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Radial Singular Solutions to Semilinear Partial Differential Equations

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Radial Singular Solutions to Semilinear Elliptic Partial Differential Equations

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Acknowledgments

This is it!!!!

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Abstract

We show the existence of countably many non-degenerate continua of singular radial solutionss to a p-subcritical, p-Laplacian Dirichlet problem on the unit ball in $\mathbb{R}^N$. This result generalizes those for the 2-Laplacian to any value p [Ardila et al. (2014)], and extends recent work on the p-Laplacian by considering solutions both radial and singular [Castro et al. (2019)].

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Background

1.1 Author's Note

*Does it even have a ****ing solution?*

– Anonymous

The new decade has been an era of complex, critical thought for humanity. Ricky Gervais' Golden Globe monologue went #1 trending on YouTube for seven consecutive hours. Netflix took our favorite creepy white-male T.V. show archetype and spawned the conflictingly-creepy, white-male, T.V. show archetype in season 2 of hit series *You* – that's right Fred from *Scooby-Doo!* and Louis C.K. from real life – Penn Badgley owns Netflix now. Our president got impeached, we're at war again, and now humanity faces extinction at the hands of a virus named after America's best selling beer. On the bright side, *Parasite* won best picture! Wooooo!

But, on a darker side, we are not here for a 2020 rundown – we are here for some 2020 mathematics. The years where corporeal punishment, strict dress codes, and wrongly chosen majors motivating us to "find x" on our 8^{th} - 18^{th} grade algebra problem sets are gone! *ring-ring*. Who is it? It's the new decade(!), and it's calling us *Into the Unknown* in classic, magical-Disney-forest-vibes fashion. We're done finding x. We're done with that. *Let it Go*. It's time we start looking for something bigger. Something faster. Something better. Because 2020 is the year we find $f(x)$.

1.2 Differential Equations for Beginners

In order to solve this differential equation, you look at it until a solution occurs to you.

– George Pòlya

Differential equations (D.E.'s) model change. Solutions to these equations describe a system's behavior as this change occurs. And this would be the end of the story if solutions were easy to come by, but solution-finding often bears complicated tasks. The real world is inherently complex, and the equations modeling physical processes reflect this complexity in their expression. Systems with abundant complexity require D.E.'s hosting even greater complexity. An ideal goal for understanding how these systems behave is by finding explicit solutions to their complicated differential equations!

This can be quite hard. Impossible most of the time. Intuitively we'd think, "why care about an equation that we can't find all solutions to?" But we'll see that even these D.E.'s are from unstudy-able. They are at times the conservation laws of physicists, the predictive (or not-so-predictive) market models of economists, the weather forecasting methods of meteorologists, and much, much more.

If D.E.'s appear to be such a universally studied field, where then does the mathematician fit in? Well, these clever people (at least the ones who research D.E.'s) frequently grapple questions lying at the foundations of the work of physicists, economists, and other friends. Like whether solutions to D.E.'s can even exist(!) (called *existence* problems) – and if they do, then how many exist(?) (*uniqueness* and/or *multiplicity* problems), and do these solutions behave "nicely" in any way (*regularity* problems)?

The main purpose of this thesis is proving that the quasilinear boundary value problem

$$\begin{aligned} \triangle_p u + g(u) = 0 \text{ in } x \in R^N;\ \|x\| < 1, \\ u(x) = 0 \text{ for } \|x\| = 1 \end{aligned} \tag{1.1}$$

with g p-subcritical, has uncountably many non-degenerate continua of singular radial solutions, see Theorem 2.6 below. In other words, this thesis project tackles the question of *multiplicity of solutions* to a very special case of a famous differential equation (Laplace's equation).

With this in mind, let's first speak on the topic of existence. `Great! Let's look for a solution!` you say. "BUT WAIT! Hold on" I say – "we can't just look for *any* ol' solution. The key is in the pudding..." Let us approach the simplest kind of differential equation.

The **ordinary differential equation** (O.D.E.) is, simply put, an equation relating two expressions of single-variable functions, their derivatives, and their arguments.

Example: A Polite Nudge Toward Picard-Lindelöf

Suppose now that we are given a Lipschitz continuous function $F : \mathbb{R} \to \mathbb{R}$ and are asked to find the function $f : \mathbb{R} \to \mathbb{R}$ satisfying the ordinary differential equation

$$F(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x), \qquad \forall x \in \mathbb{R}. \tag{1.2}$$

Remark 1. Note that this O.D.E. represents an infinite number of different O.D.E's, all depending on the chosen continuous function F. With this example, we are able to "kill uncountably many birds with one stone" so to speak.

Where would we start? If F is Riemann integrable – i.e. we can integrate using methods from high school calculus, then choosing some $a \in \mathbb{R}$, integrating over the interval $[a, x]$, and using the Fundamental Theorem of Calculus, we have

$$\begin{aligned} \int_a^x F(t)\ \mathrm{d}t &= \int_a^x \frac{\mathrm{d}}{\mathrm{d}t} f(t)\ \mathrm{d}t \\ &= f(x) - f(a), \end{aligned} \tag{1.3}$$

which tells us that

$$f(x) = f(a) + \int_a^x F(t)\ \mathrm{d}t. \tag{1.4}$$

Hooray! Our O.D.E. has a solution. In fact, it has *uncountably many* of them! This should not come as a surprise: our solutions f depend not only on x but also on our initially chosen value a. We call this type of solution to D.E.'s the **general solution**. It is the solution that encompasses the entire family of functions solving the O.D.E.!

This shows that for any given continuous function F, there exists uncountably many different choices of a for our unknown function f to satisfy the differential equation. This is a particularly nice property to have, and we will later see that even equations with much, much higher complexity than that of (1.2) can still bear similar fruit.

A nice question to ask could be "What if we *did* want a specific solution? How should a P.D.E. problem be stated so that solutions exist, or so that solutions are unique, or so that solutions behave nicely if their initial conditions are changed in small amounts?" And like the question points out, this has to do with how the problem is *posed*.

Definition 1.1. We say that a problem is **well-posed** (in the sense of Hadamard) if

1. there exists a solution,
2. the solution is unique, and
3. the solution depends continuously on initial conditions.

Point (3) refers to a very nice property of well-behaved solutions, in that small deviations made to the initial input value (e.g. a from earlier) creates correspondingly small deviations in the behavior of the solution (we'll speak more specifically on this soon ☺)!

Evidently, there are many ways to pose a D.E. problem. However we do pose it, finding an explicit solution to the D.E. is still, more often than not, an incredibly difficult task. It is the complexity of the model itself that prevents this from occurring easily.

A **partial differential equation** (P.D.E.) is a "not-so-ordinary" type of differential equation that relates expressions between functions, their possibly many arguments, and their partial derivatives.

While general solutions to O.D.E's obtain at least one free constant ($f(a)$ as we saw above), general solutions to P.D.E.'s (when they can be acquired) obtain at least one free *function*! Observe: let $F : \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying

$$\frac{\partial}{\partial x} F(x, y) = 0, \qquad \forall (x, y) \in \mathbb{R}^2. \tag{1.5}$$

Then

$$F(x, y) = f(y) \tag{1.6}$$

for *any function* $f : \mathbb{R} \to \mathbb{R}$, since

$$\ker \frac{\partial}{\partial x} = \big\{ f(y) : \ f : \mathbb{R} \to \mathbb{R} \big\}. \tag{1.7}$$

Variations of this free function can also dramatically change the characteristics or class of the solution itself! Which can get even harder to track if no explicit form of the solution can be found (and it usually can't!). Many different methods are used in finding solutions to P.D.E.'s, and since P.D.E.'s and their solutions vary so much from one to another, there is no "general theory" for approaching any. Most work is done after their *classification*, and most P.D.E.'s are classified by properties like their highest derivative term, whether their terms are linear or nonlinear, the properties of their operator, and many more.

Let us now seamlessly transition into a famous excerpt from Richard Manylegs' short story collection *What Happens to Humans Who Don't Study D.E.'s*.

An ODE to Humans

by Richard Manylegs

Magindale Pedecent was a vicious warrior centipede from the Galagar-Voroy system. Unlike her siblings, she excelled at mathematics but never pursued it. Often putting more of her time into strategizing for military soft-wear technology companies: high-paying, full-coverage companies promising happier lives in their uncushioned cubical chairs to graduating centipede-students.

One morning, while her coffee still steamed, she looked out the window and into a cloud. She imagined herself in another life: Magindale the Voroyan mathematician. World specialist in V.D.E. research (Voroyan differential equations). Stumbled upon the secret to Voroyan-interstellar-space-travel one morning before skittering into her shower. A publication two days later. Awards raining down on her like thunder-gravel does on Galagar C.

Her grip tightened on her mug. She wondered what she would do with that knowledge. A low breeze tickled her second leg and she swiped a fly away.

She would get rid of the primates. Them and their noise. She'd be doing them a favor - it's not like their planet had more than a few decades years left. At most. The academy would fund her. They'd send her to Earth with death rays and death beams and their finest flock of Voroyan warriors. The ones with a million legs. She shuffled her feet under the table.

Magindale, the most benevolent of scientists from the Galagar-Voroyan system, would arrive on Earth and test their knowledge. With her death rays and death stars, she'd make the following announcement:

"Apes - if you can provide for us solutions to the differential equation describing the motion of a ball rolling down an inclined plane 10 árjgĕns high given an initial velocity of 2 bĕmb-árs, we shall spare your tribes and go back to our colonies in peace."

Morale of the story: without further specification on how a P.D.E. problem is posed, we are doomed to become meat pies to some vastly intelligent troop of warrior centipedes.

Many interesting mathematical problems are *physically motivated*; advancements in the foundations of the physical sciences have often created a need for newer, at times more profound, mathematical infrastructure.

In this thesis, we will pose and answer several questions about partial differential equations (P.D.E.'s). Some questions might come with little to no physical motivation at all, some might. If you feel like this has occurred already – you wouldn't be wrong. But, to outright ignore the physical motivations present in this work, I believe, would be a great disservice. I mean, who knows? Maybe someone finds a great application to these studies (looking at you, star-folk in physics!) and can give this thesis a nice little nudge of recognition and applicability. Hopefully for its *readability*.

