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On the Mysteries of Interpolation Jack Polynomials

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Abstract

Interpolation Jack polynomials are certain symmetric polynomials in N variables $x_1, \dots, x_N$, with coefficients that are rational functions in another parameter κ, indexed by partitions of length at most N. Introduced first in 1996 by F. Knop and S. Sahi, and later studied extensively by Sahi, Knop-Sahi, and Okounkov-Olshanski, they have interesting connections to the representation theory of Lie algebras. Given an interpolation Jack polynomial P_λ, we would like to write

$$\frac{\partial}{\partial \kappa} P_\lambda = \sum_\mu c_\lambda^\mu P_\mu$$

where the coefficients c_λ^μ are rational functions of κ. In this thesis I present proofs of expressions for a few specific cases of these coefficients, and develop a general matrix formula which does not provide a concrete formula but can act as a starting point for computing such a formula.

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Chapter 1

Symmetric Polynomials and Partitions

In this section we provide a brief introduction to symmetric polynomials and partitions, and present a connection between them in the form of families of symmetric polynomials indexed by partitions. For further reading on symmetric polynomials and families indexed by partitions see [1, 2, 3], and for more information on partitions and their various combinatorial properties see [4].

1.1 Symmetric Polynomials

Let $P_N = \mathbb{C}[x_1, \ldots, x_N]$ be the algebra of polynomials in N variables $x_1, \ldots, x_N$ with complex coefficients. Recall that the *symmetric group* S_N is defined to be the set of bijections

$$\sigma : \{1, \ldots, N\} \to \{1, \ldots, N\}.$$

Then a polynomial $f \in \mathbb{C}[x_1, \ldots, x_N]$ is defined to be *symmetric* if it is invariant under the action of the symmetric group, i.e.,

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(N)}) = f(x_1, \ldots, x_N)$$

for all $\sigma \in S_N$. There are several fundamental examples of symmetric polynomials.

Example 1.1.1. The *elementary symmetric polynomials* $e_k := e_k(x_1, \ldots, x_N)$ in N variables are defined by the relation

$$\prod_{i=1}^{N}(1 + tx_i) = 1 + \sum_{k=1}^{N} e_k t^k. \tag{1.1}$$

That is, the k^{th} elementary symmetric polynomial is defined to be the coefficient of t^k when $\prod_{i=1}^{N}(1 + tx_i)$ is expanded out. For example, if $N = 1$, then Equation 1.1 becomes

$$1 + tx_1 = 1 + e_1 t$$

and so $e_1(x_1) = x_1$. If $N = 2$, then Equation 1.1 instead becomes

$$(1 + tx_1)(1 + tx_2) = 1 + (x_1 + x_2)t + x_1x_2t^2 = 1 + e_1t + e_2t^2$$

so that $e_1(x_1, x_2) = x_1 + x_2$ and $e_2(x_1, x_2) = x_1x_2$.

Example 1.1.2. The *complete symmetric polynomials* $h_k := h_k(x_1, \ldots, x_N)$ in N variables are defined by the relation

$$\sum_{k=1}^{\infty} h_k t^k = 1 + \prod_{i=1}^{N}(1 - x_i t)^{-1}. \tag{1.2}$$

Similarly to before, this means that the k^{th} complete symmetric polynomial is defined to be the coefficient of t^k when $\prod_{i=1}^{N}(1 - x_i t)^{-1}$ is expanded out using power series. For example, if $N = 1$ then Equation 1.2 becomes

$$\begin{aligned}\sum_{k=1}^{\infty} h_k t^k &= 1 + (1 - x_1 t)^{-1} \\ &= 1 + \sum_{j=0}^{\infty}(x_1 t)^j \\ &= 1 + x_1 t + x_1^2 t^2 + \cdots .\end{aligned}$$

So $h_1(x_1) = x_1$, $h_2(x_1) = x_1^2$, etc. If $N = 2$ then Equation 1.2 becomes

$$\begin{aligned}\sum_{k=1}^{\infty} h_k t^k &= 1 + (1 - x_1 t)^{-1}(1 - x_2 t)^{-1} \\ &= 1 + \left(\sum_{j=0}^{\infty}(x_1 t)^j\right)\left(\sum_{\ell=0}^{\infty}(x_2 t)^\ell\right) \\ &= 1 + (x_1 + x_2)t + (x_1^2 + x_1x_2 + x_2^2)t^2 + \cdots .\end{aligned}$$

Thus $h_1(x_1, x_2) = x_1+x_2$, $h_2(x_1, x_2) = x_1^2+x_1x_2+x_2^2$. In general $h_k(x_1, \ldots, x_n)$ will be the sum of all distinct monomials in $x_1, \ldots, x_n$ that have total degree k.

Example 1.1.3. The *power sums* Ψ_k in N variables are defined to be

$$\Psi_k(x_1, \ldots, x_N) := \sum_{i=1}^{N} x_i^k.$$

For example, the power sum Ψ_5 in 2 variables is

$$\Psi_5(x_1, x_2) = x_1^5 + x_2^5.$$

Another fundamental example of symmetric functions are the *monomial symmetric functions*, but we hold off on a definition of these until we have introduced the concept of a *partition*[1].

One can check that the collection of symmetric polynomials in N variables is closed under addition, multiplication and scalar multiplication and so in fact forms a subalgebra of P_N, denoted Λ_N.

An interesting question that arises now that we know Λ_N is an algebra is: which collections of symmetric polynomials form a basis for Λ_N? Each of the families of polynomials presented in Examples 1.1.1-1.1.3 do[2], and we will encounter more such families of polynomials later on after we have introduced partitions. In particular, this report will revolve around one specific basis for Λ_N, called the *interpolation Jack polynomials*.

1.2 Partitions

We now introduce the notion of a partition.

Definition 1.2.1. A *partition* of a non-negative integer m is a tuple $\lambda = (\lambda_1, \ldots, \lambda_N)$ such that $\lambda_1, \ldots, \lambda_N$ are non-negative integers, $\lambda_1 \geq \cdots \geq \lambda_N$, and $\lambda_1 + \cdots + \lambda_N = m$. The number N is called the *length* of the partition λ, and the number m is called the *size* of the partition λ, denoted $|\lambda|$.

Example 1.2.2. The partitions of length three of the integer 3 are $(1, 1, 1)$, $(2, 1, 0)$ and $(3, 0, 0)$.

[1] See Section 1.3.

[2] We will make this statement more precise in Section 1.3.

One question that arises when studying partitions is: how do we order them? There are several ways of doing this, but here we present one that is often used in the study of symmetric polynomials.

Definition 1.2.3. Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_l)$ be two partitions of the same length. Then $\lambda > \mu$ in lexicographic ordering (sometimes written $\lambda >_{lex} \mu$) if for some index i

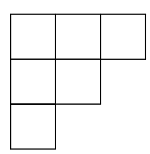
$$\lambda_j = \mu_j \text{ for } j < i \text{ and } \lambda_i < \mu_i.$$

Note that lexicographic ordering is a *total ordering* on partitions of a certain length, i.e., for each two partitions λ and μ of the same length, either $\lambda >_{lex} \mu$ or $\lambda \leq_{lex} \mu$. This is important because later on we will use lexicographic ordering to order certain families of symmetric polynomials indexed by partitions.

Another important and notorious concept related to partitions is that of the *Young diagram* associated to a partition.[3]

Definition 1.2.4. Let $\lambda = (\lambda_1, \ldots, \lambda_N)$ be a partition. Then the *Young diagram associated to* λ, which we will denote $\widetilde{\lambda}$[4], is an array of boxes with N left-justified rows, where row i has λ_i boxes. We also say that such a Young diagram has *shape* λ.

Example 1.2.5. If $\lambda = (3, 2, 1)$, then the Young diagram of shape λ is



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Young diagrams arise often in combinatorial settings, and provide an insightful approach to many problems due to their visual nature. In particular, they are integral in the study of the representation theory of the symmetric group, and will show up later in this document playing a key role in finding a concrete formula for interpolation Jack polynomials.

Young diagrams are also useful in defining the *conjugate* or *transpose* of a partition λ.

[3]Despite the name, these diagrams have in fact been around since 1900, and so are not particularly young.

[4]This is not standard notation.

Definition 1.2.6. Let $\lambda = (\lambda_1, \ldots, \lambda_N)$ be a partition. Then the *conjugate* or *transpose* of λ is the partition $\lambda^\dagger = (\lambda_1^\dagger, \ldots, \lambda_\ell^\dagger)$, where $\lambda_i^\dagger$ is the number of boxes in column i of the Young diagram of shape λ.

Note that the Young diagram of $\lambda^\dagger$ can be obtained from the Young digram of λ by flipping the boxes over the diagonal that goes from the upper left to the bottom right, i.e., taking the transpose of a Young diagram is like taking the transpose of a matrix. For example, using the partition $\lambda = (3, 2, 1)$ from Example 1.2.5 we see that in fact $\lambda = \lambda^\dagger$.

1.3 Symmetric Polynomials Indexed by Partitions

We can now combine what we have learned about symmetric polynomials and partitions to form the general type of object that this thesis will revolve around: families of symmetric polynomials indexed by partitions.

The concept of a family of symmetric polynomials being indexed by partitions is analogous to the idea of a sequence $a_1, a_2, \ldots$ being indexed by natural numbers. That is, there is a bijection between partitions and polynomials in that family, and in general this bijection is established through some type of formula involving the parts of λ.

Example 1.3.1. We now find ourselves in a place where we can define the *monomial symmetric functions*, as promised in Section 1.1. If $\lambda = (\lambda_1, \ldots, \lambda_N)$ is a partition, then we define the *monomial symmetric function* associated to λ to be

$$m_\lambda(x_1, \ldots, x_N) = \sum x_1^{a_1} \cdots x_N^{a_N}$$

where the summation is over all distinct permutations $a = (a_1, \ldots, a_N)$ of $\lambda = (\lambda_1, \ldots, \lambda_N)$. For example, the monomial symmetric function in three variables associated to $\lambda = (6, 5, 4)$ is

$$m_{(6,5,4)}(x_1, x_2, x_3) = x_1^6x_2^5x_3^4 + x_1^6x_2^4x_3^5 + x_1^5x_2^6x_3^4 + x_1^5x_2^4x_3^6 + x_1^4x_2^6x_3^5 + x_1^4x_2^5x_3^6.$$

Monomial symmetric functions are particularly important in the study of symmetric polynomials because they are a useful way of consolidating notation. That is, if $p(x_1, \ldots, x_N)$ is a symmetric polynomial, then every term in the monomial symmetric function $m_\lambda(x_1, \ldots, x_N)$ will have the same coefficient in p, so instead of writing out all of the terms in p, we can simply write it in terms of the monomial symmetric functions. For example, the symmetric polynomial

$$p(x_1, x_2) = 3x_1^2 + 3x_2^2 + x_1 + x_2$$

can be written instead as

$$p(x_1, x_2) = 3m_{(2,0)}(x_1, x_2) + m_{(1,0)}(x_1, x_2).$$

Thus instead of thinking about symmetric polynomials in terms of the *monomials* that they contain, it is useful to think of them in terms of the *monomial symmetric functions* that they contain. We shall see later on that monomial symmetric functions play an integral role in our study of interpolation Jack polynomials.

Example 1.3.2. Another prominent example of a family of symmetric polynomials indexed by partitions is the *Schur functions* or *Schur polynomials*. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_N)$ the *Schur function*, s_λ, associated to that partition is the polynomial

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det\left(x_i^{\lambda_j+N-j}\right)_{1\le i,j\le N}}{\det\left(x_i^{N-j}\right)_{1\le i,j\le N}}.$$

Although it is not clear that this will always be a polynomial in $x_1, \ldots, x_N$ (as opposed to merely a rational function), this turns out to be the case. As a concrete example, let $N = 2$ and $\lambda = (2, 1)$. Then we see that

$$\begin{aligned}\det\left(x_i^{\lambda_j+N-j}\right)_{1\le i,j\le N} &= \det\begin{pmatrix} x_1^{2+2-1} & x_1^{1+2-2} \\ x_2^{2+2-1} & x_2^{1+2-2}\end{pmatrix} \\ &= \det\begin{pmatrix} x_1^3 & x_1 \\ x_2^3 & x_2\end{pmatrix} \\ &= x_1^3x_2 - x_1x_2^3\end{aligned}$$

and

$$\begin{aligned}\det\left(x_i^{N-j}\right)_{1\le i,j\le N} &= \det\begin{pmatrix} x_1^{2-1} & x_1^{2-2} \\ x_2^{2-1} & x_2^{2-2}\end{pmatrix} \\ &= \det\begin{pmatrix} x_1 & 1 \\ x_2 & 1\end{pmatrix} \\ &= x_1 - x_2.\end{aligned}$$

Thus

$$s_{(2,1)}(x_1, x_2) = \frac{x_1^3 x_2^1 - x_1^1 x_2^3}{x_1 - x_2} = x_1^2 x_2 + x_1 x_2^2.$$

So we see that $s_{(2,1)}(x_1, x_2)$ is indeed a symmetric polynomial in the variables x_1 and x_2. It is also worth noting that when $\lambda = (\lambda_1)$ we have $s_\lambda = h_{\lambda_1}$, where h_{λ_1} is the complete symmetric polynomial mentioned in Example 1.1.2. When $\lambda = (1, \ldots, 1)$, s_λ is the elementary symmetric function e_N mentioned in Example 1.1.1 (where N is the number of 1's in λ).

We can also define variations on the symmetric polynomials presented in Examples 1.1.1, 1.1.2 and 1.1.3, as follows. If $\lambda = (\lambda_1, \ldots, \lambda_N)$ is a partition, then the elementary symmetric polynomial indexed by λ is $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$; and the complete symmetric polynomial indexed by λ and the power sum indexed by λ are defined analogously. These families of symmetric polynomials are of interest because for each family, the subset of that family that is indexed by partitions of length N (where we allow some of the λ_i to be zero) forms a basis for the space Λ_N[5]. As mentioned previously, this thesis is focused on a particular basis of symmetric polynomials indexed by partitions, called the *interpolation Jack polynomials*.

[5]To be more precise, the sets $\{e_\lambda(x_1, \ldots, x_r) \,|\, r \leq N\}$, $\{h_\lambda(x_1, \ldots, x_r) \,|\, r \leq N\}$ and $\{p_\lambda(x_1, \ldots, x_r) \,|\, r \leq N\}$ each are a basis for Λ_N.

Chapter 2

Interpolation Jack Polynomials: What Are They?

In this section we introduce the interpolation Jack polynomials, first through an abstract definition introduced by Knop and Sahi and then through an explicit combinatorial formula developed by Okounkov. For more reading on each, see respectively [5] and [6], and for further reading on similar combinatorial formulas see [1]. Note also that parts of this section follow closely the discussion in [7] (also written by the author).

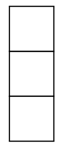
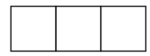
2.1 First Definition

Interpolation Jack polynomials were first introduced in 1996 by F. Knop and S. Sahi [5], and were defined by certain vanishing conditions. In order to develop the definition, we must first introduce some notation. Let $\lambda = (\lambda_1, \ldots, \lambda_N)$ be a partition and let $\rho = (\rho_1, \ldots, \rho_N)$ be a vector in $\mathbb{C}^N$. Then we say that ρ is *dominant* if $\rho_i - \rho_j \neq -1, -2, -3, \ldots$ for all $i < j$. We can then define the *ρ-hooklength* of a box $s = (i, j)$ in the Young diagram associated to the partition λ to be

$$c_\lambda^\rho(s) := (\lambda_i - j + 1) + (\rho_i - \rho_{\lambda_j^\dagger})$$

where we recall from Section 1.2 that $\lambda^\dagger = (\lambda_1^\dagger, \ldots, \lambda_\ell^\dagger)$ denotes the transpose of the partition λ.

Example 2.1.1. Let $\lambda = (1, 1, 1)$ and $\rho = (3, 2, 1)$. Then the Young diagram associated to λ and its transpose are:

 and  .

