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Reflections on Critical Math Education in the College Classroom: Critical Pedagogy and Modular Approaches

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Abstract

We provide an introduction to critical theory, critical education theory, and critical math education in particular, including a literature review and reflect on pedagogical practices that are informed by these theories, attempting to synthesize them. We propose an examination of modular approaches to critical education, describing one in detail. Our module implements some of these practices in a fifty minute undergraduate lecture.

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Chapter 1

Introduction

> And blue so far above us comes so high,
> it only gives our wish for blue a whet.
> –Robert Frost

In the first period of my first day of high school, my math class, Math 23X, met for the first time. None of us knew each other, having arrived in Exeter, NH from all different parts of the world only two days prior. So we sat there in silence, tracing the names carved into the huge oblong table, or staring out the window toward the sunny church courtyard.

After what felt like much more than a couple minutes, our instructor, Mr. Wolfson, arrived, introduced himself, and told us to look at problem 1: A 5×5 and a 3×3 square can be cut into pieces that will fit together to form a third square. . .

We worked in small groups at the table or at the boards, finishing about five problems.

"Alright, time to go," Mr. Wolfson told us.

"Why leave early?" we asked.

We were informed that 50 minutes had passed, and that if we stuck around any longer, we would be late to our next class. In disbelief, we returned to our withdrawn, first-day-of-school selves, and I left to search for my next class.

Math 23X met 50 times in the next 10 weeks, and we were in love with the class. We started coming at 1:30, 1:20, 1:15, leaving lunch early to work on problems on the board together long before Mr. Wolfson came in. We never turned in a homework assignment, our tests were returned, corrected, but never graded, and we worked together on everything. At the beginning of September, I had not known how to find the point where two lines intersected. The next September, I began my first calculus class.

And we loved the work. Parametric equations were so much more exciting to us than the last cupful of hot soup from the dining hall. Snell's law, the Pythagorean theorem, the quadratic formula, the exterior-angle theorem, the crossed-chords theorem, the Apollonian circle were the most interesting things in the world to us that first trimester, and we were always thirsting for more.

I have taken many math classes since then. Some I loved; most I dreaded or found boring. But this was a classroom filled with joy even through our fourteen-year-old homesickness and the dark New Hampshire winter. This was the first of a small handful of classrooms that made me believe that mathematics was beautiful and interesting, and that mathematics education should be a life-affirming and community-building endeavour. This class formed in me a belief that the boredom, anxiety, and despair that mathematics so often creates, especially in black and brown people, disabled people, women and nonbinary people, queer people, and socioeconomically disadvantaged people, is not endemic to mathematics education.

This thesis is a product both of my faith in the beautiful side of math education, and my concern at the discrimination and inequity that so often characterizes math classes in reality. My hope is to better understand the institutional forces and structures that make the mathematics classroom a boring or unpleasant or even a violent place, and to approach a view of what a better mathematical environment might look like.

Chapter 2

Background

O, yes,
I say it plain,
America never was America to me,
And yet I swear this oath–
America will be!
–Langston Hughs

In order to change mathematics education, we must first understand what its present problems are, and how they came to be. To this end, we will introduce first the analytic methods of poststructuralism and its connection to social justice topics. We then discuss Critical Education Theory, which uses those analytic methods to address education in particular, and finally, we will introduce the contemporary work of mathematics educators in applying these theories in a specifically mathematical environment.

2.1 Post-Structural Analysis

This section primarily draws on Belsey (2002), which is an excellent source for non-philosophers intrested in Post-structuralism.

Postmodernism came about in the mid 20th century in Europe and the US in reaction to the modernist movement in literature, art, and philosophy. In contrast to realist thinking, which strives for scientific objectivity and accuracy, postmodernism posits that all knowledge is inherently self-referential, and thus a product not only of the systems of reality itself but also the aesthetics and biases of its observers.

Post-structural analysis is a subfield of postmodern philosophy that analyzes systematic structures within a society with the understanding that these systems are all interdependent, and that dominant elements within them are defined largely in terms of subordinate elements. For example, analyzing gender as a structure, we might note that while femininity is often defined in terms of its passivity and subordination to masculinity, masculinity is, in practice, perscriptively defined as the opposite of femininity. Furthermore, Post-structuralism requires an acknowledgement that these structural dynamics also depend on other related structures, such as racial hierarchy, class status, &c.

Critical theories take this analysis a step further. Recognizing that objectivity is generally an illusion that hides one's biases, critical theory takes a normative approach to Post-structural analysis. That is, critical theorists analyzing gender would note that the dynamic between masculinity and femininity is specifically oppressive and harmful to all people within the gender system in different ways, and that emancipation from that system is also necessary to attaining liberation in a larger sense.

2.1.1 Critical Race Theory

Rooted in postmodern philosophy and poststructural analysis, Critical Race Theory (CRT) emerged from within legal studies in the 1980s to analyze structural racial inequalities that persisted in American society in spite of formal legal color-blindness, cf Omi and Winant (1986). CRT begins with the assumptions that American society is largely organized by various racial projects, and that White society defines itself by its contrast and relationship to Blackness.

The concept of a racial project is first formalized in *Racial Formation in the United States* as "Simultaneously an interpretation, representation, or explanation of racial dynamics and an effort to reorganize or redistribute resources along particular racial lines," Omi and Winant (1986). Racial projects change in nature through the course of time, but racial projects in recent US history center around attempts to disenfranchise Black voters, neuter Black political movements, and hide or privatize racial discourses, Martin (2013).