We tackle the question of multiplicity of solutions to a *very special case* of *very* famous differential equations, similar to that of our first example. Over the course of the background section, I aim to introduce this objective as clearly as possible, still attempting to pay close attention to the connective tissue beneath the subject matter we may pass through.

Finally, from the words of my previous summer research advisor:

1.3 The Initial Value Problem (I.V.P.)

> *This project is in pure mathematics, and its most direct applications are also of theoretical nature. Remember that Einstein's General Relativity theory remained as an essentially purely theoretical mind practice for almost a century before humankind came up with a profound application of it, which shapes our daily lives today – the GPS.*
>
> – Dr. Zhirayr Avetisyan

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index where each α_i is a nonnegative integer. Denote $|\alpha| = \sum_{i=1}^{n} \alpha_i$ and define

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u. \tag{1.8}$$

Additionally, if k is a nonnegative integer, define the collection of all partial derivatives of order k as

$$D^k u = \{D^\alpha u : |\alpha| = k\}, \tag{1.9}$$

which we will call the **$\mathbf{k^{th}}$-order derivative operator**.

Definition 1.2 (Partial Differential Equation)**.** Let Ω be an open subset of $\mathbb{R}^n$ and $u : \Omega \longrightarrow \mathbb{R}$. We call an equation of the form

$$F(D^k u(x),\ D^{k-1}u(x)\ , \dots , Du(x), u(x), x) = 0 \tag{1.10}$$

a **$\mathbf{k^{th}}$-order partial differential equation** (P.D.E.) if the k^{th} order derivative term is the highest order derivative term present, where

$$F : \mathbb{R}^{\sum_{i=1}^{k} n^i} \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R} \tag{1.11}$$

is given, and u is the unknown function. We say that u *solves* the P.D.E. if it satisfies the differential equation in (1.10).

Definition 1.3 (Linear and Nonlinear P.D.E.)**.** A k^{th}-order P.D.E. is **linear** if it can be written as

$$\sum_{|\alpha| \le k} a_\alpha(x)\, D^\alpha u(x) = f(x). \tag{1.12}$$

Alternatively, call a P.D.E. is **nonlinear** if it is not linear [Levandosky (25 Sept. 2002)].

Definition 1.4 (Types of Nonlinear P.D.E.)**.** A P.D.E. is **quasilinear** if it can be written as

$$\sum_{|\alpha|=k} a_\alpha\big(D^{k-1}u, \dots , Du, u, x\big)\, D^\alpha u(x) + a_0\big(D^{k-1}u, \dots , Du, u, x\big) = 0, \tag{1.13}$$

or in other words, if its highest order derivative terms occur linearly.

We call it **semilinear** if it can be written as

$$\sum_{|\alpha|=k} a_\alpha(x)\, D^\alpha u(x) + a_0\big(D^{k-1}u, \dots , Du, u, x\big) = 0, \tag{1.14}$$

or in other words, if it is quasilinear and if the coefficients of the highest order derivative terms do not depend on the unknown function or its lower order derivatives.

Finally, we call it **fully nonlinear** if the highest-order derivatives of u appear nonlinearly in the equation [Levandosky (25 Sept. 2002)].

Below are common examples and names of famous P.D.E's a physics or mathematics student would encounter in their studies (or in their thesis).

Example: Partial Differential Equations

1. Laplace's equation (2nd order; linear):

$$\triangle u = \sum_{i=1}^{n} u_{x_i x_i} = 0.$$

2. Heat equation (2nd order; linear):

$$u_t - \triangle u = 0.$$

3. Wave equation (2nd order; linear):

$$u_{tt} - \triangle u = 0.$$

4. Schrödinger's equation for a free particle (see Heat equation):

$$\imath u_t + \triangle u = 0.$$

5. Dimensionless Newton's second law (2nd order):

$$F\big(u(t), t\big) = u_{tt}.$$

6. p-Laplace equation (quasilinear):

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0.$$

Like mentioned before, finding general solutions explicitly is not an easy task once an equation becomes even *slightly* more complex (let's all thank computers for still trying though). But this task can be simplified if we choose to go at it a different direction.

Instead of finding the general solution of a P.D.E. explicitly then checking to see if certain solutions lie in the span of the general one, we can instead ask:

"are there solutions to the P.D.E. that meet [*certain* requirements] in the first place?" In other words: can we have a better time finding solutions to P.D.E.'s by imposing extra requirements on the solutions we look for from the get-go?

The answer is...*drumroll*...yes! The *Initial value problem* (I.V.P.) is a type of P.D.E. problem requiring the solutions sought after to satisfy some *initial condition* (or *initial data*) given as input.

Definition 1.5 (Initial Value Problem)**.** Let $\Omega \subset \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a solution to a partial differential equation. An **initial value problem** is posed as a partial differential equation alongside a point $(t_0, u_0) \in \mathbb{R} \times \Omega$ called the **initial condition**.

A function u then *solves* the I.V.P. if it is both:

(i) a solution to the partial differential equation, and

(ii) satisfies the condition

$$u(t_0, x) = u_0(x) \qquad \forall x \in \Omega.$$

Recall that proving a P.D.E. *has* solutions is at the forefront of big questions in the field of differential equations. To be able to prove this existence, we first desire the governing functional of our P.D.E. to exhibit some kind of continuity property. The more "well-behaved" our functional is, the more we are capable saying about the solutions to the P.D.E. One of these continuity properties is global Lipschitz continuity, or simply, just Lipschitz continuity.

Definition 1.6 (Lipschitz Continuous)**.** Let $F : X \to Y$ be a function between two metric spaces (X, d_X) and (Y, d_Y). We say that F is **Lipschitz continuous** if there exists $0 \le k \in \mathbb{R}$ (called the **Lipschitz constant**) such that

$$d_Y(F(x_1), F(x_2)) \le k d_X(x_1, x_2) \qquad \forall x_1, x_2 \in X.$$

Additionally, if F maps from one metric space to itself and the Lipschitz constant is strictly less than one, we say that F is a **contraction** (or a contraction mapping).

While we may have narrowed our search from looking at *all possible solutions* of a P.D.E. to only those satisfying a provided initial condition, this added requirement allows us to prove our first theorem regarding the *existence* of these solutions to I.V.P.'s with *well-behaved* P.D.E.'s!

Theorem 1.1 (Picard-Lindelöf). *Let $D \subset \mathbb{R} \times \mathbb{R}^n$ be an open set, and let $F : D \mapsto \mathbb{R}^n$ be continuous in the first variable and Lipschitz continuous in its second variable. Then, for any $(t_0, y_0) \in D$, there exists an interval $I := (t^-, t^+)$ containing t_0 and at least one solution $y : I \longrightarrow \mathbb{R}^n$ to the initial value problem.*

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = F(t, y(t)), \tag{1.15}$$

$$y(t_0) = y_0. \tag{1.16}$$

The source of many techniques in P.D.E.'s comes from the construction of an equivalent integral equation. Below, we outline and motivate the proof of Picard-Lindelöf. Let $(t_0, y_0) \in D$. Integrating both sides by $[t_0, t]$, we have:

$$y(t) = y_0 + \int_{t_0}^{t} F(t, y(t))\ \mathrm{d}t. \tag{1.17}$$

The goal now is showing there exists a continuous function y satisfying the integral equation above. Note that the solution appears as a fixed point of the operator!

One way to prove existence without obtaining an explicit solution is by constructing a sequence of functions that converges to our desired solution. We shall do this using what's called *Picard iteration*. Starting with the initial data,

$$y(t_0) = y_0 \tag{1.18}$$

iteratively define:

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} F(t, y_n(t))\ \mathrm{d}t. \tag{1.19}$$

If $\{y_n\}$ is a convergent sequence of functions such that $y_n \longrightarrow y$, then we're done, since y is a solution to the integral equation (and consequently, a solution exists to the initial value problem).

Definition 1.7 (Space of Continuous Functions). Let $\mathbf{C[a, b]}$ denote the set of all functions that are continuous on $[a, b]$.

If $y \in C[a, b]$, then the **norm** of y (i.e. the distance from the function y to the zero function) is $\|y\| := \max_{x \in [a,b]} |y(x)|$.

Furthermore, one can show that $C[a,b]$ satisfies the properties of a Banach space with the equipped norm. That is, $C[a,b]$ is a *complete* normed linear space (meaning that all sequences whose terms eventually get really close together converge).

Fix $\alpha \in \mathbb{R}^{(+)}$ and let $\mathcal{X} := \{y(t) \in C[I] : \|y - y_0\| \leq \alpha\}$. One can then show that the operator $\Gamma : \mathcal{X} \to \mathcal{X}$ defined by

$$\Gamma[y](t) := y_0 + \int_{t_0}^{t} F(s, y(s))\, \mathrm{d}s \tag{1.20}$$

is a contraction mapping. Then, our integral equation becomes:

$$\Gamma[y] = y \tag{1.21}$$

which means that the solutions to the integral equation are exactly the fixed points of our operator Γ! Luckily, we know there always exists a fixed point via the following theorem (also known as the contraction mapping principle), proven by Stefan Banach and Renato Caccioppoli.

Theorem 1.2 (Banach Fixed Point Theorem)**.** *Let (X, d) be a non-empty, complete metric space and $T : X \to X$ a contraction. Then T admits a unique fixed-point $x^* \in X$. In other words, there exists a unique point $x^* \in X$ such that $T(x^*) = x^*$.*

Which proves there exists a solution to (1.21), and thus, a solution to the initial value problem. For a proof of this theorem, the reader is referred to [Renardy and Rogers (2004)].

2

Introduction

Consider the linear Dirichlet problem

$$\begin{aligned} \triangle u + g(u) &= q(\|x\|), \quad && x \in B_1(0) \subset \mathbb{R}^N, \\ u(x) &= 0, && x \in \partial B_1(0), \end{aligned} \tag{2.1}$$

where $B_1(0)$ denotes the unit ball around the origin, $g : \mathbb{R} \mapsto \mathbb{R}$ is a locally Lipschitz continuous function (meaning it is Lipschitz continuous on any interval), $q : B_1(0) \mapsto \mathbb{R}$ is a continuous, bounded, radially symmetric function, and $\triangle$ denotes the Laplace operator.

