Radial Solutions to Semipositone Dirichlet Problems

Ethan Sargent

Follow this and additional works at: https://scholarship.claremont.edu/hmc_theses

Part of the Analysis Commons, and the Other Mathematics Commons

Recommended Citation
https://scholarship.claremont.edu/hmc_theses/229

This Open Access Senior Thesis is brought to you for free and open access by the HMC Student Scholarship at Scholarship @ Claremont. It has been accepted for inclusion in HMC Senior Theses by an authorized administrator of Scholarship @ Claremont. For more information, please contact scholarship@cuc.claremont.edu.
Radial Solutions to Semipositone Dirichlet Problems

Ethan Sargent

Alfonso Castro, Advisor

ShuZhi Song, Reader

Department of Mathematics

May, 2019
Copyright © 2019 Ethan Sargent.

The author grants Harvey Mudd College and the Claremont Colleges Library the nonexclusive right to make this work available for noncommercial, educational purposes, provided that this copyright statement appears on the reproduced materials and notice is given that the copying is by permission of the author. To disseminate otherwise or to republish requires written permission from the author.
Abstract

We study the problem

\[
\begin{aligned}
\Delta u + \lambda f(u) &= 0, \\
u &= 0 \text{ when } |x| = 1.
\end{aligned}
\]  

(1)

when \( f(0) \leq 0 \), \( f \) increases monotonically, and \( \lim_{u \to \infty} \frac{f(u)}{u} = \infty \). Specifically, we provide a nonexistence result for a particular choice of \( f \), a nonexistence result for large \( \lambda \), and an existence result for the case \( f(u) = e^u - 1 \).

We make use of Pohozaev identities, energy arguments, and bifurcation from a simple eigenvalue.
# Contents

Abstract iii

Acknowledgments vii

1 Historical Notes 1

2 A Nonexistence Result 5

3 Large $\lambda$ Solutions to a Semipositone Problem 9
   3.1 Definitions ........................................ 9
   3.2 Useful Lemmas ...................................... 10
   3.3 Bounding $r_0$ ...................................... 10
   3.4 Energy Arguments ................................... 11

4 An Existence Result 15
   4.1 Setup ............................................ 15
   4.2 $T$ is Twice Continuously Differentiable ............ 17
   4.3 Applying the Theorem ................................ 19

Bibliography 23
Acknowledgments

I have benefited from conversations with my second reader ShuZhi Song, and in particular from her meticulous assistance with important calculations. I am indebted to Professor Lisette de Pillis, Molly Reeves, and Jocelyn Olds-Mcspadden, for their good humor and flexibility in helping me navigate this complicated project. Lastly I am deeply grateful to my advisor, Professor Alfonso Castro, for his patience, guidance, and insight.
Chapter 1

Historical Notes

We are interested in positive solutions $u$ to a Dirichlet problem on the unit disc in $\mathbb{R}^n$. We study the problem

$$\begin{cases}
\Delta u + \lambda f(u) = 0, |x| < 1, \\
u = 0 \text{ when } |x| = 1.
\end{cases} \tag{1.1}$$

We say that a solution to (1.1) is radial if, for each $r \geq 0$, $u$ is constant on the surface of the ball of radius $r$ centered at the origin. In [Gidas et al., 1979], it is proved that if $u$ is a positive solution to (1.1), then $u$ is radial. So it suffices to study radial solutions $u(r)$ of (1.1). This allows us to convert (1.1) to an ordinary differential equation using hyper-spherical coordinates. Assuming $u$ is radial, we have the formula

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial^2 u}{\partial r^2} \left( \frac{\partial r}{\partial x_i} \right)^2 + \frac{\partial u}{\partial r} \frac{\partial^2 r}{\partial x_i^2}.$$ 

since $\frac{\partial u}{\partial \phi_i} = 0$ for each of the angular variables $\phi_i$. Thus

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial r^2} \left( \frac{\partial r}{\partial x_i} \right)^2 + \frac{\partial u}{\partial r} \sum_{i=1}^{n} \frac{\partial^2 r}{\partial x_i^2}.$$ 

$$= \frac{\partial^2 u}{\partial r^2} \sum_{i=1}^{n} \left( \frac{\partial r}{\partial x_i} \right)^2 + \frac{\partial u}{\partial r} \sum_{i=1}^{n} \frac{\partial^2 r}{\partial x_i^2}$$

$$= \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r} \frac{\partial u}{\partial r}.$$
Since the new differential equation has no dependence on the angular variables $\phi_i$, the problem becomes an ordinary differential equation with boundary condition $u(1) = 0$; concretely, we have

$$
\begin{aligned}
\left\{ 
(r^{n-1}u')' + \lambda r^{n-1}f(u) &= 0, \\
u(1) &= u'(0) = 0.
\right. 
\end{aligned}
$$

(1.2)

In [Castro and Shivaji (1989)], Castro and Shivaji extended the result of Gidas, Ni, and Nirenberg to nonnegative solutions to (1.1), by establishing in particular that a nonnegative solution is in fact a positive solution, arguing that nonnegative solutions may not have zeros on the interior of the region.