Thus the transpose of λ is $\lambda^\dagger = (3, 0, 0)$. Now consider the box $s = (2, 1)$ in this Young diagram. This is the middle box, and its ρ-hooklength is

$$\begin{aligned} c^{(3,2,1)}_{(1,1,1)}(2,1) &= (\lambda_2 - 1 + 1) + (\rho_1 - \rho_{\lambda^\dagger_1}) \\ &= 1 + (3 - \rho_3) \\ &= 1 + (3 - 0) \\ &= 4. \end{aligned}$$

For each partition λ we can then further define

$$c^\rho_\lambda := \prod_{s \in \lambda} c^\rho_\lambda(s).$$

Finally, for non-negative integers N and d (note that N is actually strictly positive) let $S(N, d)$ denote the set of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ with $\sum_i \lambda_i = d$. We then present the following definition from [5].

Definition 2.1.2. Let $\rho \in \mathbb{C}^N$ be dominant and let d and N be non-negative integers. Then for any partition $\lambda \in S(N, d)$, the *interpolation Jack polynomial* P^ρ_λ associated to that partition is the unique polynomial in N variables such that

(1) P^ρ_λ is symmetric

(2) $\deg P^\rho_\lambda \leq d$

(3) $P^\rho_\lambda(\mu + \rho) = 0$ for all partitions μ of d with $\mu \neq \lambda$

(4) $P^\rho_\lambda(\lambda + \rho) = c^\rho_\lambda$.

In other words, these polynomials are defined to be the unique symmetric polynomials in N variables satisfying a degree condition (2), a vanishing condition (3) and a normalization condition (4).

It is proven in [5] that these polynomials will in fact exist and be unique for each dominant $\rho \in \mathbb{C}^N$ and each $\lambda \in S(N, d)$. Note that they are also sometimes referred to as *shifted* Jack polynomials (see [3]).

In this report, we will focus on a particular sub-family of these polynomials, indexed by vectors of the form $\rho = \kappa\delta$, where κ is a formal parameter and $\delta = (n-1, n-2, \ldots, 1, 0)$. This sub-family is of particular interest because for certain values of κ, the interpolation Jack polynomial $P_\lambda^{\kappa\delta}$ reduces to certain other well-known symmetric polynomials. For example, let us define the *falling factorial polynomials*

$$x^{\underline{b}} := x(x-1)\cdots(x-b+1).$$

Then we can obtain the *factorial monomial symmetric functions* and the *factorial Schur functions* by replacing each x_i^b by $x_i^{\underline{b}}$ in the monomial symmetric functions and Schur functions, respectively[1]. Then for $\kappa = 0$ we obtain the factorial monomial symmetric functions and for $\kappa = 1$ we obtain the *factorial Schur functions* (see [5]).

The above definition is all very well if we want to know only about the defining properties of the interpolation Jack polynomials, but it is not very helpful in giving us a concrete expression to work with. In fact, in the paper [5], no such expression is given. However, Proposition 3.4 of [5] does give an explicit formula for $P_\lambda^{\kappa\delta}(x_1, \ldots, x_N)$ when $\lambda = (\lambda_1, 0, \ldots, 0)$, which is as follows:

$$P^{\kappa\delta}_{(\lambda_1,0,\ldots,0)}(x_1,\ldots,x_N) = \binom{-\kappa}{\lambda_1} \sum_{i_j} \prod_{j=1}^{N} \left[\binom{-\kappa}{i_{j-1}-i_j} (x_j - \kappa\delta_j - i_j)^{\underline{i_{j-1}-i_j}} \right] \quad (2.1)$$

where the sum runs through all integer sequences $\lambda_1 = i_0 \geq i_1 \geq \cdots \geq i_{n-1} \geq i_N \geq 0$. Also note that

$$\binom{-\kappa}{D} := \frac{(-\kappa - D + 1)(-\kappa - D + 2)\cdots(-\kappa)}{D!} = \frac{(-1)^D \kappa^{\underline{D}}}{D!}$$

and $\binom{-\kappa}{i_{j-1}-i_j}$ is defined similarly[2]. This formula will be useful in Chapter 4 when we look specifically at interpolation Jack polynomials of this form. However, we would also like to look at interpolation Jack polynomials that are not of this form, and so for a general formula we turn once again to Young diagrams.

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[1]See Section 1.3 for the definition of the monomial symmetric functions and the Schur functions.

[2]This is a natural extension of the usual binomial coefficient.

2.2 Combinatorial Formula

We now present a series of definitions, which build up to a combinatorial formula for the interpolation Jack polynomials $P_\lambda^{\kappa\delta}$. We present the procedure for interpolation Jack polynomials in three variables, but it can be generalized to N variables with only slight modification. Recall that for a partition λ we denote the Young diagram of shape λ by $\widetilde{\lambda}$.

Definition 2.2.1. Let $\mu = (\mu_1, \mu_2, \mu_3)$ be a partition. Then for a box s at position (i, j) in $\widetilde{\mu}$, we define

$$l_\mu(s) = \left|\{k > i \mid \mu_k \geq j\}\right| \qquad l'(s) = i - 1$$
$$a_\mu(s) = \mu_i - j \qquad a'(s) = j - 1.$$

Note that $l_\mu(s)$, $l'(s)$, $a_\mu(s)$, and $a'(s)$ are respectively the number of boxes in the Young diagram to the south, north, west and east of the box s.

Example 2.2.2. Let $\mu = (2, 0, 0)$ and s be the box at position $(1, 2)$ in $\widetilde{\mu}$. That is, s is the starred box in the following diagram:

	⋆

.

Then

$$l_\mu(s) = \left|\{k > i \mid \mu_k \geq j\}\right| = \left|\{k > 1 \mid \mu_k \geq 2\}\right| = 0,$$
$$l'(s) = 1 - 1 = 0,$$
$$a_\mu(s) = \mu_1 - 2 = 2 - 2 = 0,$$
$$a'(s) = 2 - 1 = 1.$$

Definition 2.2.3. Let $\mu = (\mu_1, \mu_2, \mu_3)$ be a partition. Then for a box s in $\widetilde{\lambda}$, we define

$$b_\mu(s, \kappa) = \frac{a_\mu(s) + \kappa(l_\mu(s) + 1)}{a_\mu(s) + \kappa l_\mu(s) + 1}.$$

Example 2.2.4. Using the same μ and s from Example 2.2.2, we see that

$$b_\mu(s, \kappa) = \frac{0 + \kappa(0 + 1)}{0 + \kappa(0) + 1} = \kappa.$$

Definition 2.2.5. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a partition. Then a *reverse tableau* of shape λ is a filling of the Young diagram $\widetilde{\lambda}$ by the numbers 1, 2, 3 such that the rows are weakly decreasing from left to right and the columns are strictly decreasing from top to bottom.

Example 2.2.6. Let $\lambda = (3, 2, 1)$. Then a few reverse fillings of the Young diagram $\widetilde{\lambda}$ are

3	3	1
2	2	
1		

3	3	1
2	1	
1		

3	2	2
2	1	
1		

For a box s in position (i, j) in a reverse tableau T, we denote the entry in s by $T(s)$ or $T(i, j)$. For example, if T is the reverse tableau on the left in Example 2.2.6 then $T(1, 1) = 3$.

Definition 2.2.7. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a partition, and let T be a reverse tableau of shape λ. Then for $i = 0, 1, 2$, we define $\widetilde{\lambda^{(i)}}$ to be the reverse tableau obtained from T by keeping only the boxes containing the numbers $i + 1, \ldots, 3$.

Example 2.2.8. Again let T be the reverse tableau on the left in Example 2.2.6. Then $\widetilde{\lambda^{(2)}}$ and $\widetilde{\lambda^{(1)}}$ are

3	3

and

3	3
2	2

respectively, and $\widetilde{\lambda^{(0)}}$ is just the original reverse tableau. Note that $\widetilde{\lambda^{(0)}}$ will always simply be the original reverse tableau.

Definition 2.2.9. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a partition, and let T be a reverse tableau of shape λ. Then for $i = 1, 2$, we define $(R/C)_{\lambda^{(i-1)}/\lambda^{(i)}}$ to be the set of boxes s in the Young diagram $\widetilde{\lambda^{(i)}}$ with the following two properties:

- *Property 1*: The row that contains s also contains a box from $\widetilde{\lambda^{(i-1)}}$ that is not in $\widetilde{\lambda^{(i)}}$.
- *Property 2*: The column that contains s does not contain a box from $\widetilde{\lambda^{(i-1)}}$ that is not in $\widetilde{\lambda^{(i)}}$.

Example 2.2.10. Again let T be the reverse tableau on the left in Example 2.2.6. Then

$$(R/C)_{\lambda^{(1)}/\lambda^{(2)}} = \emptyset, \text{ and}$$
$$(R/C)_{\lambda^{(0)}/\lambda^{(1)}} = \{(1, 2)\}.$$

Definition 2.2.11. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a partition and let T be a reverse tableau of shape λ. Then we define

$$\psi_T(\kappa) = \prod_{i=1}^{2} \prod_{s \in (R/C)_{\lambda^{(i-1)}/\lambda^{(i)}}} \frac{b_{\lambda^{(i)}}(s,\kappa)}{b_{\lambda^{(i-1)}}(s,\kappa)}$$

where the function b is as in Definition 2.2.3.

Actual calculations of $\Psi_T(\kappa)$ can be somewhat lengthy, so we hold off on a concrete example until after the next definition, when we will provide a full example calculation of an interpolation Jack polynomial.

Finally, for a box s in a reverse tableau T define the function φ as follows:

$$\varphi(s,\kappa) = \begin{cases} -2\kappa & \text{if } T(s) = 1 \\ -\kappa & \text{if } T(s) = 2 \\ 0 & \text{if } T(s) = 3. \end{cases}$$

Then we can write a combinatorial formula for the interpolation Jack polynomials that we wish to work with.

Formula 2.2.12. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a partition. Then the *interpolation Jack polynomial* associated with λ is

$$P_\lambda^{\kappa\delta}(x_1, x_2, x_3) = \sum_{\substack{T \text{ a reverse} \\ \text{tableau} \\ \text{of shape } \lambda}} \psi_T(\kappa) \prod_{s \in T} (x_{T(s)} + \varphi(s,\kappa) - a'(s) + l'(s)\kappa). \qquad (2.2)$$

Example 2.2.13. Now let's use the above Formula 2.2 to find the interpolation Jack polynomial for $\lambda = (2,0,0)$. This is a bit of a complicated calculation, so we have broken it down into several steps.

Step 1: The first step is to find all reverse tableaux of shape $\lambda = (2,0,0)$. The Young diagram associated to $\lambda = (2,0,0)$ is

$$\begin{array}{|c|c|}\hline & \\ \hline \end{array}$$

so the reverse tableaux of shape $\lambda = (2,0,0)$ are

$$T_1 = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline \end{array}\,, \quad T_2 = \begin{array}{|c|c|}\hline 3 & 1 \\ \hline \end{array}\,, \quad T_3 = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline \end{array}\,,$$

$$T_4 = \begin{array}{|c|c|}\hline 3 & 3 \\ \hline \end{array}\,, \quad T_5 = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}\,, \quad T_6 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}\,.$$

Step 2: The next step is to find $\Psi_{T_i}(\kappa)$ for each reverse tableau T_i.

Let's consider T_1. We first calculate the $\widetilde{\lambda^{(i)}}$. We have

$$\widetilde{\lambda^{(0)}} = \widetilde{\lambda^{(1)}} = \boxed{3}\boxed{2}\,, \qquad \widetilde{\lambda^{(2)}} = \boxed{3}\,.$$

Then we can use the $\widetilde{\lambda^{(i)}}$ to calculate the $(R/C)_{\lambda^{(i-1)}/\lambda^{(i)}}$. We see that

$$(R/C)_{\lambda^{(0)}/\lambda^{(1)}} = \emptyset, \qquad (R/C)_{\lambda^{(1)}/\lambda^{(2)}} = \{(1,1)\}.$$

Now for each i we calculate $b_{\lambda^{(i)}}(s,\kappa)$ and $b_{\lambda^{(i-1)}}(s,\kappa)$ for all s in $(R/C)_{\lambda^{(i-1)}/\lambda^{(i)}}$. In this case, there are no s in $(R/C)_{\lambda^{(0)}/\lambda^{(1)}}$, so we only need to worry about $(R/C)_{\lambda^{(1)}/\lambda^{(2)}}$. There is only one box in $(R/C)_{\lambda^{(1)}/\lambda^{(2)}}$, namely $s = (1,1)$, and we see that

$$a_{\lambda^{(1)}}(s) = \widetilde{\lambda_1^{(1)}} - 1 = 2 - 1 = 1, \qquad l_{\lambda^{(1)}}(s) = \left|\left\{k > 1 \mid \widetilde{\lambda_k^{(1)}} \geq 1\right\}\right| = 0,$$

$$a_{\lambda^{(2)}}(s) = \widetilde{\lambda_1^{(2)}} - 1 = 1 - 1 = 0, \qquad l_{\lambda^{(2)}}(s) = \left|\left\{k > 1 \mid \widetilde{\lambda_k^{(2)}} \geq 1\right\}\right| = 0,$$

so

$$b_{\lambda^{(1)}}(s,\kappa) = \frac{a_{\lambda^{(1)}}(s) + \kappa(l_{\lambda^{(1)}}(s)+1)}{a_{\lambda^{(1)}}(s) + \kappa l_{\lambda^{(1)}}(s) + 1} = \frac{\kappa+1}{2}$$

$$b_{\lambda^{(2)}}(s,\kappa) = \frac{a_{\lambda^{(2)}}(s) + \kappa(l_{\lambda^{(2)}}(s)+1)}{a_{\lambda^{(2)}}(s) + \kappa l_{\lambda^{(2)}}(s) + 1} = \kappa.$$

Putting this all together we have

$$\begin{aligned}\psi_{T_1}(\kappa) &= \prod_{i=1}^{2} \prod_{s \in (R/C)_{\lambda^{(i-1)}/\lambda^{(i)}}} \frac{b_{\lambda^{(i)}}(s,\kappa)}{b_{\lambda^{(i-1)}}(s,\kappa)} \\ &= \frac{\kappa}{\left(\frac{\kappa+1}{2}\right)} \\ &= \frac{2\kappa}{\kappa+1}.\end{aligned}$$

Since the calculation of $\Psi_{T_i}(\kappa)$ doesn't depend on the actual values in T_i but only on the relative sizes of those values, in this case $\Psi_{T_i}(\kappa)$ will be the same for $i = 1,2,3$.

Now consider T_4. Again, we first calculate the $\widetilde{\lambda^{(i)}}$:

$$\widetilde{\lambda^{(0)}} = \widetilde{\lambda^{(1)}} = \widetilde{\lambda^{(2)}} = T_4.$$

So

$$(R/C)_{\lambda^{(0)}/\lambda^{(1)}} = (R/C)_{\lambda^{(1)}/\lambda^{(2)}} = \emptyset.$$

Thus $\Psi_{T_4}(\kappa) = 1$. As before, Ψ_{T_i} is the same for $i = 4, 5, 6$.
Step 3: The third step is to find the entire term contributed by each reverse tableau T_i. First note that all the T_i have the same two boxes, $(1, 1)$ and $(1, 2)$. Since for a box s in a reverse tableau T_i, $a'(s)$ and $l'(s)$ depend only on the position of s, we see that for all T_i we have

$$\begin{aligned} a'(1,1) &= 1-1 = 0, & a'(1,2) &= 2-1 = 1 \\ l'(1,1) &= 1-1 = 0, & l'(1,2) &= 1-1 = 0. \end{aligned}$$

Thus for each reverse tableau T_i the term contributed is

$$\Psi_{T_i}(\kappa)\left[x_{T_i(1,1)} + \varphi((1,1),\kappa)\right]\left[x_{T_i(1,2)} + \varphi((1,2),\kappa) - 1\right].$$

We now consider each reverse tableau individually.

