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Enhancing the Quandle Coloring Invariant for Knots and Links

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Abstract

Quandles, which are algebraic structures related to knots, can be used to color knot diagrams, and the number of these colorings is called the quandle coloring invariant. We strengthen the quandle coloring invariant by considering a graph structure on the space of quandle colorings of a knot, and we call our graph the quandle coloring quiver. This structure is a categorification of the quandle coloring invariant. Then, we strengthen the quiver by decorating it with Boltzmann weights. Explicit examples of links that show that our enhancements are proper are provided, as well as background information in quandle theory.

Contents

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Chapter 1

Introduction

How can we tell when two knots are fundamentally different? This is the primary question in knot theory. Knots are defined to be equivalent if we can move one into the other without breaking and re-gluing the strand.

Reidemeister (1927) showed that knots are equivalent precisely when we can move from the diagram of one to the diagram of the other by a sequence of **Reidemeister moves** (or R-moves for short), which are shown below.

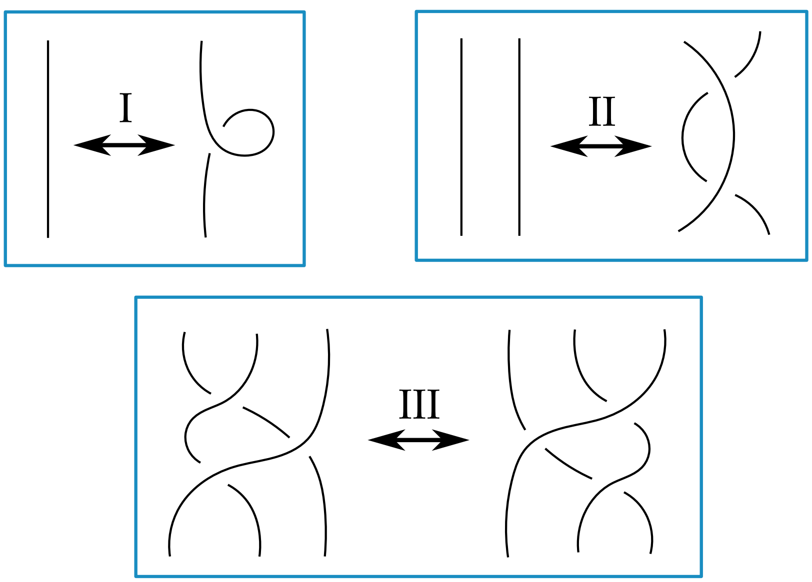


Figure 1

For example, consider the following three knot diagrams:

Figure 2

The left two knots can be deformed into one another, while the knot on the right (which is called the trefoil knot) is distinct. We can prove that the left two are equivalent by using R-moves:

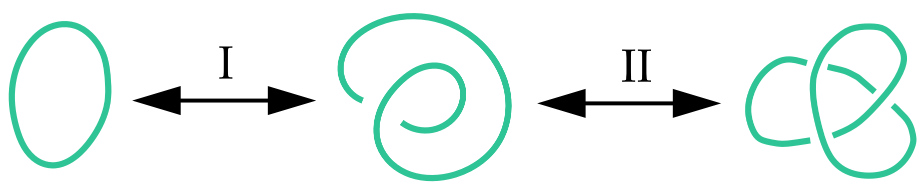


Figure 3

However, we won't be able to prove that the trefoil is distinct from the unknot (circle) using R-moves since those can only be used to show knot equivalence. This brings us back to the original question of how we can show nonequivalence of knots. The answer: knot invariants.

A **knot invariant** is a value that we can ascribe to any knot such that that value will be the same for equivalent knots. This value could be an integer, a polynomial, or anything else as long as it obeys that condition. One way to show that something is a knot invariant is to show that the value is unchanged by R-moves. In other words, we would want to show that for any knot, the value before and after performing any R-move is the same.

We are interested in a particular knot invariant known as the quandle coloring invariant. A quandle is an algebraic structure motivated by knots, much like how a group is an algebraic structure motivated by symmetry. It is defined in such a way that the fundamental quandle of a knot (discussed in Section 2.2) is an invariant. Suppose we have a quandle Q with finitely many elements. We can think of each quandle element as a color. Given some knot diagram K, we can assign colors from Q to arcs of K in a way that "respects the quandle structure" (this will be made precise in the next chapter). Then there will be some finite number of ways we can validly color K by elements of Q. We call this number the quandle coloring invariant of K with respect to Q, which is denoted by $\Phi_Q^{\mathbb{Z}}(K)$.

Quandle colorings can be nicely visualized:

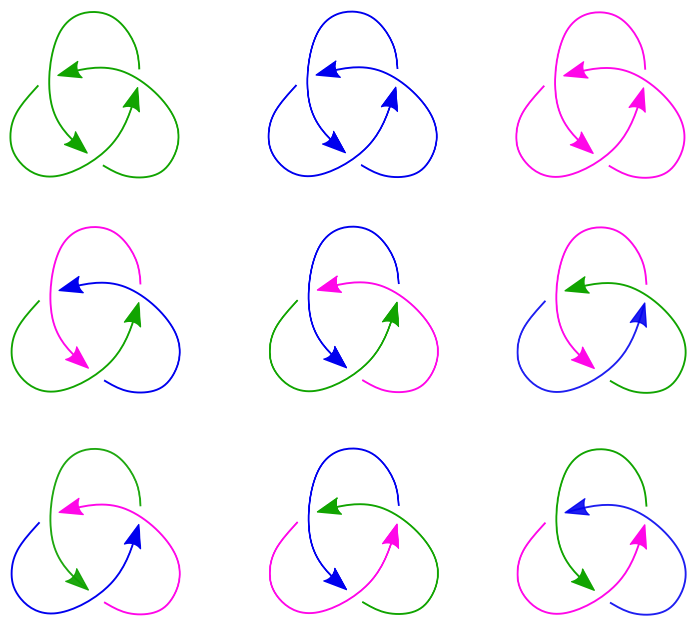


Figure 4

Above, we have the colorings of the trefoil by a particular 3 element quandle Q. Since there are 9 colorings, $\Phi_Q^{\mathbb{Z}}(\text{trefoil}) = 9$. We will discuss this particular case in more detail in Example 4.

In this thesis, we are interested in improving the quandle coloring invariant. Note that $\Phi_Q^{\mathbb{Z}}(K)$ is an integer-valued invariant, which means each coloring is essentially considered as an individual and counted up. As we will discuss in the next chapter, quandle colorings actually represent quandle homomorphisms, which are maps the respect the quandle structure. The importance of this is that the colorings are inherently algebraic, which begs the question: is there a way to impose a structure on the space of colorings?

The answer is yes! We will define the quandle coloring quiver (see Section 2.3) as an enhancement of $\Phi_Q^{\mathbb{Z}}(K)$ that takes into account the structure of the coloring space, which yields a whole family of knot invariants. The quandle coloring quiver allows us to visualize relationships between different colored knot diagrams in a way that is useful in telling knots apart.

It turns out that not all knot invariants are helpful in distinguishing knots. In fact, assigning the value 0 to every knot satisfies the definition of a knot invariant, but it cannot tell any two knots apart. Thus, it is important to provide examples of knots that are distinguishable by a given invariant, and, in the case of enhancements, to distinguish knots with the enhancement that cannot be distinguished by the original invariant.

In the rest of the thesis, we will present such examples that prove that our enhancements are strict and derive further enhancements from the quandle coloring quiver.

Chapter 2

Quandles and Knot Colorings

Here, we will introduce some relevant definitions and examples. We'll start with the fundamentals of knot theory, and work our way through quandle theory, with the aim of understanding the quandle coloring invariant. Sections 2.2 and onward will be best appreciated by readers with some experience with first semester abstract algebra and group theory. We will end the chapter by discussing our enhancement of the quandle coloring invariant: the quandle coloring quiver.

2.1 Knot Basics

Imagine you have a long piece of thread. You take your thread and twist and tie it around itself as much as you'd like. Then you light a match, melt the two ends of the thread and weld them together. Maybe it looks something like this (where the gaps indicate the strands passing over one another):

Figure 5

You have just created a knot! In math terminology, we can describe what a knot is in the following definition.

Definition 1. A **knot** is a simple closed curve embedded in $\mathbb{R}^3$.

Although the knot is embedded in $\mathbb{R}^3$, we can nearly flatten the knot and imagine it laying in $\mathbb{R}^2 \times [0, 1]$, a "thickened" plane. As mentioned in Chapter 1, we can describe a knot by drawing its knot diagram. The knot diagram is essentially a visualization of what the knot would look like from an aerial view of the thickened plane.

We will not be considering the geometry of the knot. It may be helpful to think of the knot as a long elastic strand that can be stretched and moved, as long as it is not cut or glued at any point. This intuition can be made into something more precise in the following definition:

Definition 2. We say that two knots K, K' are **equivalent** if there exists an ambient isotopy from K to K'.

We may imagine an isotopy from K to K' as a continuous animation where at time $t = 0$, we have K and at time $t = 1$, we have K'.

Thankfully, we don't need to worry about the details of isotopies, because Reidemeister (1927) established the following theorem:

Theorem 1. *Two knot diagrams K, K' represent equivalent knots if and only if they are related by a sequence of Reidemeister moves* (*R-moves*).

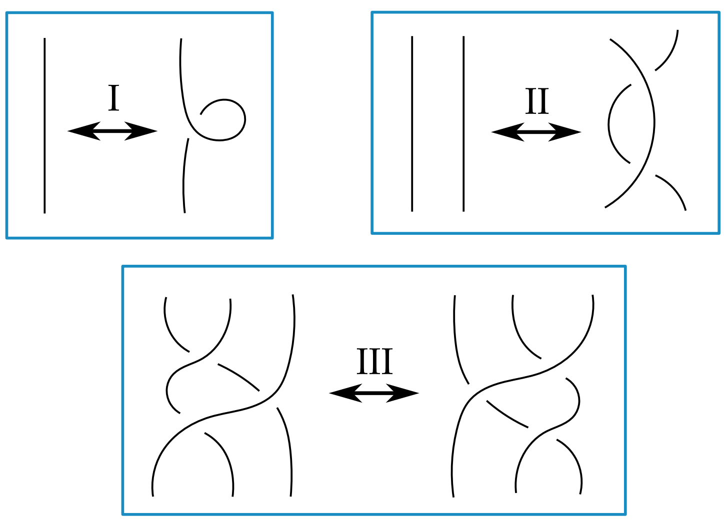


Figure 6

The fact that the three R-moves fully characterize equivalence of knots is very useful in that it turns the continuous problem of dealing with knots (can we push around knot K and eventually get to knot K'?) into a discretized one (which sequence of R-moves takes us from K to K'? Does such a sequence exist?). Thus, for the rest of this thesis, when we say two knots are equivalent, we mean that the knot diagram of one can be altered by these R-moves and made into the other.