A solution to (1.2) is an ordered pair $(u, \lambda)$ which satisfies the above equations for a given function $f$. This function $f$ is called the nonlinearity of the Dirichlet problem.

Joseph and Lundgren [Joseph and Lundgren (1970)] studied (1.2) for $f(u) = (1 + \alpha u)^\beta$ with $\alpha, \beta$ real numbers and $f(u) = e^u$. They proved the following existence and uniqueness results. When $f(u) = e^u$, there exists $\lambda_* > 0$ such that (1.1) has

1. no solutions when $\lambda > \lambda_*$, $(n \geq 1)$
2. one solution when $\lambda = \lambda_*$, $(n \geq 1)$
3. two solutions when $0 < \lambda < \lambda_*$, $(n = 1, 2)$,
4. an infinite number of solutions when $\lambda = 2(n = 3)$,
5. a finite but large number of solutions when $|\lambda - 2| \neq 0$ is small $(n = 3)$,
6. an infinite number of solutions when $\lambda = 2(n = 2)(n < 10)$
7. a finite but large number of solutions when $|\lambda - 2(n = 2)| \neq 0$ is small $(n < 10)$, and
8. one solution for each $\lambda < 2(n = 2) (n \geq 10)$.

Similar properties are proved for when $f(u) = (1 + \alpha u)^\beta$.

In [Jacobsen and Schmitt (2002)], the problem (1.1) was considered for a larger class of partial differential operators. They provided existence and multiplicity results for these problems. They also motivated the study of
(1.1) as a limiting case of the problem
\[
\begin{align*}
v_t &= \Delta v + \lambda (1 - \varepsilon v)^m e^v (1 + \varepsilon v), \ x \in \Omega, \\
v &= 0, \ x \in \partial \Omega.
\end{align*}
\] (1.3)

Per Jacobsen and Schmitt (2002), (1.3) is the solid fuel ignition model which arises in combustion theory. Nontrivial solutions to (1.1) are the steady-state solutions to (1.3). A detailed exposition of the existence and uniqueness results for (1.1) established in Joseph and Lundgren (1970) may be found in Bebernes and Eberly (1989), entitled Mathematical Problems from Combustion Theory.

Cohen and Keller (1967) defined a positone Dirichlet problem to be one where the nonlinearity \( f(u) \) is positive-valued, continuous, and monotonically increasing. For example, the problem
\[
\begin{align*}
\Delta u + \lambda e^u &= 0, \ |x| < 1, \\
u &= 0 \text{ when } |r| = 1.
\end{align*}
\] (1.4)

is positone. We are particularly interested in the case where \( f(u) \) is continuous and monotonically increasing, but not necessarily positive at 0. For example, we study the case where \( f(u) = e^u - 1 - \varepsilon \) for \( \varepsilon > 0 \). Such a Dirichlet problem is called semipositone. We restrict our investigation to \( f(u) \) with the property \( \lim_{u \to \infty} f(u)/u = \infty \). Such a function is called superlinear.
Chapter 2

A Nonexistence Result

Joseph and Lundgren studied problem (1.1) for the nonlinearity \( f(u) = (1 + \alpha)^\beta \). We are interested in an example of a similar semipositone problem which in fact has no solutions. We study the nonlinearity \( f(u) = |−1 + \alpha u|^{\beta - 1}(-1 + \alpha u) \), \( \alpha, \beta > 0 \), in the supercritical case \( (\beta > \frac{N+2}{N-2}) \). This function \( f \) is monotonically increasing, yet satisfies \( f(0) = -1 < 0 \). Thus the problem

\[
\begin{cases}
(r^{n-1}u')' + \lambda r^{n-1} - 1 + \alpha u|^{\beta-1}(-1 + \alpha u) = 0 \\
u(1) = u'(0) = 0
\end{cases}
\tag{2.1}
\]

is semipositone. Under the change of variables \( v = -1 + \alpha u, x = r \), (2.1) becomes

\[
\begin{cases}
(x^{n-1}v')' + \lambda x^{n-1}|v|^{\beta-1}v = 0 \\
v(1) = -1, v'(0) = 0
\end{cases}
\tag{2.2}
\]

This form is advantageous since for a function \( v(x) \) satisfying the differential equation

\[
(x^{N-1}v')' + \hat{\tau} x^{N-1}v^\beta = 0,
\tag{2.3}
\]

we may apply a convenient form of Pohozaev’s identity.