Recall that in Cartesian coordinates, the action of the 3-dimensional Laplacian on a twice-differentiable function $u : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}. \tag{2.2}$$

The modeling of spherically symmetric systems play a large role in fields such as physics and chemistry. In spherical coordinates, the action of the 3-dimensional Laplacian can be expressed as

$$\triangle u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}, \tag{2.3}$$

an expression arguably greater in complexity than its Cartesian sibling. This complexity continues to trend in the N-dimensional case. The action of the n-dimensional **Laplace operator** in spherical coordinates is

$$\triangle u = \frac{\partial^2 u}{\partial r^2} + \frac{N-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Delta_{S^{N-1}} u, \tag{2.4}$$

where $\Delta_{S^{N-1}}$ is an operator depending only on the $N-1$ angular coordinates acting on S^{N-1}, the unit sphere embedded on $\mathbb{R}^{N-1}$. This operator is known as the Laplace-Beltrami operator – a generalization of the Laplace operator to Riemannian manifolds.

We will call a function **radial** if it is independent of all angular coordinates. If the function u is radial, then equation (2.4) becomes

$$\triangle u = \frac{\partial^2 u}{\partial r^2} + \frac{N-1}{r}\frac{\partial u}{\partial r}. \tag{2.5}$$

Call u a **radial solution** to the Dirichlet problem in (2.1) if u satisfies

$$\begin{gathered} u'' + \frac{n-1}{r}u' + g(u) = q(r), \qquad r \in [0, T], \\ u(T) = 0. \end{gathered} \tag{2.6}$$

For additional information on radial solutions to quasilinear problems, the reader is referred to [Castro and Jacobsen (2020)] and [Jacobsen and Schmitt (2000)]. This boundary value problem has been studied extensively through the study of the associated initial value problem:

$$\begin{gathered} u'' + \frac{N-1}{r}u' + g(u) = q(r), \qquad r \in [0, 1], \\ u'(0) = 0 \quad \text{and} \quad u(0) = d. \end{gathered} \tag{2.7}$$

Existence and uniqueness of solutions to the initial-value problem for every $d \in \mathbb{R}$, as well as continuity with respect to these initial conditions, can then be demonstrated using an application of Banach's fixed-point theorem.

Because our goal is to find solutions to a boundary value problem, we wish to better understand how the solutions of the initial value problem behave at and near our specified boundary. To help do so, we will define an "energy" quantity to these solution functions that will help measure their oscillatory behavior. Let

$$E(r; d) \doteq \frac{1}{2}(u'(r; d))^2 + G(u(r; d)), \qquad \text{where } G(u) = \int_0^u g(v)\,\mathrm{d}v, \tag{2.8}$$

denote the total **classical energy** (analogous to the total *mechanical energy* of a physical system) a solution u to the initial-value problem in (2.7) would have, associated to the initial condition d, at a point $r \in [0, 1]$ in its domain.

Sufficient conditions for the classical energy to tend to infinity, as initial conditions get very large, were given by the work of Castro and Kurepa in [Castro and Kurepa (1987a)] as follows. For $\kappa \in (0,1]$, $\rho > 0$, g strictly increasing, and $g(0) = 0$, they define, a priori, the functions

$$\Lambda(\kappa, u) := nG(\kappa u) - \frac{N-2}{2}\, u g(u)\,, \tag{2.9}$$

$$\Lambda_{\pm}(\kappa) := \lim_{u \to \pm\infty} \Lambda(\kappa, u) \left(\frac{u}{g(u)}\right)^{N/2}, \text{ and} \tag{2.10}$$

$$F(d) := \left(\frac{d}{g(d)}\right)^{N+\rho-1} G(\kappa d)\,, \tag{2.11}$$

to prove the following theorem:

Theorem 2.1 (Castro, Kurepa [1987]). *Suppose that $|g(u)| \geq a|u|$ holds for some $a > 0$ when u sufficiently large. If*

i) $\Lambda(1, u)$ is bounded below and $\Lambda_+(\kappa) = \infty$ (respectively $\Lambda_-(\kappa) = \infty$) for some $\kappa \in (0, 1)$, or

ii) $F(d) \to \infty$ as $d \to \infty$ (respectively $F(d) \to \infty$ as $d \to -\infty$),

then

$$\lim_{d\to\infty} E(r; d) = \infty, \qquad (\textit{respectively} \lim_{d\to-\infty} E(r; d) = \infty) \tag{2.12}$$

uniformly for $r \in [0, 1]$.

This result, alongside the phase-plane analysis in [Castro and Kurepa (1987a)], was used to show that there are sufficiently large initial conditions such that the initial value problem (2.7) is a solution to the Dirichlet problem (2.1). From [Castro and Kurepa (1987b)], the theorem is stated as follows:

Theorem 2.2 (Castro, Kurepa [1987]). *Suppose $\lim_{|u|\to\infty} \frac{g(u)}{u} = \infty$. If*

i) $\Lambda(1, u)$ is bounded below and $\Lambda_+(\kappa) = \infty$ (respectively $\Lambda_-(\kappa) = \infty$) for some $\kappa \in (0, 1)$, or

ii) $F(d) \to \infty$ as $d \to \infty$ (respectively $F(d) \to \infty$ as $d \to -\infty$),

then the boundary value problem

$$\begin{aligned} \triangle u + g(u) &= q(\|x\|), \quad & x &\in B_1(0) \subset \mathbb{R}^N, \\ u(x) &= 0, & x &\in \partial B_1(0), \end{aligned}$$

has infinitely many radial solutions with $u(0) > 0$ (respectively $u(0) < 0$).

Note that we can also speak of ways to categorize different types of the Dirichlet problem in (2.1). One way to differentiate between different versions of the problem is by the growth rate of the nonlinear (or perturbation) function g. We say that g is a **superlinear nonlinearity** if it satisfies the growth rate condition

$$\lim_{|u|\to\infty} \frac{g(u)}{u} = \infty. \tag{2.13}$$

Another way to "categorize" the problem is by changing the types of solutions we look for. For example, by weakening the requirements for a function to be a solution for our Dirichlet problem, we may investigate properties of the new solution space and see if holds any similarities to the old one. Call u a **singular radial** solution to the Dirichlet problem if u is radial, the functions $r^{N-1}u'$ and $r^{N-1}g(u)$ are integrable on $[0,1]$, $u(r) \to \pm\infty$ as $r \to 0$, and if u satisfies

$$\int_{B_1(0)} \big(u'\varphi' + (q(r) - g(u))\varphi\big)\ \mathrm{d}r = 0. \tag{2.14}$$

In other words, we would call u a **weak solution** to the Dirichlet problem in (2.1), for all functions $\varphi : B_1(0) \mapsto \mathbb{R}$ of class C^∞ having compact support in $B_1(0)$. Such functions u satisfying (2.14) are also known as distributional solutions to the boundary value problem. The study of these solutions rely heavily on the initial value problem

$$\begin{gathered} u'' + \frac{N-1}{r}u' + g(u) = q(r), \qquad r \in [0,1], \\ u(\tilde{b}) = c_1 \quad \text{and} \quad u'(\tilde{b}) = c_2, \end{gathered} \tag{2.15}$$

where $0 < \tilde{b} < 1$ and c_1, c_2 constants.

Consider the following cases for $p \in \left(\frac{N}{N-2}, \frac{N+2}{N-2}\right)$:

1. Say g is a **subcritical nonlinearity** if

$$\begin{gathered} \lim_{u\to+\infty} \frac{g(u)}{u^p} \in (0,\infty), \quad \text{and} \\ \lim_{u\to-\infty} \frac{g(u)}{-|u|^q} \in (0,\infty), \qquad \text{for } q \in \left(1, \frac{N+2}{N-2}\right). \end{gathered} \tag{2.16}$$

2. Say g is a **sub-super critical nonlinearity** if

$$\begin{gathered} \lim_{u\to+\infty} \frac{g(u)}{u^p} \in (0,\infty), \quad \text{and} \\ \lim_{u\to-\infty} \frac{g(u)}{-|u|^q} \in (0,\infty), \qquad \text{for } q > \frac{N+2}{N-2}. \end{gathered} \tag{2.17}$$

3. Say g is a **jumping nonlinearity** if

$$\lim_{u\to+\infty}\frac{g(u)}{u^p}\in(0,+\infty), \quad \text{and} \\ \lim_{u\to-\infty}\frac{g(u)}{u}=\gamma\in(-\infty,\lambda_1), \qquad \text{for } \lambda_1>0. \tag{2.18}$$

where λ_1 is the principal eigenvalue of $-\triangle$ in $B_1(0)$ with zero Dirichlet boundary data.

Finally, a **continuum** is a nonempty compact, connected metric space, and a **non-degenerate continuum** is a contiuum containing more than a one point.

The following theorems were proven in [Ardila et al. (2014)] regarding the multiplicity of solutions satisfying both the initial value problem in (2.15) and the boundary condition $u(1)=0$, for different nonlinearity cases shown above.

Theorem 2.3 (**Ardila et al. [2014]**). *If g is subcritical or sub-super critical, then the Dirichlet problem*

$$\begin{aligned} \triangle u+g(u)&=q(\|x\|), && x\in B_1(0)\subset\mathbb{R}^N, \\ u(x)&=0, && x\in\partial B_1(0), \end{aligned} \tag{2.19}$$

has countably many non-degenerate continua of singular, radial solutions. In particular, the problem has uncountably many radial singular solutions.

Theorem 2.4 (**Ardila et al. [2014]**). *If g is a jumping nonlinearity then the Dirichlet problem*

$$\begin{aligned} \triangle u+g(u)&=q(\|x\|), && x\in B_1(0)\subset\mathbb{R}^N, \\ u(x)&=0, && x\in\partial B_1(0), \end{aligned}$$

has two non-degenerate continua of singular radial solutions.