We will refer to each R-move by its number. For example, the first box depicts the R1 move. Observe that each number matches the number of strands involved.

The fundamental question in knot theory is the following: given two knots, how can we tell if they are equivalent or not? This motivates the next definition.

Definition 3. Let $\mathcal{K}$ be the set of all knots, and S be a set. A function $I : \mathcal{K} \to S$ is a **knot invariant** if I is constant on equivalence classes of knots. In other words, if K is ambient isotopic to K', then $I(K) = I(K')$.

Thus, as Reidemeister's theorem classifies equivalences of knots, it gives us a way to determine whether or not a function is a knot invariant. In order to establish a function defined from a knot diagram as a knot invariant, one only needs to show that the function value is locally unchanged by R-moves. This is one reason why the knot diagram is the most commonly used representation of a knot (as opposed to another representation, like a Gauss code). See Kauffman (1999) for more about Gauss codes.

It will be useful for us to consider the oriented knot, in which we pick some direction for an arc (an uninterrupted line in the knot diagram), which we indicate with an arrow, and follow that through the rest of the knot.

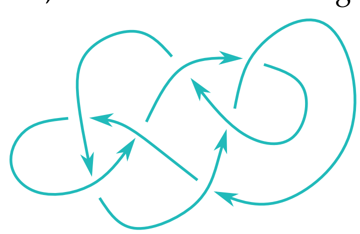


Figure 7

We will also be dealing with **links**, which are multiple component knots (two or more knots interlocked). A knot is just a link with a single component. We will refer to knots and links somewhat interchangeably.

For more on knots and links, see Elhamdadi and Nelson (2015).

2.2 Quandles

Just as groups are an algebraic structure motivated by the symmetries, quandles are algebraic structures motivated by knots. A specific case of this structure was studied by Takasaki (1943), who called it *kei*, which translates to "square jewel." We get the term quandle from Joyce (1982).

Definition 4. A set X equipped with a binary operation $\triangleright$ is a **quandle** if it satisfies

1. $x \triangleright x = x$ for all $x \in X$,
2. for each $y \in X$, the map $f_y : X \longrightarrow X$ defined by $f_y(x) = x \triangleright y$ is a bijection, and
3. $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ for all $x, y, z \in X$.

We will provide motivation for the quizzical quandle axioms in a bit. First we will draw a connection between quandles and knots. Say we have an oriented link diagram L with n arcs. Assign each arc some label, like $a, b, c, \ldots$ Then, we obtain a set of relations from the crossings of L by requiring the following relation to hold at each crossing:

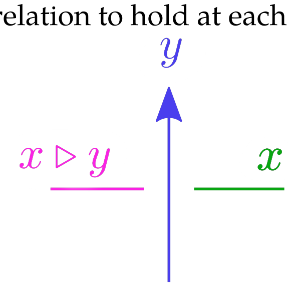


Figure 8

Note that only the orientation of the overarc matters. In words, if we assigned the label y to the overarc at a crossing, view the crossing with the y strand pointing up. Then if we have an arc labeled x on the right and an arc labeled z on the left, we require $z = x \triangleright y$.

Definition 5. The **fundamental quandle** $Q(L)$ is the set of equivalence classes of quandle words generated by the arc labels under the equivalence relations given by the crossings and the quandle axioms.

Example 1. Let us compute the fundamental quandle $Q(K_{3,1})$ for the trefoil knot. Here the arcs are labelled a, b, c.

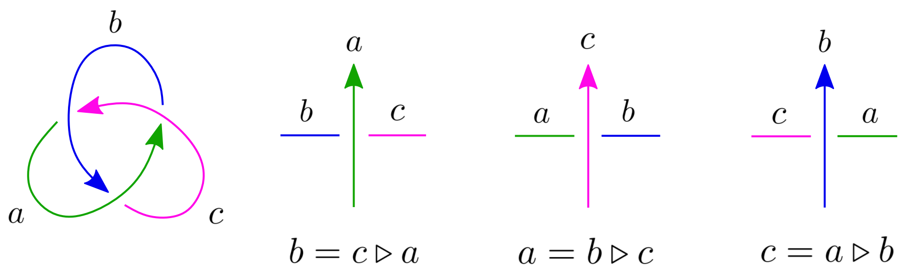


Figure 9

Each crossing yields a relationship between a, b and c, as shown in Figure 9. Then the fundamental quandle has presentation $Q(K_{3,1}) = \langle a, b, c \mid b = c \triangleright a, a = b \triangleright c, c = a \triangleright b \rangle$. Note that the fundamental quandle is infinite, since elements like $a \triangleright c$ cannot be expressed as one of a, b,or c. Just as in this example, by construction, any knot or link with finitely many arcs in its diagram (which is called a tame knot) will have a finitely generated fundamental quandle.

Now, how does the fundamental quandle relate to the quandle axioms? Well, we want the axioms to be defined in such a way that $Q(L)$ is an invariant of L. In other words, we need $Q(L)$ to be locally invariant under the R-moves! Let's look at each of the three R-moves and their relationships to the axioms.

First we have R1. Let's say the strand on the left is labelled x.

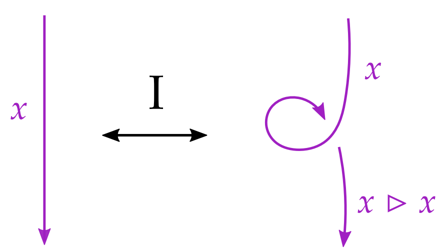


Figure 10

If we follow the quandle crossing rule after performing the R1 move, since x is passing under x, the bottom arc must be labelled $x \triangleright x$. If we want $Q(L)$ to be invariant under R1, we need the labels at the bottom on either side of the R1 move to match. Since x is on the left and $x \triangleright x$ is on the right, this means that $x \triangleright x = x$. Note that x was arbitrary, so this must hold for all labels. Hence we should require $x \triangleright x = x$ for all $x \in Q(L)$. Compare this to axiom 1 of the quandle definition.

Next, let's take a look at R2 in a similar manner.

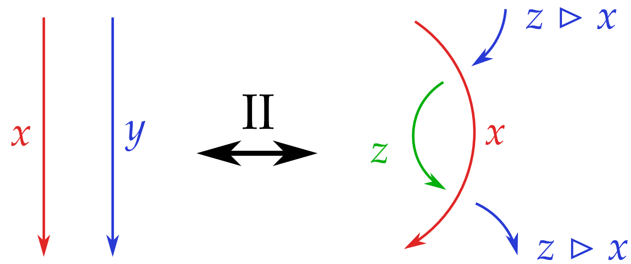


Figure 11

Comparing the left and right sides of the R2 move, we require $y = z \triangleright x$. Looking at the right side first, we see that given any $z, x \in Q(L)$ there should be a unique $y \in Q(L)$ such that $y = z \triangleright x$. This means that the map $f_x(z) : Q(L) \longrightarrow Q(L)$ defined $f_x(z) := z \triangleright x$ is injective. Since the blue strand on the left could have been given any label in $Q(L)$, this map $f_x(z)$ is surjective. Thus, it is bijective, which is precisely the condition we have in quandle axiom 2.

Finally, let's talk about R3.

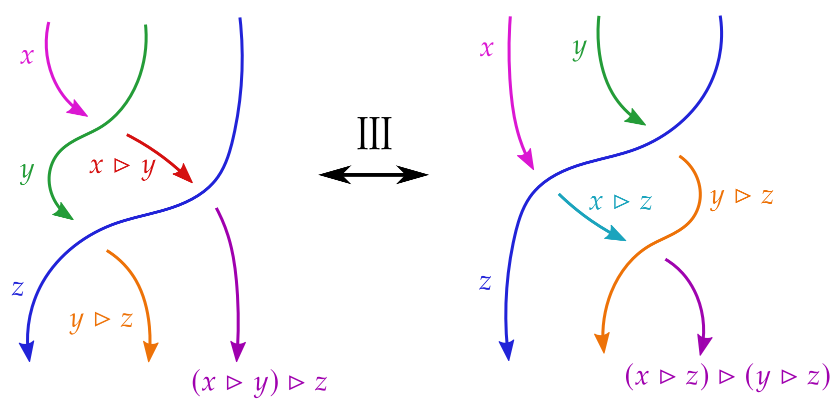


Figure 12

As before, we require that the labels at the top and bottom match on either side of the R3 move. At the top we start with x, y, z from left to right. On the bottom, note that the left and middle strand labels match already. All that's left is to require $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$, which is the right distributive condition required in quandle axiom 3. It is interesting to note that this axiom tells us that quandles are a non-associative algebraic structure.

This shows that the quandle axioms are motivated by the R-moves, and that they are defined in such a way that allows the fundamental quandle of a knot to truly be fundamental to that knot! More precisely, the fundamental quandle is a knot invariant.

Remark 1. The fundamental quandle is a complete invariant up to mirror image for non-split links, which was proved by Fenn and Rourke (1992). This means that for non-equivalent knots K, K', the fundamental quandles $Q(K)$, $Q(K')$ will not be isomorphic. If K and K' are mirror images of each other, their fundamental quandles may or may not be isomorphic. Although $Q(K)$ is a strong invariant, by itself it is not so helpful in distinguishing knots, since showing whether or not two presentations for fundamental quandles are isomorphic is just as difficult as dealing with the knot diagrams themselves. As we will see in a bit, the quandle coloring invariant uses the fundamental quandle in a way that is useful.

Remark 2. We just saw how knots are related to quandles, but it is not always the case that a quandle has a knot associated to it, as we will see in the following examples.

Example 2. Let G be a group where the $\triangleright$ operation is n-fold conjugation: $x \triangleright y = y^n x y^{-n}$. Then $(G, \triangleright)$ is a quandle.