One key observation informed by critical race theory is that Black-led political movements in US history have only been successful when they have coincided with the interests of the White state apparatus.

2.1.2 Other Critical Theories

In addition to critical race theory, critical theories are also well-developed in the fields of women's/gender studies and in queer studies. In broad terms, these theoretical apparata analyze gender and sex as institutions. The interested reader may start with Judith Butler's *Gender Trouble* (1990).

2.1.3 Intersectionality

The language of intersectionality provides us with a simple way to synthesize concepts from various critical theories. In essence, intersectionality is merely the observation that people belong to multiple groups and classes, and that all of these groups affect their experience of the world in interdependent ways. For example, a white woman's experience of gender may be materially different from a black woman's experience of gender, in addition to the expected differences in their experience of race. This harkens back to the idea from poststructuralism that systems within a society are deeply interwoven, Belsey (2002).

Kenji Yoshino, a legal scholar, attempts to synethesize some of the ideas of intersectional critical theory in a modern civil rights context in his book, *Covering*, Yoshino (2006). Drawing on his experience As a gay asian man, he identifies a number of demands that dominant society makes upon members of oppressed minorities. Most interesting in the current era of civil rights is the demand to cover. Yoshino defines covering as the process of conforming to dominant norms even while not hiding one's identity per se. For example, a gay man might be out as gay, and might even introduce his same-sex partner, but he might cover by not dressing or talking in gay-coded ways, or not being openly affectionate with his partner in public. Similarly, a Black person cannot hide the physical fact of her blackness, but she may avoid "acting Black" by using white-coded colloquial speech, styling her hair in certain ways, or avoiding associating with other Black people.

Yoshino's theory of covering expands the civil rights and critical theory discourse beyond the realm of traditional identity groups by arguing that covering demands are made on almost all people for some aspect of their identity, however small. He cites, for example, the experiences of very religious Christian students in the secular environment of Yale Law School. These students are not members of a "traditionally" oppressed group; however, they felt the need to cover their religiousness, as they felt that their "intelligence would go at a 25% discount" if their professors and classmates were aware of their beliefs. Thus, Yoshino connects the desire not to cover with the desire to be fully human, and express oneself and tell one's story on one's own terms, Yoshino (2006).

2.2 Critical Education Theory

We now turn our attention to systems of education, both as structures in their own right and as systems which reproduce existing structures.

2.2.1 Paolo Freire

Paolo Freire was an educator and philosopher of education who grew up in Bazil in the 1920s and 30s. As a child, he performed extremely poorly in school due to poverty and hunger, Freire (1998). Once his family's financial situation had improved, he was able to enroll in college, and became a high school Portuguese teacher. As an instructor, he developed an influential educational theory based on his work with illiterate students, who were disenfranchised within Brazil.

In his first major work, *Pedagogy of the Oppressed* Freire (1970), Freire identifies the "banking" system of education as a tool of oppression in the classroom that also perpetuates oppression outside of the classroom. In this system, knowledge of any kind is treated as a commodity, which teachers have and students lack. Under the banking model, students are consistently subordinated, and the biases of the instructors, societies, and institution are passed onto the students and accorded the same value as other forms of knowledge.

Freire situtates his educational model in the context of an ever-present contradiction between the oppressor and the oppressed. (While these categories are formulated as socio-economic classes, students of Freire have often noted that they can be extended to encompass gender, race, &c., see bell hooks (1994).) Within this contradiction, both classes are unable to express their full humanity, the oppressed because they are not free to do so, and the oppressor because their humanity would require them to recognize the humanity of the oppressed. Thus, both classes actually share the need, and on some level, the desire, for liberation. Freire stresses that this liberation must, on some level, be a cooperation between classes, since the alternative is an armed revolution, which he claims are rarely in fact liberatory.

Given this theory of oppression, we return to the question of education, and how it can become a tool of liberation. Freire proposes a model for deconstructing the banking system of education that begins with reconstructing and reforming student-teacher relations, as the basic organizing principle of education. He advocates for a student-teacher relationship based on mutual respect, rather than heierarchy. With such a relationship, the instructor must respect the ability of the students to think critically, and simply point their capacity for critical thought in a useful direction. This dialogic pedagogical system does not require students to memorize lectures and ideas dictated by the teacher, as they are not merely "banking" knowledge. Thus, a dialogic system is significantly less likely to be a vehicle for reproducing norms and biases than a banking model.

Freire also addresses the need for education to address generative themes. He introduces the concept of *conscientização*, or conscientization, whereby students and teachers together form a critical consciousness of their society. This consciousness in principle comes with a realization of the need for liberation, and thus critical education should not only educate students without reproducing systems of power, but actively work to deconstruct those same systems, Frankenstein (1983).

2.2.2 bell hooks

bell hooks is an American feminist philosopher and educational theorist whose work critiques and expands on Freirean epistemology and educational praxis. In particular, she expands on Freire's discourse of oppressor and oppressed to involve race and gender dynamics, as opposed to a monolithic view of class as the singular main axis of oppression.