**Lemma 1 (Pohozaev’s Identity):** If \( v(x) \) satisfies

\[
(x^{N-1}v')' + \hat{\tau} x^{N-1}v^\beta = 0,
\]

then \( v(x) \) satisfies the integral identity

\[
x^N \left( \frac{(v'(x))^2}{2} + \frac{\hat{\tau} v^{\beta+1}}{\beta+1} \right) + \frac{N-2}{2} x^{N-1}v v' = \hat{\tau} \int_0^x s^{N-1} \left( \frac{N}{\beta+1} - \frac{N-2}{2} \right) v^{\beta+1} \, dt.
\]
Proof. We first multiply the differential equation on both sides by \(v\), and integrate by parts on \([0, x]\). This yields

\[
vv'x^{N-1} - \int_0^x t^{N-1}(v')^2 \, dt + \int_0^x t^{N-1} \dot{v}v^{\beta+1} \, dt = 0.
\]

We next multiply the differential equation on both sides by \(xv'\), and again integrate by parts on \([0, x]\). This yields

\[
\frac{x^N(v')^2}{2} + \frac{N-2}{2} \int_0^x t^{N-1}(v')^2 \, dt + \int_0^x v't^N \dot{v}^{\beta} \, dt = 0.
\]

Multiplying the first equation by \(\frac{N-2}{2}\) and adding it to the second, we get

\[
\frac{x^N(v')^2}{2} + \frac{N-2}{2} vv'x^{N-1} = -\int_0^x \hat{\beta} t^{N-1}v^{\beta+1} \, dt - \frac{N-2}{2} \int_0^x t^{N-1}v^{\beta+1} \, dt.
\]

The first integral on the right may be integrated by parts with \(\lambda = t^N\),

\[
du = v^{\beta}v' \, dt.
\]

We have

\[
-\int_0^x \hat{\beta} t^{N-1}v^{\beta+1} \, dt = -\frac{\hat{\beta} x^N v^{\beta+1}}{\beta + 1} + \frac{N}{\beta + 1} \int_0^x \hat{\beta} t^{N-1}v^{\beta+1} \, dt.
\]

Substituting (2.5) into (2.4) yields

\[
\frac{x^N(v')^2}{2} + \frac{N-2}{2} vv'x^{N-1} = -\frac{\hat{\beta} x^N v^{\beta+1}}{\beta + 1} + \frac{N}{\beta + 1} \int_0^x \hat{\beta} t^{N-1}v^{\beta+1} \, dt.
\]

Rearranging yields Pohozaev’s identity, that is,

\[
x^N \left(\frac{(v'(x))^2}{2} + \frac{\hat{\beta} v^{\beta+1}}{\beta + 1} \right) + \frac{N-2}{2} x^{N-1} vv' = \hat{\beta} \int_0^x t^N v^{\beta+1} \, dt.
\]

We can only apply Pohozaev’s identity to (2.2) in a region in which \(v = |v|\); that is, \(v\) is nonnegative. Suppose \(u\) is a positive solution to (2.1) satisfying \(-1 + au(0) > 0\). Then \(v(0) > 0\), while \(v(1) = -1 < 0\), ensuring that there is some \(x_0 \in (0, 1)\) where \(v(x_0) = 0\) by the continuity of \(v\), and \(v(x) \geq 0\) for \(0 \leq x \leq x_0\). Then \(v\) satisfies (2.3) for \(x \in [0, x_0]\), and we can apply the lemma.

Put \(x = x_0\) in (2.6). We have \(v'(x_0)^2 \geq 0\), while \(v(x_0) = 0\), so the left
side is nonnegative. In the supercritical case the integrand is strictly negative on \([0, x_0)\), therefore the right hand side is strictly negative. This is a contradiction, so it must be that \(-1 + \alpha u(0) \leq 0\). In this case, (2.2) implies

\[ xv'' + (N - 1)v' = \tilde{t}|v|^{\beta} \]  

(2.7)

while integrating (2.2) implies

\[ v'(x) = -\frac{\alpha \lambda}{x^{N-1}} \int_{0}^{x} t^{N-1}|v|^{\beta} \, dt. \]  

(2.8)

(2.8) demonstrates that \(v'\) is always negative, unless \(v\) is the constant function, which is not a solution. Therefore at any point \(x_0\) where \(v\) is nonzero, \(v'(x_0) < 0\). The right hand side of (2.7) is always nonnegative. Therefore \(v''(x) \geq 0\). So \(v'(0) = 0, v'(x_0) < 0\), while \(v''(x) \geq 0\), a contradiction. So there are no solutions.
Chapter 3

Large $\lambda$ Solutions to a Semipositone Problem

3.1 Definitions

We consider the semipositone problem

$$\begin{cases} (ru^{n-1}u')' + \lambda ru^{n-1}(e^u - (1 + \varepsilon)) = 0, \\ u(1) = u'(0) = 0. \end{cases}$$

(3.1)

for $\varepsilon > 0$. We are interested in the existence of solutions to (3.1) for large $\lambda$.

