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A Coherent Proof of Mac Lane's Coherence Theorem

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Abstract

Mac Lane's Coherence Theorem is a subtle, foundational characterization of monoidal categories, a categorical concept which is now an important and popular tool in areas of pure mathematics and theoretical physics. Mac Lane's original proof, while extremely clever, is written somewhat confusingly. Many years later, there still does not exist a fully complete and clearly written version of Mac Lane's proof anywhere, which is unfortunate as Mac Lane's proof provides very deep insight into the nature of monoidal categories. In this thesis, we provide brief introductions to category theory and monoidal categories, and we offer a precise, clear development of Mac Lane's ideas towards a complete proof of the coherence theorem. This thesis will hopefully provide future readers a thorough introduction to monoidal categories and a clearly written proof of Mac Lane's Coherence Theorem, saving those who are interested in truly understanding Mac Lane's theorem dozens of hours of their time.

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Acknowledgments

I want to thank Vin de Silva for mentoring my thesis studies and for his patience with my learning. I learned many math and non-math things from de Silva, but the most valuable thing de Silva taught me is the importance of clarity in math expositions and ways to deliver such clarity. In any writing, an author needs to simultaneously play the role of the reader, but this is challenging in mathematics due to the "binary nature" of mathematical understanding (if I may borrow a quote from Mikhael Gromov): "Either you have no inkling of an idea [of a math concept], or, once you have understood it, the very idea appears so embarrassingly obvious that you feel reluctant to say it aloud." De Silva shared with me many ways on how to write clearly, and that one should carefully consider notation, remarks, and paragraph and sentence structures. He also explained that if such efforts become too extensive, then this puts the clarity of writing at risk in becoming lengthy and confusing. De Silva explained to me that there is a balance in all of these things, and that a balance is achieved by considering an audience background. I ultimately learned to recognize when my writing has not yet achieved that balance and needs further work. I am not sure if I would have ever understood or thought of these things on my own if not for de Silva's mentorship.

I would also like to thank Dagan Karp for his years of being an extremely supportive adviser. Karp told me about REUs, graduate school, and various aspects of math academia that I had no idea about. I participated in many great opportunities by simply following Karp's wise advice, and so I am forever thankful to him for sparing his time to enlighten me.

Chapter 1

Introduction to Category Theory

Category theory is a beautiful language for many useful concepts of math. Here we present the axioms of category theory and introduce the concepts of categories, functors, and natural transformations, as well as examples of these concepts.

1.1 Categories

Definition 1.1.1. A **category** $\mathcal{C}$ consists of

- a collection of **objects** $\mathrm{Ob}(\mathcal{C})$
- a collection of **morphisms** $\mathrm{Hom}(\mathcal{C})$ of the form $f : A \to B$, with A, B objects. We write $\mathrm{Hom}(A, B)$ for the collection of all morphisms between A, B.

This data is equipped with a binary operator $\circ$ known as **composition** such that for any objects A, B, C,

$$\circ : \mathrm{Hom}(A, B) \times \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B)$$

Furthermore, the following laws must be obeyed.

(1) Identity. For each $A \in \mathrm{Ob}(\mathcal{C})$, there exists an **identity** $\mathrm{id}_A : A \to A$ in $\mathrm{Hom}(\mathcal{C})$

(2) Closed under Composition. Let A, B, C be objects on $\mathcal{C}$. If $g : A \to B$ and $f : B \to C$, then $g \circ f : A \to C$ is in $\mathrm{Hom}(A, C)$.

(3) Associativity under Composition. For objects A, B, C and D such that

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

we have the equality

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(4) Identity Action. For any $f \in \text{Hom}(\mathcal{C})$ where $f : A \to B$ we have that

$$1_B \circ f = f = f \circ 1_A.$$

We now introduce some examples of categories.

Example 1.1.2. The canonical example of a category is the **category of sets**, denoted as **Set**, which we can describe as

Objects. All sets X.[1]

Morphisms. All functions between sets $f : X \to Y$.

We can check that this construction satisfies the axioms of a category

(1) Each set X has an identity function $\text{id}_X : X \to X$ where $\text{id}_X(x) = x$ for each $x \in X$.

(2) The composition of two functions $f : X \to Y$ and $g : Y \to Z$ is again a function $g \circ f : X \to Y$ where $(g \circ f)(x) = g(f(x))$.

(3) Function composition is associative.

(4) If $f : X \to Y$ then $\text{id}_Y \circ f = y \circ \text{id}_X = f$.

Because most of mathematics is based in set theory, we shall see that while this is a fairly simple category, it is one of the most useful.

Example 1.1.3. The second canonical example is the **category of groups**, denoted as **Grp**. This can be described as

Objects. All groups $(G, \cdot)$. Here, $\cdot : G \times G \to G$ is the group operation.

[1]One must be careful in saying things like "The objects are all (blank)." We will address this later.

Morphisms. All group homomorphisms $\varphi : (G, \cdot) \to (H, \cdot)$. Specifically, set functions $\varphi : G \to H$ where $\varphi(g \cdot g') = \varphi(g) \cdot \varphi(g')$.

We again check this satisfies the axioms of a category.

(1) Each group $(G, \cdot)$ has a identity group homomorphism $\mathrm{id}_G : (G, \cdot) \to (G, \cdot)$ where $\mathrm{id}_G(g) = g$.

(2) The function composition of two group homomorphisms $\varphi : (G, \cdot) \to (H, \cdot)$ and $\psi : (H, \cdot) \to (K, \cdot)$ is again a group homomorphism where $(\psi \circ \varphi)(g) = \psi(\varphi(g))$. This is because

$$\begin{aligned}(\psi \circ \varphi)(g \cdot g') &= \psi(\varphi(g \cdot g')) \\ &= \psi(\varphi(g) \cdot \varphi(g')) \\ &= \psi(\varphi(g)) \cdot \psi(\varphi(g')) \\ &= (\psi \circ \varphi)(g) \cdot (\psi \circ \varphi)(g).\end{aligned}$$

(3) Function composition is associative; therefore, composition of group homomorphisms is associative.

(4) If $\varphi : (G, \cdot) \to (H, \cdot)$ is a group homomorphism, then $\mathrm{id}_H \circ \varphi = \varphi \circ \mathrm{id}_G = \varphi$.

Therefore we see that this is a category.

Example 1.1.4. The third canonical example is the **category of topological spaces**, denoted **Top**. We describe this as

Objects. All topological spaces (X, τ) where τ is a topology on the set X.

Morphisms. All continuous functions $f : (X, \tau) \to (Y, \tau')$.

The reader can show that this too satisfies the axioms of a category.

There is a minor issue with the presentation of the previous examples. For each of the categories, we said that the objects consisted of *all* sets, groups, or topological spaces. The first observation about this remark is that these collection of objects are not sets. For example, in **Set**, the objects do not form a set by Russell's paradox, and neither do the morphisms. Thus we need to make distinctions between categories based on their size.

Definition 1.1.5. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is

- **Finite** if there are only finitely many objects and finitely many morphisms.
- **Locally Finite** if, for every pair of objects A, B, the set $\text{Hom}_{\mathcal{C}}(A, B)$ is finite.
- **Small** if the collection of objects and collections of morphisms assemble into a set.
- **Locally Small** if $\text{Hom}_{\mathcal{C}}(A, B)$ is a set for every pair of objects A, B.
- **Large** if $\mathcal{C}$ is not locally small. That is, the objects and morphisms do not form a set.

We now introduce the concept of a *subcategory*, which is also extremely useful to include in our vocabulary.

Definition 1.1.6. Let $\mathcal{C}$ be a category. We say a category $\mathcal{S}$ is a **subcategory of $\mathcal{C}$** if

(1) $\mathcal{S}$ is a category, with composition the same as $\mathcal{C}$

(2) The objects and morphisms of $\mathcal{S}$ are contained in the collection of objects and morphisms of $\mathcal{C}$.

Furthermore, we say $\mathcal{S}$ is a **full subcategory** if we additionally have that

(3) For each pair of objects $A, B \in \mathcal{S}$, we have that $\text{Hom}_{\mathcal{S}}(A, B) = \text{Hom}_{\mathcal{C}}(A, B)$.

More informally, $\mathcal{S}$ is full if it "contains all of its morphisms."

Example 1.1.7. Let **Ab** be the category described as

Objects. All abelian groups $(G, \cdot)$

Morphisms. Group homomorphisms.

Then **Ab** is a subcategory of **Grp**. Futhermore, **Ab** is a full subcategory of **Grp**. This observation also applies to

- **FinGrp**, the category of finite groups
- **FinAb**, the category finite abelian groups
- $\mathbf{Ab_{TF}}$, the category of torsion-free abelian groups

However, none of these categories are subcategories of **Set**. In fact, many categories which are based in set theory are not actually subcategories of **Set**. This is because the objects of categories such as **Grp** or **Top** are not just sets, but are sets with extra data (such as a binary operation or a topology).

Definition 1.1.8. In a category $\mathcal{C}$, **diagrams** are directed graphs with vertices representing objects and edges representing morphisms between objects. For example, below we technically have a diagram

$$A \xrightarrow{f} B$$

although our diagrams will usually be more complicated like the one below

$$\begin{tikzcd} B \arrow[d, "g'"'] & A \arrow[l, "f'"'] \arrow[r, "f"] \arrow[dl, "h'"] \arrow[dr, "h"'] & D \arrow[d, "g"] \\ C & & E \end{tikzcd}$$

If A, B, C, D, E are sets, we can visualize the action on the elements in this diagram as

$$\begin{tikzcd} f'(a) \arrow[d, mapsto] & a \arrow[l, mapsto] \arrow[r, mapsto] \arrow[dl, mapsto] \arrow[dr, mapsto] & f(a) \arrow[d, mapsto] \\ g'(f'(a)) \stackrel{?}{=} h'(a) & & g(f(a)) \stackrel{?}{=} h(a) \end{tikzcd}$$

Note that in the above diagram it may not be the case that $g' \circ f' = h'$ or $g \circ f = h$. In the case that both hold, we would call the diagram **commutative.** We could express the commutativity by rewriting the diagram as below.

$$\begin{tikzcd} B' \arrow[d, "g'"'] & A \arrow[l, "f'"'] \arrow[r, "f"] \arrow[dl, "h'=g'\circ f'"] \arrow[dr, "h=g\circ f"'] & B \arrow[d, "g"] \\ C' & & C \end{tikzcd}$$

For a general diagram, if traversal between every pair of objects, via morphism composition, is equivalent, then the diagram is **commutative** and is said to **commute**.

Definition 1.1.9. A category $\mathcal{P}$ is said to be **thin** or a **preorder** if there is **at most** one morphism $f : A \to B$ for each $A, B \in \mathcal{P}$.

The simplest thin categories are of the form below

$$\mathcal{P}$$

$$A \longrightarrow B \longrightarrow C \longrightarrow \cdots$$

but they may also have more complex shapes such as the category below.

$$\mathcal{P}$$

$$\begin{tikzcd}
\ddots & & \vdots & & \iddots \\
& B & C & D & \\
& & A \arrow[ul] \arrow[u] \arrow[ur] \arrow[dl] \arrow[d] \arrow[dr] & & \\
& E & F & G & \\
\iddots & & \vdots & & \ddots
\end{tikzcd}$$

Thin categories are very common since we often only care about keeping track of a single, binary (on or off) type of relation between any two objects. An example of such a relation is comparison of real numbers: for any two real numbers $x, y \in \mathbb{R}$, we know that either $x \le y$ or $y \le x$.

In fact, given a thin category $\mathcal{P}$, we can define the binary relation $\le$ on the objects $\mathrm{Ob}(\mathcal{P})$ as follows. For any pair of objects $A, B \in \mathcal{P}$, we have that

$$A \le B \text{ if and only if there exists an morphism } A \to B.$$

Some things are to be said about this relation:

- For each object A, there always exists a morphism $A \to A$ (namely, the identity). This implies that $A \le A$ for all objects A, so that $\le$ is reflexive.

- If $f : A \to B$ and $g : B \to C$, then we have that $A \le B$ and $B \le C$. Since we may compose morphisms, we have that $g \circ f : A \to C$. Therefore, $A \le C$, so that $\le$ is transitive.

Hence, $\mathcal{P}$ is really just a set with a reflexive and transitive binary relation. However, this is exactly the definition of a **preorder**! Therefore, preorders P can be regarded as categories with at most one morphism between any two objects, and vice versa.

Preorders can also turn into **partial orders**, which have the axiom that

$$\text{if } p \leq p' \text{ and } p' \leq p \text{ then } p = p'.$$

or **linear orders**, where for any p, p' we have that $p \leq p'$ **or** $p' \leq p$.

Example 1.1.10. Here we introduce some examples of thin categories.

Natural Numbers. The sets $\{1, 2, \dots, n\}$ for any $n \in N$ are linear orders, each of which forms a category as pictured below.

$$\begin{tikzcd} 1 \arrow[r] \arrow[loop above] & 2 \arrow[r] \arrow[loop above] & 3 \arrow[r] \arrow[loop above] & \dots \arrow[r] & n \arrow[loop above] \end{tikzcd}$$

In this figure, the loops represent the trivial identity functions.

This example can also be generalized to include $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$.

Subsets. Let X be a set. Then one can form a category $\mathbf{Subsets}(X)$ where the objects are subsets of X and the morphisms are inclusion morphisms. Hence, there is at most one morphism between any two sets.

Since there is at most one morphism between any two objects of the category, we see that this forms a thin category, and hence a partial ordering. What this then tells us is that subset containment determines an ordering, specifically a partial ordering.

Open Sets. Let (X, τ) be a topological space. Define the category $\mathbf{Open}(X)$ to be the category whose objects are the open sets of X and morphisms $U \to V$ are inclusion morphisms $i : U \to V$ whenever $U \subseteq V$. Hence, there is at most one morphism between any two open sets, so that this also forms a preorder.

Subgroups. Let G be a group. We can similarly define the category $\mathbf{SbGrp}(G)$ to be the category whose objects consists of subgroups $H \leq G$, and whose morphisms are inclusion homomorphisms. This is just like the last example; and, as in the last example, there is at most one morphism between any two subgroups H, K of G (either $i : H \to K$ or $i : K \to H$). Hence, we can place a partial ordering on this, so that subgroup containment is a partial ordering.

Ideals. Let R be a ring. Then we can form a category **Ideals**(R) whose objects are the ideals I of R and whose morphisms are inclusion morphisms. As we've seen, this forms a thin category.

Some more examples of categories are familiar constructions we've encountered throughout our mathematical journeys. All of these may be different; but category theory exposes their similarities and relationships to one another.

Category	Objects	Morphisms
Set	Sets	Functions
FinSet	Finite Sets	Functions
FinOrd	Finite Ordinals	Functions
Grp	Groups	Group Homomorphisms
Ab	Abelian Groups	Group Homomorphisms
Rng	Rings	Ring Homomorphisms
Vct	Vector Spaces	Linear Transformations
$(\Omega, E)-$**Alg**	(Ω, E)-Algebras	(Ω, E)-Homomorphisms
Top	Topological Spaces	Continuous functions
CHaus	Compact Hausdorff Spaces	Continuous functions
$\mathbf{Man}_n$	Manifolds	n-continuously diff. functions
Met	Metric Spaces	Metric-preserving functions
Grph	Graphs	Graph homomorphisms
Mon	Monoids	Monoid homomorphisms
Th	Propositional Theories	Translations
Bool	Boolean Algebras	Boolean Alg Homomorphisms

1.2 Morphisms

Definition 1.2.1. Let $f : A \to B$ be a morphism between two objects A and B. Then we say that f is an **isomorphism** if there exists a morphism $f^{-1} : B \to A$ such that

$$f \circ f^{-1} = \mathrm{id}_A \qquad f^{-1} \circ f = \mathrm{id}_B.$$

In this case, f^{-1} is unique, and we call A, B isomorphic. For any two isomorphisms $f : A \to B$ and $g : B \to C$, we have that

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Note here that we are generalizing the definition of an isomorphism. In the contexts of sets, a bijection is enough to say that two sets are isomorphic. In the contexts of groups, invertible group homomorphisms establish isomorphisms. However, our above definition can be returned to an original category of interest to give rise to the relevant notion of an "isomorphism."

We can also generalize the concept of injective and surjective morphisms.

Definition 1.2.2. Let $f : A \to B$ be a morphism. Then

1. f is a **monomorphism** (or is monic) if

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2$$

for all $g_1, g_2 : C \to A$, with C in $\mathcal{C}$.

$$\begin{tikzcd} C \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[dr, "f\circ g_1 = f\circ g_2"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

2. f is a **epimorphism** (or is epic) if

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2$$

for all $g_1, g_2 : B \to C$, with C in $\mathcal{C}$.

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "g_1\circ f = g_2\circ f"] & \\ B \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & C \end{tikzcd}$$

We demonstrate these concepts with the following example, which demonstrates that a monic, epic morphism is not always an isomorphism.

Example 1.2.3. Consider the category **Haus**, consisting of Hausdorff topological spaces as our objects with continuous functions between them as morphisms. Let D be a dense subset of a topological space X and let

$i : D \to X$ be the inclusion map. We'll show that this function is both epic and monic.

To show it is epic, let $f_1, f_2 : X \to Y$ be continuous maps form X to another (Hausdorff) topological space Y. Suppose that

$$f_1 \circ i = f_2 \circ i.$$

Now $\text{Im}(i) = D$, so the above equation tells us that $f_1(d) = f_2(d)$ for all $d \in D$. That is, the functions agree on the dense subset. However, we know from topology that this implies that $f_1 = f_2$.

> ***Proof:*** Suppose that $f_1(x) \neq f_2(x)$ for some $x \notin D$. Since the points are distinct, and since Y is Hausdorff, there must exist disjoint open sets U, V in Y such that $f_1(x) \in U$ and $f_2(x) \in V$. Since both f_1, f_2 are continuous, there must exist open sets U', V' in X such that $f(U') \subseteq U$ and $g(V') \subseteq V$.
>
> However, since D is dense in X, both U' and V' must intersect with some portion of D; that is, there is some $y \in U'$ and $z \in V'$ such that $y, z \in D$. Therefore, we see that $f_1(y) \in U$ and $f_2(z) \in V$, and since $y, z \in D$ we have that $f_1(y) = f_2(z)$. But this contradicts the fact that $U \cap V = \emptyset$. Therefore, we have a contradiction and it must be the case that $f_1(x) = f_2(x)$ for all $x \in X$, as desired.
>
> ∎

Therefore, we see that i is epic. To show that it is monic, suppose $g_1, g_2 : Y \to A$ are two parallel, continuous functions, and that

$$i \circ g_1 = i \circ g_2.$$

Since i is nothing more than an inclusion map, we immediately have that $g_1 = g_2$. Therefore, i is also monic.

However, note that $i : D \to X$ is not an isomorphism. It is injective, but by no means is it surjective, so it is certainly not an invertible map. Hence i is a counter-example to such any claim that monic, epic morphisms are isomorphisms.

Lemma 1.2.4. The composition of monomorphisms (epimorphisms) is a (an) monomorphism (epimorphism).

Proof: Let $f : A \to B$ and $g : B \to C$ be monomorphisms, and suppose $h_1, h_2 : D \to A$ are two parallel morphisms. Suppose that $(g \circ f) \circ h_1 = (g \circ f) \circ h_2$. Note that we can rewrite the equation to obtain that

$$g \circ (f \circ h_1) = g \circ (g \circ h_1) \implies f \circ h_1 = f \circ h_2.$$

as g is monic, and hence it is left cancellable. But once again, f is monic, so we cancel on the left to obtain that $h_1 = h_2$ as desired.

The proof for epimorphisms follows similarly.