(1) First consider T_1. We see that

$$\begin{aligned} T_1(1,1) &= 3, & T_1(1,2) &= 2, \\ \varphi((1,1),\kappa) &= 0, & \varphi((1,2),\kappa) &= -\kappa. \end{aligned}$$

So the term contributed by T_1 is

$$\begin{aligned} &\Psi_{T_1}(\kappa)\left[x_{T_1(1,1)} + \varphi((1,1),\kappa)\right]\left[x_{T_1(1,2)} + \varphi((1,2),\kappa) - 1\right] \\ &= \left(\frac{2\kappa}{\kappa+1}\right)(x_3)(x_2 - \kappa - 1). \end{aligned}$$

(2) For T_2 we can calculate

$$\begin{aligned} T_2(1,1) &= 3, & T_2(1,2) &= 1, \\ \varphi((1,1),\kappa) &= 0, & \varphi((1,2),\kappa) &= -2\kappa. \end{aligned}$$

So the term contributed by T_2 is

$$\begin{aligned} &\Psi_{T_2}(\kappa)\left[x_{T_2(1,1)} + \varphi((1,1),\kappa)\right]\left[x_{T_2(1,2)} + \varphi((1,2),\kappa) - 1\right] \\ &= \left(\frac{2\kappa}{\kappa+1}\right)(x_3)(x_1 - 2\kappa - 1). \end{aligned}$$

(3) For T_3 we have

$$T_3(1,1) = 2, \qquad T_3(1,2) = 1,$$
$$\varphi((1,1),\kappa) = -\kappa, \qquad \varphi((1,2),\kappa) = -2\kappa.$$

So the term contributed by T_3 is

$$\begin{aligned}&\Psi_{T_3}(\kappa)\left[x_{T_3(1,1)} + \varphi((1,1),\kappa)\right]\left[x_{T_3(1,2)} + \varphi((1,2),\kappa) - 1\right]\\ &= \left(\frac{2\kappa}{\kappa+1}\right)(x_2 - \kappa)(x_1 - 2\kappa - 1).\end{aligned}$$

(4) For T_4 we have

$$T_4(1,1) = 3, \qquad T_4(1,2) = 3,$$
$$\varphi((1,1),\kappa) = 0, \qquad \varphi((1,2),\kappa) = 0.$$

So the term contributed by T_4 is

$$\begin{aligned}&\Psi_{T_4}(\kappa)\left[x_{T_4(1,1)} + \varphi((1,1),\kappa)\right]\left[x_{T_4(1,2)} + \varphi((1,2),\kappa) - 1\right]\\ &= (x_3)(x_3 - 1).\end{aligned}$$

(5) For T_5 we have

$$T_5(1,1) = 2, \qquad T_5(1,2) = 2,$$
$$\varphi((1,1),\kappa) = -\kappa, \qquad \varphi((1,2),\kappa) = -\kappa.$$

So the term contributed by T_5 is

$$\begin{aligned}&\Psi_{T_5}(\kappa)\left[x_{T_5(1,1)} + \varphi((1,1),\kappa)\right]\left[x_{T_5(1,2)} + \varphi((1,2),\kappa) - 1\right]\\ &= (x_2 - \kappa)(x_2 - \kappa - 1).\end{aligned}$$

(6) Finally, for T_6 we have

$$T_6(1,1) = 1, \qquad T_6(1,2) = 1,$$
$$\varphi((1,1),\kappa) = -2\kappa, \qquad \varphi((1,2),\kappa) = -2\kappa.$$

So the term contributed by T_6 is

$$\begin{aligned}&\Psi_{T_6}(\kappa)\left[x_{T_6(1,1)} + \varphi((1,1),\kappa)\right]\left[x_{T_6(1,2)} + \varphi((1,2),\kappa) - 1\right]\\ &= (x_1 - 2\kappa)(x_1 - 2\kappa - 1).\end{aligned}$$

Step 4: The final step is to add up the terms contributed by each T_i to get $P^{\kappa\delta}_{(2,0,0)}(x_1, x_2, x_3)$. From step 3 we see that

$$\begin{aligned}
P^{\kappa\delta}_{\lambda}(x_1, x_2, x_3) &= \sum_{\substack{T \text{ a reverse} \\ \text{tableau} \\ \text{of shape } \lambda}} \psi_T(\kappa) \prod_{s \in T} (x_{T(s)} + \varphi(s, \kappa) - a'(s) + l'(s)\kappa) \\
&= \left(\frac{2\kappa}{\kappa+1}\right)(x_3)(x_2 - \kappa - 1) + \left(\frac{2\kappa}{\kappa+1}\right)(x_3)(x_1 - 2\kappa - 1) \\
&\quad + \left(\frac{2\kappa}{\kappa+1}\right)(x_2 - \kappa)(x_1 - 2\kappa - 1) + (x_3)(x_3 - 1) \\
&\quad\quad + (x_2 - \kappa)(x_2 - \kappa - 1) + (x_1 - 2\kappa)(x_1 - 2\kappa - 1) \\
&= x_3^2 + x_2^2 + x_1^2 + \left(\frac{2\kappa}{\kappa+1}\right)(x_3 x_2 + x_2 x_1 + x_2 x_1) \\
&\quad - \left(\frac{6\kappa^2 + 5\kappa + 1}{\kappa+1}\right)(x_3 + x_2 + x_1) + \frac{9\kappa^3 + 10\kappa^2 + 3\kappa}{\kappa+1}.
\end{aligned}$$

Chapter 3

Summer Exploits and Thesis Goals

Over the summer of 2019, I conducted research on interpolation Jack polynomials at the Fields Institute for Research in the Mathematical Sciences with fellow undergraduate Xiaomin Li from University of Illinois at Urbana-Champaign, under the supervision of Dr. Hadi Salmasian from the University of Ottawa. During this summer research we constructed several conjectures, and my original goals for this thesis were to prove these conjectures. However, while I did work on problems related to the conjectures, I ended up going in a slightly different direction than originally planned. In this section I will go over a description of the goals of my summer research, a brief summary of our conjectures, and an indication of how my final thesis has deviated from my original hopes. Note that my research over the summer focused only on interpolation Jack polynomials in three variables, so this section will primarily discuss that case.

In this section we also go over a brief introduction to Lie algebras and representation theory, as it is closely intertwined with the study of interpolation Jack polynomals. For further reading on these topics, see respectively [8, 9] and [4]. Further exposition on the conjectures developed over the summer can be found in [7].

For the remainder of this thesis, we will drop the $\kappa\delta$ superscript, and simply denote our interpolation Jack polynomials as P_λ. However, it should be assumed that they are of the form discussed on page 11.

3.1 Some More Background

Briefly, the goal of my summer research and thesis was to differentiate an interpolation Jack polynomial P_λ with respect to the parameter κ and write the result as a linear combination of other interpolation Jack polynomials, where the coefficients in the linear combination are rational functions in κ. That is, for each interpolation Jack polynomial P_λ we would like to find rational functions of κ, $c_\lambda^\mu(\kappa)$, such that

$$\frac{\partial}{\partial\kappa} P_\lambda(x_1, \ldots, x_N) = \sum_\mu c_\lambda^\mu(\kappa) P_\mu(x_1, \ldots, x_N)$$

where the sum ranges over some finite collection of partitions μ. Note that in my summer research I focused only on the case where $N = 3$. In this section we will go over some motivation for studying this particular problem.

First we motivate our decision to take the derivative with respect to κ. To this end, we first introduce the notion of a *Lie algebra*.

Definition 3.1.1. Let F be a field. A *Lie algebra* over F is an F-vector space L together with a bilinear map

$$L \times L \to L, \quad (x, y) \mapsto [x, y]$$

that satisfies the following properties:

$$[x, x] = 0 \quad \text{for all } x \in L, \tag{3.1}$$

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = 0 \quad \text{for all } x, y, z \in L. \tag{3.2}$$

The map $(x, y) \mapsto [x, y]$ is called the *Lie bracket*.

One prominent example of a Lie algebra is the *general linear algebra* for a vector space V, denoted $\mathfrak{gl}(V)$. This consists of all linear maps from V to V, with Lie bracket given by

$$[A, B] = AB - BA$$

for all $A, B \in \mathfrak{gl}(V)$. It is straightforward to verify that this satisfies both properties of the Lie bracket. Now given a Lie algebra L, we can form its *universal enveloping algebra*, which in a sense allows us to *multiply* elements.

Definition 3.1.2. Let L be a Lie algebra over a field F. A *universal enveloping algebra* of L is a pair $(\mathcal{U}, i)$, where $\mathcal{U}$ is an associative algebra with 1 over F, $i : L \longrightarrow \mathcal{U}$ is a linear map satisfying

$$i([x, y]) = i(x)i(y) - i(y)i(x) \tag{3.3}$$

for all $x, y \in L$, and the following holds: for any associative F-algebra $\mathfrak{U}$ with 1 and any linear map $j : L \longrightarrow \mathfrak{U}$ satisfying Equation 3.3, there exists a unique homomorphism of algebras $\phi : \mathcal{U} \longrightarrow \mathfrak{U}$ (sending 1 to 1) such that $\phi \circ i = j$.

This is a rather abstract definition, and we will not deal with it directly. Instead, we consider the tensor algebra of a vector space V.

Definition 3.1.3. Let V be a finite-dimensional vector space over a field F. Let $T^0V = F, T^1V = V$, and for each $m > 0$ let $T^m = V \otimes \cdots \otimes V$ (m times), where $\otimes$ denotes the tensor product over F. The *tensor algebra* on V is then the vector space

$$\mathcal{I}(V) = \bigoplus_{i=0}^{\infty} T^i V$$

together with the product

$$(v_1 \otimes \cdots \otimes v_k)(w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_m.$$

Note that this choice of product makes $\mathcal{I}(V)$ an associative algebra with unity.

Now let L be a finite-dimensional Lie algebra, and let J be the two-sided ideal in $\mathcal{I}(L)$ generated by elements of the form $x \otimes y - y \otimes x - [x, y]$ (where $x, y \in L$). Let $\mathcal{U}(L) = \mathcal{I}(L)/J$, and let $\pi : \mathcal{I}(\mathcal{L}) \longrightarrow \mathcal{U}(L)$ be the canonical homomorphism. Then the pair $(\mathcal{U}(L), i)$, where i is the restriction of π to L, is a universal enveloping algebra for L, and furthermore it is the unique universal enveloping algebra of L up to isomorphism.

Thus for a finite-dimensional Lie algebra L we can refer to *the* universal enveloping algebra of L, which can be thought of as the quotient algebra $\mathcal{U}(L) = \mathcal{I}(L)/J$, as described above.

We will now take a brief detour into representation theory. For a vector space V let $\mathfrak{gl}(V)$ be the general linear Lie algebra associated with V, as defined above, and recall the following definitions:

Definition 3.1.4. Let L be a Lie algebra and V be a vector space. If there is a Lie algebra homomorphism $\rho : L \longrightarrow \mathfrak{gl}(V)$, then the pair (V, ρ) is called an *L-module* (or *representation*) of L. Note that sometimes the vector space itself is referred to as an L-module or representation of L.

Definition 3.1.5. Let L be a Lie algebra and let V be an L-module with Lie algebra homomorphism $\rho : L \to \mathfrak{gl}(V)$. Let $W \subseteq V$ be a subspace of V. Then W is a *submodule* of V if

$$w \in W \implies (\rho(x))(w) \in W \quad \text{for all } x \in L.$$

Definition 3.1.6. Let L be a Lie algebra and let $V \neq \{0\}$ be an L-module. Then V is *irreducible* if the only submodules of V are $\{0\}$ and V.

So where do interpolation Jack polynomials come into this? Well, given a Lie algebra L, we may wish to understand its irreducible representations. In Section 15.2.1 of [8], it is shown that there is a bijective correspondence between the representations of L and those of $\mathcal{U}(L)$, so to study the irreducible representations of L it suffices to study those of $\mathcal{U}(L)$. To do this, it is useful to study how the elements of the center of $\mathcal{U}(L)$ act on the irreducible representations. Why is this useful? By Schur's Lemma[1], the elements of the center of $\mathcal{U}(L)$ act by scalars on irreducible representations of $\mathcal{U}(L)$. Thus it is sometimes the case that we can distinguish different irreducible representations by these scalars.

In particular, if L is a *reductive* Lie algebra, there is a distinguished basis of the center of $\mathcal{U}(L)$ that is indexed by partitions. Let us denote the elements in this basis as b_λ. They are called the *Capelli elements* or *Capelli operators* and interpolation Jack polynomials tend to show up in connection to the eigenvalues of these operators. For example, consider the Lie algebra $\mathfrak{gl}(V)$ mentioned above. In this case, the irreducible representations can be indexed by partitions; let us denote each irreducible representation by V_μ. Then the eigenvalue of the Capelli element b_λ acting on the irreducible representation V_μ is simply $P_\lambda^\delta(\mu)$, i.e., for each $v \in V_\mu$ we have

$$b_\lambda \cdot v = P_\lambda^\delta(\mu)v.$$

So in this case, the interpolation Jack polynomials show up as eigenvalue polynomials. Works of Sahi, Salmasian and Serganova (see [10]) further show a connection between the derivative with respect to κ of two-variable interpolation Jack polynomials and the eigenvalues of Capelli operators of orthosymplectic Lie superalgebras. This suggests that there may be further connections between the derivatives of interpolation Jack polynomials and representation theory, and this is our motivation for studying them.

[1]See Lemma 7.13 in [8].

Now let's return to the "real world" and consider what actually happens we we differentiate with respect to κ. To make things more tangible, let's again work with a concrete example.

Example 3.1.7. Consider the interpolation Jack polynomial in three variables indexed by the partition $\lambda = (2, 1, 1)$:

$$P_{(2,1,1)}(x_1, x_2, x_3) = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_2 x_2 x_3^2 - 3(\kappa + 1) x_1 x_2 x_3.$$

When we differentiate with respect to κ we get

$$\frac{\partial}{\partial \kappa} P_{(2,1,1)}(x_1, x_2, x_3) = -3 x_1 x_2 x_3$$

which is again a symmetric polynomial.

This will in fact be the case in general. That is, when we differentiate an interpolation Jack polynomial with respect to κ we will be left with a symmetric polynomial. It is straightforward to see why this is the case. Let f be a symmetric polynomial in N variables with coefficients that are rational functions of κ. Then we can write f as

$$f(x_1, \ldots, x_N) = \sum_{\mu} c_\lambda(\kappa) m_\lambda(x_1, \ldots, x_N)$$

where the m_λ are the monomial symmetric functions introduced in Section 1.3. If we differentiate with respect to κ we get

$$\frac{\partial}{\partial \kappa} f(x_1, \ldots, x_N) = \sum_{\mu} c'_\lambda(\kappa) m_\lambda(x_1, \ldots, x_N)$$

which is again a linear combination of monomial symmetric functions. Since the monomial symmetric functions are, in fact, symmetric, the resulting linear combination is also symmetric.

Now, as alluded to previously, for each N the set of interpolation Jack polynomials $\{P_\lambda(x_1, \ldots, x_r) \,|\, r \leq N\}$ forms a basis for Λ_N. Thus for each interpolation Jack polynomial P_λ we can write $\frac{\partial}{\partial \kappa} P_\lambda$ as a linear combination of other interpolation Jack polynomials where the coefficients are rational functions of κ.

Our goal over the summer was to determine which interpolation Jack polynomials occur in these linear combinations with non-zero coefficients and to find formulas for the coefficients that are non-zero. We worked exclusively with interpolation Jack polynomials in three variables, inspired by the successful solution of the problem for two variable polynomials by Sahi, Salmasian and Serganova (see [10]). Although the approach used for the two variable case will not work exactly for the three variable case, a similar approach could be effective, and so in the next section I give a brief overview of the two variable case.

3.2 Previous Work: the Two Variable Case

In this section, we will assume that all interpolation Jack polynomial are in two variables. First, we recall that in [5] an explicit formula is given for interpolation Jack polynomials $P_{(\lambda_1,0,0)}$:

$$P_{(\lambda_1,0,\dots,0)}(x_1,\dots,x_N) = \binom{-\kappa}{\lambda_1} \sum_{i_j} \prod_{j=1}^{N} \left[\binom{-\kappa}{i_{j-1}-i_j} (x_j - \kappa\delta_j - i_j)^{\underline{i_{j-1}-i_j}} \right]$$

where the sum runs through all integer sequences $\lambda_1 = i_0 \geq i_1 \geq \cdots \geq i_{n-1} \geq i_N \geq 0$. Also, from Lemma 2.3 in [5] we have the following identity for each interpolation Jack polynomial $P_{(\lambda_1,\lambda_2)}$:

$$P_{(\lambda_1,\lambda_2)}(x_1,x_2) = x_1 x_2 P_{(\lambda_1-1,\lambda_2-1)}(x_1-1,x_2-1). \tag{3.4}$$

Using this formula repeatedly, we can write

$$\begin{aligned} P_{(\lambda_1,\lambda_2)}(x_1,x_2) &= x_1 x_2 P_{(\lambda_1-1,\lambda_2-1)}(x_1-1,x_2-1) \\ &= (x_1x_2)^2 P_{(\lambda_1-2,\lambda_2-2)}(x_1-2,x_2-2) \\ &\vdots \\ &= (x_1x_2)^{\lambda_2} P_{(\lambda_1-\lambda_2,0)}(x_1-\lambda_2,x_2-\lambda_2). \end{aligned}$$

So we can write any $P_{(\lambda_1,\lambda_2)}$ in terms of an interpolation Jack polynomial indexed by a partition with only *one* non-zero part. Thus we can find an explicit formula for all interpolation Jack polynomials in two variables using Equation 3.4. Using this explicit formula, we can then write an explicit expression for the coefficient in front of $x_1^i x_2^j$ in P_λ for each i and j. Then we can differentiate these coefficients with respect to κ to get an expression for the coefficient of $x_1^i x_2^j$ in $\frac{\partial}{\partial\kappa}P_\lambda$. Let's denote this differentiated coefficient by c_{ij}.