Define $g(u) = e^u - (1 + \varepsilon)$ and

$$G(u) = \int_0^u (e^t - (1 + \varepsilon))dt = e^u - u(1 + \varepsilon) - 1.$$  

Since $g(u)$ is monotonically increasing and negative at $u = 0$, there are real numbers $\alpha, \beta$ such that $\alpha < \beta$ and $g(\alpha) = G(\beta) = 0$. Define $d = u(0)$ and an energy function

$$E(r) = \frac{(u')^2}{2} + \lambda G(u).$$

Then

$$E'(r) = u'(r)u''(r) + \lambda g(u(r))u'(r) = -\frac{n-1}{r}u'(r)^2$$

by the differential equation in (3.1). For all $r \in (0, 1)$, $-\frac{n-1}{r}(u'(r))^2 \leq 0$, thus $E'(r) \leq 0$ for all $r \in (0, 1)$. 
3.2 Useful Lemmas

Lemma 1: If $u$ is a solution to (3.1), $u$ decreases monotonically.

Suppose $u$ is a solution to (1) which does not decrease monotonically. Then there are $r_1, r_2$ so that $r_1 < r_2$ and $u(r_1) < u(r_2)$. Suppose $u(0) < u(r_2)$. Then $E(0) = \lambda G(u(0)) < \lambda G(u(r_2)) \leq E(r_2)$, a contradiction since $E'(r) \leq 0$, therefore $u(0) \geq u(r_2)$.

Since $u$ is continuous, the extremal value theorem says that $u$ must attain its minimum on $[0, r_2]$. We have $u(0) \geq u(r_2) > u(r_1)$, so the minimum is not either of the endpoints. Thus since $u$ is differentiable, there exists $r_0 < r_2$ where $u(r_0) < u(r_2)$ and $u'(r_0) = 0$. Yet $E(r_0) = \lambda G(u(r_0)) < \lambda G(u(r_2)) = E(r_2)$, a contradiction since $E'(r) \leq 0$. So $u$ decreases monotonically.

Happily, we can immediately eliminate solutions with $u(0) < \beta$ from consideration.

Lemma 2: If $u$ is a solution to (3.1), $u(0) \geq \beta$.

Suppose that $d \in [0, \beta)$. we have $E(0) = \lambda G(d) < 0$. Since $E'(r) \leq 0$ for $r \in [0, 1]$, we expect $E(1) \leq E(0) < 0$. But

$$E(1) = \frac{u'(1)^2}{2} + \lambda G(0) = \frac{u'(1)^2}{2} \geq 0.$$  

This is a contradiction. Thus no solutions exist for $d \in [0, \beta)$.

3.3 Bounding $r_0$

The above shows that if $u$ is a solution to (3.1), $u(0) = d \geq \beta$. We now seek to obtain bounds, depending on $\lambda$, on the point $r_0$ at which $u(r_0) = \frac{a+\beta}{2}$. Integrating the differential equation in (3.1), we get

$$-r^{n-1}u'(r) = \lambda \int_0^r s^{n-1} G(u(s)) \, ds.$$  (3.2)
Since $u$ decreases monotonically, we may exchange the integral above for an inequality:

$$-r^{n-1}u'(r) = \lambda \int_0^r s^{n-1} g(u(s)) \, ds \geq \lambda g\left(\frac{\alpha + \beta}{2}\right) \frac{r^n}{n}. \quad (3.3)$$

Dividing through and integrating from 0 to $r$,

$$d - u(r) \geq \frac{\lambda g\left(\frac{\alpha + \beta}{2}\right) r^2}{2n}, \quad (3.4)$$

thus

$$u(r) \leq d - \frac{\lambda g\left(\frac{\alpha + \beta}{2}\right) r^2}{2n}. \quad (3.5)$$

The equation above is an upper bound for $u(r)$ on the interval where $u(r) \geq \frac{\alpha + \beta}{2}$. This upper bound is a monotonically decreasing function.

