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# The OBJECT and the STUDY of Mathematics

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*This paper consists of my reflections on the object and the study of mathematics. One of the main issues that we tried to deal with in the 'Philosophy of Math' meetings with professor Alvin White of Harvey Mudd College was the question: "What is the object of mathematics?" This question raises many relevant questions such as: What is mathematical truth? What is the foundation of mathematics, if there exists one? How do we obtain mathematical knowledge? Is the method of mathematical proof the only conceivable method in mathematics? The questions above are all linked to one another, and let us tackle them as a whole.*

As Evert W. Beth argues: "According to the current view, mathematics is concerned with immaterial objects: points without dimensions, lines with no thickness and so on."[1]

In other words, this common view argues that mathematics is concerned with abstractions.

"Mathematics abstracts, idealizes, schematizes, constructs, simulates."[2] According to Putnam, mathematical truth comes from the fact that mathematics is an objective science, and as he states:

> Mathematics should be interpreted realistically -that is, that mathematics makes assertions that are objectively true or false, independently of the human mind, and that something answers to such mathematical notions as 'set' and 'function'. This is not to say that reality is somehow bifurcated -that is, there is one reality of material things, and then, over and above it, a second reality of 'mathematical things'.[3]

If we accept Putnam's view on mathematical truth, how can we, human beings, witness and fully conceive the foundations of mathematics? We, human beings, have access only to the material world. If mathematics has a second reality of 'mathematical things' which we cannot observe and envision, can we still believe that mathematics has its foundations in our real world?

Putnam argues that mathematics does not have a crisis in its foundations. He does not believe that mathematics either has or needs foundations. In our traditional thinking, we ascribe properties such as length, width, thickness to material objects, and therefore, when a human being thinks of a material object, she/he directly links it with the subjective object, not a real external object. If such objects ,which we call external objects, exist, we cannot visualize them, but can conceive their powers. For example, we cannot visualize a 'set' as we can visualize an 'apple', but we can use the notion of a 'set' in constructing powerful mathematical theories, and conceive the power of this construction, although it is not a subjective object.

The first step to obtaining mathematical truth is by obtaining mathematical knowledge. Because mathematical objects are non-physical realities, the common view, as Putnam points out, is that the kind of knowledge we have in mathematics is strictly a priori. However, Putnam also argues that mathematical knowledge ,in fact, resembles empirical knowledge - " that is, that the criterion of truth in mathematics just as much as in physics is success of our ideas in practice, and that mathematical knowledge is corrigible and not absolute."[4] Therefore, what matters in mathematical truth is the power of its non-physical realities in making a coherent link that is understandable by the physical world. In this sense, mathematical knowledge plays an important role in mediating between the physical world and the non-physical realities of mathematical truth. In other words, mathematical knowledge makes mathematical truth understandable by the physical world.

Mathematical knowledge starts with precise definitions and auxiliary assumptions. Then, these definitions are linked together under the assumptions, and theories are formed. Thomas Hobbes's definitions of science,in his book Leviathan, is that science is the knowledge of consequences. In the light of his definition, mathematical objects are the knowledge of the consequences of each link between given definitions, conclusions and auxiliary assumptions.

What are the criteria for a good foundation of a mathematical theory?

1) Assumptions must be realistic: Assumptions are made to simplify situations, but they must have a large scope. They should be broad enough to be applied to more complicated situations.

2) Definitions must be simple, clear and precise.

3) Assumptions and definitions must be consistent: The theoretician must use them consistently in each link of the axiomatic structure, and newly built definitions must not contradict the older ones.

4) Definitions and each step of the axiomatic argument must be fruitful: They must enable the construction of new definitions and new steps from them.

If these Kuhnian inspired conditions are satisfied, the mathematical objects which are the basic ingredients of mathematical knowledge will be based on a strong foundation. Theorems, propositions and so on which will arise from this foundation will be coherent since their build up will consist of consistent links between the knowledge of consequences of each step we take.

As Putnam argues, generally, in empirical sciences, for each theory, there exists other altemative theories, or those which are struggling to be born. He notes:

> As long as the major parts of classical logic and number theory and analysis have no alternatives in the field—alternatives which require a change in the axioms and which effect the simplicity of total science, including empirical science, so that a choice has to be made—the situation will be what it has always been. [5]

This argument suggests that once a theory which is more powerful than the already existing one appears, it is justified to accept the new theory. Putnam believes that the mathematicians can be wrong, not in the sense that their proofs are misleading, but that the auxiliary assumptions they use might be wrong. He believes that this flexible character of mathematics which allows alternatives into the field makes mathematics *'empirical'*.