We aim to extend Theorem 2.3 to the subcritical quasilinear Dirichlet problem. The quasilinear analog of the Laplace operator, the **p-Laplace** operator, is defined in n-dimensional Cartesian coordinates as

$$\begin{aligned} \triangle_p u &:= \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \qquad \text{for } 1\le p<\infty, \\ &= |\nabla u|^{p-4}\left[|\nabla u|^2\Delta u+(p-2)\sum_{i,j=1}^{N}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial x_j}\right], \end{aligned} \tag{2.20}$$

where

$$|\nabla u|^{p-4} = \left[\left(\frac{\partial u}{\partial x_1} \right)^2 + \cdots + \left(\frac{\partial u}{\partial x_N} \right)^2 \right]^{\frac{p-6}{2}}. \tag{2.21}$$

When $p = 2$, the p-Laplace operator turns into the Laplace operator. Additionally, the the p-Laplace operator turns into the $\infty-$Laplace operator as $p \longrightarrow \infty$, and the Mean Curvature operator when $p = 1$. All have their own interesting geometric and analytic properties, as well as a fair share of applications.

We study the following extension of the semilinear Dirichlet problem in (2.1):

$$\begin{aligned} \triangle_p u + g(u) = 0, \quad x \in B_1(0) \subset \mathbb{R}^N, \\ u(x) = 0, \quad x \in \partial B_1(0), \end{aligned} \tag{2.22}$$

assuming $1 < p < N$ and g is a **p-subcritical nonlinearity**

$$g(s) := \begin{cases} |s|^{q_1} & s \geq 0, \\ -|s|^{q_2} & s < 0, \end{cases} \qquad \text{with } p - 1 < q_1 < q_2 < p^* - 1 < \infty, \tag{2.23}$$

with $p^* := \frac{Np}{N-p}$ denoting the *critical Sobolev exponent*.

We say that g has **p-sub-super critical** growth if

$$g(s) := \begin{cases} |s|^{q_1} & s \geq 0, \\ -|s|^{q_2} & s < 0, \end{cases} \qquad \text{with } p - 1 < q_1 < p^* - 1 < q_2 < +\infty. \tag{2.24}$$

The existence of infinitely many radial regular solutions for a p-subcritical and a p-sub-super critical Dirichlet problem on the unit ball was shown in [Castro et al. (2019)]. The radial solutions to the boundary value problem (3.1) are the solutions to

$$\begin{aligned} (p-1)\,|u'|^{p-2}\,u'' + \frac{N-1}{r}\,|u'|^{p-2}\,u' + g(u(r)) = 0, \qquad 0 < r < 1, \\ u'(0) = 0, \quad u(1) = 0. \end{aligned} \tag{2.25}$$

The proofs in [Castro et al. (2019)] rely on the study of the initial value problem

$$\begin{aligned} (p-1)\,|u'|^{p-2}\,u'' + \frac{N-1}{r}\,|u'|^{p-2}\,u' + g(u(r)) = 0, \qquad 0 < r < 1, \\ u(0) = d, \quad u'(0) = 0. \end{aligned} \tag{2.26}$$

The results of Castro et. al are summarized below:

Theorem 2.5 (**Castro et al. [2019]**). *If g is a p-subcritical or a p-sub-super critical nonlinearity, then the problem*

$$\begin{aligned} \triangle_p u + g(u) = 0, \qquad & x \in B_1(0), \\ u(x) = 0, \qquad & x \in \partial B_1(0), \end{aligned} \tag{2.27}$$

has infinitely many radial solutions.

Their work extended the results of Theorem 2.2 to a Dirichlet problem with arbitrary p value.

The goal of this thesis is similar: we wish to extend the result shown in Theorem 2.3 for a Dirichlet problem $p \in (1, N)$. That is, we wish to study the singular radial solutions to the quasilinear, p-subcritical boundary value problem stated in (3.1). The main theorem I prove is:

Theorem 2.6 (Main Theorem). *If* (3.2) *holds, then there exists countably many non-degenerate continua of singular radial solutions to the p-subcritical problem in* (3.1).

Given a solution u to this problem, define now the following quantities:

$$\mathcal{E}(r) := \frac{p-1}{p}\,|u'(r)|^p + G\big(u(r)\big)\,, \tag{2.28}$$

$$H(r) := r\mathcal{E}(r) + \frac{N-p}{p}\,|u'(r)|^{p-2}\,u'(r)\,u(r)\,, \tag{2.29}$$

$$P(r) := \int_0^r s^{N-1}\left[NG\big(u(s)\big) - \frac{N-p}{p} g\big(u(s)\big)\,u(s)\right] \mathrm{d}s\,, \tag{2.30}$$

where

$$G(s) := \int_0^s g(t)\,\mathrm{d}t = \begin{cases} \frac{|s|^{q_1+1}}{q_1+1}, & s \geq 0, \\ \frac{|s|^{q_2+1}}{q_2+1}, & s < 0. \end{cases} \tag{2.31}$$

The quantities above are related by the **Pohozaev-type identity**

$$r^{N-1}H(r) - t^{N-1}H(t) = \int_t^r s^{N-1}\Big[NG\big(u(s)\big) - \frac{N-p}{p}\,g\big(u(s)\big)\,u(s)\Big]\,\mathrm{d}s \tag{2.32}$$

or equivalently, when $t = 0$ as

$$\begin{aligned} r^N \left[\frac{p-1}{p} \, |u'(r)|^p + G\big(u(r)\big) \right] + \frac{N-p}{p} \, r^{N-1} \, |u'(r)|^{p-2} \, u'(r) \, u(r) \\ = \int_0^r s^{N-1} \left[N G\big(u(s)\big) - \frac{N-p}{p} g\big(u(s)\big) \, u(s) \right] \, \mathrm{d}s. \end{aligned} \tag{2.33}$$

Finally, we define the **Pohozaev energy** $\mathcal{P}_E$ of a solution u as

$$\mathcal{P}_E(r) \coloneqq r^{N-1} H(r) \, . \tag{2.34}$$

3

Singular Radial Solutions

3.1 Preliminaries

We study the quasilinear Dirichlet problem

$$\begin{cases} \triangle_p u + g(u) = 0 & \text{in } B_1(0) \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial B_1(0), \end{cases} \tag{3.1}$$

where $\triangle_p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u)$, for $1 < p < N$, denotes the p-Laplacian operator (2.20) and g is the p-subcritical nonlinearity

$$g(s) := \begin{cases} |s|^{q_1} & s \geq 0, \\ -|s|^{q_2} & s < 0, \end{cases} \qquad \text{with } p-1 < q_1 < q_2 < p^* - 1 < \infty, \tag{3.2}$$

where $p^* := \frac{Np}{N-p}$ denotes the critical Sobolev exponent. Let $I := \big(0, \frac{N-p}{p-1}\big)$. For $\alpha \in I$, let

$$\Lambda(\alpha) = \Lambda(\alpha; q_1, p, N) := \left(\alpha^{p-1}(q_1+1)\left[\frac{N-p}{p} - \alpha\frac{p-1}{p}\right]\right)^{-\frac{1}{q_1-p+1}}. \tag{3.3}$$

Our results are based on the study of the solutions to the initial-value problem

$$\begin{cases} \left(r^{N-1}\, |u'|^{p-2}\, u'\right)' + r^{N-1}\, g\big(u(r)\big) = 0, & 0 < r < 1, \\ u(b; b, \alpha) = b^{-\frac{p}{q_1-p+1}}\Lambda(\alpha),\ u'(b; b, \alpha) = -b^{-\frac{q_1+1}{q_1-p+1}}\alpha\Lambda(\alpha), \end{cases} \tag{3.4}$$

for $b \in (0,1)$. Define the classical energy $\mathcal{E}$ and Pohozaev energy $\mathcal{P}$ for any solution u of (3.4) as

$$\mathcal{E}(r; u) := \frac{p-1}{p}\, |u'(r)|^p + G\big(u(r)\big), \tag{3.5}$$

$$\begin{aligned}\mathcal{P}(r;u) &:= r^N \left[\frac{p-1}{p}\,|u'(r)|^p + G\big(u(r)\big)\right] \\ &\quad + \frac{N-p}{p}\, r^{N-1}\,|u'(r)|^{p-2}\,u'(r)\,u(r)\,,\end{aligned} \tag{3.6}$$

where

$$G(s) := \int_0^s g(t)\,\mathrm{d}t = \begin{cases} \frac{s^{q_1+1}}{q_1+1}, & s \geq 0, \\ \frac{|s|^{q_2+1}}{q_2+1}, & s < 0\,. \end{cases} \tag{3.7}$$

Henceforth, fix $\alpha \in I$ and $b \in (0,1)$ and assume $u(b;b,\alpha) > 0$ (results are analogous for $u(b;\alpha,b) < 0$).

3.2 Energy Analysis

Lemma 1. *Choose $b \in (0,1)$. If*

$$\begin{aligned} u(b) &= b^{-\frac{p}{q_1-p+1}}\Lambda(\alpha)\,, \\ u'(b) &= -b^{-\frac{q_1+1}{q_1-p+1}}\alpha\Lambda(\alpha), \end{aligned} \tag{3.8}$$

then

$$\mathcal{P}(b;u) = 0. \tag{3.9}$$

Proof. Under the assumptions of (3.8),

$$\begin{aligned}\mathcal{P}(b;u) &= b^N \left[\frac{p-1}{p}\left(b^{-\frac{q_1+1}{q_1-p+1}}\alpha\Lambda(\alpha)\right)^p + \frac{\left(b^{-\frac{p}{q_1-p+1}}\Lambda(\alpha)\right)^{q_1+1}}{q_1+1}\right] \\ &\quad - \frac{N-p}{p}\, b^{N-1}\left(b^{-\frac{q_1+1}{q_1-p+1}}\alpha\Lambda(\alpha)\right)^{p-2} \cdot b^{-\frac{q_1+1}{q_1-p+1}}\alpha\Lambda(\alpha)\cdot b^{-\frac{p}{q_1-p+1}}\Lambda(\alpha).\end{aligned} \tag{3.10}$$

Factoring and combining b and $\Lambda(\alpha)$ terms, see that

$$\begin{aligned}\mathcal{P}(b;u) &= -b^{\frac{p(q_1+1)}{p-(q_1+1)}+N}\left(\Lambda(\alpha)\right)^p \\ &\qquad \left[\frac{\left(\Lambda(\alpha)\right)^{q_1-p+1}}{q_1+1} + \alpha^{p-1}\left(\frac{p-1}{p}\,\alpha - \frac{N-p}{p}\right)\right] = 0\end{aligned} \tag{3.11}$$

from (3.3). ◻

Lemma 2. *Solutions to* (3.4) *have no critical point on* $(0, b)$.

