The STEAM Journal

Volume 4
Issue 2 *The Specimen of 2020*

Article 10

December 2020

Possible Adventures in Impossible Figures

Knarik Tunyan
Purchase College, SUNY

Possible Adventures in Impossible Figures

Abstract

This interdisciplinary article aims to engage student into stepping outside traditional disciplinary boundaries by combining the arts, math, and programming. First, we explore a geometric pattern in the Penrose triangle. Then using the fundamental concepts of geometry, we find geometric relationship, calculate the vertices coordinates, and digitally reconstruct the Penrose triangle using coding. Students are encouraged to further explore this topic by finding another pattern, creating modifications of the Penrose triangle by changing measurements, or considering other impossible figures.

Keywords

the Penrose triangle, geometry, art, computer science

Possible Adventures in Impossible Figures

Knarik Tunyan

Introduction

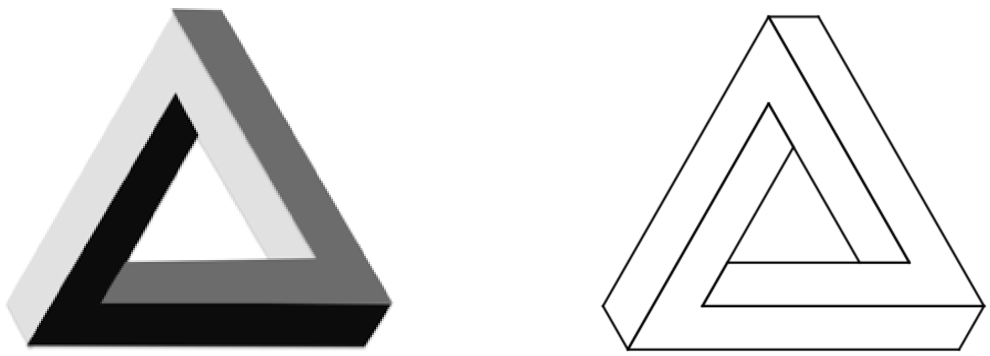
Have you ever been tricked by an optical illusion, an image that our eyes see differently from what it is in reality? Examples of optical illusions include so-called ‘moving’ pictures, i.e., static images that appear to the human eye to move due to special shapes, patterns, and colors in the image; ‘ghost images’ or afterimage illusions, i.e., illusions that one can see after staring at the original picture; impossible figures and objects, i.e., 3D objects that can be drawn in two-dimensional space, but cannot exist in three-dimensional space [4,5]. Optical illusions fascinate psychologists, philosophers, artists, and mathematicians [11]. Different aspects of optical illusions, such as spatial perspective, psychology of perception, impossible structures, or as artistic objects, have been described in numerous books and articles.

In particular, impossible figures have been used in art for centuries. O. Reutersvärd is considered to be ‘’The Father of Impossible Objects’’ because of the outstanding (more than 2500!) number of impossible objects he created [7]. His most famous is the impossible triangle, the simplest most elegant impossible figure, that he created in 1934. In 1958, unaware of Reutersvärd’s work, the psychiatrist L. Penrose and his son mathematician R. Penrose, published a detailed description of the impossibility of this triangle structure in a 3D world, and called it ‘impossibility in its purest form’ [8]. This triangle has ever since been called the Penrose triangle (Figure 1). The Penrose triangle was also used by the artist, M.C. Escher, who is famous for implementation of math principles in his art. In particular, his drawings contain complicated impossible structures [2], which inspired many artists toward a new direction in the visual arts, the ‘impossible art’. Recently, impossible objects are quite extensively used in computer graphics and even in computer video games [10].

The scope of this article is to uncover the patterns in and explore the geometric relations of the Penrose triangle. We will then use these relations to draw the Penrose triangle by utilizing Mathematica computer software [12]. Inspirational ideas for further exploration of this topic are also suggested.

Patterns in the Penrose triangle

Let us now take a closer look at the Penrose triangle (Figure 1). Note that the Penrose triangle consists of three equivalent figures, see the black, light and dark gray *L*-shaped polygons (Figure 1a). Note also that if we rotate the black polygon by 120 and 240 degrees, we will land at the other two polygons.



a) *b)*

Figure 1. The Penrose triangle by using the Mathematica code.