His understanding of mathematics resembles the Kuhnian notion of paradigms. As Kuhn suggests, many different paradigms can exist in science. He argues that if an already existing paradigm has anomalies, and contradictions, a new paradigm is formed. According to Kuhn, as long as both of the paradigms are able to generate and solve puzzles, they are equally adequate. It does not have to be that the supporters of the different paradigms are in disagreement. Kuhn's paradigms are incommensurable. Similarly, Putnam allows the existence of different paradigms in mathematics. An example to this notion in mathematics is the non-rival existence of both Euclidean and non-Euclidean geometry.

What happens if a newly formed theory is in contradiction with the already existing one? According to Popper, we would then have to test the two theories and falsify the less-satisfactory one. However, this contradicts the notion of mathematical truth that once a mathematical problem is solved, it is solved forever.[6] If a mathematical theory is formed on the basis of the criteria I have suggested, it is impossible that a proof would be wrong since each knowledge we acquire at each step of our theory is a consequence of the conclusions we acquire in the preceding step. Therefore, if it happens that a theory contradicts the other, we might want to follow Duhem's suggestion and look at the auxiliary assumptions we make. It might be that our assumptions are false, or that the assumptions of the two contradicting theories are incommensurable.

Having discussed the foundations of mathematical knowledge, next we ask the question: " Once we have all the ingredients, how can we cook our recipe to obtain mathematical knowledge?" As Putnam argues: "It does seem at first blush as if the sole method that mathematicians do use or can use is the method of mathematical proof, and as if that method consists simply in deriving conclusions from axioms which have been fixed once and for all by rules of derivation which have been fixed once and for all." [7] Putnam creates an interesting story about Martian mathematics where Martians, in testing theories, use 'quasi-empirical' methods which consist of statements generalized by induction. This method of acquiring mathematical knowledge creates a new concept which we can call: 'mathematical confirmation'. Putnam argues that our refusal to use this 'Martian' method limits the range of our proofs to only analytical ones. If we were to use quasi-empirical methods, we could also enjoy the discovery of synthetic truths in mathematics. Actually, when we look back at our history, we encounter the use of quasi-empirical methods. As Putnam suggests, the Greeks lacked the mathematical experience and mathematical sophistication, and therefore, they used generalizations in their mathematical conjectures. The simplest example is that the real numbers were not introduced through a rigorous mathematical justification. However, the use of real numbers in mathematics now enables us to construct more complicated theories.

To conclude, mathematics is a very interesting branch with lots of questions concerning its origin, study, methodology and its direction. It is, though, a unique branch because it is precise, objective and universal. It does not directly conjecture on nature as do the natural sciences, but it provides a language and foundation on which natural sciences can base their studies securely. The object of mathematics is not a subjective object, but one can feel its power through its preciseness, consistency and fruitfulness.

NOTES

1. Beth, pg: 25

2.ibid pg 24

3.Hilary Putnam, Mathematics. Matter and Method, Cambridge, 1975, pg: 60

4. ibid. pg 61

5. ibid. pg: 51

6. Rota, Mathematics and Philosophy

7. Putnam, pg: 61

**REFERENCES**

Putnam, Hilary. Mathematics. Matter and Method. Philosophical Papers, Volume I: " Mathematics without Physical Theory and Experiment al Foundations", "What is Mathematical Truth?" Cambridge University Press. 1975.

Beth, Evert W. Mathematical Thought. D. Reidel Publishing Company. Holland. 1965

Rota, Gian-Carlo. The Concept of Mathematical Truth. Essays in Humanistic Mathematics. Massachusetts Institute of Technology, Cambridge, MA.

Rota, Gian-Carlo. Mathematics and Philosophy: The story of a Misunderstanding. Essays in Humanistic Mathematics. Mathematics Association of America, Washington, DC. 1993.

Kuhn, Thomas. The Structure of Scientific Revolutions. University of Chicago Press. 1970.

Duhem, Pierre. Physical Theory and Experiment. Scientific Knowledge: Basic Issues in the Philosophy of Science, by Janet Kourany. Wadsworth Publishing Company, CA. 1987.

Popper, Karl. Science: Conjectures and Refutations. Scientific Knowledge: Basic Issues in the Philosophy of Science, by Janet Kourany. Wadsworth Publishing Company, CA 1987.