Now, as my partner and I did over the summer, Dr. Salmasian and collaborators came up with conjectures for the coefficient of P_μ in the linear combination for $\frac{\partial}{\partial\kappa}P_\lambda$. Using these conjectures and the explicit formulas for the interpolation Jack polynomials mentioned above, we can write down what the coefficient of $x_1^i x_2^j$ would be in $\frac{\partial}{\partial\kappa}P_\lambda$ *if* the conjectures were true. Let's denote this coefficient by c'_{ij}.

To prove the conjectures, then, we must simply show that $c_{ij} = c'_{ij}$ for each i, j. This equivalence, it turns out, is analogous to a hypergeometric identity proven by Dougall in 1907[2]. This proves that the conjectures are correct.

Given this approach for the two-variable case, we can ask what breaks in the three variable case. There are in fact several stages that break.

For one, we do not have an explicit formula for interpolation Jack polynomials in three variables, just one that depends on Young diagrams[3]. So we would have to re-derive an expression for the coefficient of $x_1^i x_2^j x_3^\ell$ in $P_\lambda(x_1, x_2, x_3)$, and this expression would depend on Young diagrams. Our conjectures also do not line up with those of Dr. Salmasian, so we would have to re-derive the coefficient of $x_1^i x_2^j x_3^\ell$ in $\frac{\partial}{\partial\kappa}P_\lambda(x_1, x_2, x_3)$ that we would get *if* our conjectures were correct. Lastly, equating coefficients in the three-variable case may not reduce to the same hypergeometric identity, so we would have to look around for another identity to possibly reduce it to. However, the approach taken in the two-variable place is still a good guide as to where to start for the three-variable case, and a similar approach may work.

3.3 Previous Work: Conjectures

As mentioned above, our research goal over the summer was to differentiate each interpolation Jack polynomial P_λ in three variables with respect to the variable κ and write the result as a linear combination of other interpolation Jack polynomials:

$$\frac{\partial}{\partial\kappa}P_\lambda = \sum_\mu c_\lambda^\mu(\kappa)P_\mu$$

[2]See [11].
[3]See Section 2.2.

where the coefficients $c^{\mu}_{\lambda}(\kappa)$ are rational functions of κ. The first question to ask in this scenario is: which interpolation Jack polynomials appear with non-zero coefficients in this linear combination? For example,

$$\frac{\partial}{\partial k}P_{(2,0,0)} = \left(\frac{5k+3}{k+1}\right)P_{(0,0,0)} + \left(\frac{-6k-4}{k+1}\right)P_{(1,0,0)} + \left(\frac{2}{k^2+2k+1}\right)P_{(1,1,0)}$$

so we see that only $P_{(0,0,0)}$, $P_{(1,0,0)}$ and $P_{(1,1,0)}$ appear with non-zero coefficients in the linear combination for $\frac{\partial}{\partial\kappa}P_{(2,0,0)}$. In fact, in general it will be the case that for each partition λ, there is a cutoff in size (with respect to lexicographic ordering) such that if μ is "too large" relative to λ then P_μ will not appear in the linear combination for $\frac{\partial}{\partial\kappa}P_\lambda$.

It is convenient at this point to break the question down into two cases: (1) if the partition λ has three non-zero parts (e.g., $\lambda = (3,2,1)$) and (2) if λ has fewer than three non-zero parts (e.g., $\lambda = (3,2,0)$). It turns out that the first case reduces to the second case:

Theorem 3.3.1. *Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a partition with exactly three non-zero parts and $\mu = (\mu_1, \mu_2, \mu_3)$ be a partition with at most three non-zero parts. Then*

$$c^{\mu}_{\lambda} = \begin{cases} 0 & \text{if } \mu_3 - \lambda_3 < 0 \\ c^{(\mu_1-\lambda_3,\mu_2-\lambda_3,\mu_3-\lambda_3)}_{(\lambda_1-\lambda_3,\lambda_2-\lambda_3,0)} & \text{otherwise.} \end{cases}$$

Note that we have written just c^{μ}_{λ} instead of $c^{\mu}_{\lambda}(\kappa)$ here, for ease of reading.

Proof. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a partition with exactly three non-zero parts, and let $\lambda^* = (\lambda_1 - \lambda_3, \lambda_2 - \lambda_3, 0)$. Then suppose that the linear combination for $\frac{\partial}{\partial\kappa}P_{\lambda^*}$ is

$$\frac{\partial}{\partial\kappa}P_{\lambda^*} = \sum_{\mu} c^{\mu}_{\lambda^*}P_\mu. \tag{3.5}$$

Now, from Corollary 2.3 in [5], we know that for any partition $\nu = (\nu_1, \nu_2, \nu_3)$ we have

$$P_{(\nu_1,\nu_2,\nu_3)}(x_1, x_2, x_3) = x_1x_2x_3P_{(\nu_1-1,\nu_2-1,\nu_3-1)}(x_1-1, x_2-1, x_3-1).$$

Repeated application of this formula to P_λ shows that

$$P_\lambda(x_1, x_2, x_3) = (x_1x_2x_3)^{\lambda_3}P_{\lambda^*}(x_1-\lambda_3, x_2-\lambda_3, x_3-\lambda_3).$$

Then differentiating both sides and using Equation 3.5, we see that

$$\begin{aligned}\frac{\partial}{\partial\kappa}P_\lambda(x_1,x_2,x_3) &= (x_1x_2x_3)^{\lambda_3}\frac{\partial}{\partial\kappa}P_{\lambda^*}(x_1-\lambda_3,x_2-\lambda_3,x_3-\lambda_3)\\ &= (x_1x_2x_3)^{\lambda_3}\sum_\mu c^\mu_{\lambda^*}P_\mu(x_1-\lambda_3,x_2-\lambda_3,x_3-\lambda_3).\end{aligned}$$

We can then move the $(x_1x_2x_3)^{\lambda_3}$ inside the summation and apply Corollary 2.3 from [5] in reverse:

$$\begin{aligned}\frac{\partial}{\partial\kappa}P_\lambda(x_1,x_2,x_3) &= \sum_\mu c^\mu_{\lambda^*}\left[(x_1x_2x_3)^{\lambda_3}P_\mu(x_1-\lambda_3,x_2-\lambda_3,x_3-\lambda_3)\right]\\ &= \sum_\mu c^\mu_{\lambda^*}P_{(\mu_1+\lambda_3,\mu_2+\lambda_3,\mu_3+\lambda_3)}(x_1,x_2,x_3). \end{aligned}\tag{3.6}$$

Since the linear combination for $\frac{\partial}{\partial\kappa}P_\lambda$ is unique, this shows the desired result. □

Let's take a look at an example to help digest this theorem.

Example 3.3.2.

- Let $\mu = (1,1,1)$ and $\lambda = (2,1,1)$ Then the coefficient of P_μ in $\frac{\partial}{\partial\kappa}P_\lambda$is -3. If we subtract $\lambda_3 = 1$ from each part in λ and μ we get $\mu' = (0,0,0)$ and $\lambda' = (1,0,0)$, and the coefficient of $P_{\mu'}$ in $\frac{\partial}{\partial\kappa}P_{\lambda'}$ is also -3.
- On the other hand, let $\mu = (1,1,0)$ and $\lambda = (2,1,1)$. Then if we subtract $\lambda_3 = 1$ from all the parts of μ we get $\mu' = (0,0,-1)$, which is not a valid partition. Thus P_μ doesn't appear in the linear combination for $\frac{\partial}{\partial\kappa}P_\lambda$.

Thus it is more interesting to focus on partitions λ with one or two non-zero parts. In this case, we find that in fact P_μ appears in the linear combination for $\frac{\partial}{\partial\kappa}P_\lambda$ only if μ is "small enough" with respect to λ. To be precise, we will see in Section 4.2[4] that if $\mu = (\mu_1,\mu_3,\mu_3)$ and $\lambda = (\lambda_1,\lambda_2,\lambda_3)$ then $c^\mu_\lambda = 0$ if $\mu \geq \lambda$ or if $\sum\mu_i > \sum\lambda_i$. In fact, computations in sage would suggest that this is an if and only if condition for λ with one or two non-zero parts. That is, we have the following conjecture.

Conjecture 3.3.3. Let $\lambda = (\lambda_1,\lambda_2,0)$ and $\mu = (\mu_1,\mu_2,\mu_3)$ be partitions. The interpolation Jack polynomial P_μ appears in the linear combination for $\frac{\partial}{\partial\kappa}P_\lambda$ if and only if $\mu < \lambda$ and $\sum\mu_i \leq \sum\lambda_i$.

[4]See Lemmas 4.2.4 and 4.2.5.

Once that we know *which* polynomials appear in the linear combinations, we can ask what the coefficients are, and if there is a “nice” formula for them. Over the summer we came up with several conjectures for these coefficients, but have yet to prove any of them. I will not dwell on these conjectures in this thesis, but will mention one of them briefly to give the reader a sense of their nature. The conjectures are broken down into several cases depending on the number of non-zero parts in λ and μ. For example, one of our conjectures is as follows:

Conjecture 3.3.4. The coefficient of $P_{(\mu_1,0,0)}$ in $\frac{\partial}{\partial\kappa}P_{(\lambda_1,0,0)}$, where $\mu_1 \geq 0$, is

$$c_{(\lambda_1,0,0)}^{(\mu_1,0,0)}(\kappa) = \frac{(-1)^{\lambda_1-\mu_1} \cdot \frac{\lambda_1!}{(\lambda_1-\mu_1)\ \mu_1!} \times \left[(2\kappa+\mu_1)^{\overline{\lambda_1-\mu_1}} + (\kappa+\mu_1)^{\overline{\lambda_1-\mu_1}}\right]}{(\kappa+\mu_1)^{\overline{\lambda_1-\mu_1}}}$$

where $a^{\overline{b}}$ denotes the *rising factorial*:

$$a^{\overline{b}} := (a+0)(a+1)\cdots(a+b-1)$$

which is analogous to the falling factorial introduced in Section 2.1

The original goal of my thesis was to attempt to prove some of the conjectures that we came up with over the summer. However, as I started trying to prove them, I found that it was easier to find a slightly different set of coefficients. In particular, I focused primarily on $\frac{\partial}{\partial\kappa}P_\lambda$ where λ has exactly one non-zero part, and instead of focusing on the number of nonzero parts in μ, I focused on the relationship between those parts and the single non-zero part of λ. The coefficients that I found are sub-cases of our summer conjectures, and it is worth noting that they agree with our conjectures (that is, the expressions in our conjectures don’t contradict the expressions that I found).

Chapter 4

Results

In this chapter we present the results of this thesis. In particular, we provide formulas, with proof, for the coefficient of P_μ in $P_{(D,0,0)}$ for several partitions μ, and then give a general matrix formula for finding the coefficients. The proofs of these formulas rely on finding the coefficient of various monomial symmetric functions, m_μ, in various interpolation Jack polynomials P_λ, so first we provide a general expression for a specific case of such coefficients, and then we move into the proofs of the coefficients of the interpolation Jack polynomials. We will use the following notation throughout this chapter:

- The coefficient of P_μ in $\frac{\partial}{\partial\kappa}P_\lambda$ we denote by $c^\mu_\lambda(\kappa)$ or just c^μ_λ.
- The coefficient of m_μ in P_λ we denote by $b^\mu_\lambda(\kappa)$ or just b^μ_λ.

Note also that unless stated otherwise, all partitions in this chapter have at most three non-zero parts, all polynomials are in three variables, and D always denotes a non-negative integer.

4.1 Coefficients of Monomial Symmetric Functions

4.1.1 Coefficient of $m_{(D-a,a,0)}$ in $P_{(D-b,b,0)}$

Theorem 4.1.1. *Let $a, b \in \mathbb{Z}$ be such that $0 < a < \frac{D}{2}, 0 \leq b < \frac{D}{2}$, and $D-b > D-a$. Then the coefficient of $m_{(D-a,a,0)}$ in $P_{(D-b,b,0)}$ is*

$$b^{(D-a,a,0)}_{(D-b,b,0)}(\kappa) = \frac{(D-2b)^{\underline{a-b}}(\kappa+a-b-1)^{\underline{a-b}}}{(a-b)!(\kappa+D-2b-1)^{\underline{a-b}}}.$$

Proof. Recall that from Formula 2.2 we have

$$P_{(D-b,b,0)}(x_1,x_2,x_3) = \sum_T \Psi_T(\kappa)\Bigg(\prod_{T(s)=1}(x_1 - a'(s) + (\ell'(s)-2)\kappa) \times \prod_{T(s)=2}(x_2 - a'(s) + (\ell'(s)-1)\kappa) \times \prod_{T(s)=3}(x_3 - a'(s) + \ell'(s)\kappa)\Bigg).$$

We want to find the coefficient of the $x_3^{D-a}x_2^a$ term in this expression. Thus we only need to look at terms generated by reverse tableaux with $D-a$ threes and a twos. Since reverse tableaux must be non-increasing along rows and strictly decreasing along columns, we see that the only reverse tableau we are interested in is

$$T = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 3 & \cdots & \cdots & \cdots & 3 & 2 & \cdots & 2 \\ \hline 2 & \cdots & 2 \\ \cline{1-3}\end{array}$$

where there are $D-a$ threes and $a-b$ twos in the top row, and b twos in the bottom row. Note that this way there will be a total of $D-a$ threes and $(a-b)+b = a$ twos, as desired. The coefficient of $x_3^{D-a}x_2^a$ will then be $\Psi_T(\kappa)$ for this T, so let's find this $\Psi_T(\kappa)$. We have

$$\widetilde{\lambda^{(0)}} = T = \widetilde{\lambda^{(1)}}, \qquad \widetilde{\lambda^{(2)}} = \overbrace{\begin{array}{|c|c|c|}\hline 3 & \cdots & 3 \\ \hline\end{array}}^{D-a}, \qquad \widetilde{\lambda^{(3)}} = \emptyset.$$

Thus

$$(R/C)_{\lambda^{(1)}/\lambda^{(0)}} = \emptyset, \qquad (R/C)_{\lambda^{(1)}/\lambda^{(2)}} = \{(1,b+1),\cdots,(1,D-a)\}.$$

Thus

$$\Psi_T(\kappa) = \prod_{s\in(R/C)_{\lambda^{(1)}/\lambda^{(2)}}} \frac{b_{\lambda^{(2)}}(s,\kappa)}{b_{\lambda^{(1)}}(s,\kappa)}.$$

So now let's find $b_{\lambda^{(1)}}(s,\kappa)$ and $b_{\lambda^{(2)}}(s,\kappa)$. If $s=(i,j)\in(R/C)_{\lambda^{(1)}/\lambda^{(2)}}$ then $i=1$ and $b+1\le j\le D-a$, so

$$a_{\lambda^{(1)}}(s)=\lambda_1^{(1)}-j=D-b-j, \qquad a_{\lambda^{(2)}}(s)=\lambda_1^{(2)}-j=D-a-j,$$
$$l_{\lambda^{(1)}}(s)=\left|\{k>1\mid\lambda_k^{(1)}\ge j\}\right|=0, \qquad l_{\lambda^{(2)}}(s)=\left|\{k>1\mid\lambda_k^{(2)}\ge j\}\right|=0.$$