Example 3. Let $X = \mathbb{Z}/n\mathbb{Z}$ where the $\triangleright$ operation is defined by $x \triangleright y \equiv 2y - x \pmod{n}$. Then $(X, \triangleright)$ is a quandle called the **dihedral quandle**.

Note that the dihedral quandle is an example of a finite quandle. Finite quandles can be fully represented by operation tables. The operation table for the dihedral quandle with $n = 3$ is shown below.

$\triangleright$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

Definition 6. Let X, Y be quandles with multiplication operations indicated by $\triangleright_X$ and $\triangleright_Y$ respectively. A map $f : X \to Y$ is a **quandle homomorphism** given that $f(a \triangleright_X b) = f(a) \triangleright_Y f(b)$ for any $a, b \in X$.

Definition 7. Let X, Y be quandles. The **hom-set** $\mathrm{Hom}(X, Y)$ is the set of all quandle homomorphisms $\phi : X \to Y$.

Quandle homomorphisms are defined the in same way as group homomorphisms. Since we will be working with quandles, unless otherwise stated, homomorphisms will refer to quandle homomorphisms.

We are finally ready to discuss the quandle coloring invariant, which is a central point of interest in this thesis.

Definition 8. Let L be an oriented link with fundamental quandle $Q(L)$ and X be a finite quandle called the coloring quandle. We will call $\mathrm{Hom}(Q(L), X)$ the **coloring space**. The **quandle coloring invariant** is the cardinality of the coloring space, $|\mathrm{Hom}(Q(L), X)|$, denoted $\Phi_X^{\mathbb{Z}}(L)$.

Remark 3. Combinatorially, each element $\phi \in \mathrm{Hom}(Q(L), X)$ can be represented as a "coloring" of the diagram of L by colors from X, as we will see in the following example. Using this analogy, a valid coloring is an assignment of an element from X to each arc in L's link diagram in a way that respects the quandle operation of X at each crossing. I like to think of the coloring quandle X as our crayon box, with the elements of X being crayons. Then we color the arcs of L with the crayons, making sure we follow the crossing rule to get a valid coloring.

How does a coloring correspond to a homomorphism? Recall that the arc labels of L generate the fundamental quandle $Q(L)$. For a coloring of L by X, each arc is assigned an element of X, so we can associate to that coloring a map $\phi : Q(L) \to X$ where if an arc is labeled a in the fundamental quandle and is assigned the color $x \in X$, then ϕ maps $a \mapsto x$. This map is a homomorphism due to the fact that the coloring must obey the crossing relation. Let's see why.

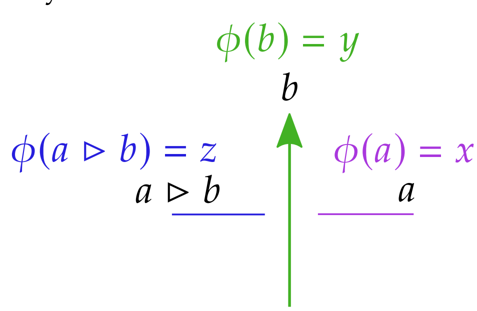


Figure 13

Suppose we have some knot diagram K colored by X with a crossing, as shown in Figure 13. Let the arcs be labelled $a, b, a \triangleright b$ in the fundamental quandle of K and suppose they are colored $x, y, z \in X$ respectively, so the corresponding map ϕ takes $a \mapsto x$, $b \mapsto y$, and $(a \triangleright b) \mapsto z$. Since the coloring must obey the crossing relation, we require $z = x \triangleright y$. Thus,

$$\phi(a \triangleright b) = z = x \triangleright y = \phi(a) \triangleright \phi(b),$$

which is the homomorphism condition. The situation at each crossing must also work out the same way, so we have $\phi(a \triangleright b) = \phi(a) \triangleright \phi(b)$ for any two generators a and b. Then, since a homomorphism from the finitely generated quandle $Q(K)$ is uniquely determined by where the generators are mapped, the map ϕ corresponding to the coloring is a homomorphism.

Example 4. Let us compute the quandle coloring invariant $\Phi_X^{\mathbb{Z}}(L)$ where L is the trefoil as orientied in example 1 and X is the dihedral quandle on 3 elements. Recall that $Q(L)$ has presentation $Q(L) = \langle a, b, c \mid b = c \triangleright a, a = b \triangleright c, c = a \triangleright b \rangle$ and X has the multiplication table shown below.

$\triangleright$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

To count the homomorphisms, we need to count the different ways we can validly map the generators a, b, c to elements of X. As seen in example 1, we must satisfy $c = a \triangleright b$, so once colors for a and b are chosen, the color for c is determined by the multiplication table. There are 3 choices each for a and b, so we see that $\Phi_X^{\mathbb{Z}}(L) = |\text{Hom}(Q(L), X)| = 3 \cdot 3 = 9$. These 9 colorings are depicted below where 0 is green, 1 is blue, and 2 is pink.

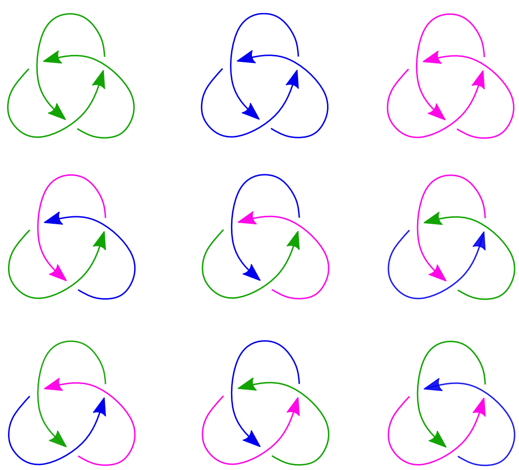


Figure 14

For example, the coloring in the center of the grid represents the homomorphism that maps $a \mapsto 0, b \mapsto 1, c \mapsto 2$.

Remark 4. Because of the first quandle axiom, which requires $x \triangleright x = x$ for all $x \in X$, the constant map $\phi_x : Q(L) \to X$ mapping $a \mapsto x$ for all $a \in Q(L)$ will always be a homomorphism for any x, so it is included in $\mathrm{Hom}(Q(L), X)$. Thus, for a quandle X with n elements, any link L will have at least n X-colorings (the constant colorings).

2.3 The Quandle Coloring Quiver

Given any link diagram, we can pick any finite quandle X and compute $\Phi_X^{\mathbb{Z}}$ to get some integer value invariant. So if we have two links L, L' where $\Phi_X^{\mathbb{Z}}(L) \neq \Phi_X^{\mathbb{Z}}(L')$, then we can distinguish L and L' from one another, and each finite quandle gives us a different invariant!

However, it ignores some potentially useful information since $\Phi_X^{\mathbb{Z}}(L) = |\mathrm{Hom}(Q(L), X)|$, the number of elements in the coloring space. Note that the set $\mathrm{Hom}(Q(L), X)$ is an invariant of L since $Q(L)$ is an invariant and $\mathrm{Hom}(Q(L), X)$ is just the set of homomorphisms from $Q(L)$ to a fixed quandle X. Also, the coloring space itself contains more information than the number of elements in the coloring space, although comparing two coloring spaces could be pretty cumbersome.

This is our motivation for this thesis project: How can we enhance the quandle coloring invariant? When we say a knot invariant E is an "enhancement" of another knot invariant I, we mean that evaluating E of a link L will give us at least as much information as evaluating I of L. Or that given $E(L)$ we can extract the information contained in $I(L)$. For example, $\mathrm{Hom}(Q(L), X)$ is an enhancement of $\Phi_X^{\mathbb{Z}}$.

We approached our research question by considering a quiver structure. A quiver is a directed graph that allows multiple edges and loops. The following definition is adapted from Cho and Nelson (2019b).

Definition 9. Let X be a finite quandle and L an oriented link. For any set of quandle endomorphisms $S \subset \mathrm{Hom}(X, X)$, the associated **quandle coloring quiver**, denoted $\mathcal{Q}_X^S(L)$, is the directed graph with a vertex for every element $f \in \mathrm{Hom}(Q(L), X)$ and an edge directed from f to g when $g = \phi f$ for an element $\phi \in S$. Important special cases include the case $S = \mathrm{Hom}(X, X)$, which we call the **full quandle coloring quiver** of L with respect to X, denoted $\mathcal{Q}_X(L)$, and the case when $S = \{\phi\}$ is a singleton, which we will denote by $\mathcal{Q}_X^{\phi}(L)$.

Theorem 2. *For X, L, and S as described above, $\mathcal{Q}_X^S(L)$ is an invariant of L.*

Proof. $Q(L)$ is an invariant of L and $\text{Hom}(Q(L), X)$ is fixed for L, X. □

Let's unpack this definition. As with the quandle coloring invariant, our ingredients include a link L and a finite coloring quandle X. The vertices correspond to elements of $\text{Hom}(Q(L), X)$, which we know to be colorings of L by X. Thus, we know that $\mathcal{Q}_X^S(L)$ is an enhancement of $\Phi_X^{\mathbb{Z}}(L)$, since the cardinality of the vertex set of $\mathcal{Q}_X^S(L)$ will be equal to $\Phi_X^{\mathbb{Z}}(L)$ by definition.

The significance of considering endomorphisms is that these endomorphisms will reveal something about the structure of the coloring space. Recall that an endomorphism is simply a homomorphism from X to itself. Suppose we have a vertex $v_1 \in \text{Hom}(Q(L), X)$ and an endomorphism $e \in \text{Hom}(X, X)$. Note that v_1 and e are both homomorphisms, and the codomain of v_1 matches the domain of e, so we can compose them. Since the composition of homomorphisms is a homomorphism, $ev_1 : Q(L) \to X$ is also a homomorphism, so since each element of $\text{Hom}(Q(L), X)$ is a vertex, we must have $v_2 \in \text{Hom}(Q(L), X)$ so that $ev_1 = v_2$.

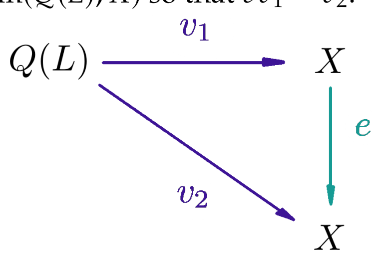


Figure 15

Whenever we have an endomorphism $e \in S$ as in definition 9, and v_1, v_2 as described, we will draw a directed edge from v_1 to v_2.