■

1.3 Functors

In singular homology, we know that given a topological space X, we can generate an associated n-th singular homology group $H_n(X)$ and a chain of maps:

$$\cdots \longrightarrow H_{n-1}(X) \longrightarrow H_n(X) \longrightarrow H_{n+1}(X) \longrightarrow \cdots$$

where the maps are the reduced boundary maps. However, suppose we have a continuous mapping $f : X \to Y$ of topological spaces. Then it turns out for each n, we obtain a mapping $f_* : H_n(X) \to H_n(Y)$ induced from f. If we have another continuous map $g : Y \to Z$ between topological spaces, which gives us $g_* : H_n(Y) \to H_n(Z)$, it is true that

$$(g \circ f)_* = g_* \circ f_* : H_n(X) \to H_n(Y).$$

This behavior consistently arose among different constructions in the early development of algebraic topology. It was in 1945 that Eilenberg and Mac Lane observed that these mappings could be described as a *functor*, which we define below.

Definition 1.3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **(covariant) functor** $F : \mathcal{C} \to \mathcal{D}$ is a "mapping" such that

- Every $C \in \text{Ob}(\mathcal{C})$ is assigned uniquely to some $F(C) \in \mathcal{D}$
- Every morphism $f : C \to C'$ in $\mathcal{C}$ is assigned uniquely to some morphism $F(f) : F(C) \to F(C')$ in $\mathcal{D}$ such that

$$F(1_C) = 1_{F(C)} \qquad F(g \circ f) = F(g) \circ F(f)$$

If you have seen a graph homomorphism before, this definition might seem similar. This is no coincidence, and the concepts are similar given the relationship between graphs and categories. But with that intuition in mind, we can visualize the action of a functor. Below we have arbitrary categories $\mathcal{C}, \mathcal{D}$ with $F : \mathcal{C} \to \mathcal{D}$ a functor.

$$\begin{tikzcd}[column sep=small] \mathcal{C} & & \vdots & & \\ & \cdots & A \arrow[dl, "f"'] \arrow[dr, "g\circ f"] & \cdots & \\ \cdots & B \arrow[rr, "g"'] & & C & \cdots \\ & \vdots & & \vdots & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] \mathcal{D} & & \vdots & & \\ & \cdots & F(A) \arrow[dl, "F(f)"'] \arrow[dr, "F(g\circ f)"] & \cdots & \\ \cdots & F(B) \arrow[rr, "F(g)"'] & & F(C) & \cdots \\ & \vdots & & \vdots & \end{tikzcd}$$

Example 1.3.2. In the introduction, we saw that topological spaces X can correspond to abelian groups $H_n(X)$. This actually satisfies the properties of a functor, so we can construct an n-th **Singular Homology functor** $H_n : \mathbf{Top} \to \mathbf{Ab}$ which acts as

$$\begin{aligned} X &\longmapsto H_n(X) \\ f : X \to Y &\longmapsto f_* : H_n(X) \to H_n(Y) \end{aligned}$$

where $f : X \to Y$ is a continuous mapping of topological spaces.

Example 1.3.3. Consider the power set $\mathcal{P}(X)$ on a set X. Then we can create a functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ as follows.

For any set X, $\mathcal{P}(X)$ is of course another set. So define the action on objects X as

$$X \longmapsto \mathcal{P}(X).$$

As for morphisms, let $f : X \to Y$ be a function between two sets X and Y. Then we define $\mathcal{P}f : \mathcal{P}(X) \to \mathcal{P}(Y)$ to be the function which sends a set $S \subseteq X$ to its image $f(S) \in \mathcal{P}(Y)$. Now we must show that this function respects identity and composition properties.

Identity. Consider the identity function on a set X, i.e. $\mathrm{id}_X : X \to X$. Then observe that for any $S \in \mathcal{P}(X)$, we have that

$$\mathcal{P}(\mathrm{id}_X)(S) = \mathrm{id}_X(S) = S$$

Therefore, $\mathcal{P}(\mathrm{id}_X) = 1_{\mathcal{P}X}$ so that $\mathcal{P}$ respects identities.

Composition. Let X, Y, Z be sets and $f : X \to Y$ and $g : Y \to Z$ be functions. Let $S \in \mathcal{P}(X)$. Observe that

$$\begin{aligned} \mathcal{P}(g \circ f) = \mathcal{P}(g \circ f)(S) &= (g \circ f)(S) \\ &= g(f(S)) \\ &= \mathcal{P}(g) \circ \mathcal{P}(f)(S). \end{aligned}$$

Therefore we see that $\mathcal{P}(g \circ f) = \mathcal{P}(g) \circ \mathcal{P}(f)$, so that $\mathcal{P}()$ describes a functor from **Set** to **Set**.

Example 1.3.4. Let G be a group. Denote the commutator subgroup as $[G, G]$. One can also show that $[G, G] \trianglelefteq G$ for any group G, which indicates we may talk about the quotient group $G/[G, G]$.

We'll show now that the projection

$$p_G : \mathbf{Grp} \to \mathbf{Ab}$$

which acts on groups and group homomorphisms as

$$\begin{aligned} G &\longmapsto G/[G, G] \\ \varphi : G \to H &\longmapsto \varphi_* : G/[G, G] \to H/[H, H] \end{aligned}$$

satisfies the conditions of a functor.

To do this, for a group homomorphism $\varphi : G \to H$, define $\varphi_* : G/[G, G] \to H/[H, H]$ as

$$\varphi_*(g + [G, G]) = \varphi(g) + [H, H].$$

Note that this is well-defined since $\varphi([G, G]) \subseteq [H, H]$.

Now, in total, what we have is that (1) $p_G : \mathbf{Grp} \to \mathbf{Ab}$ successfully sends objects G of **Grp** to objects $G/[G, G]$ of **Ab**, and (2) p_G sends homomorphisms $\varphi : G \to H$ in **Grp** to homomorphisms $p_G(\varphi) = \varphi_* : G/[G, G] \to H/[H, H]$ in **Ab**. Now we just have to check that $p_G(\text{id}_G) = \text{id}_{G/[G,G]}$ and $p_G(f \circ g) = p_G(f) \circ p_G(g)$ for all groups $G \in \mathbf{Grp}$ and morphisms $f, g \in \text{Hom}(\mathbf{Grp})$.

Identity Let G be a group and 1_G the identity on G. Let $g + [G, G] \in G/[G, G]$. Then

$$\begin{aligned} p_G(\text{id}_G)(g + [G, G]) &= \text{id}_G(g) + [G, G] \\ &= g + [G, G]. \end{aligned}$$

Therefore, $p_g(\text{id}_G) = \text{id}_{G/[G,G]}$; that is, it is the identity on $G/[G, G]$.

Composition. Now let $\varphi : G_1 \to H$ and $\psi : H \to G_2$ be homomorphisms in **Grp.** Suppose $g + [G, G] \in G/[G, G]$. Then

$$\begin{aligned} p_G(f_2 \circ f_1)(g + [G, G]) &= f_2 \circ f_1(g + [G, G]) + [G_2, G_2] \\ &= p_G(f_2)(f_1(g + [G, G]) + [H, H])) \\ &= p_G(f_2) \circ p_G(f_1)(g + [G, G]). \end{aligned}$$

Hence, we see that $p_G(f_2 \circ f_1) = p_G(f_2) \circ p_G(f_1)$.

Therefore, $p_G : \mathbf{Grp} \to \mathbf{Ab}$ is a functor.

The action of a functor can also be dualized to give rise to the notion of a *contravariant* functor.

Definition 1.3.5. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **contravariant functor** is a mapping $F : \mathcal{C} \to \mathcal{D}$ where

- Every object C of $\mathcal{C}$ is mapped to some object $F(C)$ in $\mathcal{D}$
- Every morphism $f : C \to C'$ is mapped to some morphism $F(f) : F(C') \to F(C)$ in $\text{Hom}(\mathcal{D})$. We also require the identity and composition laws:

$$F(1_C) = 1_{F(C)} \qquad F(g \circ f) = F(f) \circ F(g).$$

Note that a contravariant functor is just a covariant functor with the "arrows turned around." That is, it is the opposite definition, and this is overall an example of the inherent duality present in category theory.

Duality arises because category theory abstracts mathematics by considering *arrows* between *objects*. But when writing down definitions, the direction in which one decides to points arrows eventually gets arbitrary. That is, in every definition, one has two choices in deciding where to point arrows. This constantly leads to two dual notions in almost every concept available in category theory.

A nice example of a contravariant functor is as follows.

Example 1.3.6. Let X be a topological space. Let us begin constructing a functor $C : \mathbf{Top} \to \mathbf{Rng}$, by assigning topological spaces X to the ring $C(X)$ of continuous real-valued functions $h : X \to \mathbb{R}$.

To do this, consider a continuous function $f : X \to Y$ between topological spaces in **Top**. Then observe that f induces a function $C(f) : C(Y) \to C(X)$ where

$$C(f)(h) = h \circ f$$

where $h : Y \to \mathbb{R}$ is in $C(Y)$. Note that $h \circ f : X \to \mathbb{R}$ for all $h \in C(Y)$.

Now we check the identity and composition rules.

Let $f = \text{id}_X$ for a topological space X. Then for any $h \in C(X)$,

$$C(\text{id}_x)(h) = h \circ \text{id}_X = h$$

so that $C(\mathrm{id}_X) = \mathrm{id}_{C(X)}$. Further, suppose $f : X \to Y$ and $g : Y \to Z$. Then for $h \in C(Z)$,

$$\begin{aligned} C(g \circ f)(h) &= h \circ (g \circ f) \\ &= (h \circ g) \circ f \\ &= C(f)(h \circ g) \\ &= C(f) \circ C(g)(h). \end{aligned}$$

Hence we see that $C(g \circ f) = C(f) \circ C(g)$, so we see that $C(-) : \mathbf{Top} \to \mathbf{Rng}$ behaves as a contravariant functor from **Top** to **Rng**.

Many contravariant functors can be found in algebraic geometry; we introduce three such functors.

Example 1.3.7. Let R be a commutative ring. Recall that $\mathrm{Spec}(R)$ is the set of all prime ideals of R. In addition, recall that if $\varphi : R \to S$ is a ring homomorphism and if P is a prime ideal of S, then $\varphi^{-1}(P)$ is also a prime ideal in R. This then allows us to define a functor

$$\mathbf{Spec} : \mathbf{CRing} \to \mathbf{Set}$$

where on objects $R \mapsto \mathrm{Spec}(R)$ and on morphisms $\varphi : R \to S \mapsto \varphi^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ where $\varphi^*(P) = \varphi^{-1}(P)$.

However, we can go even deeper than this. Recall from algebraic geometry that $\mathrm{Spec}(R)$ can be turned into a topological space, using the Zariski topology. However, because $\varphi^{-1}(P)$ is a prime ideal whenever P is, we see that $\varphi^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is actually a continuous function between the topological spaces. Hence we can view this as a functor

$$\mathbf{Spec} : \mathbf{CRing} \to \mathbf{Top}.$$

Usually this is phrased more naturally as a functor $\mathbf{Spec} : \mathbf{CRing} \to \mathbf{Sch}$ where **Sch** is the category of schemes.

Example 1.3.8. Let k be an algebraically closed field. Recall that $A^n(k)$ is the set of tuples $(a_1, a_2, \dots, a_n)$ with $a_i \in k$. In algebraic geometry, it is of interest to associate each subset $S \subseteq A^n(k)$ with the ideal

$$I(S) = \{f \in k[x_1, \dots, k_n] \mid f(s) = 0 \text{ for all } s \in S\}.$$

of $k[x_1, \dots, x_n]$. Observe that this is always non-empty since $0 \in I(S)$ for any S. In additional, it is clearly an ideal of $k[x_1, \dots, x_n]$, since for any $p \in k[x_1, \dots, x_n]$,$q \in I(S)$, we have that

$$(p \cdot q)(s) = p(s) \cdot q(s) = p(s) \cdot 0 = 0 \text{ for all } s \in S.$$

so that $p \cdot q \in I(S)$. Now it's usually an exercise to show that if $S_1 \subseteq S_2$ are two subsets of $A^n(k)$, then one has that $I(S_2) \subseteq I(S_1)$. Hence this defines a functor

$$I : \mathbf{Subsets}(A^n(k))^{\mathrm{op}} \to \mathbf{Ideals}(k[x_1, \dots, x_n]).$$

where $\mathbf{Subsets}(A^n(k))$ is the category of subsets with inclusion morphisms, and $\mathbf{Ideals}(k[x_1, \dots, x_n])$ is the category of ideals with inclusion ring homomorphisms; that is, these are partial orders.

Example 1.3.9. Consider again k as an algebraically closed field. In algebraic geometry, one often wishes to associate each ideal of $k[x_1, \dots, x_n]$ with its "zero set"

$$Z(I) = \{s = (a_1, \dots, a_n) \in A^n(k) \mid f(s) = 0 \text{ for all } s \in I\}.$$

It is usually an exercise to show that if $I_1 \subseteq I_2$ are two ideals, then $Z(I_2) \subseteq Z(I_1)$. Hence we see that this defines a functor

$$Z : \mathbf{Ideals}(k[x_1, \dots, x_n]) \to \mathbf{Subsets}(A^n(k)).$$

Functors can be composed, just like functions.

Definition 1.3.10. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be categories and F, G functors as below.

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$$

Then we can define the **composite functor**

$$\mathcal{A} \xrightarrow{G \circ F} \mathcal{C}$$

where

$$C \mapsto G(F(C)) \in \mathcal{C} \qquad f \mapsto G(F(f)) \in \mathrm{Hom}(\mathcal{C}).$$

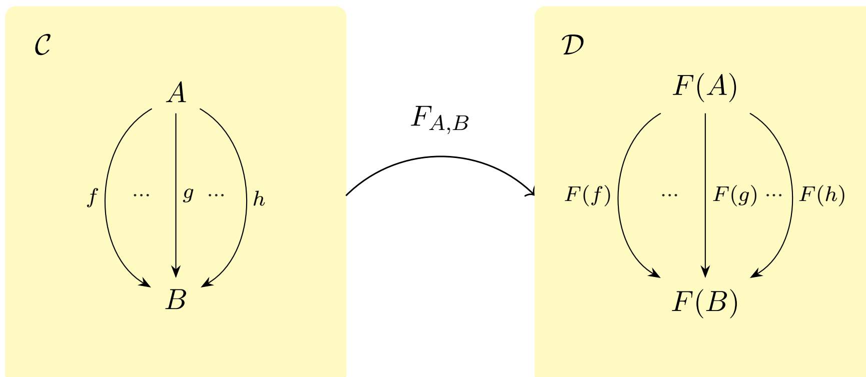
Composition of functors is associative. Hence we may form the **large** category of all categories **Cat**, whose objects are categories and whose morphisms are functors between them.

Definition 1.3.11. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is said to be an **isomorphism** if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F$ is the identity on $\mathcal{C}$ and $F \circ G$ is the identity on $\mathcal{D}$.

Next, we introduce the notion of **full** and **faithful** functors. Towards that goal, consider a functor $F : \mathcal{C} \to \mathcal{D}$ between locally small categories. Then for every pair of objects $A, B \in \mathcal{C}$, there is a function

$$F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$$

where a morphism $f : A \to B$ is sent to its image $F(f) : F(A) \to F(B)$ under the functor F.



As we have a family of functions $F_{A,B}$, we can ask: when is this function surjective or injective? This motivates the following definitions.

Definition 1.3.12. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between locally small categories. We say F is

- **Full** if $F_{A,B}$ is surjective
- **Faithful** if $F_{A,B}$ is injective

for all $A, B \in \mathcal{C}$. If $F_{A,B}$ is an isomorphism, we say F is **fully faithful**.

Now we completely ignored the situation for when $\mathcal{C}, \mathcal{D}$ are not locally small. This was just out of pedagogical interest; if $\mathcal{C}, \mathcal{D}$ are not locally small then we do not have the function described above. However, the concept of full and faithful can still be described; it's just not as nice of a description as before.

Definition 1.3.13. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

- **Full** if for all A, B, every morphism $g : F(A) \to F(B)$ in $\mathcal{D}$ is the image of some $f : A \to B$ in $\mathcal{C}$

- **Faithful** if for all A, B, we have that if $f_1, f_2 : A \to B$ with $F(f_1) = F(f_2)$, then$f_1 = f_2$.

We then say F is a **fully faithful** if it is both full and faithful.

Example 1.3.14. Let $(G, \cdot)$ and $(H, \cdot)$ be a group. Regard both groups as one object categories $\mathcal{C}$ and $\mathcal{D}$ with objects $\bullet$ and $\bullet$ where we set

$$\mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) = G \qquad \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) = H$$

so that each $g \in G$ is now a morphism $g : \bullet \to \bullet$, and vice versa for every $h \in G$, so that composition is given by the group structure. If we have a functor $F : \mathcal{C} \to \mathcal{D}$ between these categories, then the function we introduced simply becomes a set function

$$F_{\bullet,\bullet} : \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) \to \mathrm{Hom}_{\mathcal{D}}(\bullet, \bullet).$$

However, the functorial properties allow this to extend to a group homomorphism from G to H. Therefore, we see that if $F : \mathcal{C} \to \mathcal{D}$ is full, it extends to a surjective group homomorphism. If it is faithful, it extends to an injective group homomorphism.

Finally, we introduce a special and important type of functor in category theory.

Definition 1.3.15. Let $U : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is said to be **forgetful** whenever F does not preserve some axioms or structure present in $\mathcal{C}$ (whether it be algebraic or some kind of ordering) in mapping objects of $\mathcal{C}$ to $\mathcal{D}$.

Example 1.3.16. Let $(R, +, \cdot)$ be a ring. Recall that $(R, +)$ (alone with its addition) is an abelian group. Hence we can forget the structure of $\cdot : R \times R \to R$ and, in a forgetful sense, treat every ring as an abelian group.

This then defines a forgetful functor $F : \mathbf{Ring} \to \mathbf{Ab}$ which simply maps a ring to its abelian group. This works on the morphisms, since every ring homomorphism $\varphi : (R, +, \cdot) \to (S, +, \cdot)$ is a group homomorphism $\varphi : (R, +) \to (S, +)$ on the abelian groups.

Example 1.3.17. Consider the category **Top**. Each object in top is a pair (X, τ) where τ is a topology on X. Moreover, continuous functions are simply functions. This forgetful process is also functorial:

$$(X, \tau) \mapsto X \qquad f : (X, \tau) \to (Y, \tau') \mapsto f : X \to Y.$$

This then gives us the forgetful functor $F : \mathbf{Top} \to \mathbf{Set}$.

1.4 Natural Transformations

Definition 1.4.1. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. Then we define a mapping[2] between the functors

$$\eta : F \to G$$

to be a **natural transformation** if it associates each $C \in \text{Ob}(\mathcal{C})$ with a morphism

$$\eta_C : F(C) \to G(C)$$

in $\mathcal{D}$ such that for every $f : A \to B$, we have that

$$\begin{tikzcd}
A \arrow[d, "f"] & & F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
B & & F(B) \arrow[r, "\eta_B"'] & G(B)
\end{tikzcd}$$

which amounts to $\eta_B \circ F(f) = G(f) \circ \eta_A$.