In reflecting on her experience in school before and after desegregation in the 1960s, hooks also discusses the importance of joy and comfort in a learning environment, bell hooks (1994). She notes that in her all-black school pre-desegregation, her educational experience was characterized by joy and curiosity, whereas under the integrationist but still discriminatory regime of the 1960s she did not feel joy in her academic life.

In particular, one of the greatest obstacles that hooks identifies to a positive learning environment is that of boredom. She writes, "the classroom should be an exciting place, never boring', bell hooks (1994). This goal in principle is largely fulfilled by embracing a dialogic pedagogy in which students are critically engaged.

2.2.3 Anti-Oppressive Education

Further developing the work of hooks and Freire, educator Kevin Kumashiro has proposed a fourfold theory of "Anti-Oppressive Education," Kumashiro (2000). Instead of focusing on pedagogical technique, however, Kumashiro writes primarily on the content required of an education in order to deconstruct systems of oppression or "othering". He identifies four axes that require address: education for the other, education about the other, education that is critical of the process of othering, and education that actively changes students within the classroom, Kumashiro (2000).

Education for the other is simply education on any normative subject matter that is specifically tailored to meet the needs of students who experience oppression. While the other three axes require separate curricular approaches, education for the other can and should be integrated into existing curricula. One example of this might be a math class specifically designed with the strengths, weaknesses, and background of female students in mind (supposing hypothetically that ordinarily, the students who are men feel more comfortable speaking up in class, and get more attention from instructors, and thus their needs are better attended to "by default").

Education about the other is education that exposes students to the culture, norms, differences, or difficulties faced by students or other people who experience oppression in some way. This requires drawing specific attention to specific groups and their experiences. It also requires care to be taken to avoid tokenizing or fetishizing minorities, and treating the history and culture of marginalized groups with the same respect and emphasis as that of dominant groups.

Education that is critical of the process of othering uses critical theory or other approaches to discuss, generally and specifically, they ways in which oppression occurs and is reproduced generationally.

Education that changes students and society seeks to challenge the biases that students have already inherited, and the dynamics of oppression that exist within the classroom. This and education that is critical of the process of othering are most similar to Freire's concept of *conscientização*, where students first form a critical consciousness and then direct and apply it to their own lives.

Kumashiro argues that a combination of these four educational approaches is necessary to combat the reproduction of oppressive systems in the classroom, Kumashiro (2000).

2.2.4 Labor Based Grading

One question that the critical educators we have mentioned do not address at any length is the question of grading and assessment. A purely dialogic pedagogical system does not require assessment because it does not require knowledge to be stored and reproduced, but rather critically generated. Writing professor Asao B. Inoue argues that all grading is inherently oppressive because it is the mode of enforcement of a banking system of education, Inoue (2019). He argues that while it is one possible mechanism for evaluation and feedback, it is neither necessary nor sufficient to those goals. Furthermore, any grading scheme necessarily encodes institutional biases and values, and assess students only based on their performance at one moment in time, ignoring learning or improvement. Students, then, are assigned a status based on their conformity to institutional norms, and this status often affects long term socioeconomic outcomes. Therefore, grades are a significant mode by which privilege (or lack thereof) is translated through childhood and into adulthood. Students born with more privilege are able to get better grades, which then gives them more privilege.

Inoue raises the possiblity of the abolition of grading altogether in favor of less normative modes of feedback, and suggests that this regime would be ideal. However, working within the context of a university where grades are required at the end of the semester, he proposes an alternative grading scheme which is fully labor-based.

Labor-based grading is a grading scheme that seeks to ameliorate some of the problems with grading while also providing feedback and the final outcome that is required by institutions of higher education.

2.3 Critical Mathematics Education

Within the field of critical mathematics education, there is a significant body of theoretical work. It focuses primarily on secondary mathematics education. There is also a great deal of interest in curricular development with social justice themes, which we will discuss in more depth in chapter 5. We focus here on the work that is specifically grounded in critical theory.

2.3.1 Mathematics Education and Racial Projects

Mathematics educator Danny Bernard Martin has done significant work in applying critical race theory to mathematics education, mostly at the high school level. He characterizes mathematics classrooms in particular as white institutional spaces, see Martin (2013). This means that they are dominated by whites, make a covering demand on nonwhites, are organized according to principles developed by whites, and assert their racial neutrality and impartiality in spite of the white dominance of the space. The last component is important, in that it serves the neoliberal and neoconservative "colorblind" racial projects, and supports a deficit-based model of black and other nonwhite learners, essentially blaming them for their absence in these spaces, Martin (2013, 2019). These deficit-based models are applied not only along racial lines, but also often along gender and class lines in mathematics. This language constitutes a form of epistemological violence against non-dominant learners, in that it denies their reality and critical capacity.

Martin also discusses the US-historical context of mathematics as a racial project. He notes that much of the funding for mathematics in the last half-century has been driven by the defense industry during the Cold War or by other corporate industrial interests post-Cold War. Insofar as these institutions serve and maintain contemporaneous racial projects, the mathematical institutions they funded also likely served the same projects. Critiques of the corporatization of STEM-oriented academia leads to the same conclusions; while efforts toward equity and inclusion may benefit some Black students, within the Neoliberal racial project, they also serve to tokenize those same students in order to legitimize racially opressive institutions, cf Omi and Winant (1986).