Example 5. Let's solidify our understanding with an example. Consider L to be the Hopf link, which is the link consisting of two interlocking unknots (circles). Let X be the quandle given below.

$\triangleright$	1	2	3
1	1	1	2
2	2	2	1
3	3	3	3

Let S be the singleton consisting of the endomorphism ϕ that maps $1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 3$. Then we can compute the quandle coloring quiver $\mathcal{Q}_X^{\phi}(L)$.

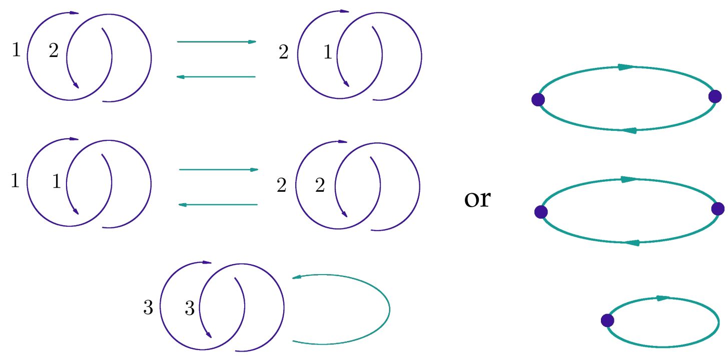


Figure 16

The five colorings of L are drawn in Figure 16. We find the edges by applying ϕ to each coloring and seeing what vertex we get sent to. ϕ swaps 1 and 2 while fixing 3, so for instance, the coloring involving 1 and 2 gets mapped to the coloring involving 2 and 1 and vice versa. This is represented by the teal arrows. Then we can shrink each Hopf link to a vertex to obtain the visualization on the right, which is a more traditional depiction of a directed graph.

In total, there are 7 endomorphisms of X,

$$\begin{aligned}
\phi_1 &: \quad [1,1,1] \\
\phi_2 &: \quad [2,2,2] \\
\phi_3 &: \quad [3,3,3] \\
\phi_4 &: \quad [2,1,3] \\
\phi_5 &: \quad [1,2,3] \\
\phi_6 &: \quad [2,2,1] \\
\phi_7 &: \quad [1,1,2]
\end{aligned}$$

where the bracketed values represent $[\phi_i(1), \phi_i(2), \phi_i(3)]$ for each i. Note that since we are dealing with finite quandles, the set of all endomorphisms can be computed exhaustively. If we let S be any subset of those, we would potentially get a quandle coloring quiver with different arrows. Note that we get arrows by applying each endomorphism to each vertex, so the out-degree of each vertex will be $|S|$, the number of endomorphisms.

The full quandle coloring quiver $\mathcal{Q}_X(L)$ is shown below, where the numbers represent multiplicities of edges.

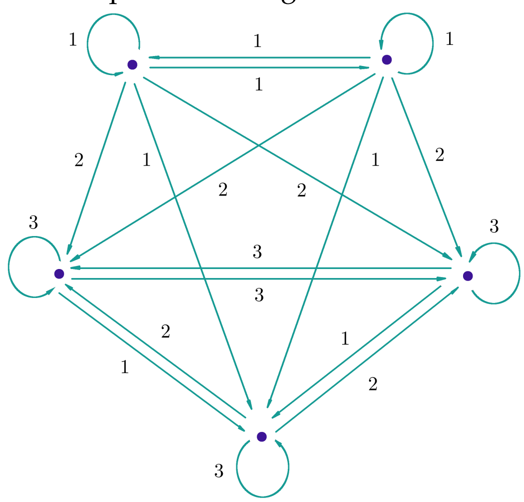


Figure 17

Remark 5. The full quiver $\mathcal{Q}_X(L)$ is a categorification of the quandle coloring invariant, with X-colorings of L as objects and elements of $\mathrm{Hom}(X, X)$ as morphisms. We will briefly discuss why.

Definition 10. Let C consist of objects and morphisms, which are maps between objects. If the following conditions hold, then C is a **category**:

1. For each object a, there exists an identity morphism $1_a : a \longrightarrow a$ so that for any morphisms $f : a \longrightarrow b$ and $g : c \longrightarrow a$, we have $f \circ 1_a = f$ and $1_a \circ g = g$.

2. For any pair of morphisms $f : a \longrightarrow b$, $g : b \longrightarrow c$, there exists a composition morphism $g \circ f : a \longrightarrow c$, and composition of morphisms is associative. In other words, $h \circ (g \circ f) = (h \circ g) \circ f$ for morphisms h, g, f with appropriate domain and codomain.

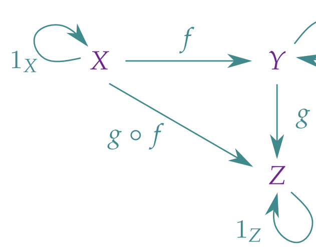


Figure 18

Consider the set $S = \mathrm{Hom}(X, X)$ of all endomorphisms of X. Note that the identity map $I_x \in S$ for any $x \in X$ is an endomorphism, which satisfies the first axiom. Then since composition of endomorphisms always results in another endomorphism, and the composition is associative, $\mathcal{Q}_X(L)$ is a category.

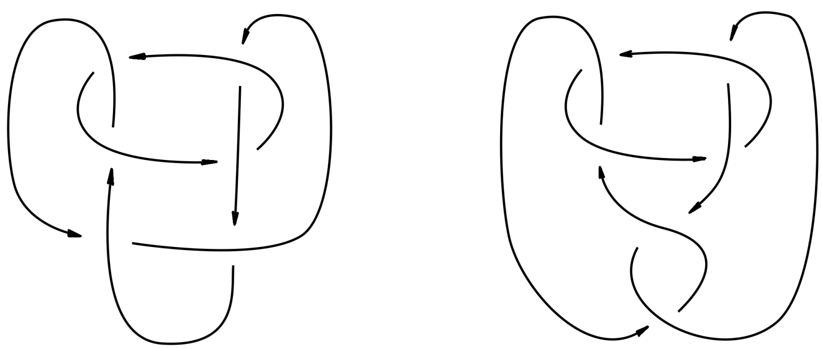
Recall that the goal of this project is to differentiate links. It turns out that we do not necessarily need to set $S = \mathrm{Hom}(X, X)$ to be able to achieve this goal. In fact, we may only need to use a single endomorphism.

We know that the quiver is an enhancement of the quandle coloring invariant. In the next example, we will demonstrate that the quiver is strictly stronger than the coloring invariant.

Example 6. Let X be the quandle given below.

$\triangleright$	1	2	3	4
1	1	3	1	3
2	4	2	4	2
3	3	1	3	1
4	2	4	2	4

Let $\phi : X \to X$ be the endomorphism mapping $1, 3 \mapsto 4$ and $2, 4 \mapsto 2$. Then consider the links $L6a1$ and $L6a5$, whose labels come from Bar-Natan et al. (2014), which are shown below.



$L6a1$ $L6a5$

Figure 19

$L6a1$ and $L6a5$ cannot be distinguished by the quandle coloring invariant, as $\Phi_X^{\mathbb{Z}}(L6a5) = \Phi_X^{\mathbb{Z}}(L6a5) = 16$, so they both have 16 X-colorings. However, consider their quandle coloring quivers, shown below.

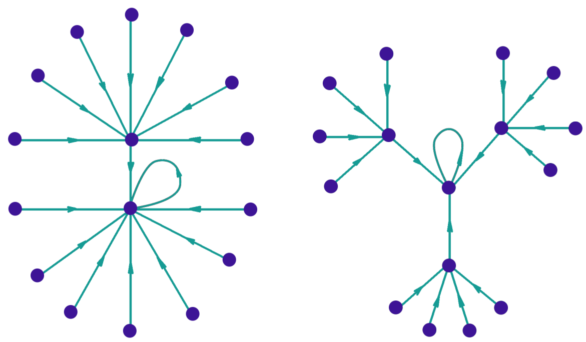


Figure 20

As the two graphs are not isomorphic, the quiver structure is enough to distinguish the links even when the quandle coloring invariant could not. This is a proof by example for the superiority of the quandle coloring quiver.

2.4 The In-degree Polynomial

As we just learned, the quandle coloring quiver is a graph valued invariant of links. Graphs can be nicely visualized, but there can be drawbacks to working with a graph valued invariant. For instance, it may be cumbersome to have to compare graphs to each other, especially if they are large.

Traditionally, polynomial valued invariants, such as the Jones polynomial due to Jones (1985), have been popular in knot theory. Thus, it was natural to try to encapsulate some information from the quandle coloring quiver into a condensed polynomial form.

We noticed from looking at several quandle coloring quivers that the out-degree of every vertex is always the same. The explanation for this is that since we get edges by applying each endomorphism in S to each vertex and seeing where the vertex gets mapped to, the out-degree of every vertex will be $|S|$. However, the out-degree varies from vertex to vertex; some colorings are more common as images of endomorphisms than others. This is what led us to create the following.

Definition 11. Let X be a finite quandle, $S \subset \mathrm{Hom}(X, X)$ a set of quandle endomorphisms, L an oriented link and $Q_X^S(L)$ the associated quandle coloring quiver with set of vertices $V(Q_X^S(L))$. Then the **in-degree quiver**

polynomial of L with respect to X is

$$\Phi_X^{\deg^+,S}(L) = \sum_{f\in V(\mathcal{Q}_X^S(L))} u^{\deg^+(f)}.$$

If $S = \{\phi\}$ is a singleton we will write $\Phi_X^{\deg^+,S}(L)$ as $\Phi_X^{\deg^+,\phi}(L)$ and if $S = \mathrm{Hom}(X,X)$ we will write $\Phi_X^{\deg^+,S}(L)$ as $\Phi_X^{\deg^+}(L)$.

Remark 6. Since $\Phi_X^{\deg^+,S}(L)$ is computed by summing over all of the vertices of $\mathcal{Q}_X^S(L)$, which represent the X-colorings of L, evaluating the $\frac{1}{|S|}\mathcal{Q}_X^S(L)$ at $u = 1$ yields the quandle coloring invariant. This means that $\Phi_X^{\deg^+,S}(L)$ is an enhancement of $\Phi_X^{\mathbb{Z}}(L)$.

