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ON THE MATRIX EQUATION $XA + AX^T = 0$, II

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ABSTRACT. The matrix equation $XA + AX^T = 0$ was recently introduced by De Terán and Dopico (Linear Algebra Appl. **434** (2011), 44–67) to study the dimension of congruence orbits. They reduced the study of this equation to a number of special cases, several of which have not been explicitly solved. In this note we obtain an explicit, closed-form solution in the difficult Type 0-I interaction case.

1. INTRODUCTION

The matrix equation

$$XA + AX^T = 0, \tag{1}$$

where A in $M_n(\mathbb{C})$ is fixed and X is unknown, was introduced in 2011 by F. De Terán and F.M. Dopico to study the dimensions of congruence orbits. In this setting, the codimension of the congruence orbit of A is given by the number of free parameters in the solution of (1). More recently, the equation (1) has also attracted the attention of a number of other authors [4–6].

Now observe that for each fixed A, the solution space to (1) is a Lie algebra, denoted $\mathfrak{g}(A)$, equipped with Lie bracket $[X,Y] = XY - YX$. Building upon the initial work of De Terán and Dopico, a complete description of such twisted matrix Lie algebras is almost at hand. The computations in [3] provide explicit descriptions, up to similarity, of all possible twisted matrix Lie algebras except in a few highly problematic cases. In [6], we provided an explicit description of the solution set in one of the most difficult cases.

We are concerned here with the particularly troublesome setting where A is the direct sum of a so-called Type 0 and Type I canonical matrix (see [7, Thm. 4.5.25], [3, Sect. 2]) These are matrices of the form $J_p(0) \oplus \Gamma_q$, where $J_p(0)$ is the $p \times p$ Jordan matrix with eigenvalue 0 and Γ_q is the $q \times q$ matrix

$$\Gamma_q := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & (-1)^{q+1} \\ 0 & 0 & 0 & \cdots & (-1)^q & (-1)^q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & -1 & -1 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

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We aim here to provide an explicit, closed form solution to the matrix equation

$$X\big(J_p(0) \oplus \Gamma_q\big) + \big(J_p(0) \oplus \Gamma_q\big)X^T = 0, \tag{2}$$

thereby obtaining a simple matricial description of the Lie algebra $\mathfrak{g}\big(J_p(0) \oplus \Gamma_q\big)$. For even p, it is not difficult to show that

$$\mathfrak{g}(J_p(0) \oplus \Gamma_q) = \mathfrak{g}\big(J_p(0)\big) \oplus \mathfrak{g}(\Gamma_q),$$

where the Lie algebras $\mathfrak{g}(J_p(0))$ and $\mathfrak{g}(\Gamma_q)$ are elegantly described in [3, p. 52, p. 54]. For odd p, however, the situation is much more complicated. Indeed, symbolic computation reveals that the matrices which arise are bewilderingly complex and typographically unwieldy. The consideration of this troublesome case takes up the bulk of this article.

Besides providing a major step in the description of twisted matrix Lie algebras, our approach is notable for the use of a number of combinatorial identities. In fact, we conclude the paper by outlining an alternate description of the solutions to (2) in which a variety of intriguing combinatorial quantities emerge.

2. Solution

We begin our consideration of the matrix equation (2) by partitioning the unknown $(p+q) \times (p+q)$ matrix X conformally with the decomposition of $J_p(0) \oplus \Gamma_q$. In other words, we write

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{3}$$

where A is $p \times p$, B is $p \times q$, C is $q \times p$, and D is $q \times q$. We should pause here to remark that the submatrix A just introduced is not the same as the fixed matrix A which appears in the general equation (1), which plays no part in what follows.

We next substitute the partitioned matrix (3) into (2) to obtain the system

$$AJ_p(0) + J_p(0)A^T = 0, \tag{4}$$

$$D\Gamma_q + \Gamma_q D^T = 0, \tag{5}$$

$$B\Gamma_q + J_p(0)C^T = 0, \tag{6}$$

$$CJ_p(0) + \Gamma_q B^T = 0, \tag{7}$$

of matrix equations which, taken together, are equivalent to (2). Fortunately, equations (4) and (5) are easily dispatched, since the explicit solutions in these cases have already been described by de Terán and Dopico [3, p. 52, p. 54].

