equation on both sides by v, and integrate by parts on $[0, x]$. This yields

$$vv'x^{N-1} - \int_0^x t^{N-1}(v')^2 dt + \int_0^x t^{N-1}\hat{\tau}v^{\beta+1}\,dt = 0.$$

We next multiply the differential equation on both sides by xv', and again integrate by parts on $[0, x]$. This yields

$$\frac{x^N(v')^2}{2} + \frac{N-2}{2}\int_0^x t^{N-1}(v')^2 dt + \int_0^x v't^N\hat{\tau}v^\beta\,dt = 0.$$

Multiplying the first equation by $\frac{N-2}{2}$ and adding it to the second, we get (moving integral terms to the right)

$$\frac{x^N(v')^2}{2} + \frac{N-2}{2}vv'x^{N-1} = -\int_0^x \hat{\tau}t^N v^\beta v'\,dt - \frac{N-2}{2}\int_0^x \hat{\tau}t^{N-1}v^{\beta+1}\,dt. \tag{2.4}$$

The first integral on the right may be integrated by parts with $\lambda = t^N$, $du = v^\beta v' dt$. We have

$$-\int_0^x \hat{\tau}t^N v^\beta v'\,dt = -\frac{\hat{\tau}x^N v^{\beta+1}}{\beta+1} + \frac{N}{\beta+1}\int_0^x \hat{\tau}t^{N-1}v^{\beta+1}dt. \tag{2.5}$$

Substituting (2.5) into (2.4) yields

$$\frac{x^N(v')^2}{2} + \frac{N-2}{2}vv'x^{N-1} = -\frac{\hat{\tau}x^N v^{\beta+1}}{\beta+1} + \hat{\tau}\int_0^x t^{N-1}\left(\frac{N}{\beta+1} - \frac{N-2}{2}\right)v^{\beta+1}\,dt.$$

Rearranging yields Pohozaev's identity, that is,

$$x^N\left(\frac{(v'(x))^2}{2} + \frac{\hat{\tau}v^{\beta+1}}{\beta+1}\right) + \frac{N-2}{2}x^{N-1}vv' = \hat{\tau}\int_0^x s^{N-1}\left(\frac{N}{\beta+1} - \frac{N-2}{2}\right)v^{\beta+1}\,dt. \tag{2.6}$$

We can only apply Pohozaev's identity to (2.2) in a region in which $v = |v|$; that is, v is nonnegative. Suppose u is a positive solution to (2.1) satisfying $-1 + \alpha u(0) > 0$. Then $v(0) > 0$, while $v(1) = -1 < 0$, ensuring that there is some $x_0 \in (0, 1)$ where $v(x_0) = 0$ by the continuity of v, and $v(x) \geq 0$ for $0 \leq x \leq x_0$. Then v satisfies (2.3) for $x \in [0, x_0)$, and we can apply the lemma.

Put $x = x_0$ in (2.6). We have $v'(x_0)^2 \geq 0$, while $v(x_0) = 0$, so the left side is nonnegative. In the supercritical case the integrand is strictly negative on $[0, x_0)$, therefore the right hand side is strictly negative. This is a contradiction, so it must be that $-1 + \alpha u(0) \le 0$. In this case, (2.2) implies

$$xv'' + (N-1)v' = \hat{\tau}|v|^{\beta} \tag{2.7}$$

while integrating (2.2) implies

$$v'(x) = -\frac{\alpha\lambda}{x^{N-1}} \int_0^x t^{N-1}|v|^{\beta}\, dt. \tag{2.8}$$

(2.8) demonstrates that v' is always negative, unless v is the constant function, which is not a solution. Therefore at any point x_0 where v is nonzero, $v'(x_0) < 0$. The right hand side of (2.7) is always nonnegative. Therefore $v''(x) \ge 0$. So $v'(0) = 0$, $v'(x_0) < 0$, while $v''(x) \ge 0$, a contradiction. So there are no solutions.

