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Once more, it is ordinary language with all its ambiguity that provides a clue that the concept of definition in mathematics might not be as monolithic as we are led to believe when the claim is made that definitions are arbitrary and we can define anything any way we wish.

"Beware the double entendre" would be a good slogan to summarize a recent article by Reuben Hersh—one that ends by enticing the reader to make up slogans with some words that have technical mathematical as well as ordinary language meanings.[1] The point of his creative exercise is to have the reader encounter and perhaps internalize what Hersh views as an important lesson that may account for difficulties students have in learning mathematics: that ordinary language is not only filled with ambiguous meanings, but that even when there is no ambiguity in ordinary language, there is generally either no connection or a tenuous one between that meaning and the mathematical one.

As an example of a tenuous connection, Hersh comments,

> If I say "I own a number of calculus books...," I don't mean *zero books*....I don't even mean *one* book....I mean two or more (p.48).

Hersh claims that he now understands that it was not mere ignorance that accounted for the comment many years ago by one of his students who asserted that zero was not a number.

Hersh offers a litany of other ordinary language expressions that are at odds with mathematical meaning: *adding* (which in ordinary language always leads to an increase in number), *difference* (signaling a comparison in ordinary language, but not necessarily subtraction), *multiplication* (repeatedly adding so that one arrives at something that is bigger than what was initially the case).

He points out that not only objects and operations but the logic of requests or demands is problematic as well. Thus when we ask someone to show that a number divisible by six is even, it is surely appropriate in ordinary usage to choose one example (like forty-two) to demonstrate the point rather than to come up with some general proof.

The connection between mathematics and ordinary language can be even more tenuous however in advanced mathematics, as Hersh points out. He comments:

> In advanced mathematics, there's more linguistic confusion. Surds (absurd), irrational and imaginary numbers, singular perturbations, degenerate kernels, strange attractors—all sound dangerous, undesirable, things to avoid (p.51).

It is true that the mismatch between mathematical and everyday meanings is significant enough to warrant our attention, and a disinclination to appreciate this observation may very well account for problems students have in appreciating mathematical meaning. There are, however, concomitant issues that are either ignored or distorted by Hersh's program to clear up the intended *entendre*—with the intention of minimizing ambiguity. They are issues that have deep consequences not only for students attempting to learn new bodies of knowledge, but for anyone attempting to appreciate the nature of mathematical thought as well as its intellectual history.

For this purpose, I would like to suggest the following complementary slogans:

1. "BE *AWARE OF* THE DOUBLE ENTENDRE."
2. "BE *AWARE OF* MULTIPLE ENTENDRE."

BE *AWARE OF* THE DOUBLE ENTENDRE

Precision of meaning is one thing. An appreciation for the evolution of ideas and the associated labor pains is another. The slogan "Be *Aware of* (rather than *Beware*) the Double Entendre" is intended to have an ameliorative rather than a dismissive quality with regard to the concept of double entendre. What do I have in mind? While Hersh has found out that some students have trouble understanding a concept like that of irrational or imaginary numbers because they seek association with such words which "sound dangerous, undesirable, things to avoid," I have discovered that many are frustrated by a disinclination to take seriously the ordinary language equivalent.

Take the case of "negative number" for example. While "negative" surely fits the bill of sounding dangerous and is something to avoid (unless of course it is associated with a biopsy), the Latin translation of that concept (which pre-dated the English translation) was just as foreboding and perhaps more revealing. These numbers were originally called *numeri ficti*—meaning *fictitious numbers*. The implication here is not only that these numbers are dangerous, but that they really do not exist—or if they do, their existence is shrouded in mystery.

What can students learn not by disassociating from an English translation, but by embracing such translations with an historical and multicultural perspective? Perhaps the deepest lesson to learn is that they are not fools if they do not immediately understand what the concept is all about. Not singly, but taken as a whole, words like "negative," "imaginary," "irrational," "complex" with regard to numbers signal something very important. That is, they suggest that these concepts evolved against considerable resistance. They may come to appreciate that in a quite deep sense, "ontogeny recapitulates phylogeny." If our students have trouble understanding how numbers are extended, then it would be a significant source of solace for them to appreciate that they are merely experiencing the labor pains of these ideas historically.

And why should these ideas have had such a labor intensive birth? Why were they not just accepted as reasonable extensions of existing knowledge? What does it mean to say, as Hersh points out, that mathematicians appreciate that zero may have meaning in the above context while ordinary language suggests the opposite? Who are these mathematicians that appreciate the meaning? Are we referring to those who gave birth to the ideas and found themselves walking on a tight-rope, or are we referring to a twentieth century embodiment of "mathematician?" Are there present day mathematicians who would have difficulty with the concept of zero defining a number of real world objects? Should there be?