Thus we can imagine that η translates the diagram produced by the functor F to a diagram produced by G. For example; if η is a natural transformation between F and G, then we also see that the diagram on the right commutes

$$\begin{tikzcd}
A \arrow[dr, "f"] \arrow[dd, "h"'] & & F(A) \arrow[rr, "\eta(A)"] \arrow[dr, "F(f)"] \arrow[dd, "F(h)"'] & & G(A) \arrow[dr, "G(f)"] \arrow[dd, "G(h)"] & \\
& B \arrow[dl, "g"] & & F(B) \arrow[rr, "\eta(B)"] \arrow[dl, "F(g)"] & & G(B) \arrow[dl, "G(g)"] \\
C & & F(C) \arrow[rr, "\eta(C)"] & & G(C) &
\end{tikzcd}$$

[2]Think "**morphism**" instead of mapping, because the word mapping here doesn't rigorously mean anything. That's because we don't really have a word to describe what a natural transformation really is. We have axioms, which we present, but we don't have a nice word. That nice word will turn out to be morphism, and you will see soon why.

if the above diagram on the left commutes. In addition, the diagram on the right commutes

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\
C \arrow[r, "k"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F(A) \arrow[rr] \arrow[dl] \arrow[dd] & & G(A) \arrow[dl] \arrow[dd] \\
F(B) \arrow[rr] \arrow[dd] & & G(B) \arrow[dd] & \\
 & F(C) \arrow[rr] \arrow[dl] & & G(C) \arrow[dl] \\
F(D) \arrow[rr] & & G(D) &
\end{tikzcd}
$$

if the above diagram on the left commutes. Note that the arrows in black in the above diagram on the right represent the morphisms η_A, η_B, η_C and η_D.

Example 1.4.2. Let K be a commutative ring in **CRing**, the category of commutative rings. We can define the functors

$$GL_n(-) : \mathbf{CRing} \to \mathbf{Grp} \quad (-)^\times : \mathbf{CRing} \to \mathbf{Grp}$$

where $GL_n(-)$ maps commutative rings to $GL_n(K)$, the group of invertible $n \times n$ matrices with entries in K, and $(-)^\times$ maps a commutative ring to its group of units $K^\times$.

Consider a commutative ring K. Recall that for matrix $M \in GL_n(K)$, we can take the determinant of K; we are usually more familiar with this concept when $K = \mathbb{R}$. However, it is a fact from ring theory that a matrix M is invertible if and only if the determinant $\det(M)$ of M is in $K^\times$. Since $GL_n(K)$ is the set of all such invertible matrices, we see that we may associate each K with its determinant function

$$\det_K : GL_n(K) \to K^\times$$

which sends an invertible $M \in GL_n(K)$ to its determinant in $K^\times$. To see that this morphism is a group homomorphism, we simply recall the determinant property

$$\det(AB) = \det(A)\det(B).$$

The claim is now that this family of morphisms assembles into a natural transformation. Specifically, that $\det : GL_n(-) \to (-)^\times$. To see, this, let $f : K \to K'$ be a homomorphism between commutative rings. Recall from ring theory that the determinant of a matrix $M = [a_{ij}]$ with $a_{ij} \in K$ is given by

$$\det(M) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$

where S_n is the symmetric group, and $\text{sgn}(\sigma)$ is the sign of a permutation. Now for det to form a natural transformation, we'll need that the diagram below commutes.

$$\begin{tikzcd} K \arrow[d, "f"] \\ K' \end{tikzcd} \qquad \begin{tikzcd} GL_n(K) \arrow[r, "\det_K"] \arrow[d, "GL_n(f)"'] & K^\times \arrow[d, "f^\times"] \\ GL_n(K') \arrow[r, "\det_{K'}"'] & K'^\times \end{tikzcd}$$

Note that $f : K \to K'$ is a commutative ring homomorphism. To show this diagram commutes, consider any $M = [a_{ij}] \in GL_n(K)$. Observe that

$$\begin{aligned}
(f^\times \circ \det_K)(M) &= f^\times(\det_K(M)) \\
&= f^\times\left(\sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}\right) \\
&= \sum_{\sigma \in S_n} \text{sgn}(\sigma) f(a_{1\sigma(1)}) \cdots f(a_{n\sigma(n)}) \\
&= \det_{K'}([f(a_{ij}])) \\
&= \det_{K'} \circ GL_n(f)(M).
\end{aligned}$$

Hence we see that the diagram commutes, so that the determinant $\det : GL_n(-) \to (-)^\times$ assembles into a natural transformation between the functors.

Example 1.4.3. Earlier, we showed that $p_G : \mathbf{Grp} \to \mathbf{Ab}$ in which $G \mapsto G/[G,G]$ was a functor. It turns out that we can formulate a natural transformation between the identity functor $1_{\mathbf{Grp}} : \mathbf{Grp} \to \mathbf{Grp}$ and the functor $p_G : \mathbf{Grp} \to \mathbf{Grp}$. Define $\eta : 1_{\mathbf{Grp}} \to p_G$ where

$$\eta_G : G \to G/[G,G]$$

is the natural projection $g \mapsto g + [G,G]$

To show this is a natural transformation, consider the morphism $f : G \to H$ in **Grp**. We know that p_G induces a morphism $f^* : G/[G, G] \to H/[H, H]$ defined as

$$f^*(g + [G, G]) = f(g) + [H, H].$$

Now let $g \in G$.

$\boldsymbol{\eta_H \circ f(g)}$**.** On one hand, observe that

$$\eta_H \circ f(g) = f(g) + [H, H].$$

$\boldsymbol{f^* \circ \eta_G(g)}$**.** On the other hand, observe that

$$f^* \circ \eta_G(g) = f^*(g + [G, G]) = f(g) + [H, H].$$

Hence, we see that

$$\eta_H \circ f = f^* \circ \eta_G$$

so that the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[d, "f"'] & G/[G,G] \arrow[d, "f^*"] \\ H \arrow[r, "\eta_H"'] & H/[H,H] \end{tikzcd}$$

and hence η is a natural transformation.

Example 1.4.4. The categories **FinOrd** and **FinSet**, are closely related categories. Recall that **FinOrd** has finite ordinals $n = \{0, 1, 2, \ldots, n-1\}$ as objects with morphisms all functions $f : m \to n$ where m, n are natural numbers, and the objects of **FinSet** are all finite sets (of some universe U) with morphisms all functions between such sets.

Obviously the objects and morphisms of **FinOrd** are in **FinSet**. Thus let S : **Finord** $\to$ $\mathbf{Set}_{\boldsymbol{F}}$ be the inclusion functor.

Define a functor $\#$: $\mathbf{Set}_{\boldsymbol{F}} \to$ **FinOrd** as follows. Assign each $X \in \mathbf{Set}_{\boldsymbol{F}}$ to the ordinal $\#X = n$, the number of elements in X. We can represent this bijective mapping as

$$\theta_X : X \to \#X.$$

Furthermore, if $f : X \to Y$ is a morphism in **FinSet**, associate f with the morphism $\#f : \#X \to \#Y$ in **FinOrd** defined by

$$\#f = \theta_Y \circ f \circ \theta_X^{-1}.$$

Thus we have that the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "\theta_X"] \arrow[d, "f"'] & \#X \arrow[d, "\#f"] \\ Y \arrow[r, "\theta_Y"] & \#Y \end{tikzcd}$$

and θ acts a natural transformation between the two functors.

Note that if X is an ordinal number, we define θ_X to be the identity function, which ensures that $\# \circ S$ is the identity functor on **FinOrd**. However, $S \circ \#$ is not the identity on **FinSet**, since the input will be X while the output will just be $\#X$ (as S is just the inclusion functor.)

Example 1.4.5. Observe that for a fixed group H, we have that

$$H \times - : \mathbf{Grp} \to \mathbf{Grp}$$

is a functor. In this case, it turns out that any group homomorphism $f : H \to K$ is a natural transformation between the functors $H \times -$ and $K \times -$.

Let $\varphi : G \to G'$ be a group homomorphism between two groups in **Grp**. Then we know that $H \times -$ and $K \times -$ induce homomorphisms

$$\varphi_{H \times G} : H \times G \to H \times G'$$

and

$$\varphi_{K \times G} : K \times G \to K \times G'.$$

Let $(h, g) \in H \times G$. If we associate $f : H \to K$ with the function $f_G : H \times G \to K \times G$ where $f_G(h, g) = (f(h), g)$, then f_G defines a natural transformation between the functors. That is, we see that on one hand,

$$\begin{aligned} \varphi_{K \times G} \circ f_G(h, g) &= \varphi_{H \times G}(f(h), g) \\ &= (f(h), \varphi(g)) \end{aligned}$$

while on the other,

$$\begin{aligned} f_{G'} \circ \varphi_{K \times G}(h, g) &= f_{G'}(h, \varphi(g)) \\ &= (f(h), \varphi(g)). \end{aligned}$$

Therefore we see that

$$\varphi_{K \times G} \circ f_G = f_{G'} \circ \varphi_{H \times G}$$

so that the following diagram commutes:

$$
\begin{tikzcd}
G \arrow[d, "f"] & H \times G \arrow[r, "f_G"] \arrow[d, "\varphi_{H\times G}"'] & K \times G \arrow[d, "\varphi_{K\times G}"] \\
G' & H \times G' \arrow[r, "f_{G'}"'] & K \times G'
\end{tikzcd}
$$

Therefore, we see that every homomorphism $f : H \to K$ forms a natural transformation between the functors $H \times -$ and $K \times -$ from **Grp** to **Grp**.

1.5 Initial, Terminal, and Zero Objects

We can also be more specific in discussing the nature of the objects of a given category $\mathcal{C}$.

Definition 1.5.1. Let $\mathcal{C}$ be a category.

- An object T of $\mathcal{C}$ is said to be **terminal** if for each object A of $\mathcal{C}$ there exists exactly one morphism $t_A : A \to T$.
- Let I be an object of $\mathcal{C}$. Then I is said to be **initial** if for each object A of $\mathcal{C}$ there exists exactly one morphism $i_A : I \to A$.
- An object Z of $\mathcal{C}$ is said to be a **zero object** if it is both terminal and initial.

  Equivalently, an object Z is zero if for any objects A, B of $\mathcal{C}$ there exists exactly one morphism $f : A \to Z$ and exactly one morphism $g : Z \to B$. Hence, for any two objects there exists a morphism between them, namely given by by $g \circ f$, called the **zero morphism** from A to B.

Note that by these definitions, a terminal or initial object are necessarily unique. Thus, a zero object is unique as well.

Example 1.5.2. Recall that in the category **Grp**, there exists a trivial group $\{e\}$. Moreover, for each group G, there exist unique group homomorphisms

$$i_G : \{e\} \to G \qquad e \mapsto e_G$$

and

$$t_G : G \to \{e\} \qquad g \mapsto e_G.$$

Note that both are group homomorphisms since they both behave on identity elements and are trivially distributive across group operations. This then shows that **Grp**, the trivial group is initial and terminal and hence a zero object.

This makes sense since for any two groups G, H, there exists a unique map

$$z : G \to H \qquad g \mapsto e_H$$

which could be factorized as

$$\begin{tikzcd}
 & \{e\} \arrow[dr, "i_H"] & \\
G \arrow[ur, "t_G"] \arrow[rr, "z"'] & & H
\end{tikzcd}$$

which demonstrates the existence of a zero object (the name "zero" makes sense now, right?), which we already know is $\{e\}$. Note in this example, we did not actually use much group theory. In fact, this could be repeated for the categories R-**Mod**, **Ab**, and other similar categories.

The next two examples demonstrate that terminal and initial objects of course don't always have to coincide like they did in the previous example.

Example 1.5.3. Let n be a finite set. Recall that we can create a category, specifically a preorder, by taking our objects to be positive integers with our morphisms being size relations.

$$\begin{tikzcd}
1 \arrow[loop above] \arrow[r] & 2 \arrow[loop above] \arrow[r] & 3 \arrow[loop above] \arrow[r] & \cdots \arrow[r] & n \arrow[loop above]
\end{tikzcd}$$

Then 1 is an initial object while n is a terminal object. This is because for any number $1 \leq m \leq n$, there exists a unique morphism from 1 to m, and a unique morphism m to n, both which may be obtained by repeated composition.

Example 1.5.4. Consider the category **Set**. Let X be a given set in this category. Then there are two unique maps which we may construct. First, there is the map

$$t_X : X \to \{*\}$$

where everything is mapped to the one element $*$ of the one point set. This is clearly a function, and hence a morphism in our category. Now secondly, we may construct

$$i_X : \varnothing \to X$$

which doesn't do anything. Thus we have that $\varnothing$ is an initial object while $\{*\}$ is a terminal object. However, one may wonder at this point: How exactly is i_X a true, set theoretic function? And doesn't this mean we can obtain a unique morphism $i'_X : X \to \varnothing$, so that $\varnothing$ is a terminal object as well?

The second question is easy to answer; if $\varnothing$ we terminal, then we'd have that $\{*\} \cong \varnothing$ which is not true. Since this is a bit of a boring answer, we'll explain both in detail.

First, recall that a function in $f : A \to X$ between two sets A and X is a relation $R \subseteq A \times X$ which satisfies two properties.

1. (Existence.) For each $a \in A$, there exists a $x \in X$ such that $(a, x) \in R$

2. (Uniqueness. Or, if you'd like, the vertical line test.) If $(a, x) \in R$ and $(a, x') \in R$ then $x = x'$.

Now observe that if $A = \varnothing$, then $R \subseteq \varnothing \times X = \varnothing$. Hence (1) and (2) are satisfied because each is trivially true. However, we don't get a function $f : X \to \varnothing$, since (1) fails. Specifically, (1) demands the existence of elements in our codomain, a demand we cannot meet if it is empty.

Thus we see that $\varnothing$ is initial, but not terminal as our intuition may suggest, and that $\{*\}$ is terminal.

1.6 Products of Categories, Functors

As one may expect, the product of categories can in fact be defined.

Definition 1.6.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then the **product category** $\mathcal{C} \times \mathcal{D}$ is the category where

Objects. All pairs (C, D) with $C \in \mathcal{C}$ and $D \in \mathcal{D}$

Morphisms. All pairs (f, g) where $f \in \text{Hom}(\mathcal{C})$ and $g \in \text{Hom}(\mathcal{D})$.

To define composition in this category, suppose we have composable morphisms in $\mathcal{C}$ and $\mathcal{D}$ as below.

$$\mathcal{C} \qquad \begin{tikzcd} \cdots \, C_1 \arrow[r, "f"] \arrow[rr, bend left, "f' \circ f"] & C_2 \arrow[r, "f'"] & C_3 \, \cdots \end{tikzcd}$$

$$\mathcal{D} \qquad \begin{tikzcd} \cdots \, D_1 \arrow[r, "g"] \arrow[rr, bend left, "g' \circ g"] & D_2 \arrow[r, "g'"] & D_3 \, \cdots \end{tikzcd}$$

Then the morphisms (f, g) and (f', g') in $\mathcal{C} \times \mathcal{D}$ are composable too, and their composition is defined as $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$.

$$\mathcal{C} \times \mathcal{D} \qquad \begin{tikzcd} \cdots \, (C_1, D_1) \arrow[r, "(f,g)"] \arrow[rr, bend left, "(f',g') \circ (f,g) = (f' \circ f, g' \circ g)"] & (C_2, D_2) \arrow[r, "(f',g')"] & (C_3, D_3) \, \cdots \end{tikzcd}$$

We also define the **projection functors** $\pi_{\mathcal{C}} : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $\pi_{\mathcal{D}} : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ with the property that

$$\pi_{\mathcal{C}}(f, g) = f \quad \pi_{\mathcal{D}}(f, g) = g.$$

Consider a pair of functors $F : \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{D}$. Then these functors determine a unique functor $H : \mathcal{B} \to \mathcal{C} \times \mathcal{D}$ where

$$\pi_{\mathcal{C}} \circ H = F \quad \pi_{\mathcal{D}} \circ H = G.$$

That is, we see that for any arrow f in $\mathcal{B}$ we have that $H(f) = (F(f), G(f))$. Hence the following diagram commutes

$$\begin{tikzcd}
 & \mathcal{B} \arrow[dl, "F"'] \arrow[d, dashed, "H"] \arrow[dr, "G"] & \\
\mathcal{C} & \mathcal{C} \times \mathcal{D} \arrow[l, "\pi_{\mathcal{C}}"] \arrow[r, "\pi_{\mathcal{D}}"'] & \mathcal{D}
\end{tikzcd}$$

and we dash the middle arrow to represent that H is induced, or defined, by this process.

We can also take the product of two different functors.

Definition 1.6.2. Let $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{D} \to \mathcal{D}'$ be two functors. Then we define the **product functor** to be the functor $F \times G : \mathcal{C} \times \mathcal{D} \to \mathcal{C}' \times \mathcal{D}'$ for which

1. If (C, D) is an object of $\mathcal{C} \times \mathcal{D}$ then $(F \times G)(C, D) = (F(C), G(D))$
2. If (f, g) is a morphism of $\mathcal{C} \times \mathcal{D}$ then $(F \times G)(f, g) = (F(f), G(g))$

   Additionally, we can compose the product of functors (of course, so long as they have the same number of factors). Thus suppose G, F and G', F' are composable functors. Then observe that

$$(G \times G') \circ (F \times F') = (G \circ F) \times (G' \circ F').$$

Note that in this formulation we have that

$$\pi_{\mathcal{C}'} \circ (F \times G) = F \circ \pi_{\mathcal{C}} \quad \pi_{\mathcal{C}'} \circ (F \times G) = G \circ \pi_{\mathcal{D}}$$

Hence, we have the following commutative diagram.

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "F"'] & \mathcal{C} \times \mathcal{D} \arrow[l, "\pi_{\mathcal{C}}"'] \arrow[r, "\pi_{\mathcal{D}}"] \arrow[d, dashed, "F \times G"] & \mathcal{D} \arrow[d, "G"] \\
\mathcal{C}' & \mathcal{C}' \times \mathcal{D}' \arrow[l, "\pi_{\mathcal{C}'}"'] \arrow[r, "\pi_{\mathcal{D}'}"] & \mathcal{D}'
\end{tikzcd}$$

Again, the dashed arrow is written to express that $F \times G$ is the functor defined by this process and makes this diagram commutative.

With all of this said, note that following: $\times$ is a function which maps categories to categories. It does this in the same that a group operation $\cdot : G \times G \to G$ maps a group to itself. Furthermore, it maps functors, which are morphisms between categories, to other categories, and it preserves composition and identity functors. Therefore, we see that $\times$ is itself a functor.

$$\times : \mathbf{Cat} \times \mathbf{Cat} \to \mathbf{Cat}$$

The functor $\times$ is mapping small categories to itself, similarly to how a group operation maps a group to itself. Since we encounter this type of situation often, we make the following definition.

Definition 1.6.3. If F is a functor such that $F : \mathcal{B} \times \mathcal{C} \to \mathcal{D}$, that is, its domain is a product category, then F is said to be a **bifunctor**.

Bifunctors are the generalization of two variable functions. It can be thought of as a functor of two variables, since if you fix either of the variables you get a regular, normal functor.

An example of a bifunctor is the cartesian product $\times$, which we can apply to sets, groups, and topological spaces. In these instances we know that value of a cartesian product is always determined uniquely by the values of the individual factors, which holds more generally for bifunctors.