Chapter 3

Large λ Solutions to a Semipositone Problem

3.1 Definitions

We consider the semipositone problem

$$\begin{cases} (r^{n-1}u')' + \lambda r^{n-1}(e^u - (1+\varepsilon)) = 0, \\ u(1) = u'(0) = 0. \end{cases} \tag{3.1}$$

for $\varepsilon > 0$. We are interested in the existence of solutions to (3.1) for large λ. Define $g(u) = e^u - (1+\varepsilon)$ and

$$G(u) = \int_0^u (e^t - (1+\varepsilon)dt = e^u - u(1+\varepsilon) - 1.$$

Since $g(u)$ is monotonically increasing and negative at $u = 0$, there are real numbers α, β such that $\alpha < \beta$ and $g(\alpha) = G(\beta) = 0$. Define $d = u(0)$ and an energy function

$$E(r) = \frac{(u')^2}{2} + \lambda G(u).$$

Then

$$E'(r) = u'(r)u''(r) + \lambda g(u(r))u'(r) = -\frac{n-1}{r}(u'(r))^2$$

by the differential equation in (3.1). For all $r \in (0,1)$, $-\frac{n-1}{r}(u'(r))^2 \leq 0$, thus $E'(r) \leq 0$ for all $r \in (0,1]$.

3.2 Useful Lemmas

Lemma 1: *If u is a solution to (3.1), u decreases montonically.*

Suppose u is a solution to (1) which does not decrease monotonically. Then there are r_1, r_2 so that $r_1 < r_2$ and $u(r_1) < u(r_2)$. Suppose $u(0) < u(r_2)$. Then $E(0) = \lambda G(u(0)) < \lambda G(u(r_2)) \leq E(r_2)$, a contradiction since $E'(r) \leq 0$, therefore $u(0) \geq u(r_2)$.

Since u is continuous, the extremal value theorem says that u must attain its minimum on $[0, r_2]$. We have $u(0) \geq u(r_2) > u(r_1)$, so the minimum is not either of the endpoints. Thus since u is differentiable, there exists $r_0 < r_2$ where $u(r_0) < u(r_2)$ and $u'(r_0) = 0$. Yet $E(r_0) = \lambda G(u(r_0)) < \lambda G(u(r_2)) = E(r_2)$, a contradiction since $E'(r) \leq 0$. So u decreases monotonically.

Happily, we can immediately eliminate solutions with $u(0) < \beta$ from consideration.

Lemma 2: *If u is a solution to (3.1), $u(0) \geq \beta$.*

Suppose that $d \in [0, \beta)$. we have $E(0) = \lambda G(d) < 0$. Since $E'(r) \leq 0$ for $r \in [0, 1]$, we expect $E(1) \leq E(0) < 0$. But

$$E(1) = \frac{u'(1)^2}{2} + \lambda G(0) = \frac{u'(1)^2}{2} \geq 0.$$

This is a contradiction. Thus no solutions exist for $d \in [0, \beta)$.

3.3 Bounding r_0

The above shows that if u is a solution to (3.1), $u(0) = d \geq \beta$. We now seek to obtain bounds, depending on λ, on the point r_0 at which $u(r_0) = \frac{\alpha+\beta}{2}$. Integrating the differential equation in (3.1), we get

$$-r^{n-1}u'(r) = \lambda \int_0^r s^{n-1} g(u(s))\, ds. \tag{3.2}$$

Since u decreases monotonically, we may exchange the integral above for an inequality:

$$-r^{n-1}u'(r) = \lambda \int_0^r s^{n-1} g(u(s))\, ds \geq \lambda g\left(\frac{\alpha+\beta}{2}\right)\frac{r^n}{n}. \tag{3.3}$$

Dividing through and integrating from 0 to r,

$$d - u(r) \geq \frac{\lambda g\left(\frac{\alpha+\beta}{2}\right) r^2}{2n}, \tag{3.4}$$

thus

$$u(r) \leq d - \frac{\lambda g\left(\frac{\alpha+\beta}{2}\right) r^2}{2n}. \tag{3.5}$$

The equation above is an upper bound for $u(r)$ on the interval where $u(r) \geq \frac{\alpha+\beta}{2}$. This upper bound is a monotonically decreasing function. Setting

$$d - \frac{\lambda g\left(\frac{\alpha+\beta}{2}\right) r^2}{2n} = \frac{\alpha+\beta}{2},$$

we obtain

$$r = \sqrt{\frac{N(2d - (\alpha+\beta))}{\lambda g(\frac{\alpha+\beta}{2})}} = C_1/\sqrt{\lambda},$$