One reason that each of the extensions of numbers (beyond natural numbers—those that Kronecker spoke of as God-given, but which Russell and Frege attempted to humanize by establishing them on a set-theoretic foundation) met with such resistance among professionals is that there was an important and healthy kind of confusion that had to be unraveled over time. It is a sort of confusion that is not easily conquered once and for all, but is perhaps built into the human mind, and reappears with each new discovery in all fields of inquiry. That is, in viewing an extension of already existing concepts, how do we connect with what exists? What do we expect of the newly emerging idea that is in common with the previous one?

Obviously a concept (of number, for example) which derives from an earlier one has *something* in common with the earlier one. Just as obviously, however, it differs from the original one. Each extension requires that we decide how much we want the emerging idea to deviate from the original. At what point is the deviation so significant that we can no longer speak of the two concepts in the same breath?

With each extension of number, mathematicians had to ask themselves what there was that was so fundamental about the concept from which it was to be derived that had to be held intact—such that letting it go would completely destroy the concept.

At early stages in the history of mathematics, extensions were characterized by mathematicians' search for a "visible" thread—something linked to the real world, or perhaps a model of some sort that might be a bit more abstract than what could be touched or seen. Just as mathematicians who were confronted with the search for some reality that linked the emerging concept of *numeri ficti* to the earthiness of the natural numbers, so our students experience discomfort when they cannot rely upon familiar models in a number system that is supposedly an extension of what is already

comfortable.

We sometimes get the impression that an axiomatic formulation of mathematics was a watershed that enabled mathematicians to resolve this problem once and for all. We thus might conclude erroneously that it is our students' inability to appreciate an axiomatic perspective that accounts for their reluctance to accept some of these extensions. We might believe that the culprit then is an overly "concrete" hold on the prior number system, and furthermore that the concrete hold is rooted in an effort to connect each idea with ordinary language usage. Thus if natural number is associated with objects you can see or touch, then it surely is understandable that our students would have a problem that mathematicians do not have with zero or negative elements being numbers at all.

But the problem does not (and did not) disappear with the creation of an axiomatic perspective. If we think of the natural numbers as a system satisfying Peano's postulates, then we know that there are certain axioms that such a system must satisfy. But as we extend this system, we find out that some of the properties must be relinquished. It is not just that we cannot "touch" negative numbers that is problematic, but rather that the extended system loses some properties of number that are associated with the positive integers and such properties are cherished by different people in different ways. If the extension from positive integers to integers enables us to solve some new equations, it also raises some eyebrows. Thus, in the extended system we can no longer hold on to mathematical induction (a loss felt perhaps more dramatically in guise of the equivalent well-ordering property). Not every subset of the new system has a least element. Similarly, an awareness that is perhaps more intuitively understood (with machinery that may sound less technical than mathematical induction) is challenged to the hilt when an extension from positive to negative rationals leads us to reject the strongly held belief that a smaller number divided by a larger number cannot equal a larger number divided by a smaller one (as in $-1/1 = 1/-1$).

When do we reach a point of no return—such that we no longer think of the newly derived system as being a number system at all? We know that the deeply embedded property of commutativity had to be relinquished under matrix multiplication. Yet, we have come to think of matrices as being a number system of sorts.

As we depict the actual evolution of number systems, we can share with our students the historical debates that took place regarding the legitimacy of purported extensions. But we can do more. If we engage them in creating alternative extensions—ones that challenge some of their own cherished properties—at what point do our students get their backs up and say that the system being created no longer reflects what numbers are "really about"?

That's the sort of question that can engage our students, once we encourage them not to by-pass the ambiguity of ordinary language and to place mathematics on a different sort of pedestal, but rather to see how the presence of language in the evolution of ideas is a testimony to the most human problems of cognition and emotion as well: How badly do we want something that opens up totally new avenues to explore, and at what price will we buy it?

BE *AWARE OF* MULTIPLE ENTENDRE

So far, we have shown how attention to double entendre can be advantageous not from the point of view of making each new concept more easily understood, but rather as a tool in enabling us to better understand the problematic nature of an entire collection of concepts.

There is however another way in which attention to ordinary language can be enlightening. This has to do less with the translation (and mistranslations) of a family of words and grammatical uses in the domains of ordinary language vs. mathematics, and more with an awareness of certain concepts that are embedded in our culture in general.

It leads us to an issue alluded to in the above section, but it puts a totally new slant on the issue. I begin with the story of a classroom event of several years ago.

I was teaching a talented group (sic) of graduate students who had previously been exposed to a number of different strategies for extending number systems. Thus, they had postulated newly extended number systems; they had derived new systems from old ones

making use of concepts such as ordered pairs of elements from the old ones; they had proved all sorts of things about the new systems in relation to the old ones; they knew what the concept of equivalence relation was all about and had seen the relevance of that concept to extensions; they had been exposed to the concept of new systems having a subset isomorphic to the old; they had been exposed to alternative historical development of the real numbers as in the case of Dedekind's cuts vs. Weierstrass' limits.