Proposition 1.6.4. Let $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ be categories. For $B \in \mathcal{B}$ and $C \in \mathcal{C}$, define the functors

$$H_C : \mathcal{B} \to \mathcal{D} \quad K_B : \mathcal{C} \to \mathcal{D}$$

such that $H_C(B) = K_B(C)$ for all B, C. Then there exists a functor $F : \mathcal{B} \times \mathcal{C} \to \mathcal{D}$ where $F(B, -) = K_B$ and $F(-, C) = H_C$ for all B, C if and and only if, for every pair of morphisms $f : B \to B'$ and $g : C \to C'$ we have that

$$K_{B'}(g) \circ H_C(f) = H_{C'}(f) \circ K_B(g).$$

Diagrammatically, this condition is

$$\begin{tikzcd}
H_C(B) = K_B(C) \arrow[r, "K_B(g)"] \arrow[d, "H_C(f)"'] & H_{C'}(B) = K_B(C') \arrow[d, "H_{C'}(f)"] \\
H_C(B') = K_{B'}(g) \arrow[r, "K_{B'}(g)"'] & H_{C'}(B') = K_{B'}(C')
\end{tikzcd}$$

Proof:

($\Longrightarrow$) Suppose such a bifunctor $F : \mathcal{B} \times \mathcal{C} \to \mathcal{D}$ exists and that it agrees with H_C and K_B; that is, $F(B, -) = K_B$ and $F(-, C) = H_C$ for all B, C. Then observe that for any $f : B \to B'$ and $g : C \to C'$,

$$\begin{aligned}(1_{B'}, g) \circ (f, 1_C) &= (1_{B'} \circ f, g \circ 1_C)\\ &= (f, g)\\ &= (f \circ 1_B, 1_{C'} \circ g)\\ &= (f, 1_{C'}) \circ (1_B, g)\end{aligned}$$

Applying the functor F to the equation, we see that

$$F(1_{B'}, g) \circ F(f, 1_C) = F(f, 1_{C'}) \circ F(1_B, g)$$

Observe that $F(B', -) = M_{B'}$, and also that

$$F(1_{B'}, g) : F(B', C) \to F(B', C').$$

However, since the first variable is fixed to B', we can write this as $K_{B'}(g) : F(B', C) \to F(B', C')$. In addition, we see that $F(f, 1_C) = F(B, C) \to F(B', C)$. In this case the second variable is fixed to C, so we see that $H_C(f) : F(B, C) \to F(B', C)$. Therefore, we see that

$$K_{B'}(g) \circ H_C(f) = H_{C'}(f) \circ K_B(g)$$

which proves that this condition is necessary. Furthermore, the equality implies the following diagram:

$$\begin{tikzcd}
F(B, C) \arrow[r, "F(1_B, g)"] \arrow[d, "F(f, 1_C)"'] & F(B, C') \arrow[d, "F(f, 1_{C'})"] \\
F(B', C) \arrow[r, "F(1_{B'}, g)"] & F(B', C')
\end{tikzcd}$$

($\Longleftarrow$) Suppose on the other hand that K_B and H_C do not constitute a unique functor. Then there exist distinct functors $F_1, F_2 : \mathcal{B} \times \mathcal{C} \to \mathcal{D}$ such that

$$\begin{aligned}F_1(B, -) = K_B = F_2(B, -)\\ F_1(-, C) = H_C = F_2(-, C).\end{aligned}$$

However, we stated that $K_B(C) = H_C(B)$ for all B, C. Therefore both equations imply that

$$F_1(B, C) = F_2(B, C)$$

for all B, C. Hence if we define $F(B, C) = K_B(C) = H_C(B)$, we obtain a consistent definition, and this does formulate a unique functor on objects. To show that this behaves on morphisms, let 1_B and 1_C be identity morphisms. Then

$$F(1_B, 1_C) = (K_B(1_B), H_C(1_C)) = \mathrm{id}_{F(1_B, 1_C)}$$

and if (f, g) is composable with (f', g'), then

$$\begin{aligned} F((f, g) \circ (f', g')) &= (K_B(f \circ f'), H_C(g \circ g')) \\ &= (K_B(f) \circ K_B(f'), H_C(g) \circ H_C(g')) \\ &= (K_B(f), H_C(g)) \circ (K_B(f'), H_C(g')) \\ &= F(f, g) \circ F(f', g'). \end{aligned}$$

Hence $F : \mathcal{B} \times \mathcal{C} \to \mathcal{D}$ is a unique bifunctor.

∎

Example 1.6.5. We now introduce what is probably one of the most important examples of a bifunctor. Note that for any (locally small) category $\mathcal{C}$, we have for each object A a functor.

$$\mathrm{Hom}(A, -) : \mathcal{C} \to \mathbf{Set}$$

We also have a functor from $\mathcal{C}^{\mathrm{op}}$ (we at the $^{\mathrm{op}}$ simply for convenience) for each $B \in \mathcal{C}^{\mathrm{op}}$.

$$\mathrm{Hom}(-, B) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

As an application of the proposition, one can see that that these two functors act as the K_B and H_C functors in the above proposition, and give rise to bifunctor

$$\mathrm{Hom} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}.$$

This is because for any $h : A \to A'$ and $k : B \to B'$, the diagram,

$$
\begin{tikzcd}
\mathrm{Hom}(A', B) \arrow[r, "h^*"] \arrow[d, "k_*"'] & \mathrm{Hom}(A, B) \arrow[d, "k_*"] \\
\mathrm{Hom}(A', B') \arrow[r, "h^*"] & \mathrm{Hom}(A, B')
\end{tikzcd}
$$

commutes. Hence the proposition guarantees that $\mathrm{Hom} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ exists and is unique.

Example 1.6.6. Recall that for an integer n and for a ring R with identity $1 \neq 0$, we can formulate the group $\mathrm{GL}(n, R)$, consisting of $n \times n$ matrices with entry values in R. As this takes in arguments, we might guess that we have a bifunctor

$$GL(-,-) : \mathbb{N} \times \mathbf{Ring} \to \mathbf{Grp}$$

where $\mathbb{N}$ is a the discrete category with elements as natural numbers. This intuition is correct: for a fixed ring R, we have a functor

$$GL(-, R) : \mathbb{N} \to \mathbf{Grp}$$

while for a fixed natural number n we have a functor

$$GL(n, -) : \mathbf{Ring} \to \mathbf{Grp}.$$

Below we can visualize the activity of this functor:

$$
\begin{array}{c|ccccc}
\vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\
R = S & GL(1,S) & GL(2,S) & \cdots & GL(k,S) & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\
R = \mathbb{Z} & GL(1,\mathbb{Z}) & GL(2,\mathbb{Z}) & \cdots & GL(k,\mathbb{Z}) & \cdots \\
\hline
 & n = 1 & n = 2 & \cdots & n = k & \cdots
\end{array}
$$

Above, we start with $\mathbb{Z}$ since the this is the initial object of the category **Ring**.

Now that we understand products of categories a functors, and we have a necessary and sufficient condition for the existence of a bifunctor, we describe necessary and sufficient conditions for the existence of a natural transformation.

Definition 1.6.7. Suppose $F, G : \mathcal{B} \times \mathcal{C} \to \mathcal{D}$ are bifunctors. Suppose that there exists a morphism η which assigns objects of $\mathcal{B} \times \mathcal{C}$ to morphisms of $\mathcal{D}$. Specifically, η assigns objects $B \in \mathcal{B}$ and $C \in \mathcal{C}$ to the morphism

$$\eta_{(B,C)} : F(B,C) \to G(B,C).$$

Then η is said to be **natural** in B if, for all $C \in \mathcal{C}$,

$$\eta_{(-,C)} : F(-,C) \to G(-,C)$$

is a natural transformation of functors from $\mathcal{B} \to \mathcal{D}$.

With the previous definition, we can now introduce the necessary condition for a natural transformation to exist between bifunctors.

Proposition 1.6.8. Let $F, G : \mathcal{B} \times \mathcal{C} \to \mathcal{D}$ be bifunctors. Then there exists a natural transformation $\eta : F \to G$ if and only if $\eta(B,C)$ is natural in B for each $C \in C$, and natural in C for each $B \in \mathcal{B}$.

Proof:

$(\implies)$ Suppose that $\eta : F \to G$ is a natural transformation. Then every object (B,C) is associated with a morphism $\eta_{(B,C)} : F(B,C) \to G(B,C)$ in $\mathcal{D}$, and this gives rise to the following diagram:

$$\begin{tikzcd}
(B,C) \arrow[d, "(f,g)"] & & F(B,C) \arrow[r, "\eta_{(B,C)}"] \arrow[d, "F(f,g)"'] & G(B,C) \arrow[d, "G(f,g)"] \\
(B',C') & & F(B',C') \arrow[r, "\eta_{(B',C')}"] & G(B',C')
\end{tikzcd}$$

Now let $C \in \mathcal{C}$ and observe that

$$\eta_{(-,C)} : F(-,C) \to G(-,c)$$

is a natural transformation for all B. On the other hand, for any $B \in \mathcal{B}$,

$$\eta_{(B,-)} : F(B,-) \to G(B,-)$$

is a natural transformation for all C. Therefore, η is both natural in B and C for all objects (B,C)

($\Longleftarrow$) Suppose on the other hand that η is a function which assigns objects (B, C) to a morphism $F(B, C) \to G(B, C)$ in $\mathcal{D}$. Furthermore, suppose that $\eta(B, C)$ is natural in B for all $C \in \mathcal{C}$ and natural in C for all $B \in \mathcal{B}$.

Consider a morphism $(f, g) : (B, C) \to (B', C')$ in $\mathcal{B} \times \mathcal{C}$. Then since η is natural for all $B \in \mathcal{B}$, we know that for all $C \in \mathcal{C}$,

$$\eta_{(-,C)} : F(-, C) \to G(-, C)$$

is a natural transformation. In addition, η is natural for all $C \in \mathcal{C}$ since for all $B \in \mathcal{B}$

$$\eta_{(B,-)} : F(B, -) \to G(B, -)$$

is a natural transformation. Hence consider the natural transformation $\eta_{(-,C)}$ acting on (B, C) and $\eta_{(B',-)}$ acting on (B', C). Then we get the following commutative diagrams.

$$\begin{tikzcd}
F(B,C) \arrow[r, "\eta_{(B,C)}"] \arrow[d, "F(f{,}1_C)"'] & G(B,C) \arrow[d, "G(f{,}1_C)"] \\
F(B',C) \arrow[r, "\eta_{(B',C)}"] & G(B',C)
\end{tikzcd}$$

$$\begin{tikzcd}
F(B',C) \arrow[r, "\eta_{(B',C)}"] \arrow[d, "F(1_{B'}{,}g)"'] & G(B',C) \arrow[d, "G(1_{B'}{,}g)"] \\
F(B',C') \arrow[r, "\eta_{(B',C')}"] & G(B',C')
\end{tikzcd}$$

Observe that the bottom row of the first diagram matches the top row of the second. Also note that $f : B \to B'$ and $g : C \to C'$, and that the diagrams imply the equations

$$G(f, 1_C) \circ \eta_{(B,C)} = \eta_{(B',C)} \circ F(f, 1_C) \tag{1.1}$$

$$G(1_{B'}, g) \circ \eta_{(B',C)} = \eta_{(B',C')} \circ F(1_{B'}, g). \tag{1.2}$$

Now suppose we compose equation (1.1) with $G(1_{B'}, g)$ on the left.

Then we get that

$$\begin{aligned}
G(1_{B'}, g) \circ G(f, 1_C) \circ \eta_{(B,C)} &= \overbrace{G(1_{B'}, g) \circ \eta_{(B',C)}}^{\text{replace via equation (2)}} \circ F(f, 1_C) \\
&= \eta_{(B',C')} \circ F(1_{B'}, g) \circ F(f, 1_C) \\
&= \eta_{(B',C')} \circ F(1_{B'} \circ f, g \circ 1_C) \\
&= \eta_{(B',C')} \circ F(f, g).
\end{aligned}$$

where in the second step we applied equation (1.2), and in the third step we composed the morphisms. Also note that we can simplify the left-hand side since

$$G(1_{B'}, g) \circ G(f, 1_C) = G(1_{B'} \circ f, g \circ 1_C) = G(f, g).$$

Therefore, we have that

$$G(f, g) \circ \eta_{(B,C)} = \eta_{(B',C')} \circ F(f, g)$$

which implies that *eta* itself is a natural transformation. Specifically, it implies the following diagram.

$$\begin{tikzcd}
(B,C) \arrow[d, "{(f,g)}"] \\
(B',C')
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F(B,C) \arrow[r, "\eta_{(B,C)}"] \arrow[d, "{F(f,g)}"'] & G(B,C) \arrow[d, "{G(f,g)}"] \\
F(B',C') \arrow[r, "\eta_{(B',C')}"] & G(B',C')
\end{tikzcd}$$

∎

Note: A way to succinctly prove the reverse implication of the previous proof is as follows. Since we know the diagrams on the left are commutative, just "stack" them on top of each other to achieve the diagram in the upper right corner, and then "squish" this diagram down to obtain the third diagram in the bottom right.

$$\begin{tikzcd}
F(B,C) \arrow[r, "\eta_{(B,C)}"] \arrow[d, "F(f{,}1_C)"'] & G(B,C) \arrow[d, "G(f{,}1_C)"] \\
F(B',C) \arrow[r, "\eta_{(B',C)}"] & G(B',C)
\end{tikzcd}$$

$$\begin{tikzcd}
F(B',C) \arrow[r, "\eta_{(B',C)}"] \arrow[d, "F(1_{B'}{,}g)"'] & G(B',C) \arrow[d, "G(1_{B'}{,}g)"] \\
F(B',C') \arrow[r, "\eta_{(B',C')}"] & G(B',C')
\end{tikzcd}$$

$$\begin{tikzcd}
F(B,C) \arrow[r, "\eta_{(B,C)}"] \arrow[d, "F(f{,}1_C)"'] & G(B,C) \arrow[d, "G(f{,}1_C)"] \\
F(B',C) \arrow[r, "\eta_{(B',C)}"] \arrow[d, "F(1_{B'}{,}g)"'] & G(B',C) \arrow[d, "G(1_{B'}{,}g)"] \\
F(B',C') \arrow[r, "\eta_{(B',C')}"] & G(B',C')
\end{tikzcd}$$

$$\begin{tikzcd}
F(B,C) \arrow[r, "\eta_{(B,C)}"] \arrow[d, "F(f{,}g)"'] & G(B,C) \arrow[d, "G(f{,}g)"] \\
F(B',C') \arrow[r, "\eta_{(B',C')}"] & G(B',C')
\end{tikzcd}$$

This is essentially what we did in the proof, although this is more crude visualization of what happened, and we were more formal throughout the process.

1.7 Functor Categories

In the proof for the last proposition, we used a trick of forming a desired natural transformation by composing two composable natural transformations. Hence, we see that natural transformations can be composed.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and consider the functors $F, G, H : \mathcal{C} \to \mathcal{D}$. Suppose further that $\sigma : F \to G$ and $\tau : G \to H$ are natural transformations between the functors. Using these natural transformations, we construct the morphism $\tau \cdot \sigma : F \to H$ which for each $C \in \mathcal{C}$ is defined as

$$(\tau \cdot \sigma)_C = \tau_C \circ \sigma_C : F(C) \to H(C).$$

What we are visually doing for a given morphism $f : c \to c'$ in $\mathcal{C}$ is defining the morphism $(\tau \cdot \sigma)_C$ as below.

$$\begin{tikzcd}
F(C) \arrow[r, "F(f)"] \arrow[d, "\sigma_C"'] \arrow[dd, bend right=60, "(\tau \cdot \sigma)_C"'] & F(C') \arrow[d, "\sigma_{C'}"] \arrow[dd, bend left=60, "(\tau \cdot \sigma)_{c'}"] \\
G(C) \arrow[r, "G(f)"] \arrow[d, "\tau_C"'] & G(C') \arrow[d, "\tau_{C'}"] \\
H(C) \arrow[r, "H(f)"] & H(C')
\end{tikzcd}$$

One way to view this is that the property of the existence of a natural transformation is transitive due to commutativity. In addition, we see that for any functor $F : \mathcal{C} \to \mathcal{D}$ there exists an identity natural transformation $1_F : F \to F$, which simply associates an object c with the arrow $1_{F(c)}$. And since natural transformations are associative under composition, we see that this actually forms a category.

Definition 1.7.1. Let $\mathcal{C}$ and $\mathcal{D}$ be small categories and consider set of all functors $F : \mathcal{C} \to \mathcal{D}$. Then the **functor category**, denoted as $\mathcal{D}^{\mathcal{C}}$ or $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, is the category where

Objects Functors $F : \mathcal{C} \to \mathcal{D}$

Morphisms Natural transformations $\eta : F \to G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$.

Example 1.7.2. Let $\mathbf{1}$ be the one element category with a single identity arrow. Then for any category $\mathcal{C}$, the functor category $\mathcal{C}^{\mathbf{1}}$ is isomorphic to $\mathcal{C}$. This is because each functor $F : \mathbf{1} \to \mathcal{C}$ simply associates the element $1 \in \mathbf{1}$ to an element $C \in \mathcal{C}$, and the identity $1_1 : 1 \to 1$ to the identity morphism 1_C in $\mathcal{C}$.

Example 1.7.3. Let $\mathbf{2}$ be the category consisting of two elements, containing the two identities and one nontrivial morphism between the objects.

$$\begin{tikzcd} 1 \arrow[loop above, "\mathrm{id}_1"] \arrow[r, "f"] & 2 \arrow[loop above, "\mathrm{id}_2"] \end{tikzcd}$$

The category $\mathbf{2}$.

Now consider the functor category $\mathcal{C}^{\mathbf{2}}$ where $\mathcal{C}$ is any category. Each functor $F : \mathbf{2} \to \mathcal{C}$ maps the pair of objects to objects $F(1)$ and $F(2)$ in $\mathcal{C}$. However, since functors preserve morphisms, we see that

$$f : 1 \to 2 \implies F(f) : F(1) \to F(2).$$

This is what each $F \in \mathcal{C}^{\mathbf{2}}$ does. Hence, every morphism $g \in \mathrm{Hom}(\mathcal{C})$ corresponds to an element in $\mathcal{C}^{\mathbf{2}}$. Hence, we call $\mathcal{C}^{\mathbf{2}}$ the category of arrows of $\mathcal{C}$.

Proof: Let $g : C \to C'$ be any morphism between objects C, C' in $\mathcal{C}$. Construct the element $G \in \mathcal{C}^{\mathbf{2}}$ as follows: $G(1) = C$, $G(2) = C'$ and $G(f) : G(1) \to G(2) = g$. Hence, $\mathrm{Hom}(\mathcal{C})$ and $\mathcal{C}^{\mathbf{2}}$ are isomorphic. Moreover, $\mathrm{Hom}(\mathcal{C})$ determines the members of $\mathcal{C}^{\mathbf{2}}$.

A crude way to visualize this proof is imaging $1 \to 2$ is a "stick" with 1 and 2 on either end, and so the action of any functor is simply taking the stick and applying it to anywhere on the direct graph generated by the category $\mathcal{C}$. Hence, this is why we say $\mathrm{Hom}(\mathcal{C})$ determines the functor category $\mathcal{C}^{\mathbf{2}}$.

∎

Example 1.7.4. Let X be a set. Hence, it is a discrete category, which if recall, it's objects are elements of X and the morphisms are just identity morphisms.

Now consider $\{0, 1\}^X$, the category of functors $F : X \to \{0, 1\}$. Then every functor assigns each element of $x \in X$ to either 0 or 1, and assigns the morphism $1_x : x \to x$ to either $1_0 : 0 \to 0$ or $1_1 : 1 \to 1$.

One way to view this is to consider $\mathcal{P}(X)$, and for each $S \in \mathcal{P}$, assign x to 1 if $x \in S$ or x to 0 if $x \notin S$. All of these mappings may be described by elements of $\mathcal{P}$, but we can also realize that each of these mappings correspond to the functors in $\{0,1\}^X$. Hence, we see that $\{0,1\}^X$ is isomorphic to $\mathcal{P}(X)$.