Therefore, in order to draw the Penrose triangle using computer programming, it would be sufficient to draw only one of the *L*-shaped figures, copy and then rotate. The problem is now reduced to determine the coordinates of vertices of one of the polygons and draw line segments between points.

Sketching the Penrose triangle

First, let us explain how to sketch the Penrose triangle. Note that there are different ways to draw it. One of the approaches is to start by drawing an equilateral triangle (Figure 2a), that will be used to form the outer triangle. Next, draw lines inside this triangle that are parallel to and equidistant from the sides of the outer triangle, thus forming the second triangle and rhombuses in each corner (Figure 2b). By repeating the procedure for the second triangle, we will have three equally spaced equilateral triangles and six equal rhombuses at the corners (Figure 2c). Finally, by connecting vertices as shown on Figure 2d, the Penrose triangle will be formed. The outer sides of the Penrose triangle form a truncated equilateral triangle.

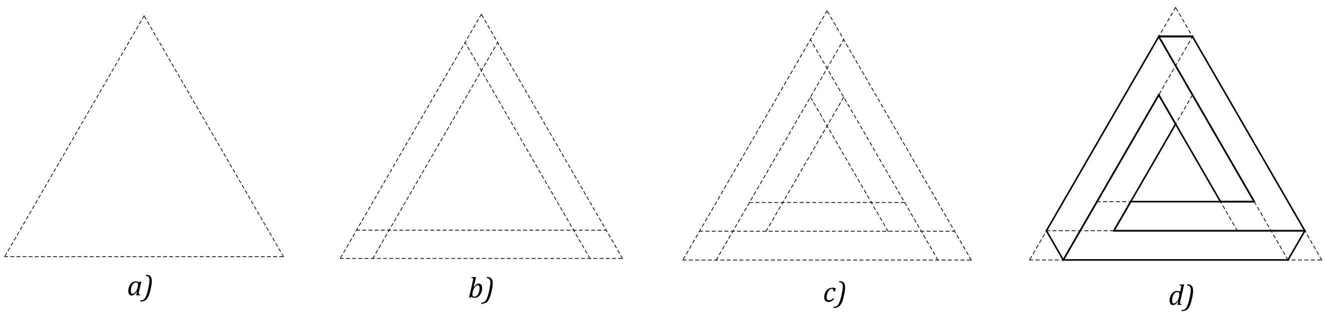


Figure 2. Sketching the Penrose triangle.

Geometric relationships and determining the coordinates

Let's now determine the coordinates of the one of the *L*-shaped figures, we will call it a primary polygon. From a calculation point of view, the best choice for the primary polygon would be the one with the most vertical and/or horizontal lines, which in our case is the figure *ABCDEF* with the bold lines (Figure 3a). It is placed on the *xy*-coordinate plane in such a way that the left corner of the outer triangle is at the origin.

Let the length of the outer triangle, l, and the distance between the sides of triangles, d, be given. These two values are used to determine the coordinates of all other vertices. In our example, $l = 10$cm and $d = 1$cm.