Setting

$$d - \frac{\lambda g\left(\frac{\alpha + \beta}{2}\right) r^2}{2n} = \frac{\alpha + \beta}{2},$$

we obtain

$$r = \sqrt{\frac{N(2d - (\alpha + \beta))}{\lambda g\left(\frac{\alpha + \beta}{2}\right)}} = \frac{C_1}{\sqrt{\lambda}},$$

So we have a positive constants $C_1$ satisfying

$$r_0 < \frac{C_1}{\sqrt{\lambda}}.$$

### 3.4 Energy Arguments

Let $r_1$ satisfy $u(r_1) = \alpha$, and let $r_2$ satisfy $u(r_2) = \frac{\alpha}{2}$. We proceed with energy arguments to show that no solutions exist for large $\lambda$. We have

$$E(1) =\frac{(u'(1))^2}{2} + \lambda G(0) = \frac{(u'(1))^2}{2} \geq 0.$$ 

Since $E'(r) \geq 0$, $E(r) \geq 0$ for all $r$. It follows that

$$\frac{(u')^2}{2} \geq -\lambda G(u) \quad (3.6)$$
for all $r$. Now suppose $r \in [r_0, r_2]$. The maximum value of $G(u(r))$ on this interval is at an endpoint since $G$ is concave up, so

$$G(u(r)) \leq G(u(r_0)) = G\left(\frac{\alpha + \beta}{2}\right)$$

or

$$G(u(r)) \leq G(u(r_2)) = G\left(\frac{\alpha}{2}\right).$$

In either case, $G(r) \leq -M_1$ for some $M_1 > 0$ on the interval $[r_0, r_2]$. Therefore on this interval, (3.6) becomes

$$\frac{(u')^2}{2} \geq \lambda M_1.$$ 

It follows that

$$-u' \geq \sqrt{\lambda} \sqrt{2M_1}$$

on this interval, where we take the negative square root since $u$ decreases monotonically. Integrating from $r_0$ to $r_2$, we get

$$u(r_0) - u(r_2) = \frac{\alpha + \beta}{2} - \frac{\alpha}{2} = \frac{\beta}{2} \geq \sqrt{\lambda} \sqrt{2M_1} (r_2 - r_0)$$
	herefore

$$r_2 - r_0 \leq \frac{M_2}{\sqrt{\lambda}}$$

for a positive constant $M_2$. From above we have that $r_0 \leq C_1 / \sqrt{\lambda}$; it follows that for some $M_3 > 0$, we have

$$r_2 \leq \frac{M_3}{\sqrt{\lambda}}.$$ 

So as $\lambda \to \infty, r_2 \to 0$. We now aim to show, toward a contradiction, as $\lambda \to \infty, r_2 \to 1$. Consider the interval $[r_2, 1]$. Define the positive constant

$$M_4 = -g\left(\frac{4}{\lambda}\right).$$

Then define a new function

$$h(x) = G(x) + M_4 x = e^x - x(1 + \varepsilon) - 1 + M_4 x = e^x + (M_4 - 1 - \varepsilon)x - 1.$$
Then \( h(0) = 0 \), and for \( x \leq \alpha/2 \), we have

\[
h'(x) = g(x) + M_4 \leq g \left( \frac{\alpha}{2} \right) + M_4 = \frac{g \left( \frac{\alpha}{2} \right)}{2} < 0.
\]

Thus \( h(u(r)) \leq 0 \) on \([r_2, 1]\). It follows that \( G(u) \leq -M_4 u \) and therefore (3.6) becomes

\[
\frac{(u')^2}{2} \geq \lambda u M_4.
\]

Thus

\[
-u' \geq \sqrt{\lambda} \sqrt{u} \sqrt{2M_4}.
\]

We can separate variables and integrate from \( r_2 \) to 1; this yields

\[
-\int_{r_2}^{1} \frac{du}{\sqrt{u}} \geq \int_{r_2}^{1} \sqrt{\lambda} \sqrt{2M_4} \, dr
\]

thus

\[
-2\sqrt{u}\bigg|_{r_2}^{1} = 2 \sqrt{\frac{\alpha}{2}} \geq \sqrt{\lambda} \sqrt{2M_4}(1 - r_2)
\]

and finally

\[
r_2 \geq 1 - \frac{M_5}{\sqrt{\lambda}}.
\]

for some positive constant \( M_5 \). Since we expect \( r_2 \leq \frac{M_3}{\sqrt{\lambda}} \), there are no solutions when

\[
1 - \frac{M_5}{\sqrt{\lambda}} > \frac{M_3}{\sqrt{\lambda}}.
\]

that is to say, there are no solutions for

\[
\lambda > (M_3 + M_5)^2.
\]
Chapter 4

An Existence Result

4.1 Setup

We study the problem

\[
\begin{cases}
(r^{n-1}u')' + \lambda r^{n-1}(e^u - 1) = 0, \\
u(1) = u'(0) = 0.
\end{cases}
\]

(4.1)

Since $e^0 - 1 = 0$, this problem is neither positone nor semipositone. This unique property will allow us to study the existence of solutions to (4.1) with help from Crandall and Rabinowitz (1971).