Example 1.7.5. Recall that, given a group G and a ring R (with identity), we can create a *group ring* $R[G]$ with identity, in a functorial way, establishing a functor

$$R[-] : \mathbf{Grp} \to \mathbf{Ring}.$$

However, we then noticed that the above functor establishes a process where we send rings R to functors $R[-] : \mathbf{Grp} \to \mathbf{Ring}$. It turns out that this process is itself a functor, and we now have the appropriate language to describe it:

$$F : \mathbf{Ring} \to \mathbf{Ring}^{\mathbf{Grp}}$$

Specifically, let $\psi : R \to S$ be a ring homomorphism. Now observe that ψ induces another ring homomorphism

$$\psi_G^* : R[G] \to S[G] \qquad \sum_{g \in G} a_g g \mapsto \sum_{g \in G} \varphi(a_g) g.$$

As a result, we see that such a ring homomorphism induces a natural transformation. To show this, let $\varphi : G \to H$ be a group homomorphism. Then observe that we get the diagram in the middle.

$$\begin{tikzcd}
G \arrow[d, "\varphi"] & R[G] \arrow[r, "\psi_G^*"] \arrow[d, "R(\varphi)"'] & S[G] \arrow[d, "S(\varphi)"] \\
H & R[H] \arrow[r, "\psi_H^*"'] & S[H]
\end{tikzcd}$$

$$\begin{tikzcd}
\sum_{g \in G} a_g g \arrow[r, mapsto] \arrow[d, mapsto] & \sum_{g \in G} \psi(a_g) g \arrow[d, mapsto] \\
\sum_{g \in G} a_g \varphi(g) \arrow[r, mapsto] & \sum_{g \in G} \psi(a_g) \varphi(g)
\end{tikzcd}$$

However, we can follow the elements as in the diagram on the right, which shows us that the diagram commutes. Hence we see that ψ^* is a natural transformation between functors $R[-] \to S[-]$. Overall, this establishes that we do in fact have a functor

$$F : \mathbf{Ring} \to \mathbf{Ring}^{\mathbf{Grp}}$$

which we wouldn't be able to describe without otherwise introducing the notion of a functor category.

Example 1.7.6. Let M be a monoid category (one object) and consider the functor category $\mathbf{Set}^M$. The objects of $\mathbf{Set}^M$ are functors $F : M \to \mathbf{Set}$, each of which have the following data:

$$F(f) : F(M) \to F(M)$$

where $f : M \to M$ is an morphism in M. Now if we interpret $\circ$ as the binary relation equipped on M, we see that for any $g : M \to M$,

$$F(g \circ f) = F(g) \circ F(f)$$

by functorial properties. Hence, each functor F maps M to a set X which induces the operation of M on X. Therefore the objects of $\mathbf{Set}^M$ are other monoids X in $\mathbf{Set}$ equipped with the same operation as M and as well as the morphisms between such monoids.

Chapter 2

Monoidal Categories

2.1 Monoidal Categories.

Definition 2.1.1. A **monoidal category** $(\mathcal{M}, \otimes, I)$ is a category $\mathcal{M}$ equipped with a bifunctor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, a special object I of $\mathcal{M}$, and three natural isomorphisms

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C \qquad \textbf{(Associator)}$$
$$\lambda_A : I \otimes A \xrightarrow{\sim} A \qquad \textbf{(Left Unit)}$$
$$\rho_A : A \otimes I \xrightarrow{\sim} A \qquad \textbf{(Right Unit)}$$

such that the following **coherence conditions** hold. For any objects A, B, C, D of $\mathcal{M}$, the following diagrams must commute.

$$\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A \otimes B, C, D}"] & \\
A \otimes (B \otimes (C \otimes D)) \arrow[ur, "\alpha_{A,B,C \otimes D}"] \arrow[d, "1_A \otimes \alpha_{B,C,D}"'] & & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha_{A, B \otimes C, D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha_{A,B,C} \otimes 1_D"']
\end{tikzcd} \tag{2.1}$$

For all A, B, the diagram below must commute.

$$
\begin{tikzcd}
A \otimes (I \otimes B) \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "1_A \otimes \lambda_B"'] & & (A \otimes I) \otimes B \arrow[dl, "\rho_A \otimes 1_B"] \\
& A \otimes B &
\end{tikzcd}
\tag{2.2}
$$

We also define some terminology within this definition.

- We call the bifunctor $\otimes$ the **monoidal product**
- We refer to I as the **identity object**

We say a **strict monoidal category** is one in which the associator, left unit and right unit are all identities.

A very natural question is: What are those mysterious coherence conditions? Unfortunately, the answer to the question is not simple. Any thorough, accurate answer to that question will be an extremely long one. Thus, the short answer is that the diagrams are the minimum requirements for Mac Lane's Coherence theorem to be true.

Purposefully being vague, we will only say at this point that the theorem guarantees that a large class of diagrams in our monoidal category will commute. We will eventually give a very precise statement of the theorem.

The modern definition of a monoidal category is somewhat deceptive since it implies many things that one would probably not guess to be implied by the initial axioms. One such result is the following proposition.

Proposition 2.1.2. Let $(\mathcal{M}, \otimes, I)$ be a monoidal category. For all $A, B \in \mathcal{M}$, the unitor diagrams

$$
\begin{tikzcd}
I \otimes (A \otimes B) \arrow[rr, "\alpha_{I,A,B}"] \arrow[dr, "\lambda_{A \otimes B}"'] & & (I \otimes A) \otimes B \arrow[dl, "\lambda_A \otimes 1_B"] \\
& A \otimes B &
\end{tikzcd}
\tag{2.3}
$$

$$\begin{tikzcd} A \otimes (B \otimes I) \arrow[rr, "\alpha_{A,B,I}"] \arrow[dr, "1_A \otimes \rho_B"'] & & (A \otimes B) \otimes I \arrow[dl, "\rho_{A \otimes B}"] \\ & A \otimes B & \end{tikzcd} \tag{2.4}$$

are commutative.

The above two unitor diagrams were initially part of the original definition of a monoidal category. Thus, the original definition had three unitor diagrams. It was GM Kelly who pointed out in [Kel64] that the the pentagon axiom and Diagram 2.2 imply the other two unitor diagrams. In fact, any one of the three unitor diagrams imply the other two, a fact one would not immediately guess.

Proof: First we show 2.3 commutes. We take the pentagon, and substitute the monoidal identity I in the first two entries:

$$\begin{tikzcd} I \otimes (I \otimes (A \otimes B)) \arrow[r, "\alpha_{I,I,A \otimes B}"] \arrow[d, "1_I \otimes \alpha_{I,A,B}"'] & (I \otimes I) \otimes (A \otimes B) \arrow[r, "\alpha_{I \otimes I,A,B}"] & ((I \otimes I) \otimes A) \otimes B \\ I \otimes ((I \otimes A) \otimes B) \arrow[rr, "\alpha_{I,I \otimes A,B}"'] & & (I \otimes (I \otimes A)) \otimes B \arrow[u, "\alpha_{I,I,A} \otimes 1_B"'] \end{tikzcd}$$

We append an instance of $I \otimes (A \otimes B)$ and $(I \otimes A) \otimes B$ in our diagram to obtain

$$\begin{tikzcd} I \otimes (I \otimes (A \otimes B)) \arrow[rr, shift left, "\alpha_{I,I,A \otimes B}"] \arrow[rr, shift right] \arrow[dd, "1_I \otimes \alpha_{I,A,B}"'] \arrow[dr, "1_I \otimes \lambda_{A \otimes B}"] & & (I \otimes I) \otimes (A \otimes B) \arrow[rr, "\alpha_{I \otimes I,A,B}"] \arrow[dl, "\rho_I \otimes (1_A \otimes 1_B)"] & & ((I \otimes I) \otimes A) \otimes B \arrow[dl, "(\rho_I \otimes 1_A) \otimes 1_B"'] \\ & I \otimes (A \otimes B) & & (I \otimes A) \otimes B & \\ I \otimes ((I \otimes A) \otimes B) \arrow[rrrr, "\alpha_{I,I \otimes A,B}"'] & & & & (I \otimes (I \otimes A)) \otimes B \arrow[ul, "(1_I \otimes \lambda_A) \otimes 1_B"] \arrow[uu, shift right] \arrow[uu, shift left, "\alpha_{I,I,A} \otimes 1_B"'] \end{tikzcd}$$

We can connect $I \otimes (A \otimes B)$ and $(I \otimes A) \otimes B$ with the morphism

$$\alpha_{I,A,B} : I \otimes (A \otimes B) \xrightarrow{\sim} (I \otimes A) \otimes B$$

In addition, we can connect $(I \otimes (I \otimes A) \otimes B)$ with $I \otimes (A \otimes B)$ with

$$1_I \otimes \lambda_{I\otimes(A\otimes B)} : I((I \otimes A) \otimes B) \xrightarrow{\sim} I \otimes (A \otimes B).$$

This results in the diagram below which may not commute; our concerns are outlined in red.

$$\begin{tikzcd}
I \otimes (I \otimes (A \otimes B)) \arrow[rr, "\alpha_{I,I,A\otimes B}"] \arrow[dr, "1_I\otimes\lambda_{A\otimes B}"] \arrow[dd, "1_I\otimes\alpha_{I,A,B}"'] & & (I \otimes I) \otimes (A \otimes B) \arrow[rr, "\alpha_{I\otimes I,A,B}"] \arrow[dl, "\rho_I\otimes(1_A\otimes 1_B)"'] & & ((I \otimes I) \otimes A) \otimes B \arrow[dl, "(\rho_I\otimes 1_A)\otimes 1_B"] \\
& I \otimes (A \otimes B) \arrow[rr, "\alpha_{I,A,B}"] & & (I \otimes A) \otimes B & \\
I \otimes ((I \otimes A) \otimes B) \arrow[ur, "1_I\otimes(\lambda_A\otimes 1_B)"'] \arrow[rrrr, "\alpha_{I,I\otimes A,B}"'] & & & & (I \otimes (I \otimes A)) \otimes B \arrow[ul, "(1_I\otimes\lambda_A)\otimes 1_B"] \arrow[uu, "\alpha_{I,I,A}\otimes 1_B"']
\end{tikzcd}$$

However, both of these red diagrams must necessarily commute due to the naturality of α. Our entire diagram almost commutes except for one final concern on the left, outlined in red.

$$\begin{tikzcd}
I \otimes (I \otimes (A \otimes B)) \arrow[rr, "\alpha_{I,I,A\otimes B}"] \arrow[dr, "1_I\otimes\lambda_{A\otimes B}"] \arrow[dd, "1_I\otimes\alpha_{I,A,B}"'] & & (I \otimes I) \otimes (A \otimes B) \arrow[rr, "\alpha_{I\otimes I,A,B}"] \arrow[dl, "\rho_I\otimes(1_A\otimes 1_B)"] & & ((I \otimes I) \otimes A) \otimes B \arrow[dl, "(\rho_I\otimes 1_A)\otimes 1_B"] \\
& I \otimes (A \otimes B) \arrow[rr, "\alpha_{I,A,B}"] & & (I \otimes A) \otimes B & \\
I \otimes ((I \otimes A) \otimes B) \arrow[ur, "1_I\otimes(\lambda_A\otimes 1_B)"'] \arrow[rrrr, "\alpha_{I,I\otimes A,B}"'] & & & & (I \otimes (I \otimes A)) \otimes B \arrow[ul, "(1_I\otimes\lambda_A)\otimes 1_B"] \arrow[uu, "\alpha_{I,I,A}\otimes 1_B"']
\end{tikzcd}$$

The commutativity of this diagram in red is forced since (1) every other diagram commutes, (2) all morphisms are isomorphisms. Thus we conclude that

$$\begin{tikzcd}
I \otimes (A \otimes B) \arrow[rr, "\alpha_{I,A,B}"] \arrow[dr, swap, "\lambda_{A\otimes B}"] & & (I \otimes A) \otimes B \arrow[dl, "\rho_A \otimes 1_B"] \\
& A \otimes B &
\end{tikzcd}$$

must commute.

Next, we show that Diagram 2.4

$$\begin{tikzcd}
A \otimes (B \otimes I) \arrow[rr, "\alpha_{A,B,I}"] \arrow[dr, swap, "1_A \otimes \rho_B"] & & (A \otimes B) \otimes I \arrow[dl, "\rho_{A\otimes B}"] \\
& A \otimes B &
\end{tikzcd}$$

commutes for all objects of $\mathcal{M}$.

First we take the pentagon diagram and insert the monoidal identity I to obtain the commutative diagram below.

$$\begin{tikzcd}
A \otimes (B \otimes (I \otimes I)) \arrow[r, "\alpha_{A,B,I\otimes I}"] \arrow[d, swap, "1_A \otimes \alpha_{B,I,I}"] & (A \otimes B) \otimes (I \otimes I) \arrow[r, "\alpha_{A\otimes B,I,I}"] & ((A \otimes B) \otimes I) \otimes I \\
A \otimes ((I \otimes B) \otimes I) \arrow[rr, swap, "\alpha_{A,B\otimes I,I}"] & & (A \otimes (B \otimes I)) \otimes I \arrow[u, swap, "\alpha_{A,B,I} \otimes 1_I"]
\end{tikzcd}$$

We append $A \otimes (B \otimes I)$ and $(A \otimes B) \otimes I$ to obtain the diagram below.

$$\begin{tikzcd}
A \otimes (B \otimes (I \otimes I)) \arrow[rr, "\alpha_{A,B,I\otimes I}"] \arrow[dr, "1_A \otimes (1_B \otimes \rho_I)"] \arrow[dd, swap, "1_A \otimes \alpha_{B,I,I}"] & & (A \otimes B) \otimes (I \otimes I) \arrow[rr, "\alpha_{A\otimes B,I,I}"] \arrow[dr, swap, "(1_A \otimes 1_B) \otimes \rho_I"] & & ((A \otimes B) \otimes I) \otimes I \arrow[dl, "\rho_{(A\otimes B)\otimes I}"] \\
& A \otimes (B \otimes I) & & (A \otimes B) \otimes I & \\
A \otimes ((I \otimes B) \otimes I) \arrow[ur, swap, "1_A \otimes (\rho_B \otimes 1_I)"] \arrow[rrrr, swap, "\alpha_{A,B\otimes I,I}"] & & & & (A \otimes (B \otimes I)) \otimes I \arrow[uu, swap, "\alpha_{A,B,I} \otimes 1_I"]
\end{tikzcd}$$

We can connect $A \otimes (B \otimes I)$ with $(A \otimes B) \otimes I$ with the morphism $\alpha_{A,B,I}$, and we may connect $(A \otimes (B \otimes I)) \otimes I$ with $A \otimes (B \otimes I)$ via $(1_A \otimes \rho_B) \otimes 1_I$. However, if we do this, we don't know if certain diagrams will commute; we will outline these concerns in red.

$$\begin{tikzcd}
A \otimes (B \otimes (I \otimes I)) \arrow[rr, red, "\alpha_{A,B,I\otimes I}"] \arrow[dr, red, "1_A\otimes(1_B\otimes\rho_I)"] \arrow[dr, green] \arrow[dd, green, "1_A\otimes\alpha_{B,I,I}"'] & & (A \otimes B) \otimes (I \otimes I) \arrow[rr, green, "\alpha_{A\otimes B,I,I}"] \arrow[dr, red, "(1_A\otimes 1_B)\otimes\rho_I"'] \arrow[dr, green] & & ((A \otimes B) \otimes I) \otimes I \arrow[dl, green, "\rho_{(A\otimes B)\otimes I}"'] \arrow[dl, red] \\
& A \otimes (B \otimes I) \arrow[rr, red, "\alpha_{A,B,I}"] & & (A \otimes B) \otimes I & \\
A \otimes ((I \otimes B) \otimes I) \arrow[ur, green] \arrow[ur, red, "1_A\otimes(\rho_B\otimes 1_I)"'] \arrow[rrrr, red, "\alpha_{A,B\otimes I,I}"'] & & & & (A \otimes (B \otimes I)) \otimes I \arrow[ul, red, "(1_A\otimes\rho_B)\otimes 1_I"] \arrow[uu, red] \arrow[uu, "\alpha_{A,B,I}\otimes 1_I"']
\end{tikzcd}$$

Observe that the top and bottom diagrams must commute by naturality in α. Our diagram so far is:

$$\begin{tikzcd}
A \otimes (B \otimes (I \otimes I)) \arrow[rr, blue, "\alpha_{A,B,I\otimes I}"] \arrow[dr, blue, "1_A\otimes(1_B\otimes\rho_I)"] \arrow[dr, green] \arrow[dd, green, "1_A\otimes\alpha_{B,I,I}"'] & & (A \otimes B) \otimes (I \otimes I) \arrow[rr, green, "\alpha_{A\otimes B,I,I}"] \arrow[dr, blue, "(1_A\otimes 1_B)\otimes\rho_I"'] \arrow[dr, green] & & ((A \otimes B) \otimes I) \otimes I \arrow[dl, green, "\rho_{(A\otimes B)\otimes I}"'] \arrow[dl, red] \\
& A \otimes (B \otimes I) \arrow[rr, blue, "\alpha_{A,B,I}"] & & (A \otimes B) \otimes I & \\
A \otimes ((I \otimes B) \otimes I) \arrow[ur, green] \arrow[ur, blue, "1_A\otimes(\rho_B\otimes 1_I)"'] \arrow[rrrr, blue, "\alpha_{A,B\otimes I,I}"'] & & & & (A \otimes (B \otimes I)) \otimes I \arrow[ul, red] \arrow[ul, blue, "(1_A\otimes\rho_B)\otimes 1_I"] \arrow[uu, red] \arrow[uu, "\alpha_{A,B,I}\otimes 1_I"']
\end{tikzcd}$$

Our remaining concern is the diagram outlined in red. However, this diagram must commute since (1) every other diagram commutes and (2) each morphism is an isomorphism. Hence we see that

$$\begin{tikzcd}
A \otimes (B \otimes I) \arrow[rr, "\alpha_{A,B,I}"] \arrow[dr, "1_A\otimes\rho_B"'] & & (A \otimes B) \otimes I \arrow[dl, "\rho_{A\otimes B}"] \\
& A \otimes B &
\end{tikzcd}$$

commutes for all $A, B \in \mathcal{M}$.

■

2.2 Examples of Monoidal Categories

We now introduce a barrage of examples of monoidal categories.

Example 2.2.1. Consider the category $(\mathbf{Set}, \times, \{\bullet\})$ equipped with the cartesian bifunctor $\times : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ and the terminal object $\{\bullet\}$ (the one element set). We'll show that this forms a monoidal category.