So we have a positive constants C_1 satisfying

$$r_0 < \frac{C_1}{\sqrt{\lambda}}.$$

3.4 Energy Arguments

Let r_1 satisfy $u(r_1) = \alpha$, and let r_2 satisfy $u(r_2) = \frac{\alpha}{2}$. We proceed with energy arguments to show that no solutions exist for large λ. We have

$$E(1) = \frac{(u'(1))^2}{2} + \lambda G(0) = \frac{(u'(1))^2}{2} \geq 0.$$

Since $E'(r) \geq 0$, $E(r) \geq 0$ for all r. It follows that

$$\frac{(u')^2}{2} \geq -\lambda G(u) \tag{3.6}$$

for all r. Now suppose $r \in [r_0, r_2]$. The maximum value of $G(u(r))$ on this interval is at an endpoint since G is concave up, so

$$G(u(r)) \leq G(u(r_0)) = G\left(\frac{\alpha+\beta}{2}\right)$$

or

$$G(u(r)) \leq G(u(r_2)) = G\left(\frac{\alpha}{2}\right).$$

In either case, $G(r) \leq -M_1$ for some $M_1 > 0$ on the interval $[r_0, r_2]$. Therefore on this interval, (3.6) becomes

$$\frac{(u')^2}{2} \geq \lambda M_1.$$

It follows that

$$-u' \geq \sqrt{\lambda}\sqrt{2M_1}$$

on this interval, where we take the negative square root since u decreases monotonically. Integrating from r_0 to r_2, we get

$$u(r_0) - u(r_2) = \frac{\alpha+\beta}{2} - \frac{\alpha}{2} = \frac{\beta}{2} \geq \sqrt{\lambda}\sqrt{2M_1}(r_2 - r_0)$$

therefore

$$r_2 - r_0 \leq \frac{M_2}{\sqrt{\lambda}}$$

for a positive constant M_2. From above we have that $r_0 \leq C_1/\sqrt{\lambda}$; it follows that for some $M_3 > 0$, we have

$$r_2 \leq \frac{M_3}{\sqrt{\lambda}}.$$

So as $\lambda \to \infty, r_2 \to 0$. We now aim to show, toward a contradiction, as $\lambda \to \infty, r_2 \to 1$. Consider the interval $[r_2, 1]$. Define the positive constant

$$M_4 = -\frac{g\left(\frac{\alpha}{2}\right)}{2}.$$

Then define a new function

$$h(x) = G(x) + M_4 x = e^x - x(1+\varepsilon) - 1 + M_4 x = e^x + (M_4 - 1 - \varepsilon)x - 1.$$

Then $h(0) = 0$, and for $x \leq \alpha/2$, we have

$$h'(x) = g(x) + M_4 \leq g\left(\frac{\alpha}{2}\right) + M_4 = \frac{g\left(\frac{\alpha}{2}\right)}{2} < 0.$$

Thus $h(u(r)) \leq 0$ on $[r_2, 1]$. It follows that

$$G(u) \leq -M_4 u$$

and therefore (3.6) becomes

$$\frac{(u')^2}{2} \geq \lambda u M_4.$$

Thus

$$-u' \geq \sqrt{\lambda}\sqrt{u}\sqrt{2M_4}.$$

We can separate variables and integrate from r_2 to 1; this yields

$$-\int_{r_2}^{1} \frac{du}{\sqrt{u}} \geq \int_{r_2}^{1} \sqrt{\lambda}\sqrt{2M_4}\, dr$$

thus

$$-2\sqrt{u}|_{r_2}^{1} = 2\sqrt{\frac{\alpha}{2}} \geq \sqrt{\lambda}\sqrt{2M_4}(1 - r_2)$$

and finally

$$r_2 \geq 1 - M_5/\sqrt{\lambda}.$$

for some positive constant M_5. Since we expect $r_2 \leq \frac{M_3}{\sqrt{\lambda}}$, there are no solutions when

$$1 - \frac{M_5}{\sqrt{\lambda}} > \frac{M_3}{\sqrt{\lambda}},$$

that is to say, there are no solutions for

$$\lambda > (M_3 + M_5)^2.$$

Chapter 4

An Existence Result

4.1 Setup

We study the problem

$$\begin{cases} (r^{n-1}u')' + \lambda r^{n-1}(e^u - 1) = 0, \\ u(1) = u'(0) = 0. \end{cases} \tag{4.1}$$

Since $e^0 - 1 = 0$, this problem is neither positone nor semipositone. This unique property will allow us to study the existence of solutions to (4.1) with help from Crandall and Rabinowitz (1971).

Specifically, we apply Theorem 1 of the above paper to demonstrate the existence of a branch of nontrivial solutions. The theorem is stated with a high degree of generality and thus will require extensive setup to be useful in the context of our Dirichlet problem.