To see why the left side of the second line holds, note that $\lambda_2^{(1)}=b$ and for all s we have $j>b$. Thus

$$b_{\lambda^{(1)}}(s,\kappa)=\frac{a_{\lambda^{(1)}}(s)+\kappa(l_{\lambda^{(1)}}(s)+1)}{a_{\lambda^{(1)}}(s)+\kappa l_{\lambda^{(1)}}(s)+1}=\frac{D-b-j+\kappa}{D-b-j+1},$$
$$b_{\lambda^{(2)}}(s,\kappa)=\frac{a_{\lambda^{(2)}}(s)+\kappa(l_{\lambda^{(2)}}(s)+1)}{a_{\lambda^{(2)}}(s)+\kappa l_{\lambda^{(2)}}(s)+1}=\frac{D-a-j+\kappa}{D-a-j+1}.$$

Plugging this into our formula for $\Psi_T(\kappa)$ we get

$$\begin{aligned}\Psi_T(\kappa)&=\prod_{j=b+1}^{D-a}\frac{b_{\lambda^{(2)}}(s,\kappa)}{b_{\lambda^{(1)}}(s,\kappa)}\\&=\prod_{j=b+1}^{D-a}\frac{\left(\frac{D-a-j+\kappa}{D-a-j+1}\right)}{\left(\frac{D-b-j+\kappa}{D-b-j+1}\right)}\\&=\prod_{j=b+1}^{D-a}\frac{(D-a-j+\kappa)(D-b-j+1)}{(D-b-j+\kappa)(D-a-j+1)}\\&=\frac{(D-2b)^{\underline{a-b}}(\kappa+a-b-1)^{\underline{a-b}}}{(a-b)!(\kappa+D-2b-1)^{\underline{a-b}}}.\end{aligned}$$

The last line here is not obvious. To arrive at this expression, I used Mathematica to simplify the product in the second to last line but was not satisfied with the expression it simplified to. However, using the expression it outputted I then took a guess at an equivalent but cleaner expression might look like, and arrived at the expression in the last line. I then used Mathematica to verify that this solution is correct. This shows the desired result. □

4.2 A General Formula for the c_λ^μ

The main result of this section is the following theorem:

Theorem 4.2.1. *For* $\mu \geq \lambda$ *we have* $c_\lambda^\mu = 0$ *and for* $\mu < \lambda$ *we have*

$$\frac{d}{d\kappa} b_\lambda^\mu = \sum_{\mu \leq \nu < \lambda} c_\lambda^\nu b_\nu^\mu = \sum_{\substack{\mu \leq \nu < \lambda \\ |\nu| \leq |\lambda|}} c_\lambda^\nu b_\nu^\mu. \tag{4.1}$$

Note that the second equality holds because $c_\lambda^\nu = 0$ if $|\nu| > |\lambda|$[1]. We will present a proof of this theorem in a few pages, but first we must prove a few supplementary lemmas.

Lemma 4.2.2. *If* $\mu > \lambda$ *then* m_μ *appears in* P_λ *with a zero coefficient, that is,* $b_\lambda^\mu = 0$.

Proof. Consider the combinatorial formula for P_λ:

$$P_\lambda(x_1, x_2, x_3) = \sum_T \Psi_T(\kappa) \left(\prod_{T(s)=1} (x_1 - a'(s) + (\ell'(s) - 2)\kappa) \right.$$
$$\left. \times \prod_{T(s)=2} (x_2 - a'(s) + (\ell'(s) - 1)\kappa) \prod_{T(s)=3} (x_3 - a'(s) + \ell'(s)\kappa) \right).$$

Let's say $m_{(\mu_1,\mu_2,\mu_3)}$ is the monomial symmetric function in this expression that is highest in lexicographic ordering. Then in particular the term $x_3^{\mu_1} x_2^{\mu_2} x_1^{\mu_3}$ appears in this expression (with some non-zero coefficient). Say this term appears in the expression produced by the reverse tableau T. Then the exponent of x_3 is less than or equal to the number of threes in T. Since μ is maximal in lexicographic ordering, we see that μ_1 must *equal* the number of threes in T, and again since μ is maximal, the number of threes in T must equal λ_1. That is, the entire top row of T is filled with threes. Thus $\mu_1 = \lambda_1$. Similarly, we can show that we must have $\mu_2 = \lambda_2$ and $\mu_3 = \lambda_3$, which is the desired result. □

Lemma 4.2.3. *For any partition* λ (*with three parts*), *the monomial symmetric function* m_λ *appears in* P_λ *with coefficient* 1, *that is,* $b_\lambda^\lambda = 1$.

Note that the proof that follows incorporates Jack polynomials, which are distinct from interpolation Jack polynomials. Jack polynomials are another family of symmetric polynomials indexed by partitions, and unlike interpolation Jack polynomials, they are homogeneous. We will denote the

[1]We will show this in Lemma 4.2.5.

Jack polynomial associated to the partition λ by J_λ. The precise definition is unimportant here, but for more information about Jack polynomials and their properties see [3].

Also note that in this proof we use results from [5], which uses slightly different notation than we do. If the reader is interested in perusing [5], the important notational differences to keep in mind here are:

1. The interpolation Jack polynomial that we refer to as P_λ is denoted by $P_\lambda^{r\delta}$ in [5].

2. The Jack polynomial that we refer to as J_λ is denoted by $P_\lambda^{(1/r)}$ in [5].

Proof. From Corollary 4.7 in [5], the highest degree homogeneous part of the interpolation Jack polynomial P_λ is the Jack polynomial J_λ. Also from [5], J_λ is of the form $m_\lambda + \sum_{\mu<\lambda} a_\mu m_\lambda$. Thus m_λ appears in P_λ with coefficient one, which is the desired result. □

Lemma 4.2.4. *Each differentiated interpolation Jack polynomial $\frac{\partial}{\partial\kappa}P_\lambda$ can be written as a linear combination of P_μ with $\mu < \lambda$.*

Proof. From Lemmas 4.2.2 and 4.2.3, we know that if $\mu \geq \lambda$ then m_μ appears in $\frac{\partial}{\partial\kappa}P_\lambda$ with a zero coefficient. Now consider the following construction of a linear combination for $\frac{\partial}{\partial\kappa}P_\lambda$. First look at the leading term in lexicographic ordering, which by Lemmas 4.2.2 and 4.2.3 we can write as

$$\left(\frac{d}{d\kappa}b_\lambda^\mu(\kappa)\right)m_\mu$$

for some $\mu < \lambda$. Then define

$$P_\lambda^{(1)} = \frac{\partial}{\partial\kappa}P_\lambda - \left(\frac{d}{d\kappa}b_\lambda^\mu(\kappa)\right)P_\mu.$$

We see that $P_\lambda^{(1)}$ is symmetric and every monomial symmetric function m_ν in $P_\lambda^{(1)}$ has $\nu < \mu$. Look at the leading term in lexicographic ordering for $P_\lambda^{(1)}$, and repeat the process. This process will terminate, since at each step we reduce the degree of the polynomial and there are finitely many terms, and will produce a linear combination for $\frac{\partial}{\partial\kappa}P_\lambda$ of the desired form. □

Lemma 4.2.5. *Each differentiated interpolation Jack polynomial $\frac{\partial}{\partial\kappa}P_\lambda$ can be written as a linear combination of P_μ with $|\mu| \leq |\lambda|$.*

Proof. From the proof of Lemma 4.2.3 we know that for any μ, the highest degree homogeneous part of P_μ is a Jack polynomial with leading term m_μ. Since m_μ has degree $|\mu|$, all terms in P_μ must have degree less than or equal to $|\mu|$.

Now consider the process for constructing a linear combination for $\frac{\partial}{\partial\kappa}P_\lambda$ described in the proof of Lemma 4.2.4. From the previous paragraph all monomial symmetric functions m_ν in $\frac{\partial}{\partial\kappa}P_\lambda$ have $|\nu| \leq |\lambda|$, and at each step in this process the only new monomial symmetric functions we can introduce are m_ν such that $|\nu| \leq |\lambda|$. Thus all P_μ in this constructed linear combination must have $|\mu| \leq |\lambda|$, as desired. □

Now we find ourselves in a position to prove Theorem 4.2.1.

Proof. (Of Theorem 4.2.1.) From Lemma 4.2.4, for an interpolation Jack polynomial P_λ we can write

$$\frac{\partial}{\partial\kappa}P_\lambda = \sum_{\nu<\lambda} c_\lambda^\nu P_\nu.$$

Thus in particular $c_\lambda^\mu = 0$ if $\mu \geq \lambda$. Now consider the coefficient of m_μ on both sides of this equation. On the left side we have just $\frac{d}{d\kappa}b_\lambda^\mu$. On the other side we have $\sum_{\nu<\lambda} c_\lambda^\nu b_\nu^\mu$. Thus,

$$\frac{d}{d\kappa}b_\lambda^\mu = \sum_{\nu<\lambda} c_\lambda^\nu b_\nu^\mu.$$

Now, from Lemma 4.2.2 we know that if $\nu < \mu$ then $b_\nu^\mu = 0$. Thus we can rewrite this as the first desired equality:

$$\frac{d}{d\kappa}b_\lambda^\mu = \sum_{\mu\leq\nu<\lambda} c_\lambda^\nu b_\nu^\mu.$$

Furthermore, by Lemma 4.2.5, $c_\lambda^\mu = 0$ if $|\mu| > |\lambda|$, so we can in fact write

$$\frac{d}{d\kappa}b_\lambda^\mu = \sum_{\substack{\mu\leq\nu<\lambda \\ |\nu|\leq|\lambda|}} c_\lambda^\nu b_\nu^\mu.$$

Thus the second equality holds as well. □

In the next three sections, we will find the coefficients $c_{(D,0,0)}^{(D-1,1,0)}$, $c_{(D,0,0)}^{(D-1,0,0)}$, and $c_{(D,0,0)}^{(D-2,2,0)}$, using the method outlined below. First note that it will be convenient to rewrite Equation 4.1 as

$$c_\lambda^\mu = \frac{d}{d\kappa}b_\lambda^\mu - \sum_{\substack{\mu<\nu<\lambda \\ |\mu|\leq|\lambda|}} c_\lambda^\nu b_\nu^\mu \tag{4.2}$$

which we can do since by Lemma 4.2.3 we know that $b_\mu^\mu = 1$. Our procedure to find the coefficient of P_μ and $\frac{\partial}{\partial\kappa}P_\lambda$ is thus as follows.

1. Find the set S_λ^μ of partitions ν such that $\mu < \nu < \lambda$ and $|\mu| \leq |\lambda|$.
2. For each partition $\nu \in S_\lambda^\mu$ find c_λ^ν and b_ν^μ.
3. Find $\frac{d}{d\kappa}b_\lambda^\mu$.
4. Use the coefficients found in steps 1 and 2, along with Equation 4.2 to calculate c_λ^μ.

4.3 Coefficient of $P_{(D-1,1,0)}$ in $\frac{\partial}{\partial\kappa}P_{(D,0,0)}$

The main result of this section is the following theorem:

Theorem 4.3.1. *The coefficient of $P_{(D-1,1,0)}$ in the linear combination for $\frac{\partial}{\partial\kappa}P_{(D,0,0)}$ is*

$$c_{(D,0,0)}^{(D-1,1,0)}(\kappa) = \frac{D(D-1)}{(\kappa+D-1)^2}.$$

In the next two sections we will prove a few intermediate lemmas and find an expression for $b_{(D,0,0)}^{(D-1,1,0)}$, building up to a proof of this theorem.

4.3.1 A Method for Finding $c_{(D,0,0)}^{(D-1,1,0)}$

The main result of this section is the following theorem:

Theorem 4.3.2. *The coefficient of $P_{(D-1,1,0)}$ in the linear combination for $\frac{\partial}{\partial\kappa}P_{(D,0,0)}$ is*

$$c_{(D,0,0)}^{(D-1,1,0)} = \frac{d}{d\kappa}b_{(D,0,0)}^{(D-1,1,0)}.$$

To prove this theorem we first prove the following two lemmas:

Lemma 4.3.3. *Let D be some positive integer and let $\mu = (D-1,1,0)$. If ν is a partition such that $\nu > \mu$ then $|\nu| \geq D$.*

Proof. Let $\nu = (\nu_1, \nu_2, \nu_3)$ be a partition such that $\nu > \mu$. We have a few cases to consider. If $\nu_1 \geq D$ then we clearly have $|\nu| \geq D$. Otherwise we must have $\nu_1 = D-1$. In this case we must have $\nu_2 \geq 1$, so $|\nu| \geq D-1+1 = D$. This shows the desired result. □

Lemma 4.3.4. *Let D be some positive integer and let $\mu = (D-1,1,0)$. If $\nu = (\nu_1, \nu_2, \nu_3)$ is a partition such that $\nu > \mu$ and $|\nu| = D$ then $\nu = (D,0,0)$.*

Proof. Let $\nu = (\nu_1, \nu_2, \nu_3)$. First note that we must have $\nu_1 \geq D-1$ since $\nu > \mu$, and $\nu_1 \leq D$ since $|\nu| = D$. Thus $\nu_1 = D-1$ or $\nu_1 = D$. If $\nu_1 = D$ then $\nu_2 = \nu_3 = 0$, so $\nu = (D,0,0)$. If $\nu_1 = D-1$, then $\nu_2 = 1$ and $\nu_3 = 0$ since $|\nu| = D$. But then $\nu = \mu$, which is a contradiction.Thus $\nu = (D,0,0)$, as desired. □

We are now in a position to prove Theorem 4.3.2.

Proof. (Of Theorem 4.3.2.) From Equation 4.2 we know that

$$c_{(D,0,0)}^{(D-1,1,0)} = \frac{d}{d\kappa} b_{(D,0,0)}^{(D-1,1,0)} - \sum_{\substack{(D-1,1,0)<\nu<(D,0,0) \\ |\nu| \leq D}} c_{(D,0,0)}^{\nu} b_{\nu}^{(D-1,1,0)}.$$

Then from Lemma 4.3.3 we know that if $\nu > (D-1,1,0)$ then $|\nu| \geq D$, so we can rewrite this as

$$c_{(D,0,0)}^{(D-1,1,0)} = \frac{d}{d\kappa} b_{(D,0,0)}^{(D-1,1,0)} - \sum_{\substack{(D-1,1,0)<\nu<(D,0,0) \\ |\nu| = D}} c_{(D,0,0)}^{\nu} b_{\nu}^{(D-1,1,0)}.$$

But then by Lemma 4.3.4 we know that if $\nu > (D-1,1,0)$ and $|\nu| = D$ then $\nu = (D,0,0)$. Thus in fact there are no terms in the above sum, and so we just get

$$c_{(D,0,0)}^{(D-1,1,0)} = \frac{d}{d\kappa} b_{(D,0,0)}^{(D-1,1,0)}.$$

□

4.3.2 A Formula for $c_{(D,0,0)}^{(D-1,1,0)}$

This section is devoted to finding an explicit formula for the coefficient of $P_{(D-1,1,0)}$ in $\frac{\partial}{\partial\kappa}P_{(D,0,0)}$. From Section 4.3.1, we know that the desired coefficient is the derivative of the coefficient $b_{(D,0,0)}^{(D-1,1,0)}$ of $m_{(D-1,1,0)}$ in $P_{(D,0,0)}$, so it is sufficient to find a formula for $b_{(D,0,0)}^{(D-1,1,0)}$

A Formula for $b_{(D,0,0)}^{(D-1,1,0)}$

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Theorem 4.3.5. *For $D \geq 2$, the coefficient of $m_{(D-1,1,0)}$ in $P_{(D,0,0)}$ is*

$$b_{(D,0,0)}^{(D-1,1,0)}(\kappa) = \frac{D\kappa}{\kappa + D - 1}.$$

Proof. From Theorem 4.1.1, for $a, b \in \mathbb{Z}$ such that $0 < a < \frac{D}{2}, 0 \leq b < \frac{D}{2}$, and $D - b > D - a$ we have

$$b_{(D-b,b,0)}^{(D-a,a,0)}(\kappa) = \frac{(D-2b)^{\underline{a-b}}(\kappa + a - b - 1)^{\underline{a-b}}}{(a-b!)(\kappa + D - 2b - 1)^{\underline{a-b}}}.$$

Setting $a = 1$ and $b = 0$ we then see that

$$\begin{aligned} b_{(D,0,0)}^{(D-1,1,0)}(\kappa) &= \frac{(D-2(0))^{\underline{1-0}}(\kappa + 1 - 0 - 1)^{\underline{1-0}}}{(1-0)!(\kappa + D - 2(0) - 1)^{\underline{1-0}}} \\ &= \frac{D\kappa}{\kappa + D - 1} \end{aligned}$$

as desired. For the interested reader, I have also included a proof of this formula that doesn't rely on Theorem 4.1.1 but instead uses Formula 2.1. □

Proof of Theorem 4.3.1

From the previous section we know that

$$b_{(D,0,0)}^{(D-1,1,0)}(\kappa) = \frac{D\kappa}{\kappa + D - 1}$$

and by Theorem 4.3.2 we know that

$$c_{(D,0,0)}^{(D-1,1,0)} = \frac{d}{d\kappa} b_{(D,0,0)}^{(D-1,1,0)}.$$

Theorem 4.3.1 follows directly from these results and an application of the quotient rule for derivatives.

