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Review: Hypercyclic Pairs of Coanalytic Toeplitz Operators

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Hypercyclic pairs of coanalytic Toeplitz operators. (English summary)
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Recall that an operator A on a Hilbert space $\mathcal{H}$ is hypercyclic if there exists a vector $x \in \mathcal{H}$ such that the orbit $\{A^n x : n \in \mathbb{N}\}$ of x under A is dense in $\mathcal{H}$. Along similar lines, a pair (A, B) of commuting operators on $\mathcal{H}$ is called hypercyclic if there exists a vector $x \in \mathcal{H}$ such that $\{A^n B^k x : n, k \geq 0\}$ is dense in $\mathcal{H}$. More generally, one can consider the hypercyclicity of semigroups generated by n-tuples $(A_1, A_2, \ldots, A_n)$ of commuting operators.

The following theorem from the introduction summarizes some of the author's key findings:

Theorem. Let $f, g \in H^\infty(G)$ where G is any open set with finitely many components, and let $\mathcal{H}(G)$ denote a Hilbert space of analytic functions on G. If $\mathcal{F} = \{M^*_{f^n} M^*_{g^k} : n, k \geq 0\}$, then the following are equivalent:

1. The pair (M^*_f, M^*_g) is hypercyclic on $\mathcal{H}(G)$.
2. The semigroup $\mathcal{F}$ generated by (M^*_f, M^*_g) contains a hypercyclic operator.
3. There exist integers $n, k \geq 0$ such that $f^n g^k$ is non-constant on every component G_i of G and $(f^n g^k)(G_i) \cap \partial\mathbb{D} \neq \varnothing$ for every $i \in \{1, 2, \ldots, N\}$.

If G is connected and say $|f(z)| > 1$ and $|g(z)| < 1$ for all $z \in G$, then one may also add the following equivalent condition:

4. There does not exist a $p > 0$ such that $|f(z)|^p = 1/|g(z)|$ for all $z \in G$.

If G has infinitely many components, then the pair (M^*_f, M^*_g) is hypercyclic on $\mathcal{H}(G)$ if and only if (M^*_f, M^*_g) is hypercyclic on $\mathcal{H}(\Omega_N)$ for each $N \geq 1$, where $\Omega_N = \bigcup_{i=1}^N G_i$ and $\{G_i\}_{i=1}^\infty$ are the components of G.

One of the primary tools used is a criterion of G. Godefroy and J. H. Shapiro [J. Funct. Anal. **98** (1991), no. 2, 229–269; MR1111569 (92d:47029) (Theorem 4.9)] for the hypercyclicity of the adjoint of a multiplication operator. It is also interesting to note that the proofs of Theorems 3.1 and 4.1 of this paper require the author to employ separate arguments depending on whether the exponent p is rational or not. While the rational case is relatively straightforward, the irrational case requires a clever function theoretic proposition (Proposition 2.4), the proof of which the author attributes to Paul Bourdon.

The results are concrete enough that the author can provide numerous examples. In particular, he gives a recipe (Theorem 8.1) for constructing a hypercyclic commutative semigroup generated by a pair of pure cosubnormal operators, yet such that the semigroup does not contain a hypercyclic operator. To be more specific, the author shows that if G is a bounded open set with infinitely many components and $\mathcal{H}(G)$ is a Hilbert space of analytic functions on G, then there exist $f, g \in H^\infty(G)$ such that the pair (M^*_f, M^*_g) is hypercyclic on $\mathcal{H}(G)$, but such that the semigroup $\mathcal{F}$ generated by (M^*_f, M^*_g) contains no hypercyclic operator.

To place the preceding result in context, we mention that prior results of the author and his collaborators [N. S. Feldman, V. G. Miller and T. L. Miller, Acta Sci. Math. (Szeged) **68** (2002), no. 1-2, 303–328; MR1916583 (2004d:47021a); corrected reprint, Acta Sci. Math. (Szeged) **68** (2002), no. 3-4, 965–990; MR1954557 (2004d:47021b)] show that there exists a bounded open set G with infinitely many components such that if A is the adjoint of the operator of multiplication by z on the Bergman space $L^2_a(G)$, then A is supercyclic (i.e., there exists $x \in L^2_a(G)$ such that $\{\alpha A^n x\colon n \geq 0, \alpha \in \mathbb{C}\}$ is dense in $L^2_a(G)$) but such that no multiple of A is hypercyclic. Due to the density of $\{\frac{2^j}{3^k} e^{in\theta}\colon j,k,n \geq 0\}$ in $\mathbb{C}$ when θ is an irrational multiple of π, it follows that the semigroup generated by the quadruple $(A, 2I, \frac{1}{3}I, e^{i\theta}I)$ is hypercyclic yet contains no hypercyclic operator. The alternate construction employed in Theorem 8.1 of the present paper shows that the cardinality of the generating set of such an example may be reduced to 2.

The paper closes with a number of open questions concerning the hypercyclicity of pairs or tuples of operators.

Reviewed by *Stephan R. Garcia*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.