It therefore remains to characterize the matrices B and C which satisfy (6) and (7). This is by far the most difficult and involved portion of our work, although our task is made somewhat easier because $B = C = 0$ when p is even [3, p. 60].

In the following, we assume that p is odd. Solving for B in (7) yields

$$B = -J_p(0)^T C^T \Gamma_q^{-T} \tag{8}$$

and substituting this into (6) gives

$$J_p(0)C^T = J_p(0)^T C^T \Gamma_q^{-T}\Gamma_q. \tag{9}$$

Before proceeding any further, we require a few preliminary lemmas. The following fact is well-known amongst those familiar with the congruence canonical form (see [8, p. 1016] or [9, p. 215]).

Lemma 1. *The cosquare* $\Gamma_q^{-T}\Gamma_q$ *of* Γ_q *is given by*

$$\Gamma_q^{-T}\Gamma_q = (-1)^{q+1}\Lambda_q, \tag{10}$$

where

$$\Lambda_q := \begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{11}$$

We also require the explicit reduction of Λ_q to its Jordan canonical form since the matrix which implements this similarity will be of great interest to us. Since we could not find the following result in the literature, we feel obliged to provide a detailed proof.

Lemma 2. *The matrix* Λ_q *satisfies* $\Lambda_q = P_q\big(I_q + J_q(0)\big)P_q^{-1}$*, where* P_q *is the* $q \times q$ *upper triangular matrix whose* (i,j) *entry is given by*

$$[P_q]_{i,j} = 2^{q-j}(-1)^{j-i}\binom{j-1}{j-i}.$$

Proof. Let $K_q = I_q + J_q(0)$ and observe that if $j = 1$, then

$$[\Lambda_q P_q]_{i,1} = \delta_{i,1}2^{q-1} = [P_q K_q]_{i,1},$$

where $\delta_{i,j}$ denotes the Kronecker δ-function, since the matrices involved are all upper-triangular. For $j \geq 2$, we make note of Identity 168 in [1], which states that

$$\sum_{k=0}^{t}(-1)^k\binom{n}{k} = (-1)^t\binom{n-1}{t}, \tag{12}$$

whenever $t \geq 0$ and $n \geq 1$. This gives us

$$\begin{aligned}
[\Lambda_q P_q]_{i,j} &= \sum_{k=1}^{q}[\Lambda_q]_{i,k}[P_q]_{k,j} \\
&= \sum_{k=i}^{q}[\Lambda_q]_{i,k}[P_q]_{k,j} \\
&= [\Lambda_q]_{i,i}[P_q]_{i,j} + \sum_{k=i+1}^{q}[\Lambda_q]_{i,k}[P_q]_{k,j} \\
&= [P_q]_{i,j} + \sum_{k=i+1}^{q} 2\cdot 2^{q-j}(-1)^{j-k}\binom{j-1}{j-k} \\
&= [P_q]_{i,j} + 2^{q-j+1}\sum_{k=i+1}^{j}(-1)^{j-k}\binom{j-1}{j-k}
\end{aligned}$$

$$
\begin{aligned}
&= [P_q]_{i,j} + 2^{q-j+1}\sum_{k=0}^{j-i-1}(-1)^k\binom{j-1}{k} &(13)\\
&= [P_q]_{i,j} + 2^{q-j+1}(-1)^{j-i-1}\binom{j-2}{j-i-1} &(14)\\
&= [P_q]_{i,j}[K_q]_{j,j} + [P_q]_{i,j-1}[K_q]_{j-1,j}\\
&= \sum_{k=1}^{q}[P_q]_{i,k}[K_q]_{k,j}\\
&= [P_qK_q]_{i,j},
\end{aligned}
$$

where the passage from (13) to (14) follows by (12). Thus $[\Lambda_q P_q]_{i,j} = [P_qK_q]_{i,j}$ for each pair (i,j), whence $\Lambda_q P_q = P_q K_q$, as claimed. □