First we require an associator. To demonstrate the existence of one, we first start with the products $A \times B$ and $B \times C$ which have the usual universal diagrams

$$\begin{tikzcd}
& C' \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\
A & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& C' \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\
B & B \times C \arrow[l, "\pi'_B"'] \arrow[r, "\pi_C"'] & C
\end{tikzcd}$$

Now just like other products, the products $A \times (B \times C)$ and $(A \times B) \times C$ have projection maps to their factors

$$\pi'_A : A \times (B \times C) \to A \qquad \pi_{A\times B} : (A \times B) \times C \to A \times B$$
$$\pi_{B\times C} : A \times (B \times C) \to B \times C \qquad \pi'_C : (A \times B) \times C \to C$$

However, note that $\pi_{B\times C}$ can be composed with $\pi'_B : B \times C \to B$ to give a map $\pi_C \circ \pi_{B\times C} : A \times (B \times C) \to B$. Similarly, $\pi_{A\times B}$ can be composed with $\pi_B : A \times B \to B$ to give a map $\pi_B \circ \pi_{A\times B} : (A \times B) \times C \to B$. As a result, the universal property of both of these products yield unique maps φ and ψ such that the diagrams below commute.

$$\begin{tikzcd}
& A \times (B \times C) \arrow[dl, "\pi'_A"'] \arrow[d, dashed, "\psi"'] \arrow[dr, "\pi'_B \circ \pi_{B\times C}"] & \\
A & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"'] & B
\end{tikzcd}$$

$$\begin{tikzcd}
& (A \times B) \times C \arrow[dl, "\pi_B \circ \pi_{A\times B}"'] \arrow[d, dashed, "\varphi"'] \arrow[dr, "\pi_C"] & \\
B & B \times C \arrow[l, "\pi_B"'] \arrow[r, "\pi_C"'] & C
\end{tikzcd}$$

Now both $A \times (B \times C)$ and $(A \times B) \times C$ have their own universal properties which we can take advantage of. Using the newly created maps and the projection maps, we have

$$\varphi : A \times (B \times C) \to A \times B \qquad \psi : (A \times B) \times C \to B \times C$$
$$\pi_C \circ \pi_{B\times C} : A \times (B \times C) \to C \qquad \pi_A \circ \pi_{A\times B} : (A \times B) \times C \to A$$

which, by the universal property of both of our products, give rise to the existence of the morphisms α and α' which make the diagrams below commute.

$$\begin{tikzcd}
& A \times (B \times C) \arrow[dl, "\psi"'] \arrow[d, dashed, "\alpha"'] \arrow[dr, "\pi_C \circ \pi_{B\times C}"] & \\
A \times B & (A \times B) \times C \arrow[l, "\pi_{A\times B}"] \arrow[r, "\pi'_C"'] & C
\end{tikzcd}$$

$$\begin{tikzcd}
& (A \times B) \times C \arrow[dl, "\pi_A \circ \pi_{A\times B}"'] \arrow[d, dashed, "\alpha'"'] \arrow[dr, "\varphi"] & \\
B & A \times (B \times C) \arrow[l, "\pi'_A"] \arrow[r, "\pi_{B\times C}"'] & C
\end{tikzcd}$$

At this point it is a simple diagram chase to show that α and α' are the are inverses, and that they are natural so that they can be defined as our associator.

At this point, we have our associator

$$\alpha_{A,B,C} : A \times (B \times C) \xrightarrow{\sim} (A \times B) \times C \qquad (a, (b, c)) \mapsto ((a, b), c).$$

To demonstrate the existence of left and right unitors, first regard the identity object $\{\bullet\}$ as a terminal object T. Then for any object A, the product $T \times A$ comes with a universal diagram

$$\begin{tikzcd}
& A \arrow[dl, "T_A"'] \arrow[d, dashed, "f"] \arrow[dr, "1_A"] & \\
T & T \times A \arrow[l, "T_A"] \arrow[r, "\lambda_A"'] & A
\end{tikzcd}$$

This is because a terminal object guarantees the existence of one morphism $T_A : A \to T$. Therefore for the unique morphism f we have that $\lambda_A \circ f$. However, observe that $f \circ \lambda_A \circ T_{T\times A} : T \times A \to T$. Since this must be equal to $T_{T\times A}$, we see that $f \circ \lambda_A = 1_{T\times A}$. Hence

$$\lambda_A : T \times A \xrightarrow{\sim} A \qquad (*, a) \mapsto a$$

is an isomorphism. With a similar construction we can produce

$$\rho_A : A \times T \xrightarrow{\sim} A \qquad (a, *) \mapsto a$$

and in both cases it is simple to show that these isomorphisms are natural. One can then verify the diagrams by repeatedly using the universal properties of the product.

While we worked in **Set**, we avoided referencing the elements of our sets explicitly. As a result this can be generalized. Every category $\mathcal{C}$ with finite products and a terminal object T forms a monoidal category $(\mathcal{C}, \times, T)$. Therefore, $(\mathbf{Top}, \times, \{\bullet\})$, $(\mathbf{Ab}, \times, \{e\})$, and $(R\text{-}\mathbf{Mod}, \times, \{0\})$ form monoidal categories via the cartesian product.

Example 2.2.2. Let R be a commutative ring. Then the category of all R-modules, $(R\text{-}\mathbf{Mod}, \otimes, \{0\})$, forms a monoidal category under the tensor product. This is again the R-module which satisfies the universal diagram

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[dr, "f"'] & M \otimes N \arrow[d, dashed, "h"] \\ & K \end{tikzcd}$$

Now consider a third R-module P; then we have two ways of constructing the tensor product. To demonstrate that we may identify these objects up to isomorphism, construct the maps

$$f : (M \otimes N) \times P \to M \otimes (N \otimes P) \qquad \left(\sum_i m_i \otimes n_i, p\right) \mapsto \sum_i m_i \otimes (n_i \otimes p)$$

and

$$f' : M \times (N \otimes P) \to (M \otimes M) \otimes P \qquad \left(m, \sum_j n_j \otimes p_j\right) \mapsto \sum_j (m \otimes n_j) \otimes p_j.$$

These maps are bilinear due to the bilinearity of $\otimes$. Hence we see that the universal property of the tensor product gives us unique map α and α' such that the diagrams below commute.

$$\begin{tikzcd}
(M \otimes N) \times P \arrow[r, "\varphi"] \arrow[dr, "f"'] & (M \otimes N) \otimes P \arrow[d, dashed, "\alpha"] \\
& M \otimes (N \otimes P)
\end{tikzcd}$$

$$\begin{tikzcd}
M \otimes (N \times P) \arrow[r, "\varphi'"] \arrow[dr, "f'"'] & M \otimes (N \otimes P) \arrow[d, dashed, "\alpha'"] \\
& (M \otimes N) \otimes P
\end{tikzcd}$$

Based on how we defined f and f', and since we know that φ and φ' is, we can determine that α and α' are "shift maps", i.e,

$$\alpha\left(\sum_i (m_i \otimes n_i) \otimes p_i\right) = \sum_i m_i \otimes (n_i \otimes p_i) \qquad \alpha'\left(\sum_i m_i \otimes (n_i \otimes p_i)\right) = \sum_i (m_i \otimes n_i) \otimes p_i.$$

Hence we see that α and α' are inverses, so what we have is an associator:

$$\alpha_{M,N,P} : (M \otimes N) \otimes P \xrightarrow{\sim} M \otimes (N \otimes P).$$

Now consider the trivial R-module, denoted $I = \{0\}$. For any R-module M we have evident maps

$$\sum_i 0 \otimes m_i \mapsto m_i \qquad \sum_i m_i \otimes 0 \mapsto 0$$

which provide isomorphisms, so that we have left and right associators

$$\lambda_M : I \otimes M \xrightarrow{\sim} M \qquad \rho_M : M \otimes I \xrightarrow{\sim} M.$$

Finally, the triangular and pentagonal diagrams are commutative since shifting the tensor product on individual elements does not change (up to isomorphism) the value of the overall elements.

Example 2.2.3. Consider the category $\mathbf{GrMod}_R$ which consist of graded R-modules $M = \{M_n\}_{n=1}^{\infty}$ Then this forms a monoidal category $(\mathbf{GrMod}_R, \otimes, I)$ where $I = \{(0)_n\}_{n=1}^{\infty}$ is the trivial graded R-module and where we define the monoidal product as $M \otimes N = \{(M \otimes N)_n\}_{n=1}^{\infty}$ where

$$(M \otimes N)_n = \bigoplus_{i+j=n} M_i \otimes N_j.$$

To show that this is monoidal, the first thing we must check is that we have an associator. Towards this goal, consider three graded R-modules $M = \{M_n\}_{n=1}^{\infty}$, $N = \{N_n\}_{n=1}^{\infty}$ and $P = \{P_n\}_{n=1}^{\infty}$. Then the m-th graded module of $M \otimes (N \otimes P)$ is

$$\begin{aligned}
[M \otimes (N \otimes P)]_m = \bigoplus_{i+j=m} M_i \otimes (N \otimes P)_j &= \bigoplus_{i+j=m} M_i \otimes \left(\bigoplus_{h+k=j} N_h \otimes P_k \right) \\
&= \bigoplus_{i+h+k=m} M_i \otimes (N_h \otimes P_k) \\
&\cong \bigoplus_{i+h+k=m} (M_i \otimes N_h) \otimes P_k \\
&= \bigoplus_{l+k=m} \left(\bigoplus_{i+h=l} M_i \otimes N_h \right) \otimes P_k \\
&= \bigoplus_{l+k=m} (M \otimes N)_l \otimes P_k \\
&= [M \otimes (N \otimes P)]_m
\end{aligned}$$

where in the third step we used the fact that the tensor product commutes with direct sums and in the fourth step we used the canonical associator regarding the tensor products of three elements. Thus we see that we have an associator

$$\alpha : M \otimes (N \otimes P) \xrightarrow{\sim} (M \otimes N) \otimes P$$

which as a graded module homomorphism, acts on each level as

$$\alpha_m : [M \otimes (N \otimes P)]_m \xrightarrow{\sim} [(M \otimes N) \otimes P]_m$$

where in each coordinate of the direct sums we apply an instance of the associator α' between the tensor product of three R-modules. The naturality of this associator is inherited from α'. In addition, we have natural left and right unitors

$$\lambda_M : I \otimes M \xrightarrow{\sim} M \qquad \rho_M : M \otimes I \xrightarrow{\sim} M$$

where on each level we utilize the natural left and right unitors for non-graded R-modules.

Example 2.2.4. Let $(M, \otimes, I, \alpha, \rho, \lambda)$ be a monoidal category, $\mathcal{C}$ any other category. Then the functor category $\mathcal{C}^M$ is a monoidal category. We treat the constant functor $I : \mathcal{C} \to M$ where

$$I(A) = I \text{ for all } A$$

as the identity element, and we can define a tensor product on this category as follows: on objects $F, G : \mathcal{C} \to M$, we define $F \boxtimes G$ as the composite

$$F \boxtimes G : \mathcal{C} \xrightarrow{\Delta} \mathcal{C} \times \mathcal{C} \xrightarrow{(F \times G)} M \times M \xrightarrow{\otimes} M$$

which can be stated pointwise as $(F \boxtimes G)(C) = F(C) \otimes G(C)$. On morphisms, we have that if $\eta : F_1 \to F_2$ and $\eta' : G_1 \to G_2$ are natural transformations, then we say $\eta \boxtimes \eta' : F_1 \boxtimes G_1 \to F_2 \boxtimes G_2$ is a natural transformation, where we define

$$(\eta \boxtimes \eta')_A = \eta_A \otimes \eta'_A : F_1(A) \otimes G_1(A) \to F_2(A) \otimes G_2(A).$$

Note that such a natural transformation is well-defined as the diagram below commutes

$$\begin{tikzcd}
A \arrow[d, "f"] & F_1(A) \otimes G_1(A) \arrow[r, "\eta_A \otimes \eta'_A"] \arrow[d, "F_1(f) \otimes G_1(f)"'] & F_2(A) \otimes G_2(A) \arrow[d, "F_2(f) \otimes G_2(f)"] \\
B & F_1(B) \otimes G_1(B) \arrow[r, "\eta_B \otimes \eta'_B"'] & F_2(A) \otimes G_2(A)
\end{tikzcd}$$

since $\otimes : M \times M \to M$ is a bifunctor. Finally, for functors $F, G, H : \mathcal{C} \to M$ define the associator $\alpha'_{F,G,H} : F \boxtimes (G \boxtimes H) \xrightarrow{\sim} (F \boxtimes (G \boxtimes H))$ as the natural transformation where for each object A

$$(\alpha'_{F,G,H})_A = \alpha_{F(A),G(A),H(A)} : F(A) \otimes (G(A) \otimes H(A)) \to (F(A) \otimes G(A)) \otimes H(A)$$

and the unitors $\lambda'_F : I \boxtimes F \to F$ and $\rho'_F : F \boxtimes I \to F$ as the natural transformations where for each object A

$$(\lambda'_F)_A = \lambda_A : I \otimes F(A) \to F(A) \qquad (\rho'_F)_A = \rho_A : F(A) \otimes I \to F(A).$$

One can then show that these together satisfy the pentagon and unit axioms.

Example 2.2.5. Let M be a monoid with identity e and multiplication $\cdot : M \times M \to M$. Suppose we treat M as discrete category, with all arrows being identity arrows. Then we can trivially turn this into a monoidal category by setting the identity object to e and defining the tensor product $\otimes$ on objects to be $m \otimes m' = m \cdot m$, while identity morphisms are trivially sent to identity morphisms. Then M forms a monoidal category.

Example 2.2.6. Consider the category $\mathbb{P}$ whose objects are the natural numbers (with 0 included) and whose morphisms are the symmetric groups S_n. That is,

Objects. The objects are $n = 0, 1, 2, \dots$.

Morphisms. For any objects n, m we have that

$$\mathrm{Hom}_{\mathbb{P}}(n, m) = \begin{cases} S_n & \text{if } n = m \\ \varnothing & \text{if} n \neq m. \end{cases}$$

Note that there are many ways of constructing this category; we just present the simplest. In general terms this is the countable disjoint union of the symmetric groups. Even more generally, this can be done for any family of groups (or rings, monoids, semigroups).

What is interesting about this category is that it intuitively forms a strict monoidal category. That is, we can formulate a bifunctor $+ : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ on objects as addition of natural numbers and on morphisms as

$$\sigma \otimes \tau \in S_{n+m}$$

where $\sigma \in S_n$ and $\tau \in S_m$ and where $\sigma \otimes \tau$ denotes the **direct sum permutation**. I could tell you in esoteric language and notation what that is, or I could just show you: σ and τ, displayed as below

$$\begin{array}{ccc} (1, 2, \dots, n) & \qquad & (1, 2, \dots, m) \\ \big\downarrow & & \big\downarrow \\ (\sigma(1), \sigma(2), \dots, \sigma(n)) & & (\tau(1), \tau(2), \dots, \tau(m)) \end{array}$$

become $\sigma \otimes \tau$ which is displayed as below.

$$\begin{array}{c} (1, 2, \dots, n, n+1, n+2, \dots, n+m) \\ \big\downarrow \\ (\sigma(1), \sigma(2), \dots, \sigma(n), n + \tau(1), n + \tau(2), \dots, n + \tau(m)) \end{array}$$

To make this monoidal, we specify that 0 is our identity element whose associated identity morphism is the empty permutation. Now clearly this operation is strict on objects. On morphisms, it is also strict in the same way that stacking three Lego pieces together in the two possible different ways are equivalent. Hence the associators and unitors are all identities and this forms a strict monoidal category.

2.3 Monoidal Functors and Examples

We now end this chapter by discussing the concept of a monoidal functor. Various names are associated with different types of monoidal functors, since the degree to which you ask a functor between monoidal categories to "preserve" the monoidal structure can be varied and hence give rise to different types of monoidal functors. These concepts are of particular importance to the rest of the study of monoidal categories, since many proofs are achieved by constructing these types of functors, and many theorems are stated in terms of monoidal functors.

Definition 2.3.1. Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \odot, J)$ be monoidal categories. A **(lax) monoidal functor** is a functor $F : \mathcal{C} \to \mathcal{D}$ equipped with

- For each pair A, B in $\mathcal{C}$, we have a natural morphism

  $$\varphi_{A,B} : F(A) \odot F(B) \to F(A \otimes B)$$

  such that for any third object C, the diagram below commutes. (Note that we suppress the subscripts for clarity.)

$$\begin{tikzcd}
F(A) \odot \big(F(B) \odot F(C)\big) \arrow[r, "\alpha"] \arrow[d, "1\odot\varphi"'] & \big(F(A) \odot F(B)\big) \odot F(C) \arrow[d, "\varphi\odot 1"] \\
F(A) \odot F(B \otimes C) \arrow[d, "\varphi"'] & F(A \otimes B) \odot F(C) \arrow[d, "\varphi"] \\
F\Big(A \otimes \big(B \otimes C\big)\Big) \arrow[r, "F(\alpha)"'] & F\Big(\big(A \otimes B\big) \otimes C\Big)
\end{tikzcd}$$

- A unique morphism $\varepsilon : J \to F(I)$ such that, for any object A of $\mathcal{C}$, the diagrams below commutes. (Again, we suppress the subscripts for clarity.)

$$\begin{tikzcd}
F(A) \odot J \arrow[r, "\rho"] \arrow[d, "1\odot\varepsilon"'] & F(A) & J \odot F(A) \arrow[r, "\lambda"] \arrow[d, "\varepsilon\odot 1"'] & F(A) \\
F(A) \odot F(I) \arrow[r, "\varphi"'] & F(A \otimes I) \arrow[u, "F(\rho)"'] & F(J) \odot F(A) \arrow[r, "\varphi"'] & F(I \otimes A) \arrow[u, "F(\lambda)"']
\end{tikzcd}$$

We say the F is **strict** if φ and ε are identities and **strong** if φ and ε are isomorphisms.

We also define a **monoidal natural transformation** between two monoidal functors $\eta : F \to G$ to be a natural transformation between the functors such that, for every A, B, the diagram below commutes.

$$\begin{tikzcd}
F(A) \odot F(B) \arrow[r, "\varphi_F"] \arrow[d, "\eta_A \odot \eta_B"'] & F(A \otimes B) \arrow[d, "\eta_{A \otimes B}"] \\
G(A) \odot G(B) \arrow[r, "\varphi_G"'] & G(A \otimes B)
\end{tikzcd}$$

Example 2.3.2. Consider the power set functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ which associates each set X with its power set $\mathcal{P}(X)$. We may ask if this yields a monoidal functor

$$\mathcal{P} : (\mathbf{Set}, \times, \{\bullet\}) \to (\mathbf{Set}, \times, \{\bullet\})$$

in any sense of lax, strong, or strict. It turns out that we may define a lax monoidal functor, but not a strong or strict.

Towards defining a lax monoidal functor, let A, B two sets. Define $\varphi_{A,B} : \mathcal{P}(A) \times \mathcal{P}(B) \to \mathcal{P}(A \times B)$ to be a function where if U, V are subsets of A, B respectively, then

$$\varphi_{A,B}(U, V) = U \times V.$$

In addition, we define the function $\varepsilon : \{\bullet\} \to P(\{\bullet\})$ where

$$\varepsilon(\bullet) = \{\bullet\}.$$

Observe that with this data we have that for any sets A, B, C, the diagram below commutes

$$\begin{tikzcd}
\mathcal{P}(A) \times \Big(\mathcal{P}(B) \times \mathcal{P}(C)\Big) \arrow[r, "\alpha"] \arrow[d, "1 \times \varphi"'] & \Big(\mathcal{P}(A) \times \mathcal{P}(B)\Big) \times \mathcal{P}(C) \arrow[d, "\varphi \times 1"] \\
\mathcal{P}(A) \times \mathcal{P}(B \times C) \arrow[d, "\varphi"'] & \mathcal{P}(A \times B) \times \mathcal{P}(C) \arrow[d, "\varphi"] \\
\mathcal{P}\Big(A \times \Big(B \times C\Big)\Big) \arrow[r, "\mathcal{P}(\alpha)"'] & \mathcal{P}\Big(\Big(A \times B\Big) \times C\Big)
\end{tikzcd}$$

and that for any set A the diagrams below commute.

$$\begin{tikzcd} \mathcal{P}(A) \times \{\bullet\} \arrow[r, "\rho"] \arrow[d, "1\times\varepsilon"'] & \mathcal{P}(A) \\ \mathcal{P}(A) \times \mathcal{P}(\{\bullet\}) \arrow[r, "\varphi"'] & \mathcal{P}(A \times \{\bullet\}) \arrow[u, "\mathcal{P}(\rho)"'] \end{tikzcd} \quad \begin{tikzcd} \{\bullet\} \times \mathcal{P}(A) \arrow[r, "\lambda"] \arrow[d, "\varepsilon\times 1"'] & \mathcal{P}(A) \\ \mathcal{P}(\{\bullet\}) \times \mathcal{P}(A) \arrow[r, "\varphi"'] & \mathcal{P}(\{\bullet\} \times A) \arrow[u, "\mathcal{P}(\lambda)"'] \end{tikzcd}$$

Note that our choice that $\varepsilon(\bullet) = \{\bullet\}$ was necessary in order for the above two diagrams to commute.