Theorem 4.1 (Crandall and Rabinowitz (1971)). *Let W and Y be Banach spaces, Ω an open subset of W and $G : \Omega \to Y$ be twice continuously differentiable. Let $w : [-1, 1] \to \Omega$ be a simple continuously differentiable arc in Ω such that $G(w(t)) = 0$ for $|t| \leq 1$. Suppose*

a) $w'(0) \neq 0$,

b) $dim N(G'(w(0)) = 2, codim(R(G'(w(0)))) = 1$,

c) $N(G'(w(0)))$ *is spanned by* $w'(0)$ *and* v, *and*

d) $G''(w(0))(w'(0), v) \notin R(G'(w(0)))$.

Then $w(0)$ is a bifurcation point of $G(w) = 0$ with respect to $C = \{w(t) : t \in [-1, 1]\}$ and in some neighborhood of $w(0)$ the totality of solutions of $G(w) = 0$ form two continuous curves intersecting only at $w(0)$.

Let U be the open unit ball in $\mathbb{R}^n$. Then (4.1) is equivalent to

$$\begin{cases} -\Delta u = \lambda(e^u - 1) = 0, \text{ in } U, \\ u = 0 \text{ on } \delta U. \end{cases} \tag{4.2}$$

Suppose u is a solution to (4.2). Then from Evans (1998), we have the following theorem.

Theorem 4.2 (Evans (1998)). (Representation formula using Green's function.) *If $u \in C^2(\overline{U})$ solves problem*

$$\begin{cases} -\Delta u = f, \; in \; U, \\ u = g \; on \; \delta U. \end{cases} \tag{4.3}$$

then

$$u(x) = -\int_{\delta U} g(y) \frac{\partial G}{\partial \nu}(x, y)\, dS(y) + \int_U f(y) G(x, y) dy, x \in U.$$

Here, again referencing Evans (1998), G is Green's function and $\frac{\partial G}{\partial \nu}(x, y)$ is the outer normal derivative of G with respect to the variable y; the latter will be irrelevant in our final calculation.

We see u satisfies the preconditions of Theorem 4.2 with $f = \lambda(e^u - 1)$ and $g = 0$. So for solutions of (4.2), we have

$$u(x) = \lambda \int_U G(x, y)(e^{u(y)} - 1) dy.$$

Define a new function $T : C^2(U) \times \mathbb{R} \to C^2(U)$ by

$$T(u, \lambda) = u(x) - \lambda \int_U G(x, y)(e^{u(y)} - 1) dy.$$

We are now ready to discuss the theorem. Let $Y = C^2(U)$ be the Banach space of twice continuously differentiable functions, and let W be the subset of $C^2(U) \times \mathbb{R}$ where the continuous function in the first slot satisfies the

Dirichlet boundary condition. Ω is simply all of W, and T takes the place of G in the theorem, to avoid confusion with Green's function. Let λ_1 be the first Dirichlet eigenvalue of U. Define $w(t) = (0, t + \lambda_1)$, the element of $C^2(U) \times \mathbb{R}$ where the first slot is the constant function 0. Then for all t,

$$T(w(t)) = T(0, \lambda_1 + t) = 0 - (\lambda_1 + t) \int_U G(x,y)(e^0 - 1)\, dy = 0.$$

4.2 T is Twice Continuously Differentiable

We begin by showing that T is twice continuously differentiable in the sense of the Fréchet derivative. We begin with a guess for the Fréchet derivative and will prove it. Define

$$T'(u, \lambda)(h, \gamma) = h - \lambda \int_U G(x,y) e^u h(y)\, dy - \gamma \int_U G(x,y)(e^u - 1).$$

We must prove that for each (h, γ), the above defines a bounded linear operator satisfying

$$\lim_{s,t \to 0} \frac{1}{|t| + |s|} \left(T(u + th, \lambda + s\gamma) - T(u, \lambda) - T'(u)(th, s\gamma) \right) = 0.$$

The above equals

$$\lim_{s,t \to 0} \frac{1}{|t| + |s|} \left(\lambda \int_U G(x,y) e^u (1 - e^{th(y)} + h(y))\, dy + s\gamma \int_U G(x,y) e^u (1 - e^{th(y)})\, dy \right).$$