We next require an explicit description of the inverse of P_q. For the sake of illustration, for $q = 5$ we obtain the corresponding matrices

$$
P_5 = \begin{bmatrix} 16 & -8 & 4 & -2 & 1\\ 0 & 8 & -8 & 6 & -4\\ 0 & 0 & 4 & -6 & 6\\ 0 & 0 & 0 & 2 & -4\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad P_5^{-1} = \frac{1}{16}\begin{bmatrix} 1 & 1 & 1 & 1 & 1\\ 0 & 2 & 4 & 6 & 8\\ 0 & 0 & 4 & 12 & 24\\ 0 & 0 & 0 & 8 & 32\\ 0 & 0 & 0 & 0 & 16 \end{bmatrix}.
$$

Lemma 3. *The inverse P_q^{-1} of the matrix P_q is the $q \times q$ upper triangular matrix whose (i,j) entry is given by*

$$
[P_q^{-1}]_{i,j} = 2^{i-q}\binom{j-1}{j-i}. \tag{15}
$$

Proof. Letting P_q' denote the $q \times q$ upper triangular matrix whose entries are given by (15), we find that $[P_qP_q']_{i,j} = 0$ for $j < i$ since the matrices involved are both upper-triangular. For $j \geq i$, we appeal to Identity 169 in [1], which we rewrite as

$$
\sum_{k=0}^{n}(-1)^{k-m}\binom{k}{m}\binom{n}{k} = \delta_{m,n}
$$

for $n \geq m$, to conclude that

$$
\begin{aligned}
[P_qP_q']_{i,j} &= \sum_{k=1}^{q}[P_q]_{i,k}[P_q']_{k,j}\\
&= \sum_{k=1}^{q} 2^{q-k}(-1)^{k-i}\binom{k-1}{k-i}\cdot 2^{k-q}\binom{j-1}{j-k}\\
&= \sum_{k=1}^{q}(-1)^{k-i}\binom{k-1}{k-i}\binom{j-1}{j-k}\\
&= \sum_{k=1}^{j}(-1)^{k-i}\binom{k-1}{k-i}\binom{j-1}{j-k}\\
&= \sum_{k=0}^{j-1}(-1)^{k-i+1}\binom{k}{k-i+1}\binom{j-1}{j-k-1}
\end{aligned}
$$

$$= \sum_{k=0}^{j-1} (-1)^{k-(i-1)} \binom{k}{i-1} \binom{j-1}{k}$$
$$= \delta_{i-1,j-1}$$
$$= \delta_{i,j}.$$

Thus P_q is invertible and $P_q^{-1} = P_q'$. □

Returning to (9) and applying Lemma 2 we obtain

$$J_p(0)C^T P_q = (-1)^{q-1} J_p(0)^T C^T P_q K_q.$$

In other words, the $p \times q$ matrix

$$Y = C^T P_q \tag{16}$$

satisfies the equation

$$J_p(0)Y = (-1)^{q-1} J_p(0)^T Y\big(I_q + J_q(0)\big). \tag{17}$$

Therefore the entries of Y satisfy

$$\begin{bmatrix} y_{2,1} & y_{2,2} & \cdots & y_{2,q} \\ y_{3,1} & y_{3,2} & \cdots & y_{3,q} \\ \vdots & \vdots & \ddots & \vdots \\ y_{p,1} & y_{p,2} & \cdots & y_{p,q} \\ 0 & 0 & \cdots & 0 \end{bmatrix} = (-1)^{q-1} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ y_{1,1} & y_{1,1}+y_{1,2} & \cdots & y_{1,q-1}+y_{1,q} \\ y_{2,1} & y_{2,1}+y_{2,2} & \cdots & y_{2,q-1}+y_{2,q} \\ \vdots & \vdots & \ddots & \vdots \\ y_{p-1,1} & y_{p-1,1}+y_{p-1,2} & \cdots & y_{p-1,q-1}+y_{p-1,q} \end{bmatrix}, \tag{18}$$

from which we see that

$$y_{i,j} = (-1)^{q-1}\big(y_{i-2,j-1} + y_{i-2,j}\big) \tag{19}$$

holds for $3 \le i \le p$ and $2 \le j \le q$. This prompts the following lemma.