We now show that this cannot be a strong or strict monoidal functor. To see this, let A, B be two sets. Observe that

$$|\mathcal{P}(A) \times \mathcal{P}(B)| = 2^{|A|} \cdot 2^{|B|} = 2^{|A|+|B|}$$

while

$$|\mathcal{P}(A \times B)| = 2^{|A\times B|}.$$

We see that in general these two sets are not of the same cardinality, and therefore one cannot establish an isomorphism between these two sets for all A, B, which we would need to do to at least construct a strong monoidal functor. Hence, we cannot regard this functor as strong or strict monoidal.

Example 2.3.3. The category of pointed topological spaces $\mathbf{Top}^*$ is the category where

Objects. Pairs (X, x_0) with X a topological space and $x_0 \in X$

Morphisms. A morphism $f : (X, x_0) \to (Y, y_0)$ is given by a continuous function $f : X \to Y$ such that $f(x_0) = y_0$.

This category is what allows us to characterize the fundamental group of a topological space as a functor

$$\pi_1 : \mathbf{Top}^* \to \mathbf{Grp}$$

which sends a pointed space (X, x_0) to its fundamental group $\pi_1(X, x_0)$ with x_0 as the selected basepoint. We demonstrate that this can be regarded as a monoidal functor

$$\pi_1 : \Big(\mathbf{Top}^*, \times, (\{\bullet\}, \bullet)\Big) \to (\mathbf{Grp}, \times, \{e\})$$

where $\{e\}$ is the trivial group. The reader may be wondering how we are putting a cartesian product structure on the **Top***, so we explain: For two topological spaces X, Y, we define

$$(X, x_0) \times (Y, y_0) = (X \times Y, (x_0, y_0))$$

where $X \times Y$ is given the product topology. The identity object $(\{\bullet\}, \bullet)$ is the trivial topological space with basepoint $\bullet$.

For any two pointed topological spaces $(X, x_0), (Y, y_0)$, define the function $\varphi_{X,Y} : \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi(X \times Y, (x_0, y_0))$ where for two loops β, γ based as x_0, y_0 respectively, then

$$\varphi_{X,Y}(\beta, \gamma) = \beta \times \gamma : [0, 1] \to X \times Y$$

which is in fact a loop in $X \times Y$ based at (x_0, y_0). The above function is bijective; an inverse can be constructed by sending a loop δ in $X \times Y$ based at (x_0, y_0) to the tuple $(p \circ \delta, q \circ \delta)$ where

$$p : X \times Y \to X \qquad q : X \times Y \to Y$$

are the projection maps. It is not difficult to see that this preserves group products, so that $\varphi_{X,Y}$ establishes the isomorphism

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

a fact usually proved in a topological course. In addition, this isomorphism to be natural: for two pointed topological spaces (X, x_0) and (Y, y_0), and for a pair of base-point preserving continuous functions $f : (X, x_0) \to (W, w_0)$ and $g : (Y, y_0) \to (Z, z_0)$, the following diagram commutes.

$$\begin{tikzcd}
\pi_1(X, x_0) \times \pi_1(Y, y_0) \arrow[r, "\varphi_{X,Y}"] \arrow[d, "\pi_1(f)\times\pi_1(g)"'] & \pi_1(X \times Y, (x_0, y_0)) \arrow[d, "\pi_1(f\times g)"] \\
\pi_1(W, w_0) \times \pi_1(Z, z_0) \arrow[r, "\varphi_{W,Z}"'] & \pi_1(W \times Z, (w_0, z_0))
\end{tikzcd}$$

Thus $\varphi_{X,Y}$ is our desired natural isomorphism.

Next, define $\varepsilon : \{e\} \to \pi_1(\{\bullet\}, \bullet)$ to be the group homomorphism that takes e to the trivial loop at $\bullet$. As in the previous example, we are actually forced to define ε in this way since $\{e\}$ is initial in **Grp**.

With this data, one can easily check that the necessary diagrams are commutative, so that the fundamental group functor π_1 is strong monoidal.

Example 2.3.4. Recall that a **Lie algebra** is a vector space $\mathfrak{g}$ over a field k with a bilinear function $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

Antisymmetry. For all $x, y \in \mathfrak{g}$, $[x, y] = -[y, x]$

Jacobi Identity. For all $x, y, z \in \mathfrak{g}$ we have that

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

For every Lie algebra $\mathfrak{g}$, we may create the **universal enveloping algebra** $U(\mathfrak{g})$. This is the algebra constructed as follows: If $T(\mathfrak{g})$ is the tensor algebra of $\mathfrak{g}$, i.e.,

$$T(\mathfrak{g}) = k \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots$$

and $I(\mathfrak{g})$ is the ideal generated by elements of the form $x \otimes y - y \otimes x - [x, y]$, then

$$U(\mathfrak{g}) = T(\mathfrak{g})/I(\mathfrak{g}).$$

By Corollary V.2.2(b) of [Kas95], this construction is actually a functor

$$U : \mathbf{LieAlg} \to k\text{-}\mathbf{Alg}.$$

Both categories can be regarded monoidal: $(\mathbf{LieAlg}, \oplus, \{\bullet\})$ is the monoidal category where we apply the cartesian product between Lie algebras, and $(k\text{-}\mathbf{Alg}, \otimes, k)$ is the monoidal category where we apply tensor products between k-algebras over the field k. The associators and unitors are the same that we have encountered in previous examples of monoidal categories with cartesian and tensor products.

We demonstrate that the universal enveloping algebra functor is strong monoidal:

$$U : (\mathbf{LieAlg}, \oplus, \{\bullet\}) \to (k\text{-}\mathbf{Alg}, \otimes, k)$$

- By Corollary V.2.3 of [Kas95], we have that if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two Lie algebras then $U(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \cong U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$. One can use Corollary V.2.3(a) to show that this isomorphism is natural in both $\mathfrak{g}_1$ and $\mathfrak{g}_2$. We let this morphism be our required isomorphism

$$\varphi_{\mathfrak{g}_1,\mathfrak{g}_2} : U(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \to U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2).$$

- Note that $U(\{\bullet\}) = k$. Therefore, we let $\varepsilon : k \to k$ be the identity.

As the associators and unitors are simple for monoidal categories with cartesian and tensor products, it is not difficult to show that the required diagrams commute. In this case, what is more difficult is obtaining naturality in φ, although this is taken care of (in a long proof) in Kassel's text.

Chapter 3

Proving Mac Lane's Coherence Theorem

3.1 Step Zero: Motivation

In the next few sections, we will take many steps which will culminate in a complete proof of Mac Lane's Coherence Theorem, an important theorem that informs us of the structure of a general monoidal category.

The proof of the Coherence Theorem can be found in [Mac71], and any research paper that concerns itself with monoidal categories will usually cite Mac Lane's book (or a similar source like Joyal and Street) for the theorem. While there is no doubt as to the veracity of Mac Lane's proof, his exposition is confusingly written and more of a proof outline. Because Mac Lane's work is very in depth, it would require a huge amount of time and work to perform a clear rewrite and restructuring. This is what this thesis does and we now offer such a complete proof.

To motivate the direction of Mac Lane's approach, we will discuss the structure of a general monoidal category, and the natural questions that arise regarding this structure.

Let $(\mathcal{M}, \otimes, I, \alpha, \rho, \lambda)$ be a monoidal category. For objects A, B, C, D, E of $\mathcal{M}$, we can use the monoidal product $\otimes$ to generate various new expressions, such as $A \otimes B$. For example, there are two ways to combine three objects:

$$A \otimes (B \otimes C) \qquad (A \otimes B) \otimes C.$$

There are five ways to multiply 4 objects:

$$A \otimes (B \otimes (C \otimes D)) \quad A \otimes ((B \otimes C) \otimes D) \quad ((A \otimes B) \otimes C) \otimes D$$
$$A \otimes ((B \otimes C) \otimes D) \quad (A \otimes (B \otimes C)) \otimes D.$$

And there are 14 ways to combine 5 objects. We will not list them here.

On the surface, we don't really know what the relationship is between the various expressions we are generating. For example, do

$$A \otimes (B \otimes C) \text{ and } (A \otimes B) \otimes C$$

or

$$A \otimes (B \otimes (C \otimes D)) \text{ and } A \otimes ((B \otimes C) \otimes D)$$

have *any relation* with each other? In practice when A, B, C, D are sets, vector spaces, groups, or whatnot, the above expressions *do* have something to do with each other. That relationship is usually an isomorphism. Therefore, if we are to develop some kind of theory of monoidal categories, we ought to make sure that these objects are isomorphic in some way.

Fortunately, monoidal categories do provide isomorphisms between different choices of multiplying together a set of objects. For example, from the axioms of a monoidal category, we know that the objects $A \otimes (B \otimes C)$ and $(A \otimes B) \otimes C$ are related via the isomorphism $\alpha_{A,B,C}$.

$$A \otimes (B \otimes C) \xrightarrow{\alpha_{A,B,C}} (A \otimes B) \otimes C$$

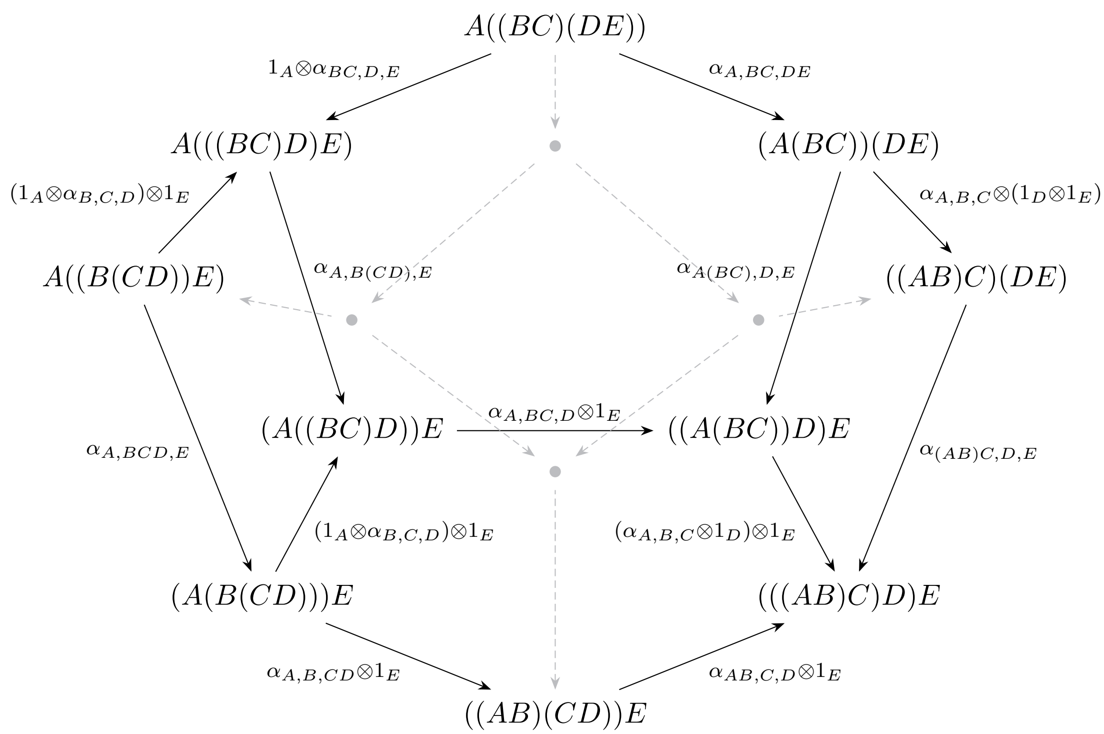
We also know from the axioms of a monoidal category that the 5 products of 4 objects are related via the diagram consisting of natural isomorphisms as below.

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "\alpha_{A,B,C\otimes D}"] \arrow[d, "1_A \otimes \alpha_{B,C,D}"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha_{A\otimes B,C,D}"] & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha_{A,B\otimes C,D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha_{A,B,C} \otimes 1_D"']
\end{tikzcd}$$

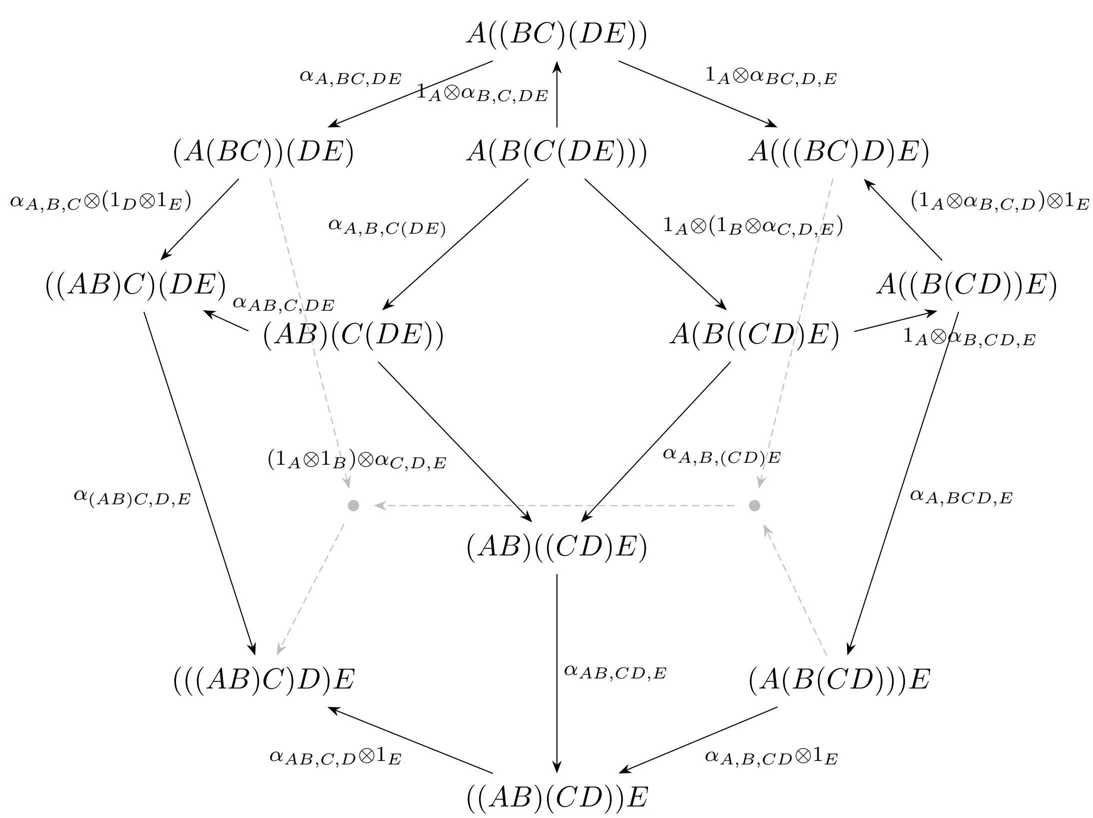
Moreover, this diagram is guaranteed to be commutative for all A,B,C,D in $\mathcal{M}$ (we will elaborate why this is a profound, useful fact).

Finally, repeatedly using instances of α, the 14 ways to multiply 5 objects are related via the 3 dimensional diagram as below.

Front. Note that the symbol $\otimes$ has been suppressed.



Back.



However, it is not an axiom of monoidal categories that this last diagram is commutative (with a ton of work, one could prove it to be commutative).

To understand what's going on, let us first understand why commutativity is important. The axioms of a monoidal category grant us the commutativity of the pentagon, which connects the five different ways of multiplying four objects A, B, C, D. This tells us the following principle: while there are 5 different ways we can multiply four objects A, B, C, D, each such choice is **canonically** isomorphic to any other choice.

To see this, suppose you and I want to multiply objects A, B, C, D together. Suppose my favorite way to do it is $(A \otimes B) \otimes (C \otimes D)$, while you choose $(A \otimes (B \otimes C)) \otimes D$. Then we might be in trouble: I have two possible ways, displayed below in blue and orange, to "reparenthesize" my product to get your object.

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[d, "1\otimes\alpha"'] & (A \otimes B) \otimes (C \otimes D) \arrow[l, "\alpha^{-1}"'] \arrow[r, "\alpha"] & ((A \otimes B) \otimes C) \otimes D \arrow[d, "\alpha^{-1}\otimes 1"] \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha"'] & & (A \otimes (B \otimes C)) \otimes D
\end{tikzcd}$$

Fortunately, the commutativity of the pentagonal diagram enures that the two paths are equal. That is,

$$\alpha \circ ((1 \otimes \alpha) \circ \alpha^{-1}) = (\alpha^{-1} \otimes 1) \circ \alpha.$$

so that, in reality, I actually have *one* unique isomorphism (i.e., a canonical isomorphism) from my object to yours, and you can also canonically get from your object to mine by inverting the unique isomorphism.

However, our choice of two different parenthesizations was arbitrary. The commutativity of the entire diagram therefore tells us that any choice of "parenthesizing" $A \otimes B \otimes C \otimes D$, the product of 4 objects in $\mathcal{M}$, is **canonically** isomorphic to any other possible choice. This brings up a few questions.

- What do we mean by "parenthesizing?"
- What about a product with n-many objects A for $n > 4$?

We will rigorously specify what we mean by parenthesizing in a bit. To answer the second question, we state that this result holds for $n > 4$; this is one version of the Coherence Theorem.

3.2 Step One: Category of Binary Words

To begin the proof of the coherence theorem, we need to first state the theorem itself. This task itself is quite laborious, although it is a worthwhile investment to establish clear terminology and notation, especially in writing the proof itself. Our primary tool will be the abstract concept of a binary word.

Definition 3.2.1. Let x_0, x_1 be two distinct symbols. A **binary word** w is an element defined recursively as follows.

- x_0 and x_1 are binary words.
- If u, v are binary words, then $(u) \otimes (v)$ is a binary word.

More precisely, a binary word is any element in the free magma $M = F(\{x_0, x_1\})$ generated by x_0, x_1, but we will see that the first definition we offered is more useful and transparent.

Example 3.2.2. Since x_0, x_1 are binary words so is the expression:

$$(x_0) \otimes (x_1)$$

Similarly, the expressions

$$(x_0) \otimes ((x_0) \otimes (x_1)) \quad ((x_0) \otimes (x_1)) \otimes x_1$$

are binary words.

From the previous example, we see that the notation is a bit clunky. On one hand, our definition, which states that $(u) \otimes (v)$ is a binary word if u, v are, is required so that we can logically manage our parentheses. On the other, it makes notation clunky.

To remedy this, we will often omit parentheses. Given an expression of a binary word, we will always omit the parentheses around individual symbols in the expression. With this rule, we have that:

$$\begin{gathered}
(x_0) \otimes (x_1) = x_0 \otimes x_1 \\
(x_0) \otimes ((x_0) \otimes (x_1)) = x_0 \otimes (x_0 \otimes x_1) \\
((x_0) \otimes (x_1)) \otimes (x_1) = (x_0 \otimes x_1) \otimes x_1
\end{gathered}$$

That is, we keep the parentheses which group together individual products, and throw away the ones which our smart human brains can don't need.

Next, we move onto an important quantity that we will often perform induction on.

Definition 3.2.3. We define the **length of a binary word** w, denoted as $\mathcal{L}(w)$, recursively as follows.