Specifically, we apply Theorem 1 of the above paper to demonstrate the existence of a branch of nontrivial solutions. The theorem is stated with a high degree of generality and thus will require extensive setup to be useful in the context of our Dirichlet problem.

**Theorem 4.1 (Crandall and Rabinowitz (1971)).** Let $W$ and $Y$ be Banach spaces, $\Omega$ an open subset of $W$ and $G : \Omega \to Y$ be twice continuously differentiable. Let $w : [-1, 1] \to \Omega$ be a simple continuously differentiable arc in $\Omega$ such that $G(w(t)) = 0$ for $|t| \leq 1$. Suppose

a) $w'(0) \neq 0$,

b) $\dim N(G'(w(0))) = 2, \text{codim}(R(G'(w(0)))) = 1$,

c) $N(G'(w(0)))$ is spanned by $w'(0)$ and $v$, and

d) $G''(w(0))(w'(0), v) \notin R(G'(w(0))).$
Then \( w(0) \) is a bifurcation point of \( G(w) = 0 \) with respect to \( C = \{ w(t) : t \in [-1, 1] \} \) and in some neighborhood of \( w(0) \) the totality of solutions of \( G(w) = 0 \) form two continuous curves intersecting only at \( w(0) \).

Let \( U \) be the open unit ball in \( \mathbb{R}^n \). Then (4.1) is equivalent to

\[
\begin{align*}
-\Delta u &= \lambda(e^u - 1), \quad \text{in } U, \\
u &= 0 \text{ on } \partial U.
\end{align*}
\]  

(4.2)

Suppose \( u \) is a solution to (4.2). Then from Evans (1998), we have the following theorem.

**Theorem 4.2 (Evans (1998)).** (Representation formula using Green’s function.) If \( u \in C^2(U) \) solves problem

\[
\begin{align*}
-\Delta u &= f, \quad \text{in } U, \\
u &= g \text{ on } \partial U.
\end{align*}
\]  

(4.3)

then

\[
u(x) = - \int_{\partial U} g(y) \frac{\partial G}{\partial n}(x, y) dS(y) + \int_U f(y) G(x, y) dy, \quad x \in U.
\]

Here, again referencing Evans (1998), \( G \) is Green’s function and \( \frac{\partial G}{\partial n}(x, y) \) is the outer normal derivative of \( G \) with respect to the variable \( y \); the latter will be irrelevant in our final calculation.

We see \( u \) satisfies the preconditions of Theorem 4.2 with \( f = \lambda(e^u - 1) \) and \( g = 0 \). So for solutions of (4.2), we have

\[
u(x) = \lambda \int_U G(x, y) (e^{u(y)} - 1) dy.
\]

Define a new function \( T : C^2(U) \times \mathbb{R} \to C^2(U) \) by

\[
T(u, \lambda) = u(x) - \lambda \int_U G(x, y) (e^{u(y)} - 1) dy.
\]

We are now ready to discuss the theorem. Let \( Y = C^2(U) \) be the Banach space of twice continuously differentiable functions, and let \( W \) be the subset of \( C^2(U) \times \mathbb{R} \) where the continuous function in the first slot satisfies the
Dirichlet boundary condition. \( \Omega \) is simply all of \( W \), and \( T \) takes the place of \( G \) in the theorem, to avoid confusion with Green’s function. Let \( \lambda_1 \) be the first Dirichlet eigenvalue of \( U \). Define \( w(t) = (0, t + \lambda_1) \), the element of \( C^2(U) \times \mathbb{R} \) where the first slot is the constant function 0. Then for all \( t \),

\[
T(w(t)) = T(0, \lambda_1 + t) = 0 - (\lambda_1 + t) \int_U G(x, y)(e^0 - 1) \, dy = 0.
\]

## 4.2 \( T \) is Twice Continuously Differentiable

We begin by showing that \( T \) is twice continuously differentiable in the sense of the Fréchet derivative. We begin with a guess for the Fréchet derivative and will prove it. Define

\[
T'(u, \lambda)(h, \gamma) = h - \lambda \int_U G(x, y)e^{uh(y)} \, dy - \gamma \int_U G(x, y)(e^u - 1).
\]

We must prove that for each \((h, \gamma)\), the above defines a bounded linear operator satisfying

\[
\lim_{s, t \to 0} \frac{1}{|t| + |s|} \left( T(u + th, \lambda + s\gamma) - T(u, \lambda) - T'(u)(th, s\gamma) \right) = 0.
\]