Lemma 4. *If i is even, then $y_{i,j} = 0$ for $1 \le j \le q$. If i is odd, then*

$$y_{i,j} = (-1)^{\frac{i-1}{2}(q-1)} \sum_{k=1}^{j} \binom{\frac{i-1}{2}}{j-k} y_{1,k} \tag{20}$$

Proof. Comparing entries in (18), we find that $y_{2,j} = 0$ for $1 \le j \le q$. In light of (19), it follows that

$$y_{4,j} = (-1)^{q-1}\big(y_{2,j-1} + y_{2,j}\big) = 0,$$

for $1 \le j \le q$. Proceeding inductively, we see that $y_{i,j} = 0$ whenever i is even.

Now suppose that i is odd. For the basis of our induction, observe that (20) holds trivially when $i + j = 2$ (i.e., when $i = j = 1$). Now suppose that $n \ge 3$ and that (20) holds if $i + j = n - 1$. Under this hypothesis, we wish to show that (20) also holds if $i + j = n$. In light of (19), we have

$$\begin{aligned} y_{i,j} &= (-1)^{q-1}\big(y_{i-2,j-1} + y_{i-2,j}\big) \\ &= (-1)^{q-1} \left[(-1)^{\frac{i-3}{2}(q-1)} \sum_{k=1}^{j-1} \binom{\frac{i-3}{2}}{j-k-1} y_{1,k} + (-1)^{\frac{i-3}{2}(q-1)} \sum_{k=1}^{j} \binom{\frac{i-3}{2}}{j-k} y_{1,k} \right] \\ &= (-1)^{\frac{i-1}{2}(q-1)} \left[\sum_{k=1}^{j-1} \binom{\frac{i-3}{2}}{j-k-1} y_{1,k} + \sum_{k=1}^{j} \binom{\frac{i-3}{2}}{j-k} y_{1,k} \right] \end{aligned}$$

$$= (-1)^{\frac{i-1}{2}(q-1)} \left[\sum_{k=1}^{j} \binom{\frac{i-3}{2}}{j-k-1} + \binom{\frac{i-3}{2}}{j-k} \right] y_{1,k}$$
$$= (-1)^{\frac{i-1}{2}(q-1)} \sum_{k=1}^{j} \binom{\frac{i-1}{2}}{j-k} y_{1,k},$$

which completes the proof. □

Armed with an explicit, entry-by-entry description of the matrix $Y = C^T P_q$, we can construct a convenient basis for the solution space of (17). Noting that Y depends upon the q free parameters $y_{1,1}, y_{1,2}, \ldots, y_{1,q}$, we introduce a family $W_1, W_2, \ldots, W_q$ of matrices by letting W_ℓ denote the matrix Y which corresponds to the initial values $y_{1,j} = \delta_{j,\ell}$ along the first row and whose remaining entries are determined by (20). To be more specific, it follows from (20) that

$$[W_\ell]_{i,j} = \begin{cases} 0 & \text{if } i \text{ is even,} \\ (-1)^{\frac{i-1}{2}(q-1)} \binom{\frac{i-1}{2}}{j-\ell} & \text{if } i \text{ is odd.} \end{cases} \tag{21}$$