Example 7. Using the links $L6a1$ and $L6a2$ with the same quandle X and endomorphism ϕ as in Example 6, we have

$$\begin{aligned}\Phi_X^{\deg^+,\phi}(L6a1) &= u^9 + u^7 + 14 \neq \\ \Phi_X^{\deg^+,\phi}(L6a2) &= 4u^4 + 12.\end{aligned}$$

Since these in-degree quiver polynomials are not equal, we are able to distinguish these links even though they have the same number of colorings (evaluating the polynomials at $u = 1$ yields 16 in both cases), which shows that the in-degree quiver polynomial is strictly stronger than the quandle coloring invariant.

Example 8. To demonstrate the effectiveness of $\Phi_X^{\deg^+,S}(L)$, the following example has been adapted from our paper, Cho and Nelson (2019b). Let X be the quandle with operation table

$\triangleright$	1	2	3	4	5	6	7	8
1	1	4	2	3	3	2	1	4
2	3	2	4	1	4	1	2	3
3	4	1	3	2	1	4	3	2
4	2	3	1	4	2	3	4	1
5	8	8	8	8	5	5	5	5
6	5	5	5	5	6	6	6	6
7	7	7	7	7	7	7	7	7
8	6	6	6	6	8	8	8	8

and let $\phi : X \to X$ be given by $\phi(1) = \phi(2) = \phi(3) = \phi(4) = 7$, $\phi(5) = \phi(6) = \phi(8) = 5$ and $\phi(7) = 6$. Below, we have a table of the in-degree polynomial for prime links with up to 7 crossings.

L	$\Phi_X^{\deg^+,\phi}(L)$
$L2a1$	$u^9 + 3u^4 + 2u^3 + u + 21$
$L4a1$	$u^{16} + u^9 + 2u^4 + 2u^3 + u + 33$
$L5a1$	$2u^{12} + u^9 + 3u^4 + 2u^3 + u + 43$
$L6a1$	$u^9 + 3u^4 + 2u^3 + u + 21$
$L6a2$	$2u^{12} + u^9 + 3u^4 + 2u^3 + u + 43$
$L6a3$	$u^{16} + 2u^{12} + u^9 + 2u^4 + 2u^3 + u + 55$
$L6a4$	$u^{27} + 3u^{12} + 3u^9 + 7u^4 + 3u^3 + u + 110$
$L6a5$	$u^{27} + 3u^9 + 7u^4 + 3u^3 + u + 77$
$L6n1$	$u^{27} + u^{16} + 3u^9 + 6u^4 + 3u^3 + u + 89$
$L7a1$	$u^{16} + 2u^{12} + u^9 + 2u^4 + 2u^3 + u + 55$
$L7a2$	$u^{16} + u^9 + 2u^4 + 2u^3 + u + 33$
$L7a3$	$u^{16} + u^{12} + u^9 + 2u^4 + 2u^3 + u + 44$
$L7a4$	$u^{16} + 2u^{12} + u^9 + 2u^4 + 2u^3 + u + 55$
$L7a5$	$u^9 + 3u^4 + 2u^3 + u + 21$
$L7a6$	$u^9 + 3u^4 + 2u^3 + u + 21$
$L7a7$	$u^{27} + u^{16} + 2u^{12} + 3u^9 + 6u^4 + 3u^3 + u + 111$
$L7n1$	$u^{16} + u^9 + 2u^4 + 2u^3 + u + 33$
$L7n2$	$u^{16} + u^{12} + u^9 + 2u^4 + 2u^3 + u + 44$

Remark 7. In this case, the polynomial is unable to distinguish certain links, such as $L5a1$ and $L6a1$. The invariant is very flexible in that choosing a different coloring quandle or set of endomorphisms will yield whole new class of invariant polynomials that could potentially distinguish different links, so there is likely a different choice of X and ϕ that we could pick that could distinguish $L5a1$ and $L6a1$.

Chapter 3

Boltzmann Weights and Quandle Cohomology

Recall that our quandle coloring quiver requires three inputs: a link L, a coloring quandle X, and a set of endomorphisms $S \subset \mathrm{Hom}(X, X)$. We showed in the previous chapter that the quandle coloring quiver is strictly stronger than the quandle coloring invariant $\Phi_X^{\mathbb{Z}}(L)$, which is somewhat intuitive as $\Phi_X^{\mathbb{Z}}(L)$ only takes in two inputs, L and X.

Can we make the quandle coloring quiver even stronger by considering a fourth input? This is the question we will explore in this chapter. We will show that considering a function called a Boltzmann weight for this fourth input enhances our quiver. In Section 3.1 we will introduce Boltzmann weights and in Section 3.2 we will discuss the relationship between Boltzmann weights and quandle cohomology.

3.1 Boltzmann Weights

Definition 12. Let A be an abelian group (usually $\mathbb{Z}$ or $\mathbb{Z}_n$). For a quandle X, a function $\phi : X \times X \to A$ is a **rack Boltzmann weight** if for all $x, y, z \in X$ we have

$$\phi(x, z) + \phi(x \triangleright z, y \triangleright z) = \phi(x, y) + \phi(x \triangleright y, z).$$

If we also have $\phi(x, x) = 0$ for all $x \in X$ then ϕ is a **quandle Boltzmann weight**.

Remark 8. We can assign Boltzmann weights to crossings of quandle colored knots by the following relations:

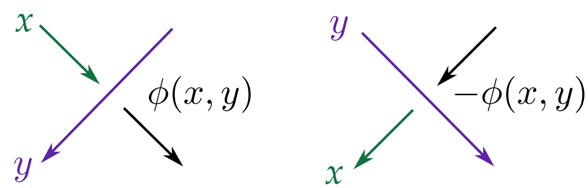


Figure 21

The left crossing is positively oriented, and the right crossing is negatively oriented. We want ϕ to behave so that for a given quandle colored knot diagram, the sum of the Boltzmann weights over all crossings is invariant under R-moves.

To show that the state sum is locally invariant under R-moves, we must show that for each quandle colored move, the sums on both sides are the same.

For the R2 move, the left side consists of two uncrossed strands, so the state sum is 0 since there are no crossings. We can get the state sum of the right side of the R2 move by stacking the images in Figure 21 on top of each other, which gives us a state sum of $\phi(x, y) + (-\phi(x, y)) = 0$, so the invariance under R2 is satisfied by the way we defined the Boltzmann weight of a crossing.

Next, we can look at R1.

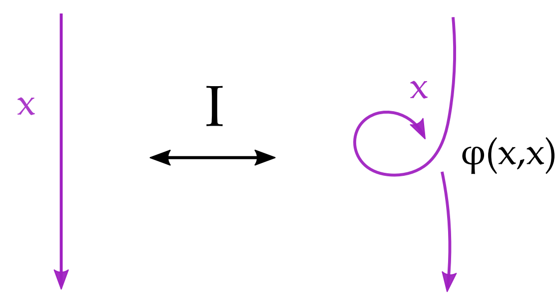


Figure 22

On the left, there are no crossings, so the state sum will be 0. On the right, we have one crossing whose Boltzmann weight is $\phi(x, x)$. Thus, we want $\phi(x, x) = 0$, which is the condition for a Boltzmann weight to be a quandle Boltzmann weight.

The situation occurring in the R3 move is the most interesting.

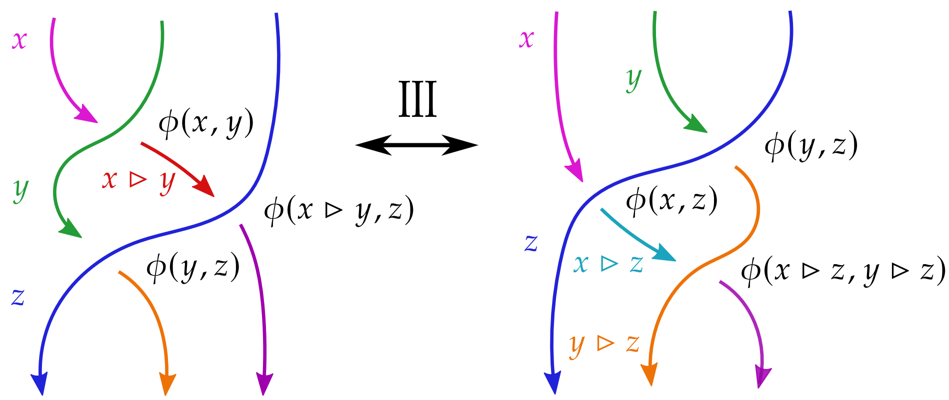


Figure 23

Note that there are three crossings on either side. The corresponding Boltzmann weights are shown, and we want the sums on either side to be equal. Note that $\phi(y,z)$ appears on both sides, so those will cancel in the sums. Then the requirement is that

$$\phi(x,y) + \phi(x \triangleright y, z) = \phi(x,z) + \phi(x \triangleright z, y \triangleright z),$$

which is precisely the condition appearing in the definition of a Boltzmann weight.

What all of this demonstrates is that given a knot diagram L and coloring quandle X, a quandle Boltzmann weight ϕ will give us an integer valued invariant for each X-coloring of L, which can be evaluated by computing the state sum of ϕ over all the crossings of an X-coloring of L.

Since Boltzmann weights involve quandle colored knot diagrams, they seem related enough to the quandle coloring quiver to be potentially useful. But how exactly can we determine a function that satisfies the Boltzmann weight conditions? This is where cohomology comes in handy. In the next section, we will introduce some relevant background.

3.2 Quandle Cohomology

Homology and cohomology groups of topological spaces are invariants that are commonly studied in algebraic topology. Here, we will introduce the basics of (co)homology and specifically discuss quandle (co)homology, which was defined and studied by Carter et al. (2003). Then we will bring it back to how it connects to Boltzmann weights and the rest of the project. As a note, this section provides a theoretical framework and motivation for a certain tool used in the project (namely quandle 2-cocycles), but understanding this framework is not necessary for understanding the rest of the project.

Definition 13. For $k \in \mathbb{N}$, let C_k be an abelian group and $\partial_k : C_k \to C_{k-1}$ be a group homomorphism such that $\partial_k \partial_{k+1} = 0$ for all $k \geq 0$.

$$0 \xleftarrow{\partial_0} C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} \cdots \xleftarrow{\partial_{n-1}} C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} \cdots$$

Then the sequence of C_k and ∂_k is called a **chain complex**. An element of C_k is called a **k-chains** and the map ∂_k is called a **boundary map**. Then the **k-th homology module** is $H_k = \ker \partial_k / \operatorname{Im} \partial_{k-1}$.