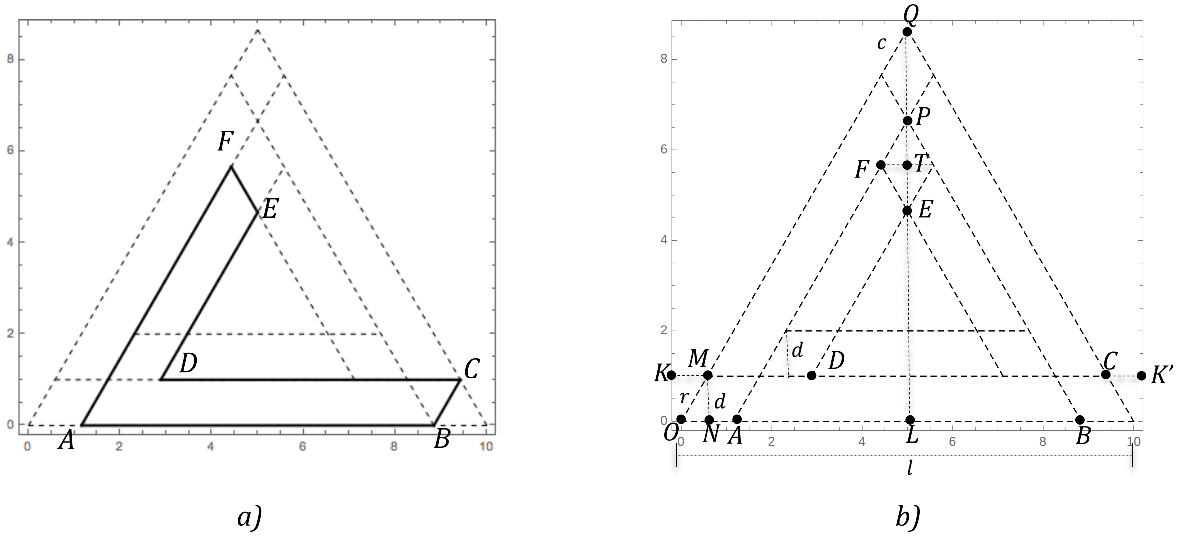


Figure 3. The Penrose triangle in the *xy*-coordinate system.

To determine the coordinates of the points A, B, C, D, E and F, we will need a few more measurements, namely, the height of the outer triangle, QL, the side of the rhombuses, r, and lengths of both diagonals. All these values can be expressed via l and d.

The side of the rhombus, r, can be evaluated by using the trigonometric ratios in the right triangle OMN (Figure 3b). Since angles in an equilateral triangle are 60°, $\angle MON = 60°$. Therefore, $\frac{MN}{OM} = \sin 60°$, or in terms of r and d, $\frac{d}{r} = \frac{\sqrt{3}}{2}$. From the last equation, we can express the side of the rhombus r in terms of d, $r = \frac{2d}{\sqrt{3}}$. Similarly, $\frac{ON}{OM} = \cos 60°$, or $ON = KM = CK' = \frac{r}{2}$.

Hence, we have the following four coordinates in terms of $l, r,$ and d: $A\ (r, 0)$, $B\ (l - r, 0)$, $C\ (l - \frac{r}{2}, d)$, $D\left(\frac{5r}{2}, d\right)$ (Figure 3b).

Since the altitude of an equilateral triangle QL bisects the base, we can calculate its height by using the Pythagorean theorem, $QL = \sqrt{OQ^2 - OL^2} = \sqrt{l^2 - \left(\frac{l}{2}\right)^2} = \frac{\sqrt{3}}{2}l$.

Let us consider the right triangles ΔOMN and ΔFPT. Note that $OM = FP$ and $\angle MON = \angle PFT$. Using the property of hypotenuse-angle congruency of right triangles, we have $\Delta OMN \cong \Delta FPT$. Therefore, $FT = ON = \frac{r}{2}$ and $PT = MN = d$. Thus, the length of the short diagonal in the rhombus is $2 \cdot \frac{r}{2} = r$, and the length of the long diagonal is $2d$.

We now determined the coordinates of the vertices E and F as follows, $E\left(\frac{l}{2}, \frac{\sqrt{3}}{2}l - 4d\right)$, $F\left(\frac{l-r}{2}, \frac{\sqrt{3}}{2}l - 3d\right)$.

We have established the coordinates for all six vertices of the primary polygon $ABCDEF$. We can now draw line segments to connect adjacent vertices, as shown on Figure 3a, by using the command Line in the Mathematica software.

The next step in drawing the Penrose triangle is to rotate the two copies of the primary figure by $120°$ and $240°$. One of the easiest way to perform this operation in Mathematica would be to find the coordinates of rotation for each of the vertices and draw connecting line segments. To find the coordinates of rotation, we can use the concept of a rotation matrix from linear algebra. Specifically, to calculate the coordinates of the rotation of the point with the coordinates (x, y) around the origin, we need to multiply the rotation matrix R_θ by the column-vector of the coordinates $\begin{bmatrix} x \\ y \end{bmatrix}$. The rotation matrix in a two-dimensional space around the origin in the counter-clockwise direction is $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, where θ is the angle of rotation.