In general, for i odd we have

$$[W_\ell]_{i,j} = (-1)^{\frac{i-1}{2}(q-1)} \binom{\frac{i-1}{2}}{j-\ell} = (-1)^{\frac{i-1}{2}(q-1)} \binom{\frac{i-1}{2}}{(j-\ell+1)-1}$$
$$= [W_1]_{i,j-(\ell-1)} = \left[W_1 \big(J_q(0)\big)^{\ell-1} \right]_{i,j},$$

so that

$$W_\ell = W_1 \big(J_q(0)\big)^{\ell-1}. \tag{22}$$

For example, if $p = q = 5$ we obtain

$$Y = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} & y_{1,5} \\ 0 & 0 & 0 & 0 & 0 \\ y_{1,1} & y_{1,1}+y_{1,2} & y_{1,2}+y_{1,3} & y_{1,3}+y_{1,4} & y_{1,4}+y_{1,5} \\ 0 & 0 & 0 & 0 & 0 \\ y_{1,1} & 2y_{1,1}+y_{1,2} & y_{1,1}+2y_{1,2}+y_{1,3} & y_{1,2}+2y_{1,3}+y_{1,4} & y_{1,3}+2y_{1,4}+y_{1,5} \end{bmatrix},$$

and

$$W_1 = \begin{bmatrix} 1&0&0&0&0\\0&0&0&0&0\\1&1&0&0&0\\0&0&0&0&0\\1&2&1&0&0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0&1&0&0&0\\0&0&0&0&0\\0&1&1&0&0\\0&0&0&0&0\\0&1&2&1&0 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 0&0&1&0&0\\0&0&0&0&0\\0&0&1&1&0\\0&0&0&0&0\\0&0&1&2&1 \end{bmatrix},$$

$$W_4 = \begin{bmatrix} 0&0&0&1&0\\0&0&0&0&0\\0&0&0&1&1\\0&0&0&0&0\\0&0&0&1&2 \end{bmatrix}, \quad W_5 = \begin{bmatrix} 0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&1 \end{bmatrix}.$$

By (21) we can see that $[W_i]_{1,j} = \delta_{i,j}$ for $1 \leq i, j \leq q$ whence the matrices $W_1, W_2, \ldots, W_q$ are linearly independent. Considering (22), we now set $W = W_1$

so that for any solution Y of (17) we have

$$Y = \sum_{\ell=1}^{q} \alpha_\ell W_\ell = W \left(\sum_{\ell=1}^{q} \alpha_\ell \left(J_q(0)\right)^{\ell-1} \right) \tag{23}$$

for some constants $\alpha_1, \alpha_2, \dots, \alpha_q$.

Returning to (16) and solving for C yields

$$\boxed{C = P_q^{-T} \left(\sum_{\ell=1}^{q} \alpha_\ell \left(J_q^T(0)\right)^{\ell-1} \right) W^T,} \tag{24}$$

where P_q^{-1} is given explicitly by (15) and $W = W_1$ is given by (21). Turning our attention now toward (8), we find that

$$\boxed{B = -J_p(0)^T C^T \Gamma_q^{-T},} \tag{25}$$

so that B is completely determined by C. In particular, the dimension of the solution space of (6) and (7) is q, which is consistent with [3, Lem. 10]. Furthermore, [3, Lem. 4] and [3, Lem. 5] tell us that the solution sets of (4) and (5) have dimensions $\lceil \frac{p}{2} \rceil$ and $\lfloor \frac{q}{2} \rfloor$, respectively. Putting everything together, we conclude that the dimension of the solution space of our original equation (2) for odd p is

$$q + \left\lceil \frac{p}{2} \right\rceil + \left\lfloor \frac{q}{2} \right\rfloor .$$

Appendix A. Combinatorial interpretation

Although we are now in possession of an explicit description of the solution to (2), it is worth examining the submatrices B and C a little bit closer since the entries of these matrices display some remarkable combinatorial properties.

Using our closed form solution (24), let us continue with our previous series of examples corresponding to the parameters $p = q = 5$. In this setting, we have