2.3.2 Rehumanization

A broader approach to critical math education is the paradigm of rehumanization of mathematics. Rehumanization and humanistic mathematics have become areas of significant attention in recent years. Humanistic mathematics education seeks to contextualize mathematics with information about the people and history that produced it, and perhaps more importantly, it assesses the affects of mathematics on history and people. Hamman (2019); Marano (2019) Humanistic mathematics, in addition to being interesting in its own right, may also help in making mathematics more interesting to students for whom theorems and proofs feel disconnected from reality or from human life.

Rehumanization also presents a view of the mathematics classroom as a racialized space. Researchers in mathematics education such as

Rochelle Gutiérrez have proposed teacher education methods grounded in the paradigm of rehumanization, which are meant to help teachers better understand the experiences of students who are subjected to marginalization (racial, linguistic) in the classroom. Gutiérrez (2015)

The field of Ethnomathematics is also closely related to the problem of rehumanization, see d'Ambrosio (1985). Ethnomathematics positions mathematics in its historical and anthropological context as a human endeavor. It studies the question of what pre-formal mathematical understanding looks like across times and cultures. This is not restricted to non-Western mathematics either: one example of ethnomathematics is an investigation of the treatment and understanding of the Dirac delta before it was formally understood and worked out as a distribution, d'Ambrosio (1985). However, ethnomathematics does seek to center mathematical developments that occured outside of the dominant academic apparatus as potentially useful ways of viewing mathematical systems.

Chapter 3

Reflections on Praxis

Philosophers
must ultimately find
their true perfection
in knowing all
the follies of mankind
– by introspection.
–Piet Hein

Critical praxis is necessarily self-reflective, because it requires an acknowledgement of ones own biases and the experiences that formed them. In this section, I reflect on my experiences as a student and educator in order to qualitatively assess these experiences in terms of the critical paradigms which we have discussed.

In talking about my personal educational experiences, I want to emphasize a couple of ideas that have defined those experiences as positive or negative. These are: the role of empathy in teaching, the role of fun and joy in learning, and the importance of community in both. I think these ideas are very closely related to some thimes in critical education theory, and especially the ideas of bell hooks in bell hooks (1994).

3.1 Phillips Exeter Academy

At Phillips Exeter Academy, where I was a student from the fall of 2012 until the spring of 2016, all courses are taught at a round table with a maximum of twelve students. This pedagogical strategy is referred to as the Harkness Method. Math courses from Math 1 (basic algebra and geometry) to Math

4a (discrete mathematics/statistics) and Math 5 (multivariable calculus and linear algebra) are taught from books of problems (and almost no non-problem text) written by the math department. Students are expected to work on problems for 50 minutes every night, which are typically not turned in, and are graded on effort. In class, students present the problems they worked on the previous night to the rest of the class, and work collaboratively on problems that they may not have finished. Problems are not organized by unit, but do progress in difficulty and subject matter. The curriculum emphasizes the use of geometric reasoning.

As a student, I took the normal curriculum through Math 5, and then took some additional mathematics electives with classmates including special topics in geometry, graph theory, real and complex analysis, and group theory. I have a lot to say about these later courses, many of which were more or less identical in content to upper-division courses taught at the Claremont Colleges, but I'll limit myself to a few observations. In my more advanced classes, the students were almost exclusively Asian and male, which was not reflective of the student body at large. Students entered ninth grade with vastly unequal amounts of knowledge and experience, and with a fairly high degree of curricular flexibility, left with even more unequal mathematical education, as many students who completed the minimal math requirements (either 4 years or equivalent to AB calculus) simply stopped taking math classes. This is consistent with trends throughout the US. As the only female student in some of my mathematics classes, I was significantly disadvantaged in that I could not seek help from my dormmates, nor could I gather with male classmates when they worked together at night in their dorms.

What was remarkable about these advanced classes, though, was the degree of humanity involved in them. While our readings and problems came from books like Rudin, just as in Claremont, we still always sat at a round table, and students still talked much more than instructors. The classes were relatively informal and low-pressure, while being extremely challenging. Many of these classes had exams that we took collaboratively. I felt a great deal of kinship with my classmates and teachers, largely for this reason. This sense was grounded in the geometry of our classroom, and very much encouraged by the instructors.

In terms of pedagogy, I have a few major takeaways from PEA math: that lectures aren't fun or even that efficient, that fun and community are really important in a math classroom, and that students can often teach each other math that none of them actually know.

3.2 Harvey Mudd College

My first semester of college was the first time in my memory that I experienced real, long-term boredom in many of my classes. This was in some ways a product of my personal failure to advocate for myself regarding placement, but also reflects in part the transition to lecture-based courses. This unexpected change illuminated in many ways for me bell hooks' emphasis on joy in learning; in the absence of joy, I have found learning much harder! bell hooks (1994) This feeling has been confirmed and tripled during the 2020 Coronavirus pandemic.