The above equals

\[
\lim_{s, t \to 0} \frac{1}{|t| + |s|} \left( \lambda \int_U G(x, y)e^{u(1 - e^{th(y)} + h(y))} \, dy + s\gamma \int_U G(x, y)e^{u(1 - e^{th(y)})} \, dy \right).
\]

In the first term we have

\[
\lim_{s, t \to 0} \frac{1}{|t| + |s|} \lambda \int_U G(x, y)e^{u(1 - e^{th(y)} + h(y))} \, dy
= \lim_{s, t \to 0} \frac{1}{t} \lambda \int_U G(x, y)e^{u} \left( h(y) \frac{1 - e^{th(y)}}{th(y)} + h(y) \right) \, dy.
\]

The fraction in the integrand approaches the derivative of \( -e^x \) at \( x = 0 \), that is \(-1\), as \( t \to 0 \), so that the above tends to 0 with \( t \). In the second term we
have

\[
\left| \lim_{s,t \to 0} \frac{1}{|t| + |s|} s \gamma \int_U G(x, y) e^{u(1 - e^{th(y)})} \, dy \right| \\
\leq |s \gamma| \left| \lim_{s,t \to 0} \frac{1}{|t|} \int_U G(x, y) e^{u(1 - e^{th(y)})} \, dy \right| \\
\leq |s \gamma| \left| \lim_{s,t \to 0} \int_U G(x, y) e^{uh(y)} \left( \frac{1 - e^{th(y)}}{th(y)} \right) \, dy \right| \\
= \lim_{s,t \to 0} |s \gamma| \left| \int_U G(x, y) e^{uh(y)} h(y) \, dy \right|
\]

so the second term tends to zero as well. \(T'(u, \lambda)(h, \gamma)\) We see that \(T'(u, \lambda)(h, \gamma)\) is linear in \(h\) and \(\gamma\). Furthermore, it is bounded; to see this, note that

\[
\|(T'(u, \lambda))(h, \gamma)\|_\infty = \frac{\|h - \lambda \int_U G(x, y) e^{uh(y)} \, dy - \gamma \int_U G(x, y)(e^u - 1)\|_\infty}{\|h\|_\infty + |\gamma|}.
\]

By the triangle inequality, the above is

\[
\leq 1 + \lambda \left\| \int_U G(x, y) e^{uh} \, dy \right\|_\infty + \left\| \int_U G(x, y)(e^u - 1) \, dy \right\|_\infty,
\]

therefore since \(u\) is continuous on \(U\), \(\|T'(u, \lambda)\|_{op}\) is bounded above. Thus \((T'(u, \lambda))(h, \gamma)\) is the Fréchet derivative of \(T\) at \((u, \lambda)\) in the direction of \((h, \gamma)\). Since we must show \(T(u, \lambda)\) is twice continuously differentiable, we must now repeat this process with \(T'(u, \lambda)\) in place of \(T(u, \lambda)\). We consider

\[
\lim_{s,t \to 0} \frac{1}{|t| + |s|} (T'(u + t \xi, \lambda + s \nu)(h, \gamma) - T'(u, \lambda)(h, \gamma))
\]

\[
= \lim_{s,t \to 0} \frac{1}{|s| + |t|} \left( (h - (\lambda + s \nu)) \int_U G(x, y) e^{uh} \, dy - \gamma \int_U G(x, y)(e^u - 1) \, dy \right)
\]

\[
- h + \lambda \int_U G(x, y) e^{uh} \, dy + \gamma \int_U G(x, y)(e^u - 1) \, dy
\]

\[
= \lim_{s,t \to 0} \frac{1}{|s| + |t|} \left( \lambda \int_U G(x, y) e^{uh} \, dy - \nu \int_U G(x, y) e^{uh} \, dy + \lambda \int_U G(x, y) e^{uh} \, dy \right)
\]

\[
= - \lambda \int_U G(x, y) e^{uh} \, dy - \nu \int_U G(x, y) e^{uh} \, dy - \gamma \int_U G(x, y) e^{uh} \, dy
\]
Applying the Theorem

which is linear in both \((h, \gamma)\) and \((\xi, \nu)\), and bounded by a similar argument to the one above. Thus \(T''(u)\) is a bilinear operator from \((C^2(U) \times \mathbb{R}) \times (C^2(U) \times \mathbb{R}) \to C^2(U)\) and takes the form

\[
(T''(u))[\langle h, \gamma \rangle, \langle \xi, \nu \rangle] = -\lambda \int_U G(x, y)e^{u}h(y)\xi(y) \, dy
- \nu \int_U G(x, y)e^{u}h(y)dy - \gamma \int_U G(x, y)e^{u}\xi(y) \, dy.
\]

and we have proven that \(T\) is twice continuously differentiable in the sense of the Frechét Derivative.