4.4 Coefficient of $P_{(D-1,0,0)}$ in $\frac{\partial}{\partial\kappa}P_{(D,0,0)}$

We now move on to the next partition in lexicographic ordering, $(D-1,0,0)$. In this section I will find a formula for the coefficient of $P_{(D-1,0,0)}$ in the linear combination for $\frac{\partial}{\partial\kappa}P_{(D,0,0)}$.

Theorem 4.4.1. *For $D \geq 1$, the coefficient of $P_{(D-1,0,0)}$ in the linear combination for $\frac{\partial}{\partial\kappa}P_{(D,0,0)}$ is*

$$c_{(D,0,0)}^{(D-1,0,0)} = \frac{-3Dk - 2D(D-1)}{\kappa + D - 1}.$$

As before, in the next few sections I will prove an intermediate lemma and find expressions for the coefficient of $m_{(D-1,0,0)}$ in $P_{(D-1,1,0)}$ and in $P_{(D,0,0)}$, building up to a proof of this theorem.

4.4.1 A Method for Finding $c_{(D,0,0)}^{(D-1,0,0)}$

The main result of this section is the following theorem:

Theorem 4.4.2. *For $D \geq 1$ we have*

$$c_{(D,0,0)}^{(D-1,0,0)}(\kappa) = \frac{d}{d\kappa}b_{(D,0,0)}^{(D-1,0,0)} - c_{(D,0,0)}^{(D-1,1,0)}b_{(D-1,1,0)}^{(D-1,0,0)}.$$

Before we prove this theorem, we must have the following lemma:

Lemma 4.4.3. *Let $\lambda = (D-1,1,0)$ and $\nu = (D-1,0,0)$. Then there are no partitions μ such that $\lambda > \mu > \nu$.*

Proof. Let $\mu = (\mu_1, \mu_2, \mu_3)$ and suppose $\mu > \nu$. Then $\mu_1 \geq D-1$. If $\mu_1 > D-1$ then $\mu > \lambda$ so we must have $\mu_1 = D-1$. Then we must also have $\mu_2 \geq 0$. If $\mu_2 = 0$ then $\mu_3 = 0$ which can't happen because $\mu > \nu$, so we must have $\mu_2 > 0$. If $\mu_2 > 1$ then $\mu > \lambda$ so in fact we must have $\mu_2 = 1$. Finally, we then have either $\mu_3 = 0$ or $\mu_3 = 1$. The latter cannot happen because then we would have $\mu > \lambda$. Thus we must have $\mu = (D-1,1,0) = \lambda$, which shows the desired result. □

Using this and the results of Section 4.3.1, we can now prove Theorem 4.4.2.

Proof. (Of Theorem 4.4.2.) By Equation 4.2, we know that

$$c_{(D,0,0)}^{(D-1,10,0)} = \frac{d}{d\kappa} b_{(D,0,0)}^{(D-1,0,0)} - \sum_{\substack{(D-1,0,0)<\nu<(D,0,0)\\ |\nu|\leq D}} c_{(D,0,0)}^{\nu} b_{\nu}^{(D-1,0,0)}.$$

Now let ν be such that $(D-1,1,0) < \nu < (D,0,0)$ and $|\nu| \leq D$. Then by Lemma 4.4.3, $\nu \geq (D-1,1,0)$. But then from the proof of Theorem 4.3.2 we know that $\nu = (D-1,1,0)$. Thus

$$c_{(D,0,0)}^{(D-1,0,0)} = \frac{d}{d\kappa} b_{(D,0,0)}^{(D-1,0,0)} - c_{(D,0,0)}^{(D-1,1,0)} b_{(D-1,1,0)}^{(D-1,0,0)}.$$

□

4.4.2 A Formula for $c_{(D,0,0)}^{(D-1,0,0)}$

This section will be dedicated to finding an explicit formula for the coefficient of $P_{(D-1,0,0)}$ in $\frac{\partial}{\partial\kappa}P_{(D,0,0)}$. It will be split into several sections for ease of reading.

A Formula for $b_{(D-1,1,0)}^{(D-1,0,0)}$

Theorem 4.4.4. *or $D \geq 2$, the coefficient of $m_{(D-1,0,0)}$ in $P_{(D-1,1,0)}$ is*

$$b_{(D-1,1,0)}^{(D-1,0,0)}(\kappa) = -\kappa.$$

Proof. Recall Formula 2.2:

$$P_\lambda(x_1,x_2,x_3) = \sum_T \Psi_T(\kappa)\Bigg(\prod_{T(s)=1}(x_1 - a'(s) + (\ell'(s)-2)\kappa) \times \prod_{T(s)=2}(x_2 - a'(s) + (\ell'(s)-1)\kappa) \prod_{T(s)=3}(x_3 - a'(s) + \ell'(s)\kappa)\Bigg).$$

Now we want to find the coefficient of the $m_{(D-1,0,0)}$ term in this polynomial. To find this coefficient it is sufficient to find the coefficient of the x_3^{D-1}, so we will focus on that here. From the combinatorial formula we see that a reverse tableau T will contribute to the x_3^{D-1} term if and only if it has $D-1$ boxes filled with 3. But our reverse tableaux are of the shape

$$\overbrace{\begin{array}{|c|c|c|c|}\hline & & \cdots & \\ \hline & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}}^{D-1}$$

where the top row has $D-1$ boxes. Reverse tableau are *strictly* decreasing down the columns, so all $D-1$ of the threes must be in the top row and the box in the second row must be either a one or a two. That is, we only need to look at the terms contributed by two specific tableaux:

$$T_1 = \begin{array}{|c|c|c|c|}\hline 3 & 3 & \cdots & 3 \\ \hline 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \qquad \text{and} \qquad T_2 = \begin{array}{|c|c|c|c|}\hline 3 & 3 & \cdots & 3 \\ \hline 2 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}\,.$$

First let's look at the term contributed by T_2. Recall that for a box $s = (i, j)$ in a reverse tableau, $a'(s) = j - 1$ and $\ell'(s) = i - 1$. Thus in this case we get

$$\begin{aligned} &\prod_{T_2(s)=1} (x_1 - a'(s) + (\ell'(s) - 2)\kappa) \times \prod_{T_2(s)=2} (x_2 - a'(s) + (\ell'(s) - 1)\kappa) \\ &\qquad \times \prod_{T_2(s)=3} (x_3 - a'(s) + \ell'(s)\kappa) \\ &\qquad = x_2 \prod_{T_1(s)=3} (x_3 - a'(s) + \ell'(s)\kappa)\,. \end{aligned}$$

Thus in fact this reverse tableau will not contribute an x_3^{D-1} term at all, since all terms it contributes will have an x_2.

So now let's look at the term contributed by T_1. In this case we get a term

$$(x_1 - a'(s_0) + (\ell'(s_0) - 2)\kappa) \prod_{T_1(s)=3} (x_3 - a'(s) + \ell'(s)\kappa) = (x_1 - \kappa) \prod_{j=0}^{D-2} (x_3 - j)$$

where s_0 denotes the box in the second row. We see that from this we will get a term $-\kappa x_3^{D-1}$. Thus the coefficient of x_3^{D-1} in $P_{(D-1,1,0)}$ is $-\kappa \Psi_{T_1}(\kappa)$. So let's find $\Psi_{T_1}(\kappa)$. We see that

$$\widetilde{\lambda^{(0)}} = T_1, \qquad \widetilde{\lambda^{(1)}} = \overbrace{\begin{array}{|c|c|c|}\hline 3 & \cdots & 3 \\ \hline \end{array}}^{D-1} = \widetilde{\lambda^{(2)}}, \qquad \widetilde{\lambda^{(3)}} = \emptyset.$$

And thus $(R/C)_{\lambda^{(i-1)}/\lambda^{(i)}} = \emptyset$ for $i = 1, 2$. Thus $\Psi_{T_1}(\kappa)$, so the coefficient of $m_{(D-1,0,0)}$ in $P_{(D-1,1,0)}$ is $-\kappa$. $\square$

A Formula for $b_{(D,0,0)}^{(D-1,0,0)}$

Theorem 4.4.5. *or $D \geq 1$, the coefficient of $m_{(D-1,0,0)}$ in $P_{(D,0,0)}$ is*

$$b_{(D,0,0)}^{(D-1,0,0)}(\kappa) = \frac{-6D\kappa^2 - 5D(D-1)\kappa - D(D-1)^2}{2(\kappa + D - 1)}.$$

From Formula 2.1 we have the following formula for $P_{(D,0,0)}$:

$$P_{(D,0,0)}(x_1, x_2, x_3) = \binom{-\kappa}{D}^{-1} \sum_{i_j} \prod_{j=1}^{3} \left[\binom{-\kappa}{i_{j-1} - i_j} (x_j - \kappa\delta_j - i_j)^{\underline{i_{j-1}-i_j}} \right]$$

where $\delta = (\delta_1, \delta_2, \delta_3) = (2, 1, 0)$, and the sum runs through all integer sequences $D = i_0 \geq i_1 \geq i_2 \geq i_3 = 0$. We want to find the coefficient of $m_{(D-1,0,0)}$, but since $P_{(D,0,0)}$ is symmetric it is sufficient to find the coefficient of x_1^{D-1}. Thus we want to find all sequences i_j where the product over j produces an x_1^{D-1} term. We see that to produce such a term we must have $i_0 - i_1 = D$ or $i_0 - i_1 = D - 1$. Let's consider the case where $i_0 - i_1 = D - 1$ first.

Case 1:

Suppose $i_0 - i_1 = D - 1$. Then $i_0 = D$ and $i_1 = 1$. This gives us two sequences to consider: $(D, 1, 1, 0)$ and $(D, 1, 0, 0)$. For the first sequence we get a term

$$\begin{aligned}
&\prod_{j=1}^{3} \left[\binom{-\kappa}{i_{j-1} - i_j} (x_j - \kappa\delta_j - i_j)^{\underline{i_{j-1}-i_j}} \right] \\
&\qquad = \left[\binom{-\kappa}{D-1} (x_1 - 2\kappa - 1)^{\underline{D-1}} \right] \left[\binom{-\kappa}{0} (x_2 - \kappa - 1)^{\underline{0}} \right] \left[\binom{-\kappa}{1} x_3^{\underline{1}} \right] \\
&\qquad = \left[\binom{-\kappa}{D-1} (x_1 - 2\kappa - 1)^{\underline{D-1}} \right] \left[\binom{-\kappa}{1} x_3 \right].
\end{aligned}$$

Every term in this product will be multiplied by x_3. Since we are only interested in x_1^{D-1} terms, we don't actually have to consider any terms produced by this sequence.

Now for the second sequence, $(D, 1, 0, 0)$ we get the term

$$\begin{aligned}
\prod_{j=1}^{3} & \left[\binom{-\kappa}{i_{j-1} - i_j} (x_j - \kappa\delta_j - i_j)^{\underline{i_{j-1}-i_j}} \right] \\
&= \left[\binom{-\kappa}{D-1} (x_1 - 2\kappa - 1)^{\underline{D-1}} \right] \left[\binom{-\kappa}{1} (x_2 - \kappa)^{\underline{1}} \right] \left[\binom{-\kappa}{0} x_3^{\underline{0}} \right] \\
&= \left[\binom{-\kappa}{i_{j-1} - i_j} (x_j - \kappa\delta_j - i_j)^{\underline{i_{j-1}-i_j}} \right] \\
&= \left[\binom{-\kappa}{D-1} (x_1 - 2\kappa - 1)^{\underline{D-1}} \right] \left[\binom{-\kappa}{1} (x_2 - \kappa) \right].
\end{aligned}$$

The coefficient of the x_1^{D-1} factor that we get from this expression is

$$\begin{aligned}
d_{(D,1,0,0)} &= \binom{-\kappa}{D-1} \binom{-\kappa}{1} (-\kappa) \\
&= \binom{-\kappa}{D-1} \left(\kappa^2 \right).
\end{aligned}$$

Case 2:
Now suppose $i_0 - i_1 = D$. Since $i_0 = D$ by definition, this forces $i_1 = 0$, which in turn forces $i_2 = 0$. Since $i_3 = 0$ by definition, we see that we only have to consider the sequence $(D, 0, 0, 0)$. The product term from this sequence is

$$\begin{aligned}
\prod_{j=1}^{3} & \left[\binom{-\kappa}{i_{j-1} - i_j} (x_j - \kappa\delta_j - i_j)^{\underline{i_{j-1}-i_j}} \right] \\
&= \left[\binom{-\kappa}{D} (x_1 - 2\kappa)^{\underline{D}} \right] \left[\binom{-\kappa}{0} (x_2 - \kappa)^{\underline{0}} \right] \left[\binom{-\kappa}{0} x_3^{\underline{0}} \right] \\
&= \binom{-\kappa}{D} (x_1 - 2\kappa)^{\underline{D}} \\
&= \binom{-\kappa}{D} (x_1 - 2\kappa)(x_1 - 2\kappa - 1) \cdots (x_1 - 2\kappa - (D-1))
\end{aligned}$$

and so the coefficient of x_1^{D-1} from this term is

$$\begin{aligned} d_{(D,0,0,0)} &= \binom{-\kappa}{D}(-2\kappa - (2\kappa+1) - \cdots - (2\kappa + (D-1)) \\ &= \binom{-\kappa}{D}\left(-2D\kappa - \sum_{\ell=0}^{D-1}\ell\right) \\ &= \binom{-\kappa}{D}\left(-2D\kappa - \frac{D(D-1)}{2}\right). \end{aligned}$$

This completes case 2.

Thus the total coefficient of x_1^{D-1} in $P_{(D,0,0)}$ is

$$\begin{aligned} \binom{-\kappa}{D}^{-1}(d_{(D,0,0,0)} + d_{(D,1,0,0)}) &= \binom{-\kappa}{D}^{-1}\left(\binom{-\kappa}{D}\left(-2D\kappa - \frac{D(D-1)}{2}\right) + \binom{-\kappa}{D-1}(\kappa^2)\right) \\ &= -2D\kappa - \frac{D(D-1)}{2} + \binom{-\kappa}{D}^{-1}\binom{-\kappa}{D-1}(\kappa^2) \\ &= -2D\kappa - \frac{D(D-1)}{2} + \left(\frac{D!}{(-1)^D\kappa^{\underline{D}}}\right)\left(\frac{(-1)^{D-1}\kappa^{\underline{D-1}}}{(D-1)!}\right)(\kappa^2) \\ &= -2D\kappa - \frac{D(D-1)}{2} - \frac{D\kappa^2}{\kappa+D-1} \\ &= \frac{-6D\kappa^2 - 5D(D-1)\kappa - D(D-1)^2}{2(\kappa+D-1)}. \end{aligned}$$

Note that the coefficients in the numerator will be even for all D, so in fact the 2 in the denominator will "cancel".