I also encountered a significant disadvantage that I as a privileged child of American-college-educated parents could not have imagined: of my twenty or so frequent social contacts (friends, neighbors, classmates, of fairly diverse gender, race, and socio-economic background), I was the only one without a parent with some kind of advanced degree, and one of only three or four without a parent who was a college professor! As a graduate of a boarding school, I felt that I had a head start on many aspects of college life, but this was a type of privilege that was, at first, completely invisible to me. My friends did not talk about their parents often, didn't mention that they had grown up in the academic world, and did not in any way imply that I or my family were in any way inferior. But they almost all aspired to careers in the academic world, as did I, and unlike me, they had a lot of information about how to achieve their goals. That all of them had parents who had applied to graduate schools, that many of their parents were friends with our professors, that they were natives with a birthright to the so-called meritocratic world of academia was a shock to me! I don't mean to overstate their advantages, or understate my own, but to discover for myself a class of hereditary academics with certain privileges, and to have no one talk about it or tell me about it was a really strange experience, more than anything else.

My parents took calculus in high school and never thought about it again, so since eleventh grade they have had no idea what I do in my math classes. I thought this was a sort of universal experience for math majors and all manner of students in technical subjects. Not so! A fairly large number of my friends are majoring in subjects closely related to their parents' PhDs.

I have no data on this matter. It's possible that my experience is completely anomalous, or that Harvey Mudd is somehow a singular pocket of children of academics or that I just somehow happened to live near and take classes with and befriend people from a very specific sub-demographic. My attempts to discuss this with other people have also met significant resistant. Just as many white people feel uncomfortable discussing and acknowledging white privilege, and men feel uncomfortable discussing male privilege, my friends felt attacked when I brought up their relative advantages in the academic world, and instead of acknowledging the trend, helpfully pointed out to me the many ways in which I am privileged and they are not. This is "I'm-not-privileged-you're-privileged" attitude is a common and well-documented defensive strategy, Yoshino (2006), but I was still somewhat surprised to encounter it in my own life, perhaps naively.

I think this is worth discussing in part because for me it shed some light on the number of challenges that all kinds of students face in postsecondary education, and in mathematics education in particular. As a disabled woman, I had sort of accepted and internalized all manner of disadvantages to the point where I don't think about them very often. All of the challenges of life with limited mobility, physical, natural, human-made, and psychological often feel like background noise to me. This realization has taught me that in questioning institutional paradigms, even the most enlightened of people can be blind when it protects their own interests. That idea is closely related to some of the ideas of racial formation theory and White Institutional Spaces in Omi and Winant (1986); Martin (2013), that the beneficiaries of oppression almost never think of themselves as such, yet act oppressively nonetheless.

3.2.1 Math 180

In the fall of 2019, I enrolled in Partial Differential Equations at Harvey Mudd with Professor Andrew Bernoff. Having taken four full courses in real and complex analysis, as well as a fair amount of physics, I had a quite comfortable foundation for the course material, in both a theoretical and a practical sense. This, however, was not the case for all students. Because of curricular changes over the last several years, many of my classmates actually felt underprepared for the material, and so problem sets designed to take 8-10 hours were taking them at times 15 or more hours–a burden particularly hard for students who were in 5-6 full classes at the time. In the first few weeks of the semester, several of these students started meeting to work together on the problem sets the evening before they were due. On a few occasions, they texted several hours into their meetings to ask for help, when the whole group was stuck, so I walked over and gave them some hints.

I enjoyed helping my classmates, and made friends with a few of them, so I offered to go to their meetings on a weekly basis and help them out. Eventually my half-hour visits turned into four-to-six hour visits, because I didn't really want to leave while anyone was struggling. (I know well the feeling of hopelessness of being stuck on a problem late into the night before it is due.) After spending some large number of Monday night hours with this group, they emailed our professor to ask if I could be paid as the official tutor for the class, since I was working a large number of hours for free. He consented to pay me, on the condition that I announce my tutoring hours and have them be open to the whole class. This was not a problem for me, as we sometimes had higher attendance in our group study hours than in lecture!

Once I was actually being paid to help my classmates, however, I felt obligated not just to be somewhat helpful, but to also prepare well. I adopted the habit of finishing the problem sets over the weekend, and then attending Professor Bernoff's office hours, to see where people were tending to have trouble, and what Professor Bernoff's strategy was in helping them to get unstuck. In many cases, this was helpful as it provided me a second way to approach a problem, or a way of communicating intuition about a problem that I had simply taken for granted.

Our actual tutoring sessions were structured quite similarly to my high school classrooms. Most of the students came in with at least partial work on every problem, and so my general strategy was first for them to spend some time thinking and pool their collective insights on the problem. In my experience this is a fairly practical and efficient way to learn, since the process of explaining a partial insight can often lead to a more solid understanding of the material. My role in this was to point out errors if they went unnoticed, answer narrow, specific questions when asked, give hints when necessary, and provide secondary explanations when students remained confused or stuck after another classmate had attempted to explain the problem. On occasion, when students were unsure how to start a problem, I would remind them of an instructive example and help to identify a general strategy.

I also consistently and strongly encouraged the group, which typically consisted of three to ten students, not to move on until everyone felt good about a given topic. I think the fact of going slowly (and frequently well past midnight) and staying together engendered a sense of community and solidarity. Furthermore, I hoped that my commitment to making sure everyone succeeded together would signal to the students who were struggling the most my empathy for them. I've often felt like the dumbest person in the class, or the person with whom no one wanted to collaborate, and it is important to me to recognize the intelligence and hard work of students in this situation.

I should mention that there were a significant number of students who were not well-served by this approach. More advanced students who were not interested in doing things slowly as a group, students who had only a couple questions but were otherwise confident, and students who for a variety of reasons couldn't come to my tutoring hours or couldn't stay at them very late are among these. For this significant minority, sometimes the professor's office hours were sufficient, and sometimes it was satisfactory for them to sit in the same room and ask me specific questions when they came up. I also frequently answered questions over text throughout the week.