4.3 Applying the Theorem

We proceed to prove the four criteria for \(w(0) = (0, \lambda_1)\) to be a bifurcation point.

a) \(w'(0) \neq 0\).

By inspection, the Frechet derivative of \(w(t)\) in the direction \(\delta\) is \(w'(t)(\delta) = (0, \delta)\) which is not identically 0.

b) \(\dim N(T'(w(0))) = 2, \text{codim}(R(T'(w(0)))) = 1\).

Note

\[
T'(w(0)) = T'(0, \lambda_1)(h, \gamma) = h - \lambda_1 \int_U G(x, y)h(y)dy
\]

\((0, 1)\) is one vector in the kernel, so the dimension is at least 1. Now pick \(h_1\) to be the eigenfunction corresponding to \(\lambda_1\); since \(\lambda_1\) is simple as the first Dirichlet eigenvalue, it has exactly one eigenfunction. So the vector \((h_1, 0)\) is an additional vector in the null space, orthogonal to \((0, 1)\), so the null space has dimension exactly 2, since there are no other eigenfunctions.

Since the cokernel contains exactly one linearly independent function \(h_1, \text{codim}(R(G'(w(0)))) = 1\). The cokernel may have no more since this would contradict the simplicity of \(\lambda_1\).

c) \(N(T'(w(0)))\) is spanned by \(w'(0)\) and \(v\).
With $v = (h_1, 0)$, this follows directly from the above. Strictly speaking, $w'(0)$ is a linear map from $[-1, 1]$ to $L(\mathbb{R}, C^2 \times \mathbb{R})$, not an element of $(C^2 \times \mathbb{R})$, so we take $\delta = 1$ at which $w'(0) = (0, 1)$.

\begin{equation}
\tag{4.5}
\end{equation}

With reference to (4.5), we have

\begin{equation*}
(T''(0, \lambda_1))[((0, 1), (h_1, 0))] = -\int_U G(x, y)h_1(y) \, dy.
\end{equation*}

We aim to show this is not contained in $R(T'(w(0)))$; that is, for $h \in W$, it is not a function of the form

$$h - \lambda_1 \int_U G(x, y)h(y) \, dy.$$ 

Suppose it were. Then there exists some $h$ where

$$-\int_U G(x, y)h_1(y) \, dy = h - \lambda_1 \int_U G(x, y)h(y) \, dy$$

and therefore

$$h = \lambda_1 \int_U G(x, y)(h(y) - h_1(y)/\lambda_1) \, dy$$

so that by the converse of (4.2), we have

$$\Delta h = -\lambda_1 (h(y) - h_1(y)/\lambda_1) = \lambda_1 h(y) - h_1(y) \, dy.$$ 

Multiply on both sides by $h_1$ and integrate over $U$ to get the identity

$$\int_U (h_1(y)\Delta h(y) + \lambda_1 h_1(y)h(y)) \, dy = -\int_U h_1^2(y)$$

\begin{equation}
\tag{4.5}
\end{equation}

We can simplify the above. First note that

$$\int_{\delta U} h \frac{\partial h_1}{\partial v} - h_1 \frac{\partial h}{\partial v} = 0,$$

since $h$ and $h_1$ are 0 on the boundary of $U$, the former because it is an element of $W$ and the latter because it is a solution to the Dirichlet problem

$$\begin{cases}
\Delta h = \lambda h \text{ in } U \\
h = 0 \text{ on } \partial U.
\end{cases}$$

\begin{equation}
\tag{4.6}
\end{equation}
Then by the statement in Evans (1998) of Green’s third identity,

\[ \int_U (h \Delta h_1 - h_1 \Delta h) = \int_{\partial U} h \frac{\partial h_1}{\partial n} - h_1 \frac{\partial h}{\partial n} = 0 \]

so that (4.6) becomes

\[ \int_U h(y) \Delta h_1(y) + \lambda_1 \int_U h_1(y) h(y) \, dy = - \lambda_1 \int_U h_1(y) h(y) \, dy + \lambda_1 \int_U h_1(y) h(y) \, dy \]

\[ = 0 = - \int_U h_1^2(y) \, dy < 0, \]

a contradiction. So the preconditions of the theorem are satisfied. Then \((0, \lambda_1)\) is a bifurcation point of \(w(t)\) and a second curve of solutions intersects \(w(t)\) at only this point, therefore this second curve is composed of nontrivial solutions since \(w(t)\) is the trivial curve. The properties of the nontrivial curve are a topic for further study.
Bibliography