Note that if $\operatorname{Im} \partial_{k+1} = \ker \partial_k$ for all k, then our sequence is exact, so homology captures how "far from exact" our chain complex is.

For some intuition about the condition that $\partial_k \partial_{k+1} = 0$, it may be helpful to contemplate the following image, which attempts to illustrate that "the boundary of a boundary is empty."

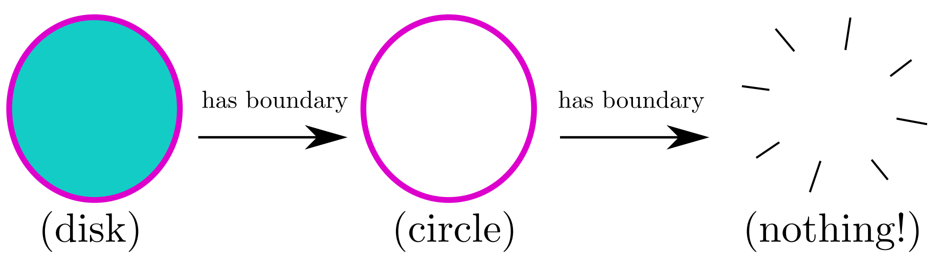


Figure 24

To get a grasp of what is going on, let us consider an example (quandle homology). For this, we will need to know what a free abelian group is.

Remark 9. The **free abelian group** generated by n-tuples of elements of a set X has elements that are formal sums of such n-tuples. For example, for

$X = \{a, b\}$, the free abelian group generated by 3-tuples would contain the element $3(a, b, b) + 5(a, a, b) - 2(b, b, b)$. The general form of an element is a finite sum of n-tuples whose coefficients are in $\mathbb{Z}$.

Definition 14. For a finite quandle X, let $\mathbf{C^R_n(X)}$ be the free abelian group generated by $(x_1, ..., x_n)$ for $x_i \in X$. We define the boundary map $\partial_n : C^R_n(X) \to C^R_{n-1}(X)$ as the following group homomorphism:

$$\begin{aligned} \partial_n(x_1, \dots, x_n) = \sum_{i=2}^{n} (-1)^i [& (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \\ & - (x_1 \triangleright x_i, x_2 \triangleright x_i, \dots, x_{i-1} \triangleright x_i, x_{i+1}, \dots, x_n)] \end{aligned} \tag{3.1}$$

for $n \geq 2$ and $\partial_n = 0$ for $n < 2$, then we extend linearly. Our chain complex is $C^R_*(X) = \{C^R_n(X), \partial_n\}$. (As an aside, the R stands for "rack." Racks are a generalization of quandles where the first quandle axiom ($x \triangleright x = x$ for all $x \in X$) is ignored. This structure can be useful for studying framed knots. See Fenn and Rourke (1992) for more about racks.)

Let $\mathbf{C^D_n(X)}$ be the subgroup of $C^R_n(X)$ generated by elements $(x_1, \dots, x_n)$ where $x_i = x_{i+1}$ for some i. (The D stands for "degeneration.")

Let $\mathbf{C^Q_n(X)} = C^R_n(X)/C^D_n(X)$. (The Q stands for "quandle." Taking the quotient by $C^D_n(X)$ allows us to satisfy the first quandle axiom that is ignored by the rack structure. This will be discussed further in Remark 12.) Then the **quandle chain complex** is $\mathbf{C^Q_*(X)} = \{C^Q_n(X), \partial^Q_n\}$ where ∂^Q_n is the induced homomorphism onto the quotient.

Remark 10. The boundary map defined by equation (3.1) seems pretty mysterious at first glance. To gain some intuition as to where this comes from, let's look at $n = 2$. We have

$$\partial_2(x, y) = [(y) - (y)] - [(x) - (x \triangleright y)] = y - y - x + (x \triangleright y).$$

Suppose we have some positively oriented crossing in a link diagram with the over-strand colored y and under-strand colored x on the right. We would compute the boundary of that crossing as $\partial(x, y)$, which corresponds to adding up the colors of the strands, with sign indicated by direction ("+"

for head and "-" for tail), as in the figure below.

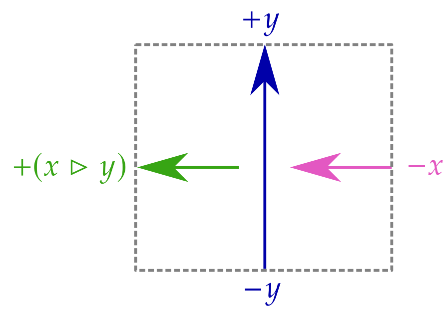


Figure 25

This intuition holds for higher dimensions, such as quandle colored knotted surfaces.

Next, we dualize to get a cochain complex.

Definition 15. Let A be a fixed abelian group. For all abelian groups B, we define $\mathrm{Hom}_{\mathbb{Z}}(\mathbf{B}, \mathbf{A})$ to be the group of homomorphisms from B to A.

For abelian groups A, B, C and group homomorphism $f : B \to C$, we define $\mathrm{Hom}_{\mathbb{Z}}(\mathbf{f}, \mathbf{A}) : \mathrm{Hom}(C, A) \to \mathrm{Hom}(B, A)$ to be the homomorphism mapping $\phi \mapsto \phi \circ f$ for all $\phi \in \mathrm{Hom}(C, A)$.

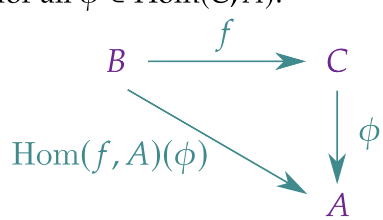


Figure 26

Remark 11. The map $\mathrm{Hom}_{\mathbb{Z}}(-, A)$, where the input can be a abelian group or group homomorphism as in Definition 15, is a contravariant functor from the category of chain complexes to the category of cochain complexes.

Definition 16. For an abelian group A (we will be using $A = \mathbb{Z}$ or $\mathbb{Z}_n$), let $\mathbf{C^n_Q(X; A)} = \mathrm{Hom}(C^Q_n(X), A)$ and $\delta^n : C^n_Q(X; A) \to C^{n+1}_Q(X; A)$ be defined by

$$(\delta^n f)(x_1, \ldots, x_{n+1}) = f \circ \partial_{n+1}(x_1, \ldots, x_{n+1}).$$

The **quandle cochain complex** is $\mathbf{C^*_Q(X; A)} = \{C^n_Q(X; A), \delta^n\}$.

The **k-th cohomology module** is $H^k = \ker \delta^k / \mathrm{Im}\, \delta^{k-1}$. Elements of $\ker \delta^k$ are called **k-cocycles** and elements of $\mathrm{Im}\, \delta^k$ are called **k-coboundaries**.

Remark 12. As we are dealing with knots, which are 2 dimensional, we will be most interested in quandle 2-cocycles. Then a quandle 2-cocycle is a map $X \times X \longrightarrow \mathbb{Z}$ that satisfies

$$\phi \circ \partial(x, y, z) = 0$$

for any $x, y, z \in X$. Expanding the sum for $\partial(x, y, z)$ yields

$$[\phi(x, z) - \phi(x \triangleright y, z)] - [\phi(x, y) - \phi(x \triangleright z, y \triangleright z)] = 0. \tag{3.2}$$

Note that Equation 3.2 can be rearranged to obtain the same condition for ϕ to be a rack Boltzmann weight from Definition 12. The additional condition needed for ϕ to be a quandle Boltzmann weight is $\phi(x, x) = 0$ for all $x \in X$, and this condition holds for quandle 2-cocycles as well. This is because by definition, the degeneration subgroup is $C_2^D(X) = \langle (x, x) | x \in X \rangle$. Thus, since $(x, x) \in C_2^D(X)$, and the quandle chain complex contains the quotient group $C_2^Q(X) = C_2^R(X)/C_2^D(X)$, the elements of the form (x, x) are in the zero set of $C_2^Q(X)$. Then, as ϕ must be a homomorphism $C_2^Q(X) \to A$, it must map group identities to each other, so it follows that $\phi(x, x) = 0$.

In summary, ϕ is a quandle 2-cocycle if and only if it is a quandle Boltzmann weight. Thus, we can define the evaluation of quandle 2-cocycles on colored knot diagrams in the same way we did for Boltzmann weights in Remark 8. We will refer to quandle 2-cocycles and Boltzmann weights interchangeably from this point forward.

Boltzmann weights can be written as linear combinations of elementary functions $\chi_{i,j} : X \times X \longrightarrow A$ where

$$\chi_{i,j}(x_1, x_2) = \begin{cases} 1, & \text{for } i = x_1 \text{ and } j = x_2 \\ 0, & \text{otherwise.} \end{cases}$$

Remark 13. Note that if a quandle 2-cocycle ϕ is also a 1-coboundary, then for a colored knot v, we must have $\phi(v) = 0$. To prove this, recall that ϕ is a 1-coboundary if there exists some homomorphism f such that $\delta^1(f) = \phi$, so

$$\phi(x, y) = \delta^1(f)(x, y) = f \circ \partial_2(x, y) = f((x) - (x \triangleright y)).$$

We will reproduce and modify the Boltzmann weight of a crossing from

Section 8 below.

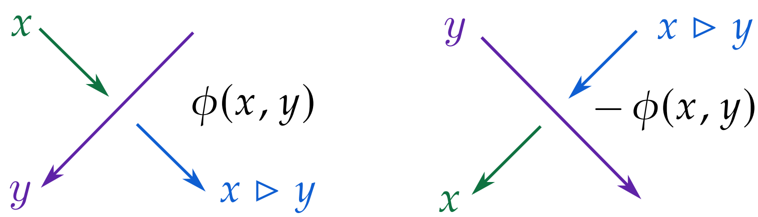


Figure 27

At a positive crossing, we have x going in ("$+x$"), $x \triangleright y$ going out ("$-x \triangleright y$"), and y going in and out (doesn't contribute to the "flux" of the crossing) and the crossing is assigned the value of $\phi(x,y)$, which equals $f((x)-(x \triangleright y))$ since it is a coboundary. Similarly, for the negative crossing, we have x going out ("$-x$"), $x \triangleright y$ going in ("$+x \triangleright y$") and the weight is $-\phi(x,y) = -f((x)-(x \triangleright y)) = f(-(x)+(x \triangleright y))$ since f is a homomorphism.