On the occasion that we had a shorter or easier problem set and finished working before 11pm, or in review sessions before the midterm or final, I or someone else would sometimes bring food for the group, and we would have time to chat. These moments were helpful for building community, and getting to know each other, as well as discussing a number of issues that were facing our college and our country at the time. In one instance that filled my heart with joy as an educator, some of the students were inspired by a conversation that came out of a tutoring session to some direct action. (For the sake of the privacy and anonymnity of those involved, I cannot describe the subject or nature of the action in any detail, but it made me quite proud as someone who had recently begun to read Friere!)

Chapter 4

Toward a Synthetic Theory of Postsecondary Mathematics Education

4.1 Mathematics and Society

As both Friere and Martin discuss, on the highest level, mathematics and mathematical education inherit their problems from ambient society. The mathematics classroom is a white institutional space in part because the university is a white institutional space–it was founded to serve white society and continues to do so. The mathematics classroom and the university are also colonial and capitalistic spaces, because they were founded to serve colonial capitalist societies. This observation motivates a sort of top-down inquiry into the nature of mathematics education: what purposes does postsecondary math education serve in our present societ? What purposes do we want it to serve? How are these ends reflected in the practices and structures of our educational systems, and how can they be restructured in service of liberatory goals?

The first question, it seems, is the easiest to answer. Students in college math classrooms belong to several groups: future educators, future professional mathematicians and statisticians, aspiring professionals in other STEM fields, including economics, medicine, computer science, engineering, &c., and other college students who take math classes for general educational requirements or for pleasure. If we view the university or college as an instrument for creating and maintaining an educated elite class, (and thus of creating and preserving hereditary privilege), then the college math classroom specifically is at the center of the endeavor to create an elite technical class. We can interpret this endeavor as an integral part of a neoliberal racial project: the class in question is, by design, majority white (and male and abled and straight and cisgendered). Even efforts to improve diversity often serve to tokenize minorities and increase the credibility of the elite; by selecting a few members of oppressed groups to become privileged, oppressive institutions can claim colorblindness to avoid addressing structural racial disparities and discrimination. Furthermore, this technical class itself often serves society in an oppressive or unequal way. During the cold war, the US Federal government increased funding for mathematics education at all levels by orders of magnitude–in order to increase the workforce involved in the military-industrial complex. This had the side effect of greatly increasing the number of mathematicians outside of the military-industrial complex, and improving math education, though unequally, for people all over the country, but as in the case of civil rights endeavors, these positive ends were reached only when they happened to coincide with dominant interests. But even the nonmilitary technological developments of the last 70 years have often served to increase inequality. Our economy is largely driven by growth in the technology sector, but capitalist technology itself is also an instrument of oppression in many cases, from the erosion of privacy and enabling of racial profiling, to simply increasing economic inequality. The first step to conscientization in a mathematics classroom, then, should be an acknowledgement of the oppressive role that mathematics can take in society. We can then turn to the question of how mathematics and mathematicians can work to deconstruct systems of oppression.

Mathematics, like art, can be studied for its pure beauty, or like philosophy, from a desire toward truth, but the majority of the time, it is used as a tool to understand other things. A humanistic view of mathematics should acknowledge the value of all of these approaches to mathematics, and thus we should hope that the mathematics classroom will always value both joy and rigor–it is a product of capitalist interests that the aesthetic and artistic roles of mathematics are undervalued. But mathematics is an analytical framework that can and should be used alongside critical theory for understanding our current society. The meaning of conscientization is the turning of our existing analytical capacities toward matters of oppression and liberation.

4.2 Normativity

Mathematics classrooms are a normative environment in the sense that students are rewarded for conforming to certain standards of behavior, writing, and thinking, and punished for failing to do so. This is not a problem ipso facto, but it is a problem in that those standards privilege white male subjectivity. That is, mathematicians really like to feel objective, but they have a lot of style requirements that they don't tell you about but judge you on. It's hard to figure these out for anyone, but also if you have different habits of speech, like how some people who aren't anglophones, or white, or straight do, your way of expressing mathematical truth is devalued. How can we combat this while still placing value on clarity and communication in mathematics? Ethnomathematics has a lot of insight here! Even within the realm of fairly mainstream society, we can ask questions like: how do blind mathematicians communicate and think? Acknowledging that different epistemic strategies are valuable is possible while maintaining mathematical rigor. Deconstructing unnecessary cultural norms within the mathematical community is necessary before we can understand mathematical insights coming out of other cultures. d'Ambrosio (1985)

4.3 Grading

The violence of the math classroom comes from its normativity with regard to language and expression. But the power of normativity comes in no small part from grading. Grading, fundamentally, extends the temporary and situational student-teacher power dynamic arbitrarily through and beyond the institutional life of a student, both materially and psychologically. This is a somewhat heavy thing, as an instructor can never really know what's on the line for students: grades can impact a students' financial status in both the short term and the long term, as well as their mental and emotional health. It's easy to hide behind some combination of objectivity and "just doing one's job," but I think these ideas are worth deconstructing. After all, if math is art, how can it be graded objectively?