In the first term we have

$$\begin{aligned} &\lim_{s,t \to 0} \frac{1}{|t| + |s|} \lambda \int_U G(x,y) e^u (1 - e^{th(y)} + h(y))\, dy \\ &= \lim_{s,t \to 0} \frac{1}{t} \lambda \int_U G(x,y) e^u \left(h(y) \frac{1 - e^{th(y)}}{th(y)} + h(y) \right) dy \end{aligned}$$

The fraction in the integrand approaches the derivative of $-e^x$ at $x = 0$, that is -1, as $t \to 0$, so that the above tends to 0 with t. In the second term we have

$$\begin{aligned}
&\left|\lim_{s,t\to 0}\frac{1}{|t|+|s|}s\gamma\int_U G(x,y)e^u(1-e^{th(y)})\,dy\right| \\
\leq&|s\gamma|\left|\lim_{s,t\to 0}\frac{1}{t}\int_U G(x,y)e^u(1-e^{th(y)})\right| \\
\leq&|s\gamma|\left|\lim_{s,t\to 0}\int_U G(x,y)e^u h(y)\left(\frac{1-e^{th(y)}}{th(y)}\right)\right| \\
=&\lim_{s,t\to 0}|s\gamma|\left|\int_U G(x,y)e^u h(y)h(y)\right| \\
=&0
\end{aligned} \tag{4.4}$$

so the second term tends to zero as well. $T'(u,\lambda)(h,\gamma)$ We see that $T'(u,\lambda)(h,\gamma)$ is linear in h and γ. Furthermore, it is bounded; to see this, note that

$$\frac{\|(T'(u,\lambda))(h,\gamma)\|_\infty}{\|h\|_\infty+|\gamma|}=\frac{\|h-\lambda\int_U G(x,y)e^u h(y)\,dy-\gamma\int_U G(x,y)(e^u-1)\|_\infty}{(\|h\|_\infty+|\gamma|}.$$

By the triangle inequality, the above is

$$\leq 1+\lambda\left\|\int_U G(x,y)e^u dy\right\|_\infty+\left\|\int_U G(x,y)(e^u-1)\,dy\right\|_\infty,$$

therefore since u is continuous on U, $\|T'(u,\lambda)\|_{op}$ is bounded above. Thus $(T'(u,\lambda))(h,\gamma)$ is the the Fréchet derivative of T at (u,λ) in the direction of (h,γ). Since we must show $T(u,\lambda)$ is twice continuously differentiable, we must now repeat this process with $T'(u,\lambda)$ in place of $T(u,\lambda)$. We consider

$$\begin{aligned}
&\lim_{s,t\to 0}\frac{1}{|t|+|s|}\left(T'(u+t\xi,\lambda+s\nu)(h,\gamma)-T'(u,\lambda)(h,\gamma)\right) \\
=&\lim_{s,t\to 0}\frac{1}{|s|+|t|}\big((h-(\lambda+s\nu)\int_U G(x,y)e^{u+t\xi}h(y)\,dy-\gamma\int_U G(x,y)(e^{u+t\xi}-1)\,dy \\
&-h+\lambda\int_U G(x,y)e^u h(y)\,dy\big)+\gamma\int_U G(x,y)(e^u-1). \\
=&\lim_{s,t\to 0}\frac{1}{|s|+|t|}\bigg(\lambda\int_U G(x,y)e^u h(y)(1-e^{t\xi})\,dy-s\nu\int_U G(x,y)e^{u+t\xi}h(y)dy \\
&+\gamma\int_U G(x,y)e^u(1-e^{t\xi})\,dy\bigg) \\
=&-\lambda\int_U G(x,y)e^u h(y)\xi(y)\,dy-\nu\int_u G(x,y)e^u h(y)dy-\gamma\int_U G(x,y)e^u\xi(y)\,dy
\end{aligned}$$

which is linear in both (h, γ) and (ξ, ν), and bounded by a similar argument to the one above. Thus $T''(u)$ is a bilinear operator from $(C^2(U) \times \mathbb{R}) \times (C^2(U) \times \mathbb{R}) \to C^2(U)$ and takes the form

$$\begin{aligned}(T''(u))[(h,\gamma),(\xi,\nu)] = &- \lambda \int_U G(x,y)e^u h(y)\xi(y)\,dy \\ &- \nu \int_u G(x,y)e^u h(y)dy - \gamma \int_U G(x,y)e^u \xi(y)\,dy.\end{aligned}$$

and we have proven that T is twice continuously differentiable in the sense of the the Fréchet Derivative.