Proof of Theorem 4.4.1

From the previous two sections we have formulas for the coefficients of $m_{(D-1,0,0)}$ in $P_{(D,0,0)}$ and in $P_{(D-1,1,0)}$. Also, we know that the coefficient of $m_{(D-1,0,0)}$ in $P_{(D-1,0,0)}$ is one, and that the coefficient of $P_{(D-1,1,0)}$ in the linear combination for $\frac{\partial}{\partial\kappa}P_{(D,0,0)}$ is

$$c_{(D,0,0)}^{(D-1,1,0)}(\kappa) = \frac{D(D-1)}{(\kappa+D-1)^2}.$$

Then by Theorem 4.4.2 we have

$$c_{(D,0,0)}^{(D-1,0,0)}(\kappa) = \frac{d}{d\kappa}b_{(D,0,0)}^{(D-1,0,0)} - c_{(D,0,0)}^{(D-1,1,0)}b_{(D-1,1,0)}^{(D-1,0,0)}.$$

Substituting our expressions for $b^{(D-1,0,0)}_{(D,0,0)}, c^{(D-1,1,0)}_{(D,0,0)}, b^{(D-1,0,0)}_{(D-1,1,0)}$ and using Mathematica to simplify the resulting expression we get

$$c^{(D-1,0,0)}_{(D,0,0)} = \frac{-3D\kappa - 2D(D-1)}{\kappa + D - 1}.$$

4.5 Coefficient of $P_{(D-2,2,0)}$ in $\frac{\partial}{\partial\kappa}P_{(D,0,0)}$

Here I present a proof of the following theorem:

Theorem 4.5.1. *For $D \geq 4$, the coefficient of $P_{(D-2,2,0)}$ in $\frac{\partial}{\partial\kappa}P_{(D,0,0)}$ is*

$$c^{(D-2,2,0)}_{(D,0,0)} = \frac{D(D-1)(D-2)(D-3)}{2(\kappa + D - 3)(\kappa + D - 2)^2(\kappa + D - 1)}.$$

Again, in the next few sections I will prove an intermediate lemma and find the coefficient of $m_{(D-2,2,0)}$ in $P_{(D,0,0)}$ and $P_{(D-1,1,0)}$, building up to a proof of this theorem.

4.5.1 A Method for Finding $c^{(D-2,2,0)}_{(D,0,0)}$

The main result of this section is the following theorem:

Theorem 4.5.2. *For $D \geq 4$ we have*

$$c^{(D-2,2,0)}_{(D,0,0)} = \frac{d}{d\kappa}b^{(D-2,2,0)}_{(D,0,0)} - c^{(D-1,1,0)}_{(D,0,0)}b^{(D-2,2,0)}_{(D-1,1,0)}.$$

To prove this theorem we will need the following lemma.

Lemma 4.5.3. *Let $\lambda = (D-1,0,0)$ and $\nu = (D-2,2,0)$. Then there are no partitions $\mu = (\mu_1, \mu_2, \mu_3)$ such that $\lambda > \mu > \nu$ and $\sum \mu_i \leq D$.*

Proof. Suppose $\mu > \nu$. Then $\mu_1 \geq D-2$. If $\mu_1 > D-2$ then $\mu > \nu$ so $\mu_1 = D-1$. Then we must have $\mu_2 = \mu_3 = 0$ because $\mu \leq \lambda$. Thus $\mu = \nu$.
Now suppose instead that we have $\mu_1 = D-2$. Then $\mu_2 \geq 2$ since $\mu \geq \nu$. If $\mu_2 > 2$ then

$$\sum \mu_i \geq \mu_1 + \mu_2 > D,$$

which is a contradiction. Thus $\mu_2 = 2$. This forces $\mu_3 = 0$ by similar reasoning, so $\mu = \nu$. This shows the desired result. □

We can now prove Theorem 4.5.2.

Proof. (Of Theorem 4.5.2.) From Equation 4.2, we have

$$c_{(D,0,0)}^{(D-2,2,0)} = \frac{d}{d\kappa}b_{(D,0,0)}^{(D-2,2,0)} - \sum_{\substack{(D-2,2,0)<\nu<(D,0,0)\\|\nu|\leq D}} c_{(D,0,0)}^{\nu} b_{\nu}^{(D-2,2,0)}.$$

So we need to find all partitions ν such that $(D-2,2,0) < \nu < (D,0,0)$ and $|\nu| \leq D$. First, from Lemma 4.5.3 we have $\nu \geq (D-1,0,0)$. Then from Lemma 4.4.3 we know that $\nu = (D-1,0,0)$ or $\nu \geq (D-1,1,0)$. Finally, from the results of Section 4.3.1 we know that if $\nu \geq (D-1,1,0)$ then $\nu = (D-1,1,0)$. Thus the only partitions ν with $(D-2,2,0) < \nu < (D,0,0)$ and $|\nu| \leq D$ are $(D-1,0,0)$ and $(D-1,1,0)$. Thus

$$c_{(D,0,0)}^{(D-2,2,0)} = \frac{d}{d\kappa}b_{(D,0,0)}^{(D-2,2,0)} - c_{(D,0,0)}^{(D-1,0,0)}b_{(D-1,0,0)}^{(D-2,2,0)} - c_{(D,0,0)}^{(D-1,1,0)}b_{(D-1,1,0)}^{(D-2,2,0)}.$$

Finally, from Lemma 4.2.5 we know that $c_{(D-1,0,0)}^{(D-2,2,0)} = 0$. Thus

$$c_{(D,0,0)}^{(D-2,2,0)} = \frac{d}{d\kappa}b_{(D,0,0)}^{(D-2,2,0)} - c_{(D,0,0)}^{(D-1,1,0)}b_{(D-1,1,0)}^{(D-2,2,0)}.$$

□

4.5.2 A Formula for $c_{(D,0,0)}^{(D-2,2,0)}$

In this section we find an expression for the coefficient of $P_{(D-2,2,0)}$ in the linear combination for $\frac{\partial}{\partial\kappa}P_{(D,0,0)}$. We break this down into a couple subsections to find the coefficient of $m_{(D-2,2,0)}$ in $P_{(D,0,0)}$ and $P_{(D-1,1,0)}$.

A Formula for $b_{(D,0,0)}^{(D-2,2,0)}$

Theorem 4.5.4. *For $D \geq 4$ the coefficient of $m_{(D-2,2,0)}$ in $P_{(D,0,0)}$ is*

$$b_{(D,0,0)}^{(D-2,2,0)}(\kappa) = \frac{D(D-1)(\kappa)(\kappa+1)}{2(\kappa+D-1)(\kappa+D-2)}.$$

Proof. From Theorem 4.1.1, for $a, b \in \mathbb{Z}$ such that $0 < a < \frac{D}{2}, 0 \leq b < \frac{D}{2}$, and $D-b > D-a$ we have

$$b_{(D-b,b,0)}^{(D-a,a,0)}(\kappa) = \frac{(D-2b)^{\underline{a-b}}(\kappa+a-b-1)^{\underline{a-b}}}{(a-b!)(\kappa+D-2b-1)^{\underline{a-b}}}.$$

Setting $a = 2$ and $b = 0$ we then see that

$$b^{(D-2,2,0)}_{(D,0,0)}(\kappa) = \frac{(D-2(0))^{\underline{2-0}}(\kappa+2-0-1)^{\underline{2-0}}}{(2-0)!(\kappa+D-2(0)-1)^{\underline{2-0}}}$$
$$= \frac{D(D-1)(\kappa)(\kappa+1)}{2(\kappa+D-1)(\kappa+D-2)}$$

as desired. For the interested reader, I have also included a proof of this formula that doesn't rely on Theorem 4.1.1 but instead uses Formula 2.1. □

A Formula for $b^{(D-2,2,0)}_{(D-1,1,0)}$

Theorem 4.5.5. *The coefficient of $m_{(D-2,2,0)}$ in $P_{(D-1,1,0)}$ is*

$$b^{(D-2,2,0)}_{(D-1,1,0)}(\kappa) = \frac{(D-2)\kappa}{D-3-\kappa}.$$

Proof. From Theorem 4.1.1, for $a, b \in \mathbb{Z}$ such that $0 < a < \frac{D}{2}$, $0 \leq b < \frac{D}{2}$, and $D - b > D - a$ we have

$$b^{(D-a,a,0)}_{(D-b,b,0)}(\kappa) = \frac{(D-2b)^{\underline{a-b}}(\kappa+a-b-1)^{\underline{a-b}}}{(a-b!)(\kappa+D-2b-1)^{\underline{a-b}}}.$$

Setting $a = 2$ and $b = 1$ we then see that

$$b^{(D-2,2,0)}_{(D-1,1,0)}(\kappa) = \frac{(D-2(1))^{\underline{2-1}}(\kappa+2-1-1)^{\underline{2-1}}}{(2-1)!(\kappa+D-2(1)-1)^{\underline{2-1}}}$$
$$= \frac{(D-2)\kappa}{\kappa+D-3}$$

as desired. For the interested reader, I have also included a proof of this formula that doesn't rely on Theorem 4.1.1 but instead uses Formula 2.1. □

Proof of Theorem 4.5.1

From the previous two sections we have formulas for the coefficients of $m_{(D-2,2,0)}$ in $P_{(D-1,1,0)}$ and in $P_{(D,0,0)}$. Also, we know that the coefficient of $m_{(D-2,2,0)}$ in $P_{(D-2,2,0)}$ is one, and that the coefficient of $P_{(D-1,1,0)}$ in the linear combination for $\frac{\partial}{\partial\kappa}P_{(D,0,0)}$ is

$$c^{(D-1,1,0)}_{(D,0,0)}(\kappa) = \frac{D(D-1)}{(\kappa+D-1)^2}.$$

Then by Theorem 4.5.2 we have

$$c_{(D,0,0)}^{(D-2,2,0)} = \frac{d}{d\kappa} b_{(D,0,0)}^{(D-2,2,0)} - c_{(D,0,0)}^{(D-1,1,0)} b_{(D-1,1,0)}^{(D-2,2,0)}.$$

Substituting in our expressions for $b_{(D,0,0)}^{(D-2,2,0)}$, $c_{(D,0,0)}^{(D-1,1,0)}$ and $b_{(D-1,1,0)}^{(D-2,2,0)}$, and using Mathematica to simplify the resulting expression we get

$$c_{(D,0,0)}^{(D-2,2,0)}(\kappa) = \frac{D(D-1)(D-2)(D-3)}{2(\kappa+D-1)(\kappa+D-2)^2(\kappa+D-3)}.$$

4.6 A Further Conjecture

In the previous sections, we found that the coefficients of $P_{(D-1,1,0)}$ and $P_{(D-2,2,0)}$ in $\frac{\partial}{\partial\kappa} P_{(D,0,0)}$ are

$$c_{(D,0,0)}^{(D-1,1,0)}(\kappa) = \frac{D(D-1)}{(\kappa+D-1)^2},$$
$$c_{(D,0,0)}^{(D-2,2,0)}(\kappa) = \frac{D(D-1)(D-2)(D-3)}{2(\kappa+D-3)(\kappa+D-2)^2(\kappa+D-1)}.$$

From this it is reasonable to form the following conjecture:

Conjecture 4.6.1. *For $0 < a < \frac{D}{2}$, the coefficient of $P_{(D-a,a,0)}$ in $\frac{\partial}{\partial\kappa} P_{(D,0,0)}$ is*

$$c_{(D,0,0)}^{(D-a,a,0)}(\kappa) = \frac{D^{\underline{2a}}}{a(\kappa+D-1)^{\underline{a}}(\kappa+D-a)^{\underline{a}}}.$$

Note that this conjecture has been tested in Sage up to $D = 15$.

4.7 A Matrix Formula for Coefficients

In this section we present a matrix formula for the coefficients c_λ^μ in terms of the coefficients b_λ^μ. First recall Theorem 4.2.1:

Theorem 4.2.1. *For $\mu \geq \lambda$ we have $c_\lambda^\mu = 0$ and for $\mu < \lambda$ we have*

$$\frac{d}{d\kappa} b_\lambda^\mu = \sum_{\mu \leq \nu < \lambda} c_\lambda^\nu b_\nu^\mu = \sum_{\substack{\mu \leq \nu < \lambda \\ |\nu| \leq |\lambda|}} c_\lambda^\nu b_\nu^\mu. \tag{4.1}$$

We can restate this theorem as a statement about matrices, in the following way. Note that since partitions of length three are totally ordered by lexicographic ordering, if we have a matrix we can index its rows and columns by partitions instead of integers. That is, we refer to the (i, j) entry as the (μ, λ) entry, where μ and λ are the i^{th} and j^{th} partitions, respectively, in lexicographic ordering.

Now let B and C be the matrices of infinite dimensions such that the (μ, λ) entries are b^{μ}_{λ} and c^{μ}_{λ}, respectively. For example, the first five partitions in lexicographic ordering are

$$(0,0,0),(1,0,0),(1,1,0),(1,1,1),(2,0,0)$$

so the $(2, 3)$ entry in B is $b^{(1,0,0)}_{(1,1,0)}(\kappa)$. Furthermore, let B' denote the entry-wise derivative of B with respect to κ. Then we have the following theorem:

Theorem 4.7.1. *We can write*

$$B' = BC.$$

Furthermore, B has a left inverse, B^{-1}, and thus

$$C = B^{-1}B'.$$

Proof. First note that by Lemmas 4.2.2 and 4.2.3, B is an upper triangular matrix with ones along the diagonal, and by Lemma 4.2.4, C is strictly upper triangular. That is, we can write

$$B = \begin{bmatrix} 1 & b^{(0,0,0)}_{(1,0,0)} & b^{(0,0,0)}_{(1,1,0)} & b^{(0,0,0)}_{(1,1,1)} & \cdots \\ 0 & 1 & b^{(1,0,0)}_{(1,1,0)} & b^{(1,0,0)}_{(1,1,1)} & \cdots \\ 0 & 0 & 1 & b^{(1,1,0)}_{(1,1,1)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & c^{(0,0,0)}_{(1,0,0)} & c^{(0,0,0)}_{(1,1,0)} & c^{(0,0,0)}_{(1,1,1)} & \cdots \\ 0 & 0 & c^{(1,0,0)}_{(1,1,0)} & c^{(1,0,0)}_{(1,1,1)} & \cdots \\ 0 & 0 & 0 & c^{(1,1,0)}_{(1,1,1)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let $D = BC$, and let d^{μ}_{λ} be the (μ, λ) entry in D. Then d^{μ}_{λ} is the μ^{th} row of B dotted with the λ^{th} column of C:

$$\begin{aligned} d^{\mu}_{\lambda} &= \left(\sum_{\nu<\mu}(0)c^{\nu}_{\lambda}\right) + c^{\mu}_{\lambda} + \left(\sum_{\mu<\nu<\lambda} b^{\mu}_{\nu}c^{\nu}_{\lambda}\right) + \left(\sum_{\nu\geq\lambda}(0)b^{\mu}_{\nu}\right) \\ &= c^{\mu}_{\lambda} + \left(\sum_{\mu<\nu<\lambda} b^{\mu}_{\nu}c^{\nu}_{\lambda}\right) \\ &= \sum_{\mu\leq\nu<\lambda} b^{\mu}_{\nu}c^{\nu}_{\lambda} \\ &= \frac{d}{d\kappa}b^{\mu}_{\lambda}. \end{aligned}$$

Thus the (μ, λ) entry of D is the (μ, λ) entry of $\frac{d}{dk}B$, and so

$$B' = BC$$

as desired. Now, we would like to multiply both sides on the left by the left inverse of B. However, before we can do this we must show that the left inverse of B exists. We can find this inverse as follows. We want to find a matrix A, with (μ, λ) entry a^{μ}_{λ}, such that $AB = I$. Consider some row of A, say the μ^{th} row. Since we want $AB = I$, we must have

$$\begin{aligned} a^{\mu}_{(0,0,0)} &= \delta_{\mu,(0,0,0)} \\ a^{\mu}_{(0,0,0)}b^{(0,0,0)}_{(1,0,0)} + a^{\mu}_{(1,0,0)} &= \delta_{\mu,(1,0,0)} \\ a^{\mu}_{(0,0,0)}b^{(0,0,0)}_{(1,1,0)} + a^{\mu}_{(1,0,0)}b^{(1,0,0)}_{(1,1,0)} + a^{\mu}_{(1,1,0)} &= \delta_{\mu,(1,1,0)} \\ &\vdots \end{aligned}$$

where $\delta_{\mu\lambda}$ is one if $\mu = \lambda$ and zero otherwise. This is a triangular system, and thus solvable for the a^{μ}_{λ}. Thus we can find a left inverse A, for B. Thus

$$C = AB'.$$

Note also that we could use the Taylor expansion of the function $f(x) = \frac{1}{x}$ to write out C explicitly in terms of B. □

Chapter 5

Further Directions

In this thesis, we have discussed the problem of differentiating an interpolation Jack polynomial P_λ in three variables with respect to the variable κ, and writing the result as a linear combination of other interpolation Jack polynomials:

$$\frac{\partial}{\partial\kappa}P_\lambda = \sum_\mu c_\lambda^\mu P_\mu.$$

We have made progress towards understanding this linear combination: we know that if μ is "large enough" with respect to λ, then $c_\lambda^\mu = 0$ (see Section 4.2), we've found explicit formulas for the c_λ^μ for a few different specific relationships between μ and λ (see Sections 4.3-4.5), and we've found an implicit matrix formula for all c_λ^μ in terms of the coefficients of monomial symmetric functions b_λ^μ. But there is plenty of work still to be done! How can we proceed from here? Well, there are a few different directions we could go in.