Sometimes, you say, math is science or engineering. And if the man about to ever-so-delicately remove my wisdom teeth does not know how to compute eigenvalues, how can I be confident in his competence? Well, I respond, ignoring the silliness of the example because I know that bridges must also be built, it does seem useful to have some measure of technical competence, for entry-level employers and beyond. But do college grades really serve this purpose? A lot of my grades mostly reflect how easy a grader the professor was, or how much global pandemic was going on at the time. Some of them are a function of varyingly flexible late-policies, or whether attendence was taken in a given class. And I'm just one disabled student! What about a good mathematician whose English wasn't great? Or a good mathematician with good English that happened to be in a non-normative dialect, who was consistently penalized for "incorrect grammar?" OK, you admit, the status quo isn't very good or consistent, but surely its better than some hippie-commune free-for-all! Ah, I respond, I never said anything about a hippie-commune free-for-all.

Labor-based grading is one option for grading mathematics-as-art, and a pass/fail system is viable for areas where competence is a critical necessity. But even if we restrict ourselves to the relative mainstream, there are differences in grading philosophies between North America and Europe that are fairly striking. In most of the United States, a passing grade means somewhere between 60% and 70% of the work has been done correctly and well, and a "good" grade usually means above 80 or 90%. In much of Europe and the UK, however, 40% is passing, with 60% being the highest mark. This is in principle not just a more lenient way of grading the same exams, but rather speaks to a pedagogical orthodoxy that values a good response to a hard challenge more than a near-perfect response to a potentially easier one. This may reduce some of the pressures of perfectionism without necessarily reducing standards of performance relative to our current system. I don't think any of these approaches solves the problem of grading wholesale, but the universe of grading systems is somewhat larger than we often make it out to be, and some creativity might help the situaion somewhat.

Chapter 5

Modular Approaches

Implementing critical educational practices in a postsecondary environment should ideally involve broad, structural change. In general, this change requires siginificant buy-in from students, professors, and administrators, and is thus not viable for implementation in most institutions in the short- to medium- term.

This suggests a practical question: is there a way to provide professors with resources such that they can integrate critical educational practices into their existing curricula with minimal effort? This question has been the subject of considerable investigation in recent years as social justice-related material has been packaged into modules for secondary and post-secondary mathematics classrooms, cf Marano (2019); Hamilton and Pfaff (2019).

In this section, we introduce a classroom module that was intended for use in MATH073, Harvey Mudd College's Introductory Linear Alebra course. Unfortunately, due to the global COVID-19 pandemic, the college moved to all-online teaching and it became impossible to use this module in practice. Thus, I've decided to provide it here as a resource for adaptation and use in future classrooms, along with the outline of an experiment by which one could evaluate this and other teaching modules.

5.1 Design Goals

The reader should note that I have never taught a class, and have no opportunity to evaluate the effectiveness of my module in reality, due to current circumstances. My hope in providing my design objectives is not to claim that the module I have designed necessarily meets those, but to set out a goal of what a module should do, and indeed, must do, before educators should consider them as a substitute or steppingstone toward more systematic changes in teaching praxis.

The goal of the module is twofold: first, to teach the dominant mathematical curricular material as well as a control section, and second, to introduce social justice themes and critical education practices to the classroom. I have attempted to do the latter in a holistic way: the module both introduces content that investigates racial disparities in the United States, and gives students the resources to discuss those disparities on their own. My hope is that these tools might also inspire students to look for other ways that their mathematical abilities might be turned toward other social justice-related topics in the future. One of the most radical and important ideas of critical education theory is the central role of the teacher's respect for their students as intellectuals, and no module can create or increase this respect where it doesn't exist, but it should ideally give the instructor an opportunity to re-center this respect if they want to do so.

5.2 Background

5.2.1 Markov Chains

The mathematical content of this module concerns Markov modelling. Markov chains are a useful tool in linear algebra for modelling the evolution of certain linear systems. They have numerous applications, including, notably, in artificial intelligence and probability theory. The study of Markov chains is pedagogically useful because it motivates some discussion of topics like matrix diagonalization and eigenvalues and eigenvectors. Because of the numerous uses of this tool, and its accessibility to first-year Harvey Mudd students, a social justice-related application of Markov chains should fit naturally into a semester-long introductory linear algebra class.

The mathematical content of a unit (be it this module or a normal lecture) is basically as follows. Suppose A is a conditional distribution matrix for a set of events or "states", $\{e_1, \cdots, e_n\}$, with $a_{ij} = \mathbb{P}[e_i|e_j]$ representing the probability that an object in state e_j will transition to state e_i in a given amount of time.

Then consider a population distribution vector p_0, where the ith element of p_0 is the fraction of the population in state e_i. Then $p_1 = Ap_0$ represents the configuration of the population after 1 time-step, and $p_k = Ap_{k-1} = A^{k-1}p_0$ represents the configuration after k time-steps. If a population dynamic can indeed be modelled in this way (linear, time-invariant, memoryless), then computing p_k for $k \gg 0$ tells us the long-term behavior of the system.

One might ask the usual questions: is it stable? How does it depend on initial conditions? How quickly does it converge? Furthermore, as matrix multiplication is computationally expensive, is there an efficient way to compute the steady-state behavior?