Notice in either case, the inputs of f corresponding to the weighted value of ϕ for the crossing record which colors are going in and out of a crossing. If we consider the entire colored knot v, note that any arc colored x in v must interact with two crossings, and it will be exiting one and entering the other. Then, since $\phi(v)$ equals the sum of the ϕ's over all of the crossings, the inputs of the corresponding f's will all cancel each other out, so we end up with

$$\phi(v) = f(0) = 0,$$

since f is a homomorphism.

For this reason, we can consider the cocycles that are not coboundaries by taking the quotient with $\operatorname{Im}\delta^1$ if we want to ignore some of the cocycles that will always evaluate trivially on colored knots. Recall that

$$\ker \delta^2 / \operatorname{Im}\delta^1 = H^2_Q,$$

the second quandle cohomology group. Thus, picking $\phi \in H^2_Q$ will yield more interesting Boltzmann weights. In general, however, finding coboundaries is difficult, so we will just say $\phi \in C^3_Q$.

Example 9. Let's try to find the possible Boltzmann weights for the quandle X into $\mathbb{Z}$ with operation table shown below.

$\triangleright$	1	2	3
1	1	1	2
2	2	2	1
3	3	3	3

Then we need to require $\phi \circ \partial(x, y, z) = 0$ for each $(x, y, z) \in C_3^Q$. Since we have taken the quotient with the degeneracy chain, these are triples in X^3 where we don't have consecutive repeated elements. We can compute the image of ∂ for each such $(x, y, z) \in C_3^Q$.

$$\begin{array}{lcl}
\partial(1,2,1), \partial(2,1,2), \partial(3,1,2), \partial(3,2,1) & = & 0 \\
\partial(1,2,3) & = & -(1,2)+(2,1) \\
\partial(1,3,1) & = & -(1,1)+(2,1) \\
\partial(1,3,2) & = & (1,2)+(2,2) \\
\partial(2,1,3) & = & -(2,1)+(1,2) \\
\partial(2,3,1) & = & (1,2)-(1,1) \\
\partial(2,3,2) & = & (2,2)-(1,2) \\
\partial(3,1,3) & = & -(3,1)+(3,2) \\
\partial(3,2,3) & = & -(3,2)+(3,1)
\end{array}$$

Note that we require $\phi(1,1) = 0$, so since we want $\phi(\partial(1,3,1)) = \phi(-(1,1)+(2,1)) = -\phi(1,1)+\phi(2,1) = 0$, this forces $\phi(1,2) = 0$. Similarly, $\phi(2,1) = 0$. Also, $\phi(\partial(3,1,3)) = \phi(-(3,1)+(3,2))$ requires $\phi(3,1) = \phi(3,2)$. The image of ∂ doesn't give us information on how ϕ maps $(1,3)$ or $(2,3)$, so those parts are free. Hence, any quandle Boltzmann weight for X must be of the form

$$\phi = a\chi_{1,3} + b\chi_{2,3} + c(\chi_{3,1} + \chi_{3,2})$$

for $a, b, c \in \mathbb{Z}$. If we had instead chosen to work over $\mathbb{Z}_n$, we would have some finite number of Boltzmann weights. In this case, we would have had n^3 Boltzmann weights.

3.3 Enhancing the Enhancement

In the previous section, we showed that the quandle Boltzmann weights introduced in Section 3.1 are equivalent to quandle 2-cocycles. Now, we will use these quandle 2-cocycles to enhance our quandle coloring quiver even further.

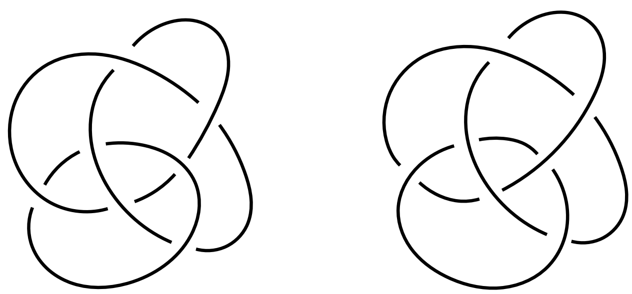
We take the following definition from our paper, Cho and Nelson (2019a).

Definition 17. Let L be an oriented link, X be a finite quandle, S a set of quandle endomorphisms of X, and ϕ a quandle 2-cocycle in $C^2_Q(X; A)$ for abelian group A. Then the **quandle cocycle quiver** $\mathcal{Q}_X^{S,\phi}(L)$ is the directed graph with vertices corresponding to X-colorings of L, edges from v_j to

v_k whenever $v_k = f(v_j)$ for some $f \in S$, and weights $\phi(v_j)$ at each vertex. When $S = \{f\}$ is a singleton we will write f instead of $\{f\}$ for simplicity.

This is almost the same as our definition for the quandle coloring quiver with the addition of the quandle 2-cocycle. For practicality, we will set $A = \mathbb{Z}$ or $\mathbb{Z}_n$. Since the cocycles give us a way to assign integers to colored knot diagrams, which are the vertices of our quiver, we can decorate the vertices with the values of the cocycle evaluated on the colorings.

Example 10. In this example, which is adapted from our paper Cho and Nelson (2019a), we show that cocycles give us information that allows us to differentiate knots even better than before.



L7n1 L7n2

Figure 28

Consider links $L7n1$ and $L7n2$ as shown above. Both have 16 X-colorings with respect to the coloring quandle given below.

$X =$

$\triangleright$	1	2	3	4
1	1	1	1	1
2	4	2	2	2
3	3	3	3	3
4	2	4	4	4

Using the single endomorphism $f(1) = 4, f(2) = f(3) = f(4) = 3$, the quandle colorings quivers for $L7n1$ and $L7n2$ are isomorphic, so neither the quandle coloring invariant nor the quandle coloring quiver are powerful enough to distinguish these links. However, we can use the quandle 2-cocycle

$$\phi = \chi_{1,2} + 2\chi_{1,3} + \chi_{1,4} + 2\chi_{2,1} + 3\chi_{3,2} + 3\chi_{3,4} + \chi_{4.1} \in C^2_Q(X; \mathbb{Z}_4)$$

in order to obtain the quandle cocycle quivers shown below, with the cocycle weights in pink.

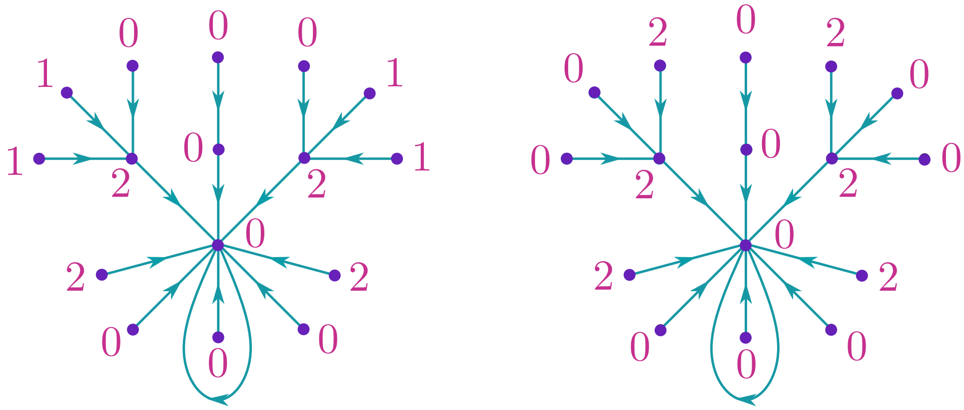


Figure 29

Since the vertex decorations differ between the two, the cocycle quivers are not equivalent, so the additional information we gain from the 2−cocycle ϕ was enough to distinguish $L7n1$ from $L7n2$. This example proves that the quandle cocycle quiver is a strict enhancement of the quandle coloring quiver.

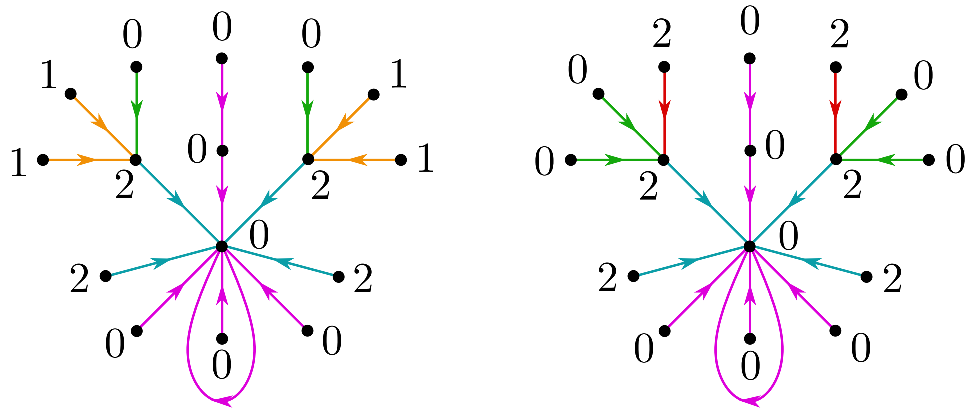
As before, we can compress some of the information of the quandle cocycle quiver into a polynomial, this time in two variables, in a way that takes the cocycle information into account.

Definition 18. Let L be a link, X a finite quandle, $S \subset \mathrm{Hom}(X,X)$, and $\phi \in C^2_Q(X;A)$. We define the **quiver enhanced cocycle polynomial** to be the polynomial

$$\Phi_X^{S,\phi}(L) = \sum_{e \in E(Q_X^S(L))} s^{\phi(v_j)} t^{\phi(v_k)}$$

where the edge e is directed from vertex v_j to vertex v_k in the quandle coloring quiver $Q_X^S(L)$.

Example 11. We can compute the quiver enhanced cocycle polynomial for the same inputs as in Example 10.