I then proposed the following (what I thought was) simple dilemma:

> The real numbers can be characterized in an axiomatic way (essentially an Archimedean ordered field, but I was careful to lay out the properties). I reviewed for them that within that system, it is possible to prove that there does not exist a number x so that $x^2 = -1$.

I then told them that one "popular" way of viewing the set of complex numbers is to define that set as a one that satisfies all the properties of the previous set, but in addition has the following property:

> There exists a number x so that $x^2 = -1$.

Question: How is such a contradiction possible?

I found their answers perplexing. Many of them claimed that the new set, the complex numbers, was a different set than the previous one—the real numbers—so that there was no implied contradiction.[2] Some people seemed to believe that the problem was resolved by *naming* the new system—as if such an act in and of itself had the power to dissolve a contradiction. Some claimed that it is not surprising to find out that what we previously held to be impossible was in fact possible since that is analogous to what growing up and being educated is all about.

Many other interesting comments were made, and in fact, encouraging students to analyze this sort of question in a non-threatening way served as a wonderful Rorschach test. By examining anomalies in a specific rather than in a global context, instructors may unearth some interesting student misconceptions. That is, if asked whether or not it would be acceptable to have a system that satisfies the two propositions X and not X simultaneously, they most likely would claim that such is not possible, and in fact is an important element in the arsenal of mathematical arguments.

Now there is a grain of truth in the students' reactions, and I perhaps misinterpreted their efforts to resolve the problem, but I still found it difficult to understand how they could not be bothered by what appeared to be an obvious contradiction. In fact, no one mentioned that the new system of complex numbers is not merely an add-on to the old system in the sense that everything that was assumed in the old system was also introduced into the new.

It is not that no one pointed out that in the new system, an important property of the old one must be relinquished (that of order), but rather that no one even entertained the possibility that *something* might be lost even if they could not name what it was.

Why is that? It took me a long time to come to appreciate what might have been going on, and I have finally come to an hypothesis that seems worth taking seriously. That is, I have come to believe that their disinclination to consider the possibility that something had to be relinquished is a function of one rather specific notion of *progress* in our culture. Adapting a phrase of Piaget's that has a slightly different connotation, I have dubbed this notion of progress *The American Phenomenon.* While there are multiple meanings of *progress* in ordinary language, a dominant one seems to assume that progress *involves getting more and more of what you find desirable* (like being able to get a solution to $x^2 = -1$ when it did not previously exist) *without ever losing anything that you previously held worthwhile.*

The fact that an extension of a number system provides you with something new and desirable but may at the same time deprive you of something you previously found desirable is not well understood. But why so? It may not be a result of the fact that the technical process of extension is poorly understood from a mathematical point of view, but rather because the concept of *progress* in general is filled with so many unexplored myths.

So, I am suggesting that it is not that we need to distinguish (and divorce) ordinary language from pre-

cise mathematical language in order to create a more accurate understanding of mathematical ideas. Rather it is worth doing some analysis of words and concepts in ordinary language that *do not at all* have mathematical counterparts, but that strongly influence the way in which our students think about mathematics and mathematical development in the first place. *Progress* is one such concept but there are others.

What is needed in order to fully appreciate that extension of systems may have a price to pay is not only an issue of mathematical logic. It requires simultaneously that we do some excavation on a concept of ordinary language that is popularly viewed as unambiguous: the concept of progress. Once more, what we need is to seek greater rather than lesser ambiguity in order to arrive finally at a view of the concept of progress that illuminates the interesting discomfort we feel when popularly held principles have to be relinquished.

I conclude with one other concept that is a meta-mathematical rather than a mathematical one. Sometimes it is our inability to appreciate fully the ambiguity of ordinary language that prevents us from understanding not only a particular mathematical concept or an array of concepts, but rather the nature of mathematical thought itself. Consider the concept of *definition*. Most of my students believe that definitions in mathematics are arbitrary. That is, they tell me that you can define things any way you want.

Holding on to a narrow and unambiguous notion of *definition*, they essentially see its application in mathematics as the replacement of one arbitrary English word with some mathematical formulation. Thus *the slope* of a line in a Cartesian co-ordinate system is meant to be a shorthand way of replacing the change in y values divided by the change in x values for any two points on a straight line.