Proof. Suppose to the contrary that there exists at least one $r^* \in (0, b)$ such that $u'(r^*) = 0$. Let $\hat{r} := \sup\{r \in (0, b) : u'(r) = 0\}$. By the continuity of u', we have $u' < 0$ over the interval $(\hat{r}, r)$. Hence, Lemma 1 and the Pohozaev-type identity (2.32) give

$$0 = \mathcal{P}\big(b; u(r; b, \alpha)\big) = \mathcal{P}\big(r; u(r; b, \alpha)\big) + \int_r^b s^{N-1}\Big[NG\big(u(s; b, \alpha)\big) - \frac{N-p}{p}\, g\big(u(s; b, \alpha)\big)\, u(s; b, \alpha)\Big]\, \mathrm{d}s \quad (3.12)$$

implying that

$$\mathcal{P}\big(r; u(r; b, \alpha)\big) = r^N G\big(u(r; b, \alpha)\big) < 0, \quad (3.13)$$

which is a contradiction, since $r^N G\big(u(r; b, \alpha)\big) > 0$ for every $r \in (\hat{r}, b)$. □

Lemma 3. *The Pohozaev energy function is a strictly increasing function on every closed interval.*

Proof. Let $[r_1, r_2] \subset [0, 1]$ be a closed interval such that $u(r) > 0$ for all $r \in [r_1, r_2]$. Employing (2.32) we have

$$\begin{aligned} \mathcal{P}(r_2; u) - \mathcal{P}(r_1; u) &= \int_{r_1}^{r_2} \Big[NG\big(u(s)\big) - \frac{N-p}{p}\, g\big(u(s)\big)\, u(s)\Big]\, \mathrm{d}s \\ &= \int_{r_1}^{r_2} \Big[\frac{N}{q_1+1}\, \big|u(s)\big|^{q_1+1} - \frac{N-p}{p}\, \big|u(s)\big|^{q_1+1}\Big]\, \mathrm{d}s \\ &\geq \min_{s \in [r_1, r_2]} \big|u(s)\big|^{q_1+1} \int_{r_1}^{r_2} s^{N-1} \Big[\frac{N}{q_1+1} - \frac{N-p}{p}\Big]\, \mathrm{d}s \\ &> 0. \end{aligned} \quad (3.14)$$

Now let $[r_1, r_2] \subset [0, 1]$ be a closed interval such that $u(r) \leq 0$ for all

$r \in [r_1, r_2]$. Observe that

$$\begin{aligned}\mathcal{P}(r_2;u) - \mathcal{P}(r_1;u) &= \int_{r_1}^{r_2} \left[\frac{N}{q_2+1}\left|u(s)\right|^{q_2+1} + \frac{N-p}{p}\left|u(s)\right|^{q_2} u(s)\right] \mathrm{d}s \\ &\geq \int_{r_1}^{r_2} \left[\frac{N}{q_2+1}\left|u(s)\right|^{q_2+1} - \frac{N-p}{p}\left|u(s)\right|^{q_2+1}\right] \mathrm{d}s \\ &\geq \min_{s\in[r_1,r_2]} \left|u(s)\right|^{q_2+1} \int_{r_1}^{r_2} s^{N-1}\left[\frac{N}{q_2+1} - \frac{N-p}{p}\right] \mathrm{d}s \\ &> 0\end{aligned} \tag{3.15}$$

as desired. □

Lemma 4. *There exists a constant $m > 1$ such that*

$$\lim_{b\to 0^+} \mathcal{P}(mb;u) = +\infty. \tag{3.16}$$

Proof. Define

$$\hat{r} := \hat{[}witch]r(b) = \sup\{r \in (b,1) \colon u(t) \geq \frac{1}{2}u(b) \text{ for all } t \in [b,r]\}. \tag{3.17}$$

Fix $r \in [b, \hat{r}]$ and integrate (3.4) on $[b, r]$:

$$\begin{aligned}-r^{N-1}\left|u'(r)\right|^{p-2}u'(r) &= -b^{N-1}\left|u'(b)\right|^{p-2}u'(b) + \int_b^r s^{N-1}\left(u(s)\right)^{q_1} \mathrm{d}s \\ &\geq -b^{N-1}\left|u'(b)\right|^{p-2}u'(b) + \left(\frac{u(b)}{2}\right)^{q_1}\frac{(r^N-b^N)}{N} \\ &\geq \left(\frac{u(b)}{2}\right)^{q_1}\frac{(r-b)\,r^{N-1}}{N},\end{aligned} \tag{3.18}$$

since $r^N - b^N \geq (r-b)r^{N-1}$ whenever $r \geq b$. Note that the first line of equation (3.18) also tells us $u'(r) < 0$ (since the right hand side is strictly positive). Dividing by a factor of r^{N-1}, we have

$$-u'(r) \geq \left(\frac{u(b)}{2}\right)^{\frac{q_1}{p-1}} \frac{(r-b)^{\frac{1}{p-1}}}{N^{\frac{1}{p-1}}}. \tag{3.19}$$

Integrating over $[b, \hat{[}witch]r]$:

$$\frac{u(b)}{2} \geq \frac{p-1}{p}\left(\frac{u(b)}{2}\right)^{\frac{q_1}{p-1}} \frac{(\hat{r}-b)^{\frac{p}{p-1}}}{N^{\frac{1}{p-1}}} \tag{3.20}$$

implying that

$$N^{\frac{1}{p-1}} \frac{p}{p-1} \left(\frac{u(b)}{2}\right)^{\frac{p-(q_1+1)}{p-1}} \geq (\hat{r} - b)^{\frac{p}{p-1}}. \tag{3.21}$$

Let $C = N^{\frac{1}{p-1}} \frac{p}{p-1} \left(\frac{1}{2}\right)^{\frac{p-(q_1+1)}{p}}$. Raising both sides to the $\frac{p-1}{p}$ power and isolating $\hat{r}$, we have

$$\hat{r} \leq b + C \cdot (u(b))^{\frac{p-(q_1+1)}{p}}. \tag{3.22}$$

Finally, substituting in (3.8) gives us

$$b \leq \hat{r} \leq (1 + C \cdot \Lambda(\alpha))\, b := mb. \tag{3.23}$$

Observe that because

$$u(t) \geq \frac{u(b)}{2} \qquad \forall t \in [b, \hat{r}], \tag{3.24}$$

Lemma 3 tells us that

$$\begin{aligned} \mathcal{P}(mb; u) &\geq \mathcal{P}(\hat{r}; u) \\ &\geq \frac{N-p}{p}\, \hat{r}^{N-1}\, |u'(\hat{r})|^{p-2}\, u'(\hat{r})\, u(\hat{r}) \\ &\geq \frac{N-p}{p}\, \hat{r}^{N-1}\, |u'(\hat{r})|^{p-2}\, u'(\hat{r})\, \frac{u(b)}{2} \\ &= \frac{N-p}{p}\, \hat{r}^{N-1}\, |u'(\hat{r})|^{p-2}\, u'(\hat{r})\, b^{\frac{p}{p-(q_1+1)}} \Lambda(\alpha) \\ &\to +\infty \end{aligned} \tag{3.25}$$

as $b \to 0^+$. Using Lemma 3 again, we conclude that $\mathcal{P}(1; u) \to +\infty$ as $b \to 0^+$. □

Corollarys 1. *The following*:

$$\lim_{b \to 0^+} \mathcal{E}(r; u) = +\infty. \tag{3.26}$$

holds for all $r \in (b, 1]$.

Proof. The energy

$$\mathcal{E}(r;u) := \frac{p-1}{p}\,|u'(r)|^p + G\big(u(r)\big)\,, \tag{3.27}$$

is a decreasing function for any solution u, since

$$\mathcal{E}'(r) = \Big[(p-1)|u'|^{p-2}u'' + g\big(u\big)\Big]\,u' \le -\frac{N-1}{r}|u'|^p \le 0 \tag{3.28}$$

for $r \in (0,1]$. Combining this with Lemma (4), we get

$$\begin{aligned} \lim_{b\to 0^+} \mathcal{P}(1;u) &= \lim_{b\to 0^+} \Big[\mathcal{E}(1;u) + \frac{N-p}{p}\,|u'(1)|^{p-2}\,u'(1)\,u(1)\Big] \\ &= \lim_{b\to 0^+} \mathcal{E}(1;u) \\ &\le \mathcal{E}(r;u) \end{aligned} \tag{3.29}$$

as desired. □

Lemma 5. *For each $b \in (0,1)$, the solutions to* (3.4) *satisfy*

$$\lim_{r\to 0} u(r) = +\infty. \tag{3.30}$$

Proof. If $u(r^*) \le 0$ for some $r \in (0,b)$, by the Mean Value Theorem, there exists $s \in (r^*, b)$ such that $u'(s) > 0$. Since $u'(b) < 0$, applying the Intermediate Value Theorem, there exists $\hat{s} \in (s,b)$ such that $u'(\hat{s}) = 0$, contradicting Lemma 2. Thus, solutions to (3.4) are positive on $(0,b)$.

By Lemma , $u(r^*) \ne 0$, so either $u'(r^*) > 0$ or $u'(r^*) < 0$. If $u'(r^*) > 0$, then by the continuity of u', there would exist $s \in (r^*, b)$ such that $u'(s) = 0$, violating Lemma 2. If $u'(r^*) < 0$, then by the continuity of u, there would exist $s_2 \in (r^*, b)$ such that $u'(s_2) > 0$. Then, by the continuity of u', there would exist $s_1 \in (r^*, s_2)$ such that $u'(s_1) = 0$, violating Lemma 2. We conclude that $u(r) > 0$ for all $r \in (0,b)$.