For example, the coordinates of the point $A(r, 0)$ rotated around the origin by $120°$ degrees are $\begin{bmatrix} \cos 120° & -\sin 120° \\ \sin 120° & \cos 120° \end{bmatrix} \cdot \begin{bmatrix} r \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{r}{2} \\ \frac{\sqrt{3}}{2}r \end{bmatrix}$. The result of performing the rotation operation is presented in Figure 4a. Note that after the rotation we also need to make the horizontal translation by l units to the right. This translation can be done by adding l to the x-coordinate, or, in matrix form by adding the column-vector $\begin{bmatrix} l \\ 0 \end{bmatrix}$ to the newly obtained coordinates. Therefore, for example, for the vertex A, the coordinates after performing rotation and translation are $\begin{bmatrix} -\frac{r}{2} \\ \frac{\sqrt{3}}{2}r \end{bmatrix} + \begin{bmatrix} l \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{r}{2} + l \\ \frac{\sqrt{3}}{2}r \end{bmatrix}$.

Thus, by multiplying the rotation matrix R by the vector of coordinates for each point of the primary figure and making the horizontal translation by l units to the left, we will obtain the second L-shape.

The new coordinates (x', y') of a polygon vertex with the coordinates (x, y) can be calculated as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 120° & -\sin 120° \\ \sin 120° & \cos 120° \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} l \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{x}{2} - \frac{\sqrt{3}}{2}y + l \\ \frac{\sqrt{3}}{2}x - \frac{1}{2}y \end{bmatrix},$$

or, in a linear system form,

$$\begin{cases} x' = -\dfrac{x}{2} - \dfrac{\sqrt{3}}{2}y + l \\ y' = \dfrac{\sqrt{3}}{2}x - \dfrac{1}{2}y \end{cases} .$$

The final result of combining the rotation by $\theta = 120°$ followed by the translation is demonstrated in Figure 4b (the dark gray part on the Figure 1a).

To obtain the third piece of the Penrose triangle (the light gray part on Figure 1a), we perform similar steps, i.e., the rotation of the primary figure by $\theta = 240°$, followed by the translation by

$l/2$ units to the right and $\frac{\sqrt{3}}{2}l$ units up. In general matrix form, the new coordinates (x'', y'') of a polygon vertex with the coordinates (x, y) are calculated as follows,

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} \cos 240° & -\sin 240° \\ \sin 240° & \cos 240° \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{l}{2} \\ \frac{\sqrt{3}}{2}l \end{bmatrix},$$

or, in a linear system form,

$$\begin{cases} x'' = -\dfrac{x}{2} + \dfrac{\sqrt{3}}{2}y + \dfrac{l}{2} \\ y'' = -\dfrac{\sqrt{3}}{2}x - \dfrac{1}{2}y + \dfrac{\sqrt{3}}{2}l \end{cases}.$$