$Q_X^{f,\phi}(L7n1)$ $Q_X^{f,\phi}(L7n2)$

Figure 30

$$\Phi_X^{f,\phi}(L7n1) = 6 + 4s^2 + 2t^2 + 4st^2$$
$$\neq \Phi_X^{f,\phi}(L7n2) = 6 + 4s^2 + 4t^2 + 2s^2t^2$$

In the figure above, edges with the same head and tail weights are given a particular color, which are then tallied up in the polynomials. For instance, the orange edges in $Q_X^{f,\phi}(L7n1)$ point from vertices weighted 1 to vertices weighted 2. These contribute s^1t^2 in the polynomial, so since there are 4 of them, the coefficient is 4. Since these polynomials are not equal, we see that in this case, the cocycle polynomial is strong enough to distinguish the links. This helps tell us that the quiver enhanced cocycle polynomial encapsulates useful information from the cocycle quiver.

Next, we will discuss a connection between the quiver enhanced cocycle polynomial and existing literature.

Remark 14. In a paper by Carter et al. (2003), the authors define the **partition function** as follows: For an abelian group A, a finite quandle X, a 2-cocycle $\phi \in Z_Q^2(X;A)$, the partition function (associated with ϕ) for a link L is

$$\sum_{v_j} \prod_{\tau} B(\tau, v_j)$$

where $B(\tau, v_j)$ is the Boltzmann weight of the crossing τ for an X-coloring v_j of L, as discussed in Remark 8.

If we set $A = \mathbb{Z}$ and treat $\mathbb{Z}$ as the multiplicative cyclic group $\mathbb{Z} = \langle s \rangle$, then notice that for a given crossing τ of a coloring v_j, we have

$$B(\tau, v_j) = s^{\pm\phi(x,y)}.$$

where x and y are the colors of the appropriate strands and the $\pm$ sign indicates the orientation of the crossing. Then taking the product over all of the crossings for a given coloring v_j yields

$$\prod_{\tau} B(\tau, v_j) = s^{\phi(v_k)}$$

since $\phi(v_k)$ is just the sum over all the colored crossings of a link. Then, the partition function is

$$\sum_{v_j} \prod_{\tau} B(\tau, v_j) = \sum_{v_j} s^{\phi(v_j)}.$$

Recall that in the quandle coloring quiver, the out-degree of each vertex is $|S|$, the number of endomorphisms used, so we see that

$$\sum_{v_j} s^{\phi(v_j)} = \frac{1}{|S|} \sum_{e \in E(\mathcal{Q}_X^S(L))} s^{\phi(v_j)},$$

which is equivalent to $\frac{1}{|S|}\Phi_X^{S,\phi}(L)$ evaluated at $t = 1$.

Thus, the quiver enhanced cocycle polynomial is a generalization of the partition function defined by Carter et al. (2003)!

Chapter 4

Conclusion

4.1 The Structure of the Coloring Space

We started off by considering the quandle coloring invariant, $\Phi^X_{\mathbb{Z}}(L) = |\text{Hom}(Q(L), X)|$, which counts the X-colorings of a knot L. Then, we removed the absolute value bars, setting the coloring space free, and began to explore the relationships between colored knot diagrams. Since the colorings are homomorphisms, these relationships came in the form of endomorphisms of the coloring quandle X, the maps that preserve the algebraic structure of the quandle.

The endomorphisms gave the colored knots, whom the quandle coloring invariant considered in isolation, a way to interact with each another. We defined the quandle coloring quiver as a way to visualize these interactions.

Let's look at the dihedral quandle on 3 elements one last time:

$$X = \begin{array}{c|ccc} \triangleright & 0 & 1 & 2 \\ \hline 0 & 0 & 2 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 2 \end{array}$$

Let the endomorphism ϕ map $0 \mapsto 2, 1 \mapsto 0, 2 \mapsto 1$. Then the quandle coloring quiver shows us how the 9 X-colorings of the trefoil map onto each other:

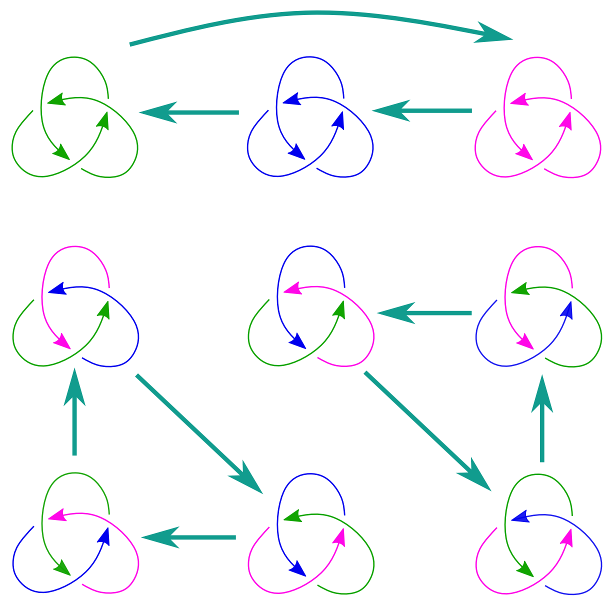


Figure 31

The image above displays the ways in which homomorphisms from the fundamental quandle of the trefoil into the coloring quandle are related by a particular endomorphism. I find this is interesting on its own, but throughout this thesis, we've gone through examples that prove that looking at these relationships are more than just beautiful, they are meaningful in a knot theoretic sense. In other words, these relationships help us distinguish knots that we could not distinguish by looking at colored knots as individuals.

In a very broad sense, this project points to the fact that **relationships are powerful,** and the metaphorical implication is encouraging to me.

4.2 Questions for Future Work

Clark et al. (2014) conjectured that for any nonequivalent links $L1, L2$, there exists a finite quandle X so that $L1$ and $L2$ can be distinguished by the quandle coloring invariant with respect to X. This has been verified computationally for prime knots with up to 12 crossings.

This seems very plausible. Due to the fact that the fundamental quandle $Q(L)$ is a complete invariant up to mirror image of the knot, intuitively, there should exist coloring quandles X for which looking at the homomorphisms into X gives us useful information. Thus, here are some questions of interest:

- Given a link L, which kinds of coloring quandles X will produce the most interesting results?

To motivate this question, we can think about the case of the trefoil and the dihedral quandle X. As we saw earlier, there are 9 different colorings. In fact, the quandle X works particularly nicely with the trefoil, as the trefoil and the dihedral quandle both have a predictable structure: the trefoil has three-fold rotational symmetry, and the dihedral quandle always follows the relation $x \triangleright y = 2y - x(\text{mod}3)$. If we picked another 3 element quandle, like

$$X' = \begin{array}{c|ccc} \triangleright & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 3 \end{array}$$

we would see that there are only 3 X'-colorings of the trefoil, the constant colorings. However $L4a1$, also called "Solomon's knot" according to the Knot Atlas, has 9 X'-colorings as shown below:

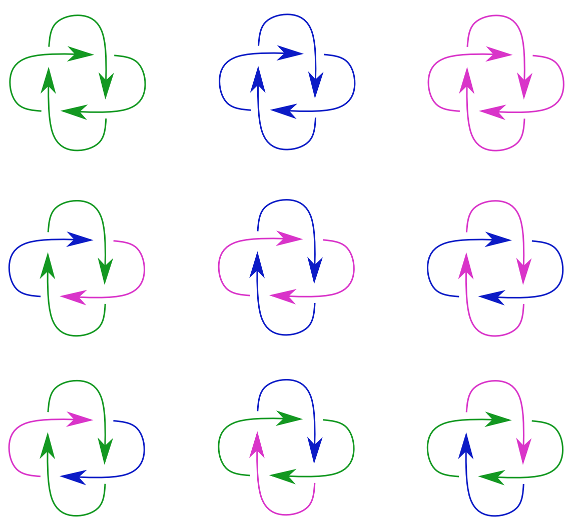


Figure 32

On the other hand, $L4a1$ only has the 3 trivial constant colorings with respect to the dihedral quandle X.

It seems that some coloring quandles produce non-trivial colorings for certain knots, but not others. Thus, it would be enlightening to find criteria for determining what kinds of quandles work well with a given knot.

In a similar manner, I am curious about the following:

- Given links L_1, L_2 with the same number of X-colorings, which endomorphisms of X will tend to be useful in distinguishing L_1, L_2 in the quandle coloring quivers? How about quandle 2-cocycles in the quandle cocycle quivers?

Perhaps we have a case where we have nontrivial colorings of $L1$ and $L2$ by X, but the same number for both. Is there a way to determine what kinds of endomorphisms will be effective in producing non-isomorphic quandle coloring quivers? For instance, if we picked ϕ to be the constant map that sends all elements of X to a particular x, then both quivers will be "star" projections, where all colorings map onto the constant coloring where the knot has been dunked into a paint bucket of color x. Thus, we know that the constant endomorphisms will never be helpful in distinguishing $L1$ from $L2$. The same goes for the identity endomorphism.

Similarly, if we are also considering 2-cocycles, it would be useful to know what which ones are good at producing useful values. We know that we only get trivial values for cocycles that are also coboundaries, but it is hard to determine whether a function is a coboundary. Is there an effective computational way to generate coboundaries? Are there other conditions that make cocycles evaluate trivially on colored knots?

Overall, our newly defined invariants are very flexible in that they can produce different results based on different choices of coloring quandles, endomorphisms, or quandle 2-cocycles. Knowing what kinds of choices are good to make would be helpful in navigating the vast space of possible outcomes.

Perhaps most importantly, this project has made me consider this question:

- Are there other knot invariants that could be enhanced by studying (algebraic) relationships that have previously been ignored? In other areas of math, how can we use relationships between objects to understand mathematical structures more deeply?

All of this began from the simple idea of paying attention to endomorphisms. I wonder if there are other counting invariants that can be improved in the same way, by taking advantage of the algebraic structure that is already there, waiting to be uncovered. For instance, since our quivers form small categories, can we gain even more insight into the nature of knots by looking at the relationships between quandle coloring quivers, in other words, functors between the categories of quandle colored knots?

The general idea of studying relationships (maps) is one that is prominent in math. In the case of knot colorings, we learn things about the mysterious fundamental quandle $Q(L)$ by looking at the homomorphisms into a finite coloring quandle X, where the homomorphisms are called colorings. Analogously, we can study a finite group G by looking at the homomorphisms into the general linear group GL_n, where those homomorphisms are called representations.

In any case, I look forward to exploring the potential of mathematical relationships wherever I go.

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