Suppose there exists M such that $u(r) \le M$ for all $r \in (0, \frac{b}{2})$. Let $r \in (0, \frac{b}{2})$. Since $u'(r) < 0$, Lemma 3 gives us

$$\begin{aligned} \mathcal{P}(b/2;u) &\ge -\frac{N-p}{p}\,r^{N-1}\,\big(u'(r)\big)^{p-1}\,u(r) \\ &\ge -\frac{N-p}{p}\,r^{N-1}\,\big(u'(r)\big)^{p-1}\cdot M\,. \end{aligned} \tag{3.31}$$

Let $K := K(M;b) = \left(-\frac{p}{N-p} \cdot \frac{\mathcal{P}(\frac{b}{2})}{M}\right)^{\frac{1}{p-1}}$. The above can be re-expressed as

$$-\,u'(r) \geq K\, r^{\frac{1-N}{p-1}}. \tag{3.32}$$

Integrating over $[r, \frac{b}{2}]$ we have

$$u(r) \geq u(b/2) + K\,\frac{p-1}{p-N}\left(\left(\frac{b}{2}\right)^{\frac{p-N}{p-1}} - r^{\frac{p-N}{p-1}}\right). \tag{3.33}$$

But $\frac{p-N}{p-1} < 0$, which means

$$\lim_{r\to 0^+} u(r) \geq \lim_{r\to 0^+}\left[u(b/2) + K\,\frac{p-1}{p-N}\left(\frac{b}{2}\right)^{\frac{p-N}{p-1}} - \frac{p-1}{p-N}\, r^{\frac{p-N}{p-1}}\right] = +\infty\,, \tag{3.34}$$

a contradiction. Thus, $\lim_{r\to 0^+} u(r) = +\infty$ by the continuity of u, concluding the proof. □

Lemma 6. *If*

$$q_1 > \max\left\{p^* - \frac{N}{N-p},\ \frac{(p-1)(N+1)}{N-(p-1)}\right\} \tag{3.35}$$

then singular solutions of (3.4) *are also weak solutions of* (3.1).

Proof. Multiplying (3.1) by $\phi \in C^\infty(B_1(0))$ of compact support and integrating over $B_1(0)$, we have the weak formulation

$$\int_{B_1(0)} \triangle_p\, u\ \phi\ \mathrm{d}x + \int_{B_1(0)} g(u)\ \phi\ \mathrm{d}x = 0. \tag{3.36}$$

We wish to show both integrals converge. Recall that $\triangle_p\, u = \nabla\cdot(|\nabla u|^{p-2}\nabla u)$. Using a Green's identity (integration by parts),

$$\int_{B_1(0)} \triangle_p\, u\ \phi\ \mathrm{d}x = -\int_{B_1(0)} |\nabla u|^{p-2}\,\nabla u\cdot\nabla\phi\ \mathrm{d}x. \tag{3.37}$$

We now claim that

$$\int_{B_1(0)} |\nabla u|^{p-2}\,|\nabla u\cdot\nabla\phi|\ \mathrm{d}x < \infty. \tag{3.38}$$

By the Cauchy-Schwarz inequality and the boundedness of test functions,

$$\begin{aligned}\int_{B_1(0)} |\nabla u|^{p-2}\,|\nabla u\cdot\nabla\phi|\;\mathrm{d}x &\le \int_{B_1(0)} |\nabla u|^{p-1}\,|\nabla\phi|\;\mathrm{d}x\\ &\le M\int_{B_1(0)} |\nabla u|^{p-1}\;\mathrm{d}x.\end{aligned}\tag{3.39}$$

Because u is a radial solution to (3.1), we have $\frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x_i}$ for $i = 1,\dots,N$. It follows that $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$, since $r = (x_1^2+\cdots+x_N^2)^{\frac{1}{2}}$, and

$$\begin{aligned}|\nabla u| &= \left(\sum_{i=1}^{N}\left(\frac{\partial u}{\partial r}\cdot\frac{x_i}{r}\right)^2\right)^{\frac{1}{2}}\\ &= \left|\frac{\partial u}{\partial r}\right|\cdot\left(\frac{1}{r^2}\sum_{i=1}^{N}x_i^2\right)^{\frac{1}{2}}\\ &= \left|\frac{\partial u}{\partial r}\right|.\end{aligned}\tag{3.40}$$

Substituting and performing a coordinate transformation, the claim in (3.38) can be re-expressed as

$$\int_0^b r^{N-1}\left|\frac{\partial u}{\partial r}\right|^{p-1}\mathrm{d}r + \int_b^1 r^{N-1}\left|\frac{\partial u}{\partial r}\right|^{p-1}\mathrm{d}r < \infty,\tag{3.41}$$

or equivalently, as the claim that $\frac{\partial u}{\partial r}\in L^{p-1}(B_1(0))$.

Since the energy

$$\mathcal{E}(r;u) := \frac{p-1}{p}\,|u'(r)|^p + G\big(u(r)\big)\,,\tag{3.42}$$

is a decreasing function for any solution u and $r\in(0,1]$, we have

$$\mathcal{E}(r) = \frac{p-1}{p}\,|u'(r)|^p + G\big(u(r)\big) \le \mathcal{E}(b) < \infty \qquad \text{for } b\le r\le 1,\tag{3.43}$$

which shows u and u' are bounded on $[b,1]$, and thus u is defined on $[b,1]$.

From Lemmas 1 and 3,

$$r^N\,\frac{p-1}{p}\,|u'(r)|^p + \frac{N-p}{p}\,r^{N-1}\,|u'(r)|^{p-2}\,u'(r)\,u(r)\le 0\tag{3.44}$$

for $r \in (0, b]$. Moving the second term and cancelling a $|u'(r)|^{p-1}$ and an r^{N-1}, we have

$$r\,\frac{p-1}{p}\,|u'(r)| \le \frac{N-p}{p}\,u(r), \tag{3.45}$$

or equivalently,

$$|u'(r)| \le \frac{N-p}{p-1} \cdot \frac{u(r)}{r}. \tag{3.46}$$

Again, from Lemmas 1 and 3,

$$r^N\,\frac{\big(u(r)\big)^{q_1+1}}{q_1+1} + \frac{N-p}{p}\,r^{N-1}\,|u'(r)|^{p-2}\,u'(r)\,u(r) \le 0 \tag{3.47}$$

for $r \in (0, b]$. Moving the second term, cancelling a factor of $u(r)$ and r^{N-1}, and utilizing (3.46) we have

$$\begin{aligned}\big(u(r)\big)^{q_1} &\le \frac{(q_1+1)(N-p)}{p} \cdot \frac{|u'(r)|^{p-1}}{r} \\ &\le \frac{(q_1+1)(N-p)}{p}\left(\frac{N-p}{p-1}\right)^{p-1} \cdot \frac{\big(u(r)\big)^{p-1}}{r^p},\end{aligned} \tag{3.48}$$

or equivalently,

$$u(r) \le \left[\frac{(q_1+1)(N-p)}{p}\left(\frac{N-p}{p-1}\right)^{p-1}\right]^{\frac{1}{q_1-(p-1)}} \cdot r^{-\frac{p}{q_1-(p-1)}} \tag{3.49}$$

Using the estimates from (3.46) and (3.49),

$$\begin{aligned}|u'|^{p-1} &\le \left(\frac{N-p}{p-1}\right)^{p-1}\,\frac{1}{r^{p-1}} \cdot \big(u(r)\big)^{p-1} \\ &\le \left(\frac{N-p}{p-1}\right)^{p-1}\left[\frac{(q_1+1)(N-p)}{p}\left(\frac{N-p}{p-1}\right)^{p-1}\right]^{\frac{p-1}{q_1-(p-1)}} r^{-\frac{p(p-1)}{q_1-(p-1)}-(p-1)} \\ &:= C_1 C_2\, r^{-\frac{(q_1+1)(p-1)}{q_1-(p-1)}}.\end{aligned} \tag{3.50}$$

Meaning that $r^{N-1}|u'|^{p-1}$ is integrable, since $q_1 > \frac{(p-1)(N+1)}{N-(p-1)}$.

Finally, term two of (3.36) is integrable if $r^{N-1}\big(u(r)\big)^{q_1}$ is integrable. Using estimate (3.49), we have

$$r^{N-1}\big(u(r)\big)^{q_1} \le C_2\, r^{N-1} \cdot r^{-\frac{pq_1}{q_1-(p-1)}}, \tag{3.51}$$

which is integrable, since $q_1 > p^* - \frac{N}{N-p}$.

□

3.3 Phase-Plane Analysis

Lemma 7. *Let $0 < b < 1$. If $x : [b,1] \to \mathbb{R}$ and $y : [b,1] \to \mathbb{R}$ are continuous functions such that*

$$\rho(t) = \sqrt{\big(x(t)\big)^2 + \big(y(t)\big)^2} > 0, \qquad \forall t \in [b,1], \tag{3.52}$$

then there exists a continuous function $\theta : [b,1] \to \mathbb{R}$ such that

$$x(t) = \rho(t)\cos\big(\theta(t)\big) \qquad \textit{and} \qquad y(t) = \rho(t)\sin\big(\theta(t)\big) \tag{3.53}$$

for all $t \in [b,1]$.

Proof. By the continuity of x, fix an $\epsilon \in (b,1]$ such that $x(t) > 0$ for all $t \in [b,\epsilon]$ and define

$$\theta_0(t) := \tan^{-1}\left(\frac{y(t)}{x(t)}\right). \tag{3.54}$$

Then

$$\cos\big(\theta_0(t)\big) = \frac{x(t)}{\rho(t)} \qquad \text{and} \qquad \sin\big(\theta_0(t)\big) = \frac{y(t)}{\rho(t)},$$

satisfying the desired relations (3.53) on $[b,\epsilon]$. Define

$$S := \big\{t \in [\epsilon,1] : \text{ there exists a continuous function } \theta : [b,t] \to \mathbb{R} \text{ with } \theta(t) = \theta_0(t) \;\; \forall t \in [b,\epsilon]\big\} \tag{3.55}$$

and let $\tau = \sup S \geq \epsilon$.