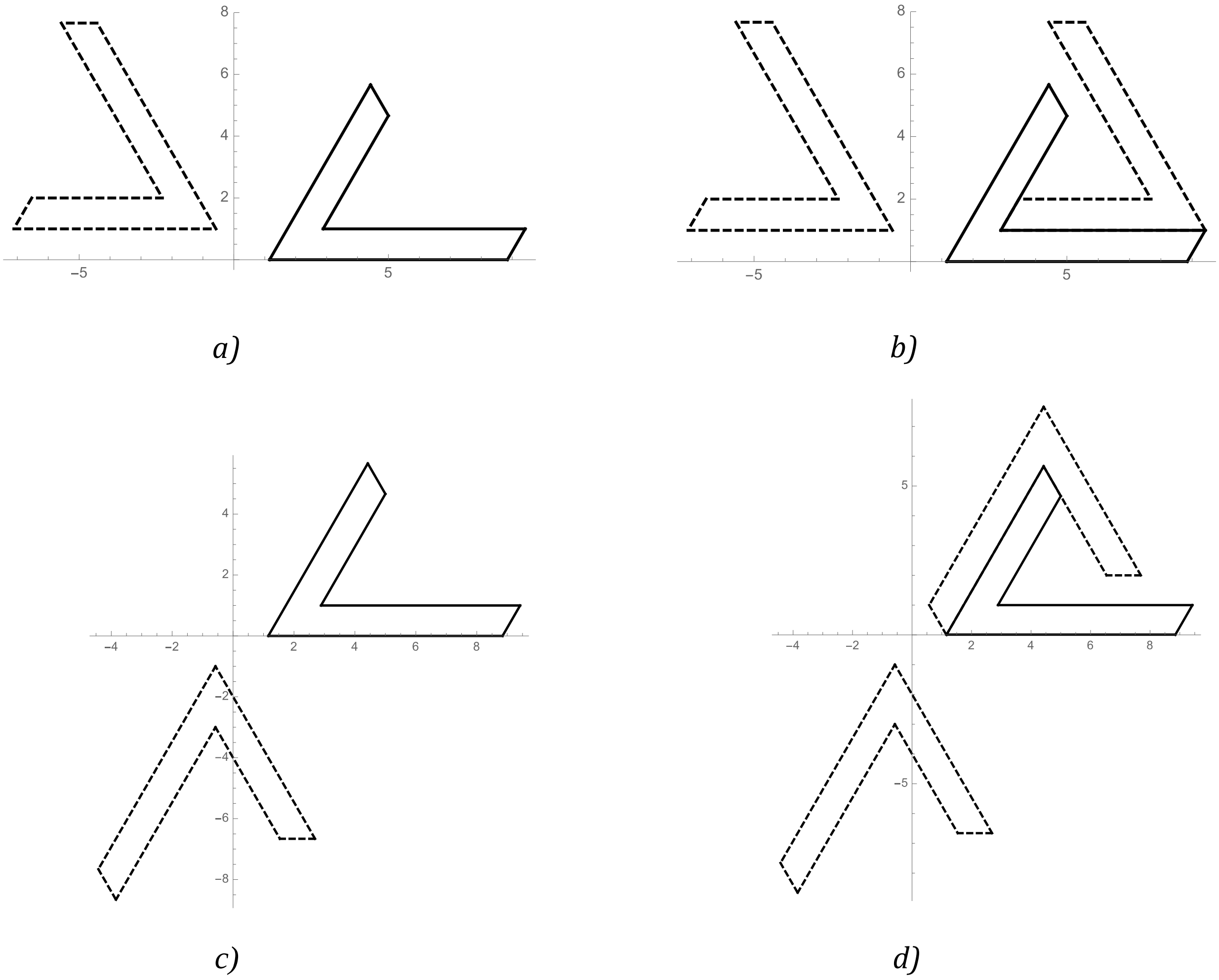


Figure 4. Rotations and translations.

The Mathematica code can be found in the Appendix. There is no unique way of writing the code, one might want to explore Mathematica for other command options or use any other programming language to draw the Penrose triangle.

Conclusion

The mathematics behind impossible figures is quite interesting. Two basic properties have been observed while working with impossible n-degree regular polygons (with equal sides and angles). First, an n-degree impossible polygon consists of the n L-like equivalent shapes and, second, the angle of rotation is $360°/n$.

This geometrical excursion into the Penrose triangle not only can be easily adapted to any math and computer science classroom, but also opens multiple possibilities for future exploration. Examples include expanding general geometrical knowledge and finding other ways of defining the coordinates of vertices; changing the values for the parameters l and d and investigating restrictions on these two parameters; reformulating the problem by assigning initial values to the lengths of the outer and inner triangles; applying a similar technique to sketch an impossible square, pentagon, hexagon, or create a new impossible figure.

In summary, this paper was developed to show the importance of geometry in computer science, and how using it creatively and in a fun way can achieve an interesting and beautiful result. Multidisciplinary approach becomes the new norm in education. This project demonstrates different aspects of the same concept, promotes appreciation of math, the beauty of the arts, and the use of coding.

References

[1] H. Anton, *Elementary Linear Algebra*, 11th ed., Wiley, USA, 2013.

[2] B. Ernst, *The Magic Mirror of M.C. Escher,* Barnes & Noble, Inc., by arrangement with Taschen America LLC, 1994.