What the concept of arbitrary definition neglects to appreciate is first of all that no one goes around just defining things arbitrarily and that considerable spade work is necessary in order to decide what is worth defining in the first place. That is, definitions single out objects with a *purpose* in mind, and frequently that purpose is arrived at as a culminating act of inquiry rather than as a first step (as most texts would have us believe). In addition, of course, there are logical criteria that need to be unearthed before definitions are accepted. For example, in most circumstances, we do not select definitions that we believe would lead to contradictions. Thus the concept of *the* slope of *a* straight line would make little sense if slope changed in value depending upon which points were selected along the line.

But there is something deeper about the concept of definition which does borrow from ordinary language use of definition. That is, there are occasions upon which definitions even in mathematics serve some function other than that of *stipulating* one expression for some other. That is, there are occasions upon which definitions are descriptive in nature.[3] Far from being arbitrary, these definitions are intended to convey with a degree of accuracy what it is that accords intuitively with our beliefs.

So, for example, there are many different ways of defining a circle in precise mathematical terms. Though, as Hersh would point out, common language usage might not distinguish carefully between points along the rim and interior points (for example), in no case would we expect that what we previously defined as slope would satisfy the definition of circle. Such a definition would not accord with our prior sense of what a circle "really is." To adopt the notion of definition in mathematics as *arbitrary* is to show a lack of appreciation for the interesting range of ways the concept of definition functions in ordinary language. It is to act as if the Socratic search for "justice" or "beauty" is a pointless venture on the grounds that any shorthand expression would do.

Once more, it is ordinary language with all its ambiguity that provides a clue that the concept of definition in mathematics might not be as monolithic as we are led to believe when the claim is made that definitions are arbitrary and we can define anything any way we wish.

CONCLUSION

So Hersh, in his delightful essay, reminds us that ordinary language can be misleading and can interfere with students' understanding of mathematical ideas. That lesson itself, however, is misleading if we do not also take into consideration that ambiguity of language can be an asset, especially when the goal is not necessarily to unearth the precise meaning of a rela-

tively narrow mathematical concept (like negative integer), but rather to appreciate how it is that an array of related concepts (like number) has evolved.

It is by looking at the array of ordinary language meanings (and concomitant emotional baggage) associated with numbers that we can begin to imagine a state of mind that was behind Kronecker's reaction to Lindemann's demonstration of the transcendental nature of pi: Just a little over a century ago, he said:

> What good is your beautiful investigation regarding pi? Why study such problems, since irrational numbers do not exist?[4]

The pedagogical issues are complicated here and I have made no effort to spell this awareness out in terms of any teaching program. Furthermore, I have intentionally focused narrowly on the concept of number rather than upon the range of interesting specific concepts that Hersh has explored. I have also not explored in general the role that ordinary language plays in thinking, nor have I delved in particular into the role of metaphorical thinking in mathematics—a thinking that might account for the variety and richness of systems described by language such as "ring," "field," "ideal," and even "manifold" and "commutator."[5]

While what I have claimed does not negate Hersh's argument, I have attempted to point out that the ambiguity of ordinary language serves a number of interesting functions beyond the antiseptic one of identifying and delimiting (sic again) its potential in understanding mathematics.

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NOTES

[1]R. Hersh, "Math Lingo vs. Plain Language," *American Mathematical Monthly*, 104 (1), (1997), 48- 51.

[2]While having some surface validity, the problem with this way of resolving the contradiction is that I did not *construct* the new system from the old (in which case, this explanation could perhaps be justified). What I did was to postulate the new system so that I was not looking at the elements in the new system as having any existence other than what could be described by the axioms themselves. Though this strategy is frequently used in extending systems, it has a slippery enough quality so that Bertrand Russell was led to comment:

> The method of "postulating' what we want has many advantages; they are the same as the advantages of theft over honest toil.

[3]I. Scheffler, *The Language of Education* (Springfield IL.: Charles C. Thomas, 1960) 11- 35.

[4]E.T. Bell. *Men of Mathematics*, (New York: Simon and Schuster, 1937).

[5]For a more general philosophical discussion of the issues raised by Hersh and myself with regard to different kinds of metaphors (extinct, dormant and active) and their uses/abuses, see

> M. Black, "More About Metaphor," *Metaphor and Thought*, Anthony Ortony, ed. (Cambridge: Cambridge University Press, 1979) 19 - 45.

For a discussion of the role in mathematics *per se*, see

> S.I. Brown, *Student Generations*, Consortium of Mathematics and Its Applications (COMAP) (Lexington, MA, 1988) 43 - 47.

> S.I. Brown and M.I. Walter, *The Art of Problem Posing*,, (Hillsdale, NJ: Lawrence Erlbaum Associates, 1990).

> D. Pimm, *Speaking Mathematically: Communication in Mathematics Classrooms* (New York: Routledge & Kegan Paul, 1987).

> *Mathematical Reasoning: Analogies, Metaphors and Images*, L.D. English ed. (Mahwah, NJ: Lawrence Erlbaum Associates, 1997).