The first direction would be to continue as we were in Chapter 4. That is, we find the coefficients c_λ^μ one-by-one going in reverse lexicographic ordering. This will give us concrete formulas for arbitrarily many coefficients, but will not solve the problem completely since if we start at an arbitrary partition λ we cannot go through all partitions $\mu < \lambda$. Furthermore, we have only started this approach for the specific case $\lambda = (D, 0, 0)$. In cases where λ has more than one non-zero part it may be more computationally intense to find coefficients in this manner. However, it is possible that as we start to compute specific coefficients, a pattern will emerge, as did in the case of finding $c_{(D,0,0)}^{(D-a,a,0)}$ (see Section 4.6). Once we have a conjecture, we can turn to other means to try to prove it, such as the use of hypergeometric functions.

The other main direction to go in is to focus on finding the coefficients b_λ^μ, of monomial symmetric functions in interpolation Jack polynomials. There are a few reasons to take this approach. First, the solution for the two-variable case relied on having expressions for these coefficients, so if we want to try a similar approach for three variables we will need to calculate them. Furthermore, we saw in Sections 4.3-4.5 that the approach outlined in the previous paragraph relies heavily on knowing the coefficients b_λ^μ, so if we want to take that approach then it will be useful to know them. Finally, the matrix formula found in Section 4.7 gives an implicit expression for the c_λ^μ in terms of the b_λ^μ, so if we want to use this implicit formula to find an explicit formula for the c_λ^μ, it would again be useful to know the b_λ^μ.

In this thesis we have made progress towards the problem of finding the coefficients c_λ^μ, but have yet to find an explicit formula, and thus the general problem remains open. It is my hope that the exploration in this thesis will aid in the solution of this problem, and foremost I hope that the relatively concrete manipulations presented here can lead to a deeper understanding of the theory behind interpolation polynomials. This brings to mind a quote by Emmy Noether[1]:

> If one proves the equality of two numbers *a* and *b* by showing first that '*a* is less than or equal to *b*' and then '*a* is greater than or equal to *b*', it is unfair; one should instead show that they are really equal by disclosing the inner ground for their equality.

Here I have shown several equalities using algebraic manipulations. However, I wonder *why* these equalities hold and what more complex mathematical structure they may be hinting at. My hope is that further study of this problem will explain the "inner ground for their equality".

[1]See [12].

Appendix A

Alternative Proofs for Various Coefficients

A.1 The Coefficient of $m_{(D-1,1,0)}$ in $P_{(D,0,0)}$

Theorem 4.3.5. *For $D \geq 2$, the coefficient of $m_{(D-1,1,0)}$ in $P_{(D,0,0)}$ is*

$$b_{(D,0,0)}^{(D-1,1,0)}(\kappa) = \frac{D\kappa}{\kappa + D - 1}.$$

Proof. Let $\lambda = (D, 0, 0)$. Then from Formula 2.1, we can write

$$P_\lambda(x_1, x_2, x_3) = \binom{-\kappa}{D} \sum_{i_j} \prod_{j=1}^{3} \left[\binom{-\kappa}{i_{j-1} - i_j} (x_j - \kappa\delta_j - i_j)^{\underline{i_{j-1} - i_j}} \right]$$

where $\delta = (\delta_1, \delta_2, \delta_3) = (2, 1, 0)$, and the sum runs through all integer sequences $D = i_0 \geq i_1 \geq i_2 \geq i_3 = 0$. Since P_λ is a symmetric polynomial, the coefficient of $m_{(D-1,1,0)}$ in P_λ is just the coefficient of $x_1^{D-1}x_2$. So let's try to find that coefficient. To this end, let's look at the product term produced by each sequence i_j. For this product to output a term $x_1^{D-1}x_2$, we must have

$$\begin{aligned} i_0 - i_1 &= D - 1 \\ i_1 - i_2 &= 1. \end{aligned}$$

Since we know $i_0 = D$, this forces $i_1 = 1$ and $i_2 = 0$. Thus it is sufficient to compute the product for these values of i_j. First, for $j = 3$ we have

$$i_{j-1} - i_j = i_2 - i_3 = 0 - 0 = 0.$$

So the $j = 3$ term is just 1. For $j = 2$ we have

$$i_{j-1} - i_j = i_1 - i_2 = 1 - 0 = 1.$$

So the $j = 2$ term is

$$\begin{aligned}\binom{-\kappa}{i_1 - i_2}(x_2 - \kappa\delta_2 - i_2)^{\underline{i_1 - i_2}} &= \binom{-\kappa}{1}(x_2 - \kappa - 0)^{\underline{1}} \\ &= -\kappa^{\underline{1}}(x_2 - \kappa)^{\underline{1}} \\ &= -\kappa(x_2 - \kappa).\end{aligned}$$

Finally, when $j = 1$ we have

$$i_{j-1} - i_j = i_0 - i_1 = D - 1.$$

So the $j = 1$ term is

$$\begin{aligned}\binom{-\kappa}{i_0 - i_1}(x_1 - \kappa\delta_1 - i_1)^{\underline{i_0 - i_1}} &= \binom{-\kappa}{D - 1}(x_1 - \kappa\delta_1 - i_1)^{\underline{D-1}} \\ &= \frac{(-1)^{D-1}\kappa^{\underline{D-1}}}{(D-1)!}(x_1 - 3\kappa - i_1)^{\underline{D-1}}.\end{aligned}$$

Thus the product we are interested in will be

$$\begin{aligned}\prod_{j=1}^{3}\left[\binom{-\kappa}{i_{j-1} - i_j}(x_j - \kappa\delta_j - i_j)^{\underline{i_{j-1} - i_j}}\right] &= \left(\frac{(-1)^{D-1}\kappa^{\underline{D-1}}}{(D-1)!}(x_1 - 3\kappa - i_1)^{\underline{D-1}}\right) \\ &\quad \times (-\kappa(x_2 - \kappa))\,(1).\end{aligned}$$

This will be a polynomial in x_1 and x_2. However, we are only interested in the term involving $x_1^{D-1}x_2$, which is

$$\left(\frac{(-1)^{D-1}\kappa^{\underline{D-1}}}{(D-1)!}x_1^{D-1}\right)(-\kappa x_2) = \frac{(-1)^D\kappa^{\underline{D-1}}\kappa}{(D-1)!}x_1^{D-1}x_2.$$

Now, looking again at Formula 2.1 we see that to get the full coefficient of $x_1^{D-1}x_2$ in P_λ we have to multiply by $\binom{-\kappa}{D}^{-1}$. Thus the final coefficient is

$$\begin{aligned}\binom{-\kappa}{D}^{-1}\frac{(-1)^D\kappa^{\underline{D-1}}\kappa}{(D-1)!} &= \left(\frac{D!}{(-1)^D\kappa^{\underline{D}}}\right)\left(\frac{(-1)^D\kappa^{\underline{D-1}}\kappa}{(D-1)!}\right) \\ &= \frac{D\kappa(\kappa)(\kappa+1)\cdots(\kappa+D-2)}{(\kappa)(\kappa+1)\cdots(\kappa+D-1)} \\ &= \frac{D\kappa}{\kappa + D - 1}.\end{aligned}$$

Thus the coefficient of $m_{(D-1,1,0)}$ in $P_{(D,0,0)}$ is $\frac{D\kappa}{\kappa+D-1}$, as desired. □

A.2 The Coefficient of $m_{(D-2,2,0)}$ in $P_{(D,0,0)}$

Theorem 4.5.4. *For $D \geq 4$ the coefficient of $m_{(D-2,2,0)}$ in $P_{(D,0,0)}$ is*

$$b_{(D,0,0)}^{(D-2,2,0)}(\kappa) = \frac{D(D-1)(\kappa)(\kappa+1)}{2(\kappa+D-1)(\kappa+D-2)}.$$

Proof. Recall that from Formula 2.1 that

$$P_{(D,0,0)} = \binom{-\kappa}{D}^{-1} \sum_{i_j} \prod_{j=1}^{3} \left[\binom{-\kappa}{i_{j-1}-i_j} (x_j - \kappa\delta_j - i_j)^{\underline{i_{j-1}-i_j}} \right]$$

where the sum runs over all integer sequences $D = i_0 \geq i_1 \geq i_2 \geq i_3 = 0$. We want to find the coefficient of $x_1^{D-2}x_2^2$ in this expression. Now, $\delta_3 = i_3 = 0$, so if $i_2 \neq i_3$ then the product term for the sequence i_j will have an x_3 in every term. Since we are only interested in the $x_1^{D-2}x_2^2$ term, we know that any sequence of interest to us will have $i_2 = i_3 = 0$. Furthermore, to get an x_2^2 term we must have $i_1 - i_2 \geq 2$, and to get an x_1^{D-2} term we must have $i_0 - i_1 \geq D-2$. Thus

$$\begin{aligned} D - i_1 \geq D-2 \quad &\text{and} \quad i_1 - 0 \geq 2, \quad \text{so} \\ 2 \geq i_1 \quad &\text{and} \quad i_1 \geq 2. \end{aligned}$$

So $i_1 = 2$. Thus the only sequence that is of interest to us is $i_0 = D, i_1 = 2, i_2 = 0, i_3 = 0$. The term we get from this sequence is

$$\begin{aligned} &\binom{-\kappa}{D-2}(x_1 - 2\kappa - 2)^{\underline{D-2}} \binom{-\kappa}{2} (x_2 - k)^{\underline{2}} \binom{-\kappa}{0} (x_3)^{\underline{0}} \\ &\qquad = \binom{-\kappa}{D-2}(x_1 - 2\kappa - 2)^{\underline{D-2}} \binom{-\kappa}{2} (x_2 - k)^{\underline{2}}. \end{aligned}$$

The $x_1^{D-2}x_2^2$ term produced by this product is

$$\binom{-\kappa}{D-2}\binom{-\kappa}{2} x_1^{D-2}x_2^2.$$

Thus the coefficient of $x_1^{D-2}x_2^2$, and thus $m_{(D-2,2,0)}$ in $P_{(D,0,0)}$ is

$$\begin{aligned} \binom{-\kappa}{D}^{-1}\binom{-\kappa}{D-2}\binom{-\kappa}{2} &= \frac{D!}{(-1)^D \kappa^{\underline{D}}} \frac{(-1)^{D-2}\kappa^{\underline{D-2}}}{(D-2)!} \frac{(-1)^2\kappa^{\underline{2}}}{2!} \\ &= \frac{D(D-1)(\kappa)(\kappa+1)}{2(\kappa+D-1)(\kappa+D-2)}. \end{aligned}$$

That is,

$$b_{(D,0,0)}^{(D-2,2,0)}(\kappa) = \frac{D(D-1)(\kappa)(\kappa+1)}{2(\kappa+D-1)(\kappa+D-2)}.$$

□

A.3 Coefficient of $m_{(D-2,2,0)}$ in $P_{(D-1,1,0)}$

Theorem 4.5.5. *The coefficient of $m_{(D-2,2,0)}$ in $P_{(D-1,1,0)}$ is*

$$b_{(D-1,1,0)}^{(D-2,2,0)}(\kappa) = \frac{(D-2)\kappa}{D-3-\kappa}.$$

Proof. Recall Formula 2.2:

$$P_{(D-1,1,0)}(x_1,x_2,x_3) = \sum_T \Psi_T(\kappa)\Bigg(\prod_{T(s)=1}(x_1 - a'(s) + (\ell'(s)-2)\kappa) \times \prod_{T(s)=2}(x_2 - a'(s) + (\ell'(s)-1)\kappa) \times \prod_{T(s)=3}(x_3 - a'(s) + \ell'(s)\kappa)\Bigg).$$

We now want to find the coefficient of the $x_3^{D-2}x_2^2$ term. Thus we see that we are only interested in terms generated by reverse tableaux where $D-2$ boxes are filled with 3's and two are filled with 2's. Since reverse tableaux must be weakly decreasing along rows and strictly decreasing along columns, the only reverse tableau that gives a product with a $x_3^{D-2}x_2^2$ term is the following:

$$T = \begin{array}{|c|c|c|c|}\hline 3 & \cdots & 3 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\,.$$

Thus the coefficient of $x_3^{D-2}x_2^2$ in $P_{(D-1,1,0)}$ is $\psi_T(\kappa)$. So let's find this $\psi_T(\kappa)$. We have

$$\lambda^{(0)} = \lambda^{(1)} = \{(1,1),\dots,(1,D-2),(2,1)\},$$
$$\lambda^{(2)} = \{(1,1),\dots,(1,D-2)\},\quad \lambda^{(3)} = \emptyset.$$

And thus

$$(R/C)_{\lambda^{(0)}/\lambda^{(1)}} = \emptyset, \quad (R/C)_{\lambda^{(1)}/\lambda^{(2)}} = \{(1,2), \dots (1, D-2)\}.$$

So $\psi_T(\kappa)$ reduces to

$$\psi_T(\kappa) = \prod_{s \in (R/C)_{\lambda^{(1)}/\lambda^{(2)}}} \frac{b_{\lambda^{(2)}}(s,\kappa)}{b_{\lambda^{(1)}}(s,\kappa)}.$$

Now let's find $b_{\lambda^{(1)}}(s,\kappa)$ and $b_{\lambda^{(2)}}(s,\kappa)$. Let $s = (i,j)$ be a box in $(R/C)_{\lambda^{(1)}/\lambda^{(2)}}$. Then $i = 1$ and $2 \le j \le D-2$. Thus

$$\begin{aligned} l_{\lambda^{(1)}}(s) &= \left|\left\{k > 1 \mid \lambda_k^{(1)} \ge j\right\}\right| = 0 \\ l_{\lambda^{(2)}}(s) &= \left|\left\{k > 1 \mid \lambda_k^{(2)} \ge j\right\}\right| = 0 \end{aligned}$$

and

$$\begin{aligned} a_{\lambda^{(1)}}(s) &= \lambda_1^{(1)} - j = D - 1 - j \\ a_{\lambda^{(2)}}(s) &= \lambda_1^{(2)} - j = D - 2 - j. \end{aligned}$$

Thus

$$\begin{aligned} b_{\lambda^{(1)}}((1,j),\kappa) &= \frac{a_{\lambda^{(1)}}(1,j) + \kappa(l_{\lambda^{(1)}}(1,j)+1)}{a_{\lambda^{(1)}}(1,j) + \kappa l_{\lambda^{(1)}}(1,j) + 1} = \frac{D-1-j+\kappa}{D-j} \\ b_{\lambda^{(2)}}((1,j),\kappa) &= \frac{a_{\lambda^{(2)}}(1,j) + \kappa(l_{\lambda^{(2)}}(1,j)+1)}{a_{\lambda^{(2)}}(1,j) + \kappa l_{\lambda^{(2)}}(1,j) + 1} = \frac{D-2-j+\kappa}{D-1-j}. \end{aligned}$$

And so the quotient is

$$\frac{b_{\lambda^{(2)}}((1,j),\kappa)}{b_{\lambda^{(1)}}((1,j),\kappa)} = \frac{\left(\frac{D-2-j+\kappa}{D-1-j}\right)}{\left(\frac{D-1-j+\kappa}{D-j}\right)} = \frac{(D-j+\kappa-2)(D-j)}{(D-j+\kappa-1)(D-j-1)}.$$

Thus our product telescopes:

$$\begin{aligned} \psi_T(\kappa) &= \prod_{s \in (R/C)_{\lambda^{(1)}/\lambda^{(2)}}} \frac{b_{\lambda^{(2)}}(s,\kappa)}{b_{\lambda^{(1)}}(s,\kappa)} \\ &= \prod_{j=2}^{D-2} \frac{(D-j+\kappa-2)(D-j)}{(D-j+\kappa-1)(D-j-1)} \\ &= \frac{(D-2)\kappa}{\kappa+D-3}. \end{aligned}$$

Note that for the last step I simplified the expression using Mathematica. Thus the coefficient of $m_{(D-2,2,0)}$ in $P_{(D-1,1,0)}$ is

$$b_{(D-1,1,0)}^{(D-2,2,0)}(\kappa) = \frac{(D-2)\kappa}{\kappa + D - 3}.$$

□

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