4.3 Applying the Theorem

We proceed to prove the four criteria for $w(0) = (0, \lambda_1)$ to be a bifurcation point.

a) $w'(0) \neq 0$.

   By inspection, the Frechet derivative of $w(t)$ in the direction δ is $w'(t)(\delta) = (0, \delta)$. Thus $w'(0)(\delta) = (0, \delta)$ which is not identically 0.

b) $\dim N(T'(w(0)) = 2, \operatorname{codim}(R(T'(w(0)))) = 1$.

   Note

$$T'(w(0)) = T'(0, \lambda_1)(h, \gamma) = h - \lambda_1 \int_U G(x,y)h(y)dy$$

   $(0, 1)$ is one vector in the kernel, so the dimension is at least 1. Now pick h_1 to be the eigenfunction corresponding to λ_1; since λ_1 is simple as the first Dirichlet eigenvalue, it has exactly one eigenfunction. So the vector $(h_1, 0)$ is an additional vector in the null space, orthogonal to $(0, 1)$, so the null space has dimension exactly 2, since there are no other eigenfunctions.

   Since the cokernel contains exactly one linearly independent function h_1, $\operatorname{codim}(R(G'(w(0)))) = 1$. The cokernel may have no more since this would contradict the simplicty of λ_1.

c) $N(T'(w(0)))$ is spanned by $w'(0)$ and v.

With $v = (h_1, 0)$, this follows directly from the above. Strictly speaking, $w'(0)$ is a linear map from $[-1, 1]$ to $L(\mathbb{R}, C^2 \times \mathbb{R})$, not an element of $(C^2 \times \mathbb{R})$, so we take $\delta = 1$ at which $w'(0) = (0, 1)$.

d) $T''(w(0))(w'(0), v) \notin R(T'(w(0)))$.

With reference to (4.5), we have

$$(T''(0, \lambda_1))[(0, 1), (h_1, 0)] = -\int_U G(x, y)h_1(y)\, dy.$$

We aim to show this is not contained in $R(T'(w(0)))$; that is, for $h \in W$, it is not a function of the form

$$h - \lambda_1 \int_U G(x, y)h(y)\, dy.$$

Suppose it were. Then there exists some h where

$$-\int_U G(x, y)h_1(y)\, dy = h - \lambda_1 \int_U G(x, y)h(y)\, dy$$

and therefore

$$h = \lambda_1 \int_U G(x, y)(h(y) - h_1(y)/\lambda_1)\, dy$$

so that by the converse of (4.2), we have

$$\Delta h = -\lambda_1(h(y) - h_1(y)/\lambda_1) = \lambda_1 h(y) - h_1(y)\, dy.$$

Multiply on both sides by h_1 and integrate over U to get the identity

$$\int_U \left(h_1(y)\Delta h(y) + \lambda_1 h_1(y)h(y)\right)\, dy = -\int_U h_1^2(y) \tag{4.5}$$

We can simplify the above. First note that

$$\int_{\delta U} h\frac{\partial h_1}{\partial \nu} - h_1\frac{\partial h}{\partial \nu} = 0,$$

since h and h_1 are 0 on the boundary of U, the former because it is an element of W and the latter because it is a solution to the Dirichlet problem

$$\begin{cases} \Delta h = \lambda h \text{ in } U \\ h = 0 \text{ on } \partial U. \end{cases} \tag{4.6}$$

Then by the statement in Evans (1998) of Green's third identity,

$$\int_U (h\Delta h_1 - h_1 \Delta h) = \int_{\delta U} h\frac{\partial h_1}{\partial \nu} - h_1 \frac{\partial h}{\partial \nu} = 0$$

so that (4.6) becomes

$$\begin{aligned} &\int_U h(y)\Delta h_1(y) + \lambda_1 \int_U h_1(y)h(y)\,dy \\ =& -\lambda_1 \int_U h_1(y)h(y)\,dy + \lambda_1 \int_U h_1(y)h(y)\,dy \\ =&0 = -\int_U h_1^2(y)\,dy < 0, \end{aligned} \tag{4.7}$$

a contradiction. So the preconditions of the theorem are satisfied. Then $(0, \lambda_1)$ is a bifurcation point of $w(t)$ and a second curve of solutions intersects $w(t)$ at only this point, therefore this second curve is composed of nontrivial solutions since $w(t)$ is the trivial curve. The properties of the nontrivial curve are a topic for further study.

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