$$C = \begin{bmatrix} \frac{\alpha_1}{16} & 0 & \frac{\alpha_1}{16} & 0 & \frac{\alpha_1}{16} \\ \frac{1}{16}(\alpha_1 + 2\alpha_2) & 0 & \frac{1}{16}(3\alpha_1 + 2\alpha_2) & 0 & \frac{1}{16}(5\alpha_1 + 2\alpha_2) \\ \frac{1}{16}(\alpha_1 + 4(\alpha_2 + \alpha_3)) & 0 & \frac{1}{16}(5\alpha_1 + 8\alpha_2 + 4\alpha_3) & 0 & \frac{1}{16}(13\alpha_1 + 4(3\alpha_2 + \alpha_3)) \\ \frac{1}{16}(\alpha_1 + 6\alpha_2 + 12\alpha_3 + 8\alpha_4) & 0 & \frac{1}{16}(7\alpha_1 + 18\alpha_2 + 20\alpha_3 + 8\alpha_4) & 0 & \frac{1}{16}(25\alpha_1 + 38\alpha_2 + 28\alpha_3 + 8\alpha_4) \\ \frac{\alpha_1}{16} + \frac{\alpha_2}{2} + \frac{3\alpha_3}{2} + 2\alpha_4 + \alpha_5 & 0 & \frac{9\alpha_1}{16} + 2\alpha_2 + \frac{7\alpha_3}{2} + 3\alpha_4 + \alpha_5 & 0 & \frac{41\alpha_1}{16} + \frac{11\alpha_2}{2} + \frac{13\alpha_3}{2} + 4\alpha_4 + \alpha_5 \end{bmatrix}.$$

In particular, the simple parametrization at the heart of (24) is obfuscated by pre- and post- multiplication with the somewhat complicated matrices P_q^{-T} and W^T, respectively. This results in the rather overwhelming complexity apparent in the preceding example. However, upon performing an appropriate reparametrization, we can obtain the significantly simpler matrix

$$C = \begin{bmatrix} \beta_1 & 0 & \beta_1 & 0 & \beta_1 \\ \beta_2 & 0 & \beta_2 + 2\beta_1 & 0 & \beta_2 + 4\beta_1 \\ \beta_3 & 0 & \beta_3 + 2\beta_2 + 2\beta_1 & 0 & \beta_3 + 8\beta_1 + 4\beta_2 \\ \beta_4 & 0 & \beta_4 + 2\beta_3 + 2\beta_2 + 2\beta_1 & 0 & \beta_4 + 12\beta_1 + 8\beta_2 + 4\beta_3 \\ \beta_5 & 0 & \beta_5 + 2\beta_4 + 2\beta_3 + 2\beta_2 + 2\beta_1 & 0 & \beta_5 + 16\beta_1 + 12\beta_2 + 8\beta_3 + 4\beta_4 \end{bmatrix}. \tag{26}$$

We sketch here an independent combinatorial argument for determining the exact coefficients which arise in this manner. Since our motivation for doing so is purely aesthetic, we leave many of the tedious details to the reader.

We begin anew, expanding (9) and using Lemma 1 to obtain

$$\begin{bmatrix} c_{1,2} & c_{2,2} & \cdots & c_{q,2} \\ c_{1,3} & c_{2,3} & \cdots & c_{q,3} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1,p} & c_{2,p} & \cdots & c_{q,p} \\ 0 & 0 & \cdots & 0 \end{bmatrix} = (-1)^{q-1} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ c_{1,1} & c_{2,1}+2c_{1,1} & \cdots & c_{q,1}+\sum_{i=1}^{q-1} 2c_{i,1} \\ c_{1,2} & c_{2,2}+2c_{1,2} & \cdots & c_{q,2}+\sum_{i=1}^{q-1} 2c_{i,2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1,p-1} & c_{2,p-1}+2c_{1,p-1} & \cdots & c_{q,p-1}+\sum_{i=1}^{q-1} 2c_{i,p-1} \end{bmatrix}. \tag{27}$$

At this point, a short inductive argument tells us that $c_{i,j} = 0$ whenever j is even. On the other hand, the odd columns of C are governed by a much more intricate recurrence. We begin by noting that there are no restrictions placed upon the entries in the first column of C. We therefore set

$$c_{1,1} = \beta_1, \quad c_{2,1} = \beta_2, \ldots, \quad c_{q,1} = \beta_q,$$

where $\beta_1, \beta_2, \ldots, \beta_q$ are free parameters. The remaining odd columns can be computed using (27), which provides us with the recursion

$$c_{i,2k+1} = (-1)^{q-1}\big(c_{i,2k-1} + \sum_{\ell=1}^{i-1} 2c_{\ell,2k-1}\big). \tag{28}$$