The key to all of this is diagonalization. We find matrices of the form $A = PDP^{-1}$ such that D is diagonal, i.e. D consists of the eigenvalues λ_i of A on the diagonal entries, and has zeroes in the off-diagonal entries, and P is invertible. So $A^k = PDP^{-1}PDP^{-1} \cdots PDP^{-1} = PD^kP^{-1}$. Since D is diagonal, D^k is easy to compute: it is a diagonal matrix with entries λ_i^k. So if $\{v_i\}$ are a corresponding eigenbasis, then we can write $p_k = \sum c_i \lambda_i^k v_i$, where $p_0 = \sum c_i v_i$. Thus we can characterize the behavior of this system pretty fully in terms of its eigenvalues and eigenvectors, which is computationally very useful, and we can see that it will converge and that the initial conditions will not matter in the steady-state (unless $c_i = 0$ for some i).

5.2.2 MATLAB

MATLAB is a programming language and environment widely used in sciences and engineering for numerical data analysis. Students at Harvey Mudd's Math 73 course use MATLAB for worksheets and exercises to gain intuition for the material they are learning, see applications of that material, and to build familiarity with data-oriented programming. In this module, some pre-processed data and functions are given to students to investigate numerically and visually using MATLAB. For courses in which MATLAB is not a central component, this computational component can be done by the instructor in advance as a demonstration, decreasing some of the interactive content of the class.

5.2.3 Race and Class in the US

One of the primary theses of Critical Race Theory is that race and racism (and specifically anti-Black racism) are among the most basic driving forces of United States history and society. This holds true today and can be seen in political movements on the right and left, in the national struggle with mass incarceration, in ongoing public health crises, and in the daily experiences of Black and Brown people. But in discussions about any issues related to race, I very frequently hear the refrain, "It's not about race, it's about class," from both white and nonwhite peers. The argument goes that Black people are more likely to be poor because of historical discrimination, but now that racial discrimination is illegal (is it?) poor people of all races are facing the same hurdles. This argument is actually incorrect along many axes, but it is a belief that many left-leaning white people who consider themselves nonracist or antiracist profess. (I call it a belief rather than a misconception because not only is there ample counterevidence, but as a myth, it serves to alleviate white racial discomfort and the guilt associated with racial privilege.) This module uses data and the tools of linear algebra to investigate this claim on a factual basis, and introduces an opportunity for students to discuss both the myth and the reality on a theoretical level as well.

5.2.4 The Dataset

The data I have used comes from a huge longitudinal study by Chetty, Hendren, Jones, and Porter of almost every person in the United States between 1989 and 2015. Chetty et al. (2019) The study contains enormously nuanced analysis and finds some surprising results as well as many unsurprising results regading race, gender, and generational wealth. I use fairly small subset of the data they have collected in the hopes that it is tractable for students, but I would recommend that the interested reader take a look at the original paper for a detailed look at gender differences among different racial groups, neighborhood effects, and so on. The income data comes from the US Census Bureau and federal income tax returns, and is thus quite accurate and comprehensive; the authors estimate that 90% of the US population is accurately accounted for on an individual basis.

The data we use consists of three components. We first have the income distribution, separated by race and gender, of household incomes stratified by quintile (N.B. the first quintile here means the lowest income) in 1989, and then the same data for 2015. The third component is the transition matrix between the two groups. Note that the transition matrix cannot be inferred from the two distributions, as it tracks the changes in household income for individuals over the course of a generation.

5.3 Classroom Materials

The full course materials which I have prepared can be found at https://github.com/lefriedberg/Math73, where the reader can also find detailed documentation of the matlab scripts I have written. Interested educators are more than welcomed to fork, share, or use these materials however they would like. This section contains an outline of the classroom materials, which can also be found on the github.

5.3.1 Suggested Class Schedule

This module was designed for a 50-minute class using a partially-flipped classroom. As such, a possible model for a class schedule would be:

- 10 minute review of mathematical material
- 5 minute introduction to the dataset and exercises
- 20 minute individual/partner work on exercises
- 5 minute small group/pair discussion of exercises
- 10 minute full group discussion of exercises.

5.3.2 Suggested Exercises

- Without using the transition matrices, think of a few ways income changes could be modelled by demographic (eg simple mean income rising/falling). Using this method, project the distribution a few generations into the future. Do your results seem reasonable?
- Verify the relation

  ```
  <race>_<sex>_trmat * <race>_<sex>_pdist == <race>_<sex>_kdist
  ```

  for a few groups.
- Either using `rungens` or by hand, use the transition matrices to project distributions a few generations into the future. How do the results compare to other methods?
- Using `rungens` or by hand, use the transition matrix for a certain group with an imaginary initial distribution (eg a flat distribution, or one where everyone starts rich or poor). What does this tell us? How does this change the transient and steady-state behavior?

- Focus on two demographic groups of interest. How do their distributions compare now? How do their steady-state distributions compare? Can you think of some economic or other structural reasons why this might be the case?
- Compute the steady-state distributions using eigenvalues and eigenvectors. How quickly do the generations converge to within 1% of the steady-state values.
- (for students with slightly more programming experience) For what values of n is it faster to use eigenvalue methods to compute the nth generation than using iterative matrix methods?
- Think of a metric or upward (or downward) mobility. Which groups have the highest or lowest of each?
- Think of another (hypothetical) dataset where this kind of modelling might be useful. Is the data readily available?

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