By way of contradiction, suppose $\tau < 1$. Then either $x(\tau) = 0$ or $x(\tau) \neq 0$. If $x(\tau) \neq 0$, then $\left|\frac{y(\tau)}{\rho(\tau)}\right| < 1$. Via an elementary calculation, it follows that

$$\left|\frac{y(\tau)}{\rho(\tau)}\right| < \frac{1}{2}\left(1 + \left|\frac{y(\tau)}{\rho(\tau)}\right|\right). \tag{3.56}$$

Hence, there exists a $\delta > 0$ such that for $t \in (\tau - \delta, \tau + \delta)$,

$$\left|\frac{y(t)}{\rho(t)}\right| < \frac{1}{2}\left(1 + \left|\frac{y(\tau)}{\rho(\tau)}\right|\right). \tag{3.57}$$

Because $x(\tau) \neq 0$, let k be an integer such that $\theta(\tau) \in \left(\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi\right)$. Define for all $t \in (\tau, \tau + \delta)$

$$\theta(t) := \sin_k^{-1}\left(\frac{y(t)}{\rho(t)}\right) \tag{3.58}$$

where $\sin_k^{-1}$ is the inverse of $\sin x$ on the open interval $\left(\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi\right)$. Observe that θ satisfies (3.53) and is continuous since $\sin_k^{-1}$ is continuous. Thus, we have a contradiction on the fact that τ is the least upper bound of S.

If $x(\tau) = 0$, then $\left|\frac{y(\tau)}{\rho(\tau)}\right| = 1$ and proceed likewise (i.e. if $x \equiv 0$ in a small neighborhood around τ, then define (3.58) over that neighborhood. If $x \not\equiv 0$ around a small neighborhood of τ, repeat the prior argument).

□

3.4 Proof of Theorem 2.6 (Main Theorem)

Let $u(r;\alpha,b)$ and $u'(r;\alpha,b)$ be as in (3.4), and define

$$\rho(r;\alpha,b) := \sqrt{\Big(u(r;\alpha,b)\Big)^2 + \Big(u'(r;\alpha,b)\Big)^2}. \tag{3.59}$$

By Lemma 7 and Lemma 5, there exists $b_0 > 0$ such that if $\alpha \in I$ and $b \in (0, b_0)$, then $\Big(u(r;\alpha,b), u'(r;\alpha,b)\Big) \neq (0,0)$ for all $r \in [b,1]$ and there exists a unique continuous argument function $\theta(r;\alpha,b)$ for all $r \in (0,1]$ such that

$$u(r;\alpha,b) = \rho(r;\alpha,b)\cos\big(\theta(r;\alpha,b)\big), \tag{3.60}$$

$$u'(r;\alpha,b) = -\rho(r;\alpha,b)\sin\big(\theta(r;\alpha,b)\big). \tag{3.61}$$

Differentiating (3.60) with respect to r,

$$\begin{aligned} u'(r;\alpha,b) = {} & \rho'(r;\alpha,b)\cos\big(\theta(r;\alpha,b)\big) \\ & - \rho(r;\alpha,b)\sin\big(\theta(r;\alpha,b)\big)\cdot\theta'(r;\alpha,b). \end{aligned} \tag{3.62}$$

Lemma 8. *If $\hat{r} > b$ and $u'(\hat{r};\alpha,b) = 0$, then*

$$\lim_{r\to\hat{r}} \rho'(r) = 0 \qquad \textit{and} \qquad \lim_{r\to\hat{r}} \theta'(r) = 1. \tag{3.63}$$

Proof. Suppose $u(\hat{r}) > 0$ and $u'(\hat{r}) = 0$ for $\hat{r} \in (b, 1)$ and let $r < \hat{r}$ be such that $u(s) > 0$ for $s \in (r, \hat{r})$. From (3.4),

$$r^{N-1} |u'(r;\alpha,b)|^{p-2} u' = \int_r^{\hat{r}} s^{N-1} g\big(u(s)\big) \, ds. \tag{3.64}$$

Since $u'(r;\alpha,b) > 0$ and

$$\big(u'(r;\alpha,b)\big)^2 = \left[\frac{1}{r^{N-1}} \int_r^{\hat{r}} s^{N-1} g\big(u(s)\big) \, ds\right]^{\frac{2}{p-1}} \tag{3.65}$$

we have

$$\Big(\big(u'(r;\alpha,b)\big)^2\Big)' = \frac{2}{p-1} \left[\frac{1}{r^{N-1}} \int_r^{\hat{r}} s^{N-1} g\big(u(s)\big) \, ds\right]^{\frac{3-p}{p-1}} \cdot \left[\frac{1-N}{r^N} \int_r^{\hat{r}} s^{N-1} g\big(u(s)\big) \, ds - g\big(u(r)\big)\right], \tag{3.66}$$

which means that

$$\lim_{r \to \hat{r}^-} \Big(\big(u'(r;\alpha,b)\big)^2\Big)' = 0. \tag{3.67}$$

From (3.59), observe now that

$$\lim_{r \to \hat{r}^-} \rho'(r) = \lim_{r \to \hat{r}^-} \left[\frac{u(r;\alpha,b)\, u'(r;\alpha,b) + \Big(\big(u'(r;\alpha,b)\big)^2\Big)'}{\rho(r;\alpha,b)}\right] = 0, \tag{3.68}$$

and from (3.62),

$$\lim_{r \to \hat{r}^-} u'(r;\alpha,b) = \lim_{r \to \hat{r}^-} \Big[-\rho(r;\alpha,b) \sin\big(\theta(r;\alpha,b)\big)\Big] \cdot \theta'(r;\alpha,b) \tag{3.69}$$

we conclude that

$$\lim_{r \to \hat{r}^-} \theta'(r;\alpha,b) = 1 \tag{3.70}$$

using (3.61), proving the lemma. Similarly, the same can be shown for

$$\lim_{r \to \hat{r}^-} \rho'(r) = \lim_{r \to \hat{r}^-} \left[\frac{u(r;\alpha,b)\, u'(r;\alpha,b) + \Big(\big(u'(r;\alpha,b)\big)^2\Big)'}{\rho(r;\alpha,b)}\right] = 0, \tag{3.71}$$

$$\lim_{r \to \hat{r}^+} u'(r;\alpha,b) = \lim_{r \to \hat{r}^-} \Big[-\rho(r;\alpha,b) \sin\big(\theta(r;\alpha,b)\big)\Big] \cdot \theta'(r;\alpha,b) \tag{3.72}$$

we conclude that

$$\lim_{r \to \hat{r}^-} \theta'(r; \alpha, b) = 1 \tag{3.73}$$

The case of $u(\hat{r}) < 0$ follows similarly. □

By Lemma 8 and equating (3.61) and (3.62), we have

$$\theta'(r; \alpha, b) = \sin^2\left(\theta(r; \alpha, b)\right) - \frac{u''}{\rho(r; \alpha, b)} \cos\left(\theta(r; \alpha, b)\right) \tag{3.74}$$

for $u'(\hat{r}) \neq 0$, and $\theta'(\hat{r}) = 1$ at $u'(\hat{r}) = 0$.

Now we are ready to prove Theorem 2.6.

Proof of Theorem 2.6. Let

$$q := q(u) = \begin{cases} q_1 & u \geq 0 \\ q_2 & u < 0 \end{cases} \tag{3.75}$$

Fix L and positive integers j and γ such that

$$L < \min\left\{ \left(\frac{q+1}{2}\right)^{\frac{q+1-p}{q+1}},\ \left(\frac{p}{2(p-1)}\right)^{\frac{q+1-p}{p}} \right\} \tag{3.76}$$

and

$$j > \max\left\{ 2(N-1) + \sqrt{4(N-1)^2 + 1},\ \frac{4(N-1)^2}{L},\ j_0 \right\} \tag{3.77}$$

with $j > j_0$ satisfying

$$\frac{\pi - \frac{2}{j}}{\frac{Lj}{N-1} - 4(N-1)} < \frac{1}{4\lfloor\sqrt{j}\rfloor} \qquad \text{and} \qquad \frac{2}{j - 4(N-1) - \frac{1}{j}} < \frac{1}{4\lfloor\sqrt{j}\rfloor} \tag{3.78}$$

and

$$\gamma > \max\left\{ \frac{p(q+2)}{q-p+1},\ \frac{(p+1)(q+1)}{q-p+1} \right\} \tag{3.79}$$

so that

$$\frac{\gamma - p}{p} \cdot (q+1-p) - p > 1 \qquad \text{and} \qquad \frac{\gamma}{q+1} \cdot (q+1-p) - p > 1. \tag{3.80}$$

By Corollary 1 there exists $b_0 > 0$ such that $\mathcal{E}(r; \alpha, b) \geq j^\gamma$ for all $r \in [b, 1]$ for $b < b_0$.

Next we note that either $u(r;\alpha,b) > 0$ for all $r \in (b, \frac{1}{4})$, or there exists $\bar{r} \in (b, \frac{1}{4})$ such that $u(\bar{r};\alpha,b) = 0$ and $u'(r;\alpha,b) < 0$ in $(b,\bar{r})$. If $u(r;\alpha,b) > 0$ for all $r \in (b, \frac{1}{4})$, then $\theta(r) > 0$ for all $r \in (b, \frac{1}{4})$. On the other hand, if there exists $\bar{r} \in (b, \frac{1}{4})$ such that $u(\bar{r};\alpha,b) = 0$, we may assume $u(r) > 0$ for all $r \in [b,\bar{r}]$. Hence $\theta(\bar{r}) = \frac{\pi}{2}$. Since $\theta'(\bar{r}) > 0$, then $\theta(r;\alpha,b) \geq \frac{\pi}{2}$ for any $r > \bar{r}$.