From this one can deduce that the entries of C satisfy

$$c_{i,j} = (-1)^{\frac{p-1}{2}(q-1)} \sum_{\ell=0}^{i-1} S_{\lfloor \frac{j}{2} \rfloor}(\ell)\beta_{i-\ell}, \tag{29}$$

where $S_k(n)$ denotes the coordination sequence for a k-dimensional cubic lattice (see Table 1). To be more specific, $S_k(n)$ represents the number of lattice points in $\mathbb{Z}^k$ at distance n from the origin. For instance, the first row of Table 1 is $1, 0, 0, \ldots$ since the trivial lattice contains only the origin. The second row of Table 1 is $1, 2, 2, \ldots$ since the lattice $\mathbb{Z}^1 = \mathbb{Z}$ contains exactly one point at distance one from 0 (namely 0 itself), and exactly two points at distance n from 0 (namely $\pm n$) for $n \geq 1$. See [2] for more information about the numbers $S_k(n)$.

Although we do not indulge in the somewhat lengthy derivation of (29), it is possible to highlight the main points. First observe the Toeplitz matrix Λ_q, given by (11), which appears in (9) is a truncation of the infinite Toeplitz matrix having symbol

$$\frac{1+x}{1-x} = 1 + 2x + 2x^2 + 2x^3 + \cdots.$$

This gives rise to the coefficients evident in the recursion (28). More generally, we have the generating function identity

$$\left(\frac{1+x}{1-x}\right)^k = \sum_{n=0}^{\infty} S_k(n)x^n, \tag{30}$$

which can be used to explain the appearance of the $S_k(n)$ which arise through repeated applications of the recursion (28).

n	0	1	2	3	4	5	6	7	8	9
$S_0(n)$	1	0	0	0	0	0	0	0	0	0
$S_1(n)$	1	2	2	2	2	2	2	2	2	2
$S_2(n)$	1	4	8	12	16	20	24	28	32	36
$S_3(n)$	1	6	18	38	66	102	146	198	258	326
$S_4(n)$	1	8	32	88	192	360	608	952	1408	1992
$S_5(n)$	1	10	50	170	450	1002	1970	3530	5890	9290
$S_6(n)$	1	12	72	292	912	2364	5336	10836	20256	35436
$S_7(n)$	1	14	98	462	1666	4942	12642	28814	59906	115598
$S_8(n)$	1	16	128	688	2816	9424	27008	68464	157184	332688
$S_9(n)$	1	18	162	978	4482	16722	53154	148626	374274	864146
$S_{10}(n)$	1	20	200	1340	6800	28004	97880	299660	822560	2060980

TABLE 1. The initial terms of the sequences $S_0(n), S_1(n), \ldots, S_{10}(n)$.

$n =$	0	1	2	3	4	5	6	7	8	9
$G_0(n)$	1	1	1	1	1	1	1	1	1	1
$G_1(n)$	1	3	5	7	9	11	13	15	17	19
$G_2(n)$	1	5	13	25	41	61	85	113	145	181
$G_3(n)$	1	7	25	63	129	231	377	575	833	1159
$G_4(n)$	1	9	41	129	321	681	1289	2241	3649	5641
$G_5(n)$	1	11	61	231	681	1683	3653	7183	13073	22363
$G_6(n)$	1	13	85	377	1289	3653	8989	19825	40081	75517
$G_7(n)$	1	15	113	575	2241	7183	19825	48639	108545	224143
$G_8(n)$	1	17	145	833	3649	13073	40081	108545	265729	598417
$G_9(n)$	1	19	181	1159	5641	22363	75517	224143	598417	1462563
$G_{10}(n)$	1	21	221	1561	8361	36365	134245	433905	1256465	3317445

TABLE 2. The initial terms of the sequences $G_0(n), G_1(n), \ldots, G_{10}(n)$.