[3] E.O. Harris, C. Smith, and A. Carpenter, *Geometry in the walnut grove: an applied mathematical approach to art*, Journal of Mathematics and the Arts, 13, 1-2 (2019), pp. 152-172.

[4] *Impossible Object*. (n.d.); retrieved September 8, 2019, from https://en.wikipedia.org/wiki/Impossible_object.

[5] S. Martinez-Conde and S. Macknik, *Champions of Illusion: The Science Behind Mind-Boggling Images and Mystifying Brain Puzzles,* New York: Scientific American / Farrar, Straus and Giroux, 2017.

[6] D.E. Norton and C.A. Yackel, *Introduction to the PRIMUS Special Issue on Mathematics and the Arts in the Udergraduate Classroom*, Primus, 26, 4 (2016), pp. 269-273.

[7] *Oscar Reutersvärd*. (n.d.); retrieved September 8, 2019, from https://en.wikipedia.org/wiki/Oscar_Reutersv%C3%A4rd.

[8] R. Penrose and L.S. Penrose, *Impossible Objects: A Special Type of Visual Illusion,* British Journal of Psychology, 49 (1958), pp. 31-33.

[9] P. Portaankorva-Koivisto and M. Havinga, *Integrative phenomena in visual arts and mathematics*, Journal of Mathematics and the Arts, 13, 1-2, (2019), pp. 4-24.

[10] USTWO. (n.d.). *Monument Valley*. *A surreal exploration through fantastical architecture and impossible geometry;* retrieved September 8, 2019, from https://www.ustwo.com/work/monument-valley/.

[11] L.R. Wiest, A. Ayebo, and M.D. Dornoo, *Engaging all students with 'impossible geometry'*, Australian Senior Mathematics Journal , 24, 1 (2010), pp. 57-63.

[12] Wolfram. (n.d.). *Mathematica*; retrieved September 8, 2019, from http://www.wolfram.com/mathematica/.

Appendix . Drawing the Penrose Triangle in Mathematica.

```
l = 10; (*length of the outer triangle*)
d = 1; (*distnace between triangles*)
r = 2 d / Sqrt[3]; (*side of rhombuses*)
pA = {r, 0}; (*coordinates of the vertices A,B,C,D,E,F*)
pB = {l - r, 0};
pC = {l - r / 2, d};
pD = {5 r / 2, d};
pE = {l / 2, Sqrt[3] / 2 l - 4 d};
pF = {(l - r) / 2, Sqrt[3] / 2 l - 3 d};
(*drawing line segments for a primary polygon*)
L = Line[{{pA, pB}, {pB, pC}, {pC, pD}, {pD, pE}, {pE, pF}, {pF, pA}}];
RM1 = RotationMatrix[2 Pi / 3]; (*rotation matrix for theta=120 degrees*)
TM1 = {l, 0}; (*translation matrix for theta=120 degrees*)
(*rotation of each vertex followed by translation for each vertex*)
pA1 = RM1.pA + TM1;
pB1 = RM1.pB + TM1;
pC1 = RM1.pC + TM1;
pD1 = RM1.pD + TM1;
pE1 = RM1.pE + TM1;
pF1 = RM1.pF + TM1;
(*drawing line segments for the second polygon*)
L1 = Line[{{pA1, pB1}, {pB1, pC1}, {pC1, pD1}, {pD1, pE1}, {pE1, pF1}, {pF1, pA1}}];

RM2 = RotationMatrix[4 Pi / 3]; (*rotation matrix for theta=240 degrees*)
TM2 = {l / 2, Sqrt[3] * l / 2}; (*translation matrix for theta=240 degrees*)
(*rotation of each vertex followed by translation for each vertex*)
pA2 = RM2.pA + TM2;
pB2 = RM2.pB + TM2;
pC2 = RM2.pC + TM2;
pD2 = RM2.pD + TM2;
pE2 = RM2.pE + TM2;
pF2 = RM2.pF + TM2;
(*drawing line segments for the third polygon*)
L2 = Line[{{pA2, pB2}, {pB2, pC2}, {pC2, pD2}, {pD2, pE2}, {pE2, pF2}, {pF2, pA2}}];
Graphics[{Thick, L, L1, L2}] (*drawing the Penrose Triangle*)
```