Suppose that $r \in [\frac{1}{4}, \frac{3}{4}]$ and

$$\theta(r;\alpha,b) \in \left[k\pi + \frac{\pi}{2} - \frac{1}{j}, k\pi + \frac{\pi}{2} + \frac{1}{j}\right] \tag{3.81}$$

for some nonnegative integer k. Then $|\cos\left(\theta(r;\alpha,b)\right)| \leq \frac{1}{j}$ and consequently

$$\left|\sin\left(\theta(r;\alpha,b)\right)\right| \geq \sqrt{\frac{j^2-1}{j^2}}. \tag{3.82}$$

Thus, from (3.74) we have

$$\begin{aligned}
\theta'(r;\alpha,b) &= \sin^2\left(\theta(r;\alpha,b)\right) \\
&\quad + \frac{g(u)\,u}{\rho^2(r;\alpha,b)\cdot(p-1)\,|u'|^{p-2}} + \frac{(N-1)\,u'\,\cos\left(\theta(r;\alpha,b)\right)}{r\cdot\rho(r;\alpha,b)} \\
&\geq \frac{j^2-1}{j^2} - \frac{4(N-1)\left|u'\,\cos\left(\theta(r;\alpha,b)\right)\right|}{\rho(r;\alpha,b)} \\
&\geq \frac{j^2-1}{j^2} - 4(N-1)\left|\frac{u'}{\rho(r;\alpha,b)}\right|\cdot\frac{1}{j} \\
&\geq \frac{j^2-1}{j^2} - 4(N-1)\cdot\frac{1}{j} \\
&> 0.
\end{aligned} \tag{3.83}$$

Assuming that $\theta(s;\alpha,b) \in [\theta(r), k\pi + \frac{\pi}{2} + \frac{1}{j}]$ for $s \in [r, r_1]$,

$$\theta(r_1;\alpha,b) \geq \theta(r;\alpha,b) + \left(r_1 - r\right)\left[\frac{j^2-1}{j^2} - 4(N-1)\cdot\frac{1}{j}\right]. \tag{3.84}$$

Since $\theta(r_1;\alpha,b) \leq k\pi + \frac{\pi}{2} + \frac{1}{j}$ and $\theta(r;\alpha,b) \geq k\pi + \frac{\pi}{2} - \frac{1}{j}$,

$$k\pi + \frac{\pi}{2} + \frac{1}{j} \geq k\pi + \frac{\pi}{2} - \frac{1}{j} + \left(r - \frac{1}{4}\right)\left[\frac{j^2-1}{j^2} - 4(N-1)\cdot\frac{1}{j}\right], \tag{3.85}$$

implying that

$$\left(r_1 - r\right) \le \frac{2}{j - 4(N-1) - \frac{1}{j}} < \frac{1}{4\lfloor\sqrt{j}\rfloor} \tag{3.86}$$

using (3.78). That is, if r satisfies (3.81) then there exists $r_1 \in \left[r, r + \frac{1}{4\lfloor\sqrt{j}\rfloor}\right]$ such that

$$\theta(r_1) = k\pi + \frac{\pi}{2} + \frac{1}{j}. \tag{3.87}$$

Now suppose

$$\theta(r;\alpha,b) \in \left[k\pi + \frac{\pi}{2} + \frac{1}{j}, (k+1)\pi + \frac{\pi}{2} - \frac{1}{j}\right]. \tag{3.88}$$

Then $|\cos\theta(r;\alpha,b)| \ge \frac{1}{j}$ and

$$(u')^2 \le (j^2 - 1)\,u^2 \tag{3.89}$$

follows from (3.60). Consequently,

$$|u'| \le \rho(r;\alpha,b) = (u^2 + (u')^2)^{\frac{1}{2}} \le j|u|. \tag{3.90}$$

Using these estimates on (3.74),

$$\begin{aligned}
\theta'(r;\alpha,b) &= \sin^2\left(\theta(r;\alpha,b)\right)\\
&\quad + \frac{g(u)\,u}{\rho^2(r;\alpha,b)\cdot(p-1)\,|u'|^{p-2}} + \frac{(N-1)\,u'\,\cos\left(\theta(r;\alpha,b)\right)}{r\cdot\rho(r;\alpha,b)}\\
&\ge \frac{|u|^{q+1}}{\rho^p(r;\alpha,b)\cdot(N-1)} - \frac{(N-1)\,|u'\,\cos\left(\theta(r;\alpha,b)\right)|}{r\cdot\rho(r;\alpha,b)}\\
&\ge \frac{1}{j^p\,(N-1)}\cdot|u|^{q+1-p} - 4(N-1).
\end{aligned} \tag{3.91}$$

We now wish to bound $|u|$ by a factor of j. Recall that $\mathcal{E}(r;\alpha,b) \ge j^\gamma$ for all $r \in [b,1]$. Hence

$$|u| \ge \left(\frac{p}{2(p-1)}\right)^{\frac{1}{p}} \cdot j^{\frac{\gamma-p}{p}} \qquad \text{or} \qquad |u| \ge \left(\frac{q+1}{2}\right)^{\frac{1}{q+1}} \cdot j^{\frac{\gamma}{q+1}}. \tag{3.92}$$

Since both cases in (3.92) are addressed by our choices of γ and j in (3.77), (3.79), and (3.80), we conclude that

$$\theta'(r;\alpha,b) > \frac{L\,j}{N-1} - 4(N-1) > 0. \tag{3.93}$$

Thus by (3.93), if (3.88) holds, then there exists $r_2 \in \left[r_1, r_1 + \frac{1}{4\lfloor\sqrt{j}\rfloor}\right]$ such that

$$r_2 - r < \frac{\pi - \frac{2}{j}}{\frac{Lj}{N-1} - 4(N-1)} < \frac{1}{4\lfloor\sqrt{j}\rfloor}, \tag{3.94}$$

and

$$\theta(r_2; \alpha, b) = (k+1)\pi + \frac{\pi}{2} - \frac{1}{j}. \tag{3.95}$$

Next we estimate $\theta(1; \alpha, b)$. Suppose that $\left|\theta(\frac{1}{4}; \alpha, b) - (k\pi + \frac{\pi}{2} - \frac{1}{j})\right| < \frac{1}{j}$. By (3.83), there exists $r_1 \in \left(\frac{1}{4}, \frac{1}{4} + \frac{2}{j-4(N-1)-\frac{1}{j}}\right)$ such that

$$\theta(r_1; \alpha, b) = k\pi + \frac{\pi}{2} + \frac{1}{j}. \tag{3.96}$$

Now consider any $\theta(s; \alpha, b) \in [k\pi + \frac{\pi}{2} + \frac{1}{j}, (k+1)\pi + \frac{\pi}{2} - \frac{1}{j}]$ for $s \in [r_1, t]$ and $t > r_1$. From (3.94), we see that $\theta(s; \alpha, b)$ cannot remain in the interval for longer than $\frac{\pi - \frac{2}{j}}{\frac{Lj}{N-1} - 4(N-1)}$. Thus there exists $r_2 \in \left(r_1, r_1 + \frac{\pi - \frac{2}{j}}{\frac{Lj}{N-1} - 4(N-1)}\right)$ such that

$$\theta(r_2; \alpha, b) = (k+1)\pi + \frac{\pi}{2} - \frac{1}{j}. \tag{3.97}$$

The existence of an $r_3 \in \left(r_2, r_2 + \frac{2}{j-4(N-1)-\frac{1}{j}}\right)$ such that

$$\theta(r_3; \alpha, b) = (k+1)\pi + \frac{\pi}{2} + \frac{1}{j} \tag{3.98}$$

can then be shown by iterating on this procedure, imitating the argument used to establish the existence of r_1. Since $r \geq \frac{1}{4}$ and we assumed that $\left|\theta(\frac{1}{4}, \alpha, b) - (k\pi + \frac{\pi}{2} - \frac{1}{j})\right| < \frac{1}{j}$, we obtain

$$\frac{1}{4} < r_3 < \frac{1}{4} + \frac{4}{j - 4(N-1) - \frac{1}{j}} + \frac{\pi - \frac{2}{j}}{\frac{Lj}{N-1} - 4(N-1)} \tag{3.99}$$
$$\theta(r_3; \alpha, b) - \theta(r; \alpha, b) > \pi.$$

By repeating the existence argument for r_1 and r_2 a total of $\lfloor\sqrt{j}\rfloor$ times, we see that there will exist some $\overline{r}$ such that

$$\frac{1}{4} < \overline{r} < \frac{1}{4} + \lfloor\sqrt{j}\rfloor \left[\frac{2}{j - 4(N-1) - \frac{1}{j}} + \frac{\pi - \frac{2}{j}}{\frac{Lj}{N-1} - 4(N-1)}\right] \tag{3.100}$$
$$\text{and} \quad \theta(\overline{r}; \alpha, b) \geq \lfloor\sqrt{j}\rfloor\, \pi.$$

From (3.78), we get $\overline{r} \leq \frac{3}{4}$. Finally, since $\theta'(r;\alpha,b) > 0$ for $\theta(r;\alpha,b) = k\pi + \frac{\pi}{2}$, we have proven that

$$\theta(1;\alpha,b) = \lfloor\sqrt{j}\rfloor\,\pi. \tag{3.101}$$

In the case that $\left|\theta(\frac{1}{4},\alpha,b) - (k\pi + \frac{\pi}{2} - \frac{1}{j})\right| \geq \frac{1}{j}$, we provide a similar set of arguments by first proving the existence of an $r_1 \in \left(\frac{1}{4}, \frac{1}{4} + \frac{\pi - \frac{2}{j}}{\frac{Lj}{N-1} - 4(N-1)}\right)$ with $\theta(r_1;\alpha,b) = (k+1)\pi + \frac{\pi}{2} - \frac{1}{j}$ and proceeding likewise, such that repeating these arguments $\lfloor\sqrt{j}\rfloor$ total times gives (3.100), and consequently,

$$\theta(1;\alpha,b) \geq \lfloor\sqrt{j}\rfloor\,\pi, \tag{3.102}$$

proving the theorem. □

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