Recalling that $B = -J_p(0)^T C^T \Gamma_q^{-T}$ one can rapidly deduce that the odd indexed rows of B are identically zero. For the even indexed rows of B, we may appeal to the generating function identity

$$\left(\frac{1+x}{1-x}\right)^k \cdot \frac{1}{1-x} = \left(\sum_{n=0}^{\infty} S_k(n)x^n\right)\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} G_k(n)x^n,$$

where $G_k(n) = \sum_{\ell=0}^{n} S_k(\ell)$ denotes the number of lattice points in $\mathbb{Z}^k$ at distance $\leq n$ from the origin (see Table 2), to eventually obtain

$$[B]_{i,j} = \begin{cases} 0 & \text{if } i \text{ is odd,} \\ (-1)^{j+(q+1)(\frac{p-1}{2})} \displaystyle\sum_{\ell=0}^{q-j+1} G_{k-1}(\ell)\beta_{i-\ell} & \text{if } i = 2k. \end{cases}$$

For example when $p = 7$ and $q = 5$, we have

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -(\beta_5+\beta_4+\beta_3+\beta_2+\beta_1) & \beta_4+\beta_3+\beta_2+\beta_1 & -(\beta_3+\beta_2+\beta_1) & \beta_2+\beta_1 & -\beta_1 \\ 0 & 0 & 0 & 0 & 0 \\ -(\beta_5+3\beta_4+5\beta_3+7\beta_2+9\beta_1) & \beta_4+3\beta_3+5\beta_2+7\beta_1 & -(\beta_3+3\beta_2+5\beta_1) & \beta_2+3\beta_1 & -\beta_1 \\ 0 & 0 & 0 & 0 & 0 \\ -(\beta_5+5\beta_4+13\beta_3+25\beta_2+41\beta_1) & \beta_4+5\beta_3+13\beta_2+25\beta_1 & -(\beta_3+5\beta_2+13\beta_1) & \beta_2+5\beta_1 & -\beta_1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

References

1. Arthur T. Benjamin and Jennifer J. Quinn, *Proofs that really count*, The Dolciani Mathematical Expositions, vol. 27, Mathematical Association of America, Washington, DC, 2003, The art of combinatorial proof. MR 1997773 (2004f:05001)
2. J. H. Conway and N. J. A. Sloane, *Low-dimensional lattices. VII. Coordination sequences*, Proc. Roy. Soc. London Ser. A **453** (1997), no. 1966, 2369–2389. MR 1480120 (98j:11051)
3. Fernando De Terán and Froilán M. Dopico, *The solution of the equation $XA + AX^T = 0$ and its application to the theory of orbits*, Linear Algebra Appl. **434** (2011), no. 1, 44–67. MR 2737231
4. Andrii R. Dmytryshyn, Vyacheslav Futorny, and Vladimir V. Sergeichuk, *Miniversal deformations of matrices of bilinear forms*, Linear Algebra Appl. **436** (2012), no. 7, 2670–2700. MR 2890026
5. Andrii R. Dmytryshyn, Bo Kagström, and Vladimir V. Sergeichuk, *The solution of a pair of matrix equations $(X^TA + AX, X^TB + BX) = (0,0)$ with skew-symmetric A and B*, UMEA University Department of Computing Science Technical Report, SE-901 87 UMEA, Sweden.
6. Stephan Ramon Garcia and Amy L. Shoemaker, *On the matrix equation $XA + AX^T = 0$*, Linear Algebra Appl. **438** (2013), no. 6, 2740–2746. MR 3008530
7. Roger A. Horn and Charles R. Johnson, *Matrix analysis*, second ed., Cambridge University Press, Cambridge, 2013. MR 2978290
8. Roger A. Horn and Vladimir V. Sergeichuk, *Canonical forms for complex matrix congruence and *congruence*, Linear Algebra Appl. **416** (2006), no. 2-3, 1010–1032. MR 2242477 (2007c:15018)
9. ______, *Canonical matrices of bilinear and sesquilinear forms*, Linear Algebra Appl. **428** (2008), no. 1, 193–223. MR 2372588 (2009a:15040)

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