- $\mathcal{L}(x_0) = 0$ and $\mathcal{L}(x_1) = 1$
- If $w = u \otimes v$ for two binary words u, v, we set:

$$\mathcal{L}(w) = \mathcal{L}(u) + \mathcal{L}(v)$$

Example 3.2.4. The binary words

$$(x_1 \otimes x_0) \otimes x_1, \qquad (x_1 \otimes x_1) \otimes x_0, \qquad (x_0 \otimes (x_1 \otimes x_1)) \otimes x_0$$

all have length 2.

More informally, the length of binary word is simply the number of x_1 symbols that appear in its expression.

Example 3.2.5. For any binary word w, we have that

$$\mathcal{L}(w \otimes x_0) = \mathcal{L}(x_0 \otimes w) = \mathcal{L}(w).$$

If additionally u, v are binary words, we also have that

$$\begin{aligned}\mathcal{L}(u \otimes (v \otimes w)) &= \mathcal{L}(u) + (\mathcal{L}(v) + \mathcal{L}(w))\\ &= (\mathcal{L}(u) + \mathcal{L}(v)) + \mathcal{L}(w)\\ &= \mathcal{L}((u \otimes v) \otimes w).\end{aligned}$$

We will use the observations made in the previous example later in this section. We now demonstrate that these binary words assemble into a category.

Definition 3.2.6. The **category of binary words** is the category $\mathcal{W}$ where

Objects. All binary words w of length $n = 0, 1, 2, \dots,$

Morphisms. For any two binary words w and v, we have that

$$\mathrm{Hom}_{\mathcal{W}}(v, w) = \begin{cases} \{\bullet\} & \text{if } v, w \text{ are the same length} \\ \varnothing & \text{otherwise.} \end{cases}$$

where $\{\bullet\}$ denotes the one point set.

What the above definition tells us is that any two binary words share a morphism if and only if they are of the same length. Moreover, they will only ever share *exactly one* morphism. Since there is always at most one morphism between any two objects in $\mathcal{W}$, we see that $\mathcal{W}$ **is a thin category**. Moreover, it is monoidal. To prove that it is monoidal, we will need the following small lemma.

Lemma 3.2.7. The multiplication of binary words extends to a bifunctor $\otimes : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$.

Proof: First, we explain how $\otimes : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ operates on objects and morphisms. If (u, v) is an object of $\mathcal{W} \times \mathcal{W}$, we set $\otimes(u, v) = u \otimes v$. Next, consider two morphisms in $\mathcal{W}$.

$$\gamma : u \to u' \qquad \beta : v \to v'.$$

Note that this implies $\mathcal{L}(u) = \mathcal{L}(u')$ an $\mathcal{L}(v) = \mathcal{L}(v')$, which also imply that

$$\mathcal{L}(u \otimes v) = \mathcal{L}(u) + \mathcal{L}(v) = \mathcal{L}(u') + \mathcal{L}(v') = \mathcal{L}(u' \otimes v').$$

Therefore, we define the image of (γ, β) under the functor, $\otimes(\gamma, \beta)$, which we more naturally denote as $\gamma \otimes \beta$, to be the unique morphism between $u \otimes v \to u' \otimes v'$.

We can picture the action of this functor on objects and morphisms more clearly as below.

$\mathcal{W} \times \mathcal{W}$ $\quad (u_1, v_1) \xrightarrow{(\gamma,\beta)} (u_2, v_2)$ maps to $\mathcal{W}$ $\quad u \otimes v \xrightarrow{\gamma \otimes \beta} u' \otimes v'$

In addition, for any (u, v) in $\mathcal{W} \times \mathcal{W}$, the identity morphism $1_{(u,v)}$: $(u, v) \to (u, v)$ is mapped to the identity $1_{u \otimes v} : u \otimes v \to u \otimes v$. Finally, to demonstrate that this respects composition, suppose that (γ, β) is composable with (γ', β') as below.

$\mathcal{W} \times \mathcal{W}$

$$(u_1, v_1) \xrightarrow{(\gamma,\beta)} (u_2, v_2) \xrightarrow{(\gamma',\beta')} (u_3, v_3)$$

As both $(\gamma', \beta') \otimes (\gamma, \beta)$ and $(\gamma' \circ \gamma) \otimes (\beta' \circ \beta)$ are parallel morphisms acting as $(u_1, v_1) \to (u_3, v_3)$, they must be equal because $\mathcal{W}$ is a thin category (and hence parallel morphisms are equal).

Therefore, we see that $\otimes : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ is a bifunctor.

$\blacksquare$

We now show that $\mathcal{W}$ assembles into a monoidal category.

Proposition 3.2.8. $(\mathcal{W}, \otimes, x_0)$ is a monoidal category with monoidal product $\otimes : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ and identity object x_0.

Proof: First, we define our product to be given by the bifunctor $\otimes : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$. Second, we define our identity object to be x_0. With these two conditions we now need to find unitors, an associator, and check that the necessary diagrams commute.

Now as any two binary words of the same length share a *unique* morphism, all morphisms are isomorphisms. Therefore, by Example 3.2.5, the isomorphisms

$$\begin{aligned}
\alpha_{u,v,w} &: u \otimes (v \otimes w) \xrightarrow{\sim} (u \otimes v) \otimes w \\
\lambda_w &: x_0 \otimes w \xrightarrow{\sim} w \\
\rho_w &: w \otimes x_0 \xrightarrow{\sim} w
\end{aligned}$$

are forced to exist. Further, these isomorphisms are natural because all diagrams commute in a thin category. In addition, since $\mathcal{W}$ is a thin category, all diagrams commute, and so, in particular, the required diagrams

$$\begin{tikzcd}
u \otimes (x_0 \otimes v) \arrow[rr, "\alpha_{u,x_0,v}"] \arrow[dr, "1_u \otimes \lambda_v"'] & & (u \otimes x_0) \otimes v \arrow[dl, "\rho_u \otimes 1_v"] \\
& u \otimes v &
\end{tikzcd}$$

$$\begin{tikzcd}
u \otimes (v \otimes (w \otimes z)) \arrow[r, "\alpha_{u,v,w\otimes z}"] \arrow[d, "1_u \otimes \alpha_{v,w,z}"'] & (u \otimes v) \otimes (w \otimes z) \arrow[r, "\alpha_{u\otimes v,w,z}"] & ((u \otimes v) \otimes w) \otimes z \\
u \otimes ((v \otimes w) \otimes z) \arrow[rr, "\alpha_{u,v\otimes w,z}"'] & & (u \otimes (v \otimes w)) \otimes z \arrow[u, "\alpha_{u,v,w} \otimes 1_z"']
\end{tikzcd}$$

also commute, so that $(\mathcal{W}, \otimes, x_0)$ satisfies the axioms of a monoidal category.

■

We now make a few important comments on how to interpret α, ρ, and λ.

- Each $\alpha_{u,v,w} : u \otimes (v \otimes w) \to (u \otimes v) \otimes w$ can be thought of as an operator which **shifts** the parentheses to the left. Dually, $\alpha_{u,v,w}^{-1}$ shift them to the right.

- Each $\lambda_w : x_0 \otimes w \xrightarrow{\simeq} w$ can be thought of as an operator that **removes** an identity from the left. Dually, λ_w^{-1} adds an identity to the left.
- Each $\rho_w : w \otimes x_0 \xrightarrow{\simeq} w$ can be thought of as an operator that **removes** an identity from the right. Dually, ρ_w^{-1} adds an identity to the right.

Hence, this very primitive monoidal category $\mathcal{W}$ encodes some basic and useful operators on binary words.

3.3 Step Two: Pure Binary Words

In this section we begin discussing a specific subset of binary words, namely the one which lack an identity x_0. As the theorem is quite complex, this initial restriction allows us to develop intuition and some tools that simplify the proof later.

Definition 3.3.1. A **pure binary word** w of length n is a binary word w of length n which has no instance the empty word x_0.

Example 3.3.2. The only pure binary word of length 1 is x_1. There is also only one pure binary word of length 2, which is $x_1 \otimes x_1$. The pure binary words of length 3 are

$$x_1 \otimes (x_1 \otimes x_1) \qquad (x_1 \otimes x_1) \otimes x_1$$

and the pure binary words of length 4 are as below.

$$x_1 \otimes (x_1 \otimes (x_1 \otimes x_1)) \quad x_1 \otimes ((x_1 \otimes x_1) \otimes x_1) \quad ((x_1 \otimes x_1) \otimes x_1) \otimes x_1$$
$$x_1 \otimes ((x_1 \otimes x_1) \otimes x_1) \quad (x_1 \otimes (x_1 \otimes x_1)) \otimes x_1$$

Lemma 3.3.3. For each $n \geq 1$, there are finitely many pure binary words of length n.

Proof: First, there is clearly only one pure binary word of length 1. Now note that every pure binary word of length $n > 1$ will always have the form

$$w = u \otimes v$$

where u, v are binary words with $\mathcal{L}(u) = i$, $\mathcal{L}(v) = n - i$, for some $i = 1, 2, \dots, n-1$. Therefore, if there are B_i many binary words of length i, and B_{n-i} many of length $n - i$, then there are

$$B_n = \sum_{i=1}^{n-1} B_i \cdot B_{n-i}$$

many words of length B_n. Since $B_1 = 1$ is finite, we see that B_n must also be finite for all $n > 1$.

■

Interestingly, the above recurrence can generate the Catalan Numbers. Specifically, B_{n+1} is exactly the n-th Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n} \qquad 1, 2, 5, 14, 42, 132, 429, \cdots$$

However, we make no critical use of this fact in our proofs.

Next, we form a category of pure binary words.

Definition 3.3.4. The **category of pure binary words** $\mathcal{W}_{\mathrm{P}}$ is the full subcategory of $\mathcal{W}$ constructed by restricting the objects of $\mathcal{W}$ to its pure binary words.

More explicitly, $\mathcal{W}_{\mathrm{P}}$ is the category defined as:

Objects. All pure binary words w of length $n = 0, 1, 2, \dots,$

Morphisms. For any two pure binary words u, v of the same length, we have that $\mathrm{Hom}_{\mathcal{W}_A}(u, v) = \{\bullet\}$, the one point set. No other morphisms are allowed.

We now focus on a particular set of morphisms in $\mathcal{W}_{\mathrm{P}}$. Recall that we may think of each $\alpha_{u,w,v}$ as a "shift map"

$$\alpha_{u,w,v} : u \otimes (v \otimes w) \to (u \otimes v) \otimes w$$

which makes a single change in the parenthesis of a binary word. However, α itself does not characterize all possible always in which we make a single change of parentheses within a larger, more complex binary word. An example of this is the morphism

$$1_s \otimes \alpha_{u,v,w} : s \otimes (u \otimes (v \otimes w)) \to s \otimes ((u \otimes v) \otimes w)$$

which makes an *internal* change of parentheses. As we will need to focus on these more complicated morphisms, we rigorously define them below.

Definition 3.3.5 (α-arrows)**.** A **forward $\boldsymbol{\alpha}$-arrow** of $\mathcal{W}_{\mathrm{P}}$ is a morphism in $\mathcal{W}_{\mathrm{P}}$ which we recursively define as follows.

- For any triple of pure binary words w_1, w_2, w_3 in $\mathcal{W}_{\mathrm{P}}$, the morphism

$$\alpha_{w_1,w_2,w_3} : w_1 \otimes (w_2 \otimes w_3) \to (w_1 \otimes w_2) \otimes w_3$$

  is a forward α-arrow.

- If $\beta : w \to w'$ is a forward α-arrow, and u is an arbitrary pure binary word, then the morphisms

$$1_u \otimes \beta : u \otimes w \to u \otimes w' \qquad \beta \otimes 1_u : w \otimes u \to w' \otimes u$$

  are forward α-arrows.

We also define a **backward α-arrow** to be the inverse of a forward α-arrow.

Example 3.3.6. Below are a few simple examples of α-arrows. The first two are forward, while the third is backward.

$$\begin{tikzcd} x_1 \otimes (x_1 \otimes x_1) \arrow[d, "\alpha_{x_1,x_1,x_1}"'] \\ (x_1 \otimes x_1) \otimes x_1 \end{tikzcd} \quad \begin{tikzcd} x_1 \otimes (x_1 \otimes (x_1 \otimes x_1)) \arrow[d, "1_{x_1} \otimes \alpha_{x_1,x_1,x_1}"'] \\ x_1 \otimes ((x_1 \otimes x_1) \otimes x_1) \end{tikzcd} \quad \begin{tikzcd} (x_1 \otimes x_1) \otimes (x_1 \otimes x_1) \arrow[d, "\alpha^{-1}_{x_1,x_1,x_1 \otimes x_1}"] \\ x_1 \otimes (x_1 \otimes (x_1 \otimes x_1)) \end{tikzcd}$$

We can have even more complicated examples; for example, the morphism below

$$\begin{tikzcd} (u \otimes (x_1 \otimes (x_1 \otimes x_1))) \otimes v \arrow[d, "(1_u \otimes \alpha_{x_1,x_1,x_1}) \otimes 1_v"'] \\ (u \otimes ((x_1 \otimes x_1) \otimes x_1) \otimes v \end{tikzcd}$$

is an α-morphism for any pure binary words u, v. For example, setting $u = (x_1 \otimes x_1) \otimes x_1$ and $v = x_1 \otimes x_1$, we obtain the forward α-arrow as below.

$$\begin{tikzcd} ((x_1 \otimes x_1) \otimes x_1 \otimes (x_1 \otimes (x_1 \otimes x_1))) \otimes (x_1 \otimes x_1) \arrow[d, "(1_{(x_1 \otimes x_1) \otimes x_1} \otimes \alpha_{x_1,x_1,x_1}) \otimes 1_{(x_1 \otimes x_1)}"'] \\ ((x_1 \otimes x_1) \otimes x_1 \otimes ((x_1 \otimes x_1) \otimes x_1) \otimes (x_1 \otimes x_1) \end{tikzcd}$$

We emphasize that α-arrows only ever involve a single instance of α or α^{-1} in their expression.

Next, we introduce a particularly important instance of a pure binary word that will become essential to our proof.

Definition 3.3.7. We define the **terminal word** $w^{(n)}$ of length n recursively as follows.

- x_1 is the terminal word of length 1.

- If $w^{(k)}$ is the terminal word of length k, then $w^{(k+1)} = w^k \otimes x_1$ is the terminal word of length $k+1$.

More informally, the terminal word is the unique pure binary word of length n for which all parentheses begin on the left.

Example 3.3.8. Below we list the terminal words by length.

Length	Terminal Word
1	x_1
2	$x_1 \otimes x_1$
3	$(x_1 \otimes x_1) \otimes x_1$
4	$((x_1 \otimes x_1) \otimes x_1) \otimes x_1$
5	$(((x_1 \otimes x_1) \otimes x_1) \otimes x_1) \otimes x_1$

We now introduce a quantity which provides a "distance-measure" between a pure binary word of length n and the terminal word $w^{(n)}$.

Definition 3.3.9. We (recursively) define the **rank** of a binary word as follows.

- $r(x_1) = 0$.
- For a pure binary word of the form $w = u \otimes v$, we set

$$r(u \otimes v) = r(u) + r(v) + \mathcal{L}(v) - 1.$$

Example 3.3.10. We compute the ranks on the pure binary words of length 4.

$$\begin{aligned} &r(x_1(x_1(x_1x_1))) = 3 && r(x_1((x_1x_1)x_1)) = 2 \\ &r((x_1x_1)(x_1x_1)) = 1 && r((x_1(x_1x_1))x_1) = 1 \\ &r(((x_1x_1)x_1)x_1) = 0 \end{aligned}$$

Note that $w^{(4)} = ((x_1x_1)x_1)x_1$ and $r(((x_1x_1)x_1)x) = 0$. Hence we see that our intuition of the rank being a distance measure from $w^{(n)}$ so far makes sense.

An important property of distance-measuring functions is nonnegativity, which we will now see is satisfied by the rank function.

Lemma 3.3.11. Let w be a pure binary word of length n. Then $r(w) \geq 0$.

Proof: We prove this by induction on n. First observe that this clearly holds for $n = 0$ since $r(x_1) = 0$.

Now let w be a pure binary word of length k, and suppose the statement is true for all pure binary words with length less than k. Since $k > 1$, we may write $w = u \otimes v$ for some pure binary words u, v, in which case

$$r(w) = \overbrace{r(u) + r(v)}^{\geq 0 \text{ by induction}} + \mathcal{L}(v) - 1.$$

Since $\mathcal{L}(v) \geq 1$, we see that $r(w) \geq 0$ as desired.

■

Keeping with the analogy of the rank being a distance measure, we ought to verify that it is zero if and only if the input, which is being measured from $w^{(n)}$, is $w^{(n)}$ itself. We verify that this is the case for the rank function.

Proposition 3.3.12. Let w be a pure binary word of length n. Then $r(w) = 0$ if and only if $w = w^{(n)}$.

Proof: We proceed by induction. In the simplest case, when $n = 1$, we have that $r(x_1) = 0$ by definition. As $x_1 = w^{(1)}$, we see that this satisfies the statement.

Let w be a pure binary word of length k, and suppose the statement is true for all pure binary words with length less than k. Then we may write our word in the form $w = u \otimes v$, and we have that

$$r(w) = r(u) + r(v) + \mathcal{L}(v) - 1.$$

By Lemma 3.3.11 we know that $r(u), r(v) \geq 0$. Therefore, if $\mathcal{L}(v) > 1$ then $r(w) \neq 0$. Hence, consider the case for when $\mathcal{L}(v) = 1$, so that $v = x_1$. Then

$$r(u \otimes v) = r(u) + r(x_1) + \mathcal{L}(x_1) - 1 = r(u)$$

Therefore, $r(w) = 0$ if and only if if $r(u) = 0$. But by induction, this holds if and only if $u = w^{(k-1)}$. So we see that $w = w^{(k-1)} \otimes x_1 = w^{(k)}$, which proves our result for all n.

■

Lemma 3.3.13. Let $\beta : v \to w$ be a forward α-arrow. Then $r(v) < r(w)$. In other words, forward α-arrows decrease rank.

Proof: To demonstrate this, we perform induction on the structure of forward α-arrows.

Our base case is $\beta = \alpha_{u,v,w} : u \otimes (v \otimes w) \xrightarrow{\sim} (u \otimes v) \otimes w$ for some arbitrary words u, v, w. With this case, observe that

$$\begin{aligned} r(u \otimes (v \otimes w)) &= r(u) + r(v \otimes w) + \mathcal{L}(v \otimes w) - 1 \\ &= r(u) + (r(v) + r(w) + \mathcal{L}(w) - 1) \\ &+ \mathcal{L}(v \otimes w) - 1 \end{aligned}$$

while

$$\begin{aligned} r((u \otimes v) \otimes w) &= r(u \otimes v) + r(w) + \mathcal{L}(w) - 1 \\ &= r(u) + r(v) + r(w) + \mathcal{L}(v) - 1 + r(w) \\ &+ \mathcal{L}(w) - 1. \end{aligned}$$

If we subtract the quantities, we observe that

$$r(u \otimes (v \otimes w)) - r((u \otimes v) \otimes w) = \mathcal{L}(v \otimes w) - \mathcal{L}(w) > 0$$

since v has at least length 1. Therefore $\alpha_{u,v,w}$ decreases length as desired.

Next, we reach our inductive step: let $\beta = 1_u \otimes \gamma : u \otimes v \to u \otimes w$ where $\gamma : v \to w$ is a forward α-arrow for which the statement is already true. In this case we have that

$$r(u \otimes v) = r(u) + r(v) + \mathcal{L}(v) - 1.$$

while

$$r(u \otimes w) = r(u) + r(w) + \mathcal{L}(w) - 1.$$

Since $\mathcal{L}(v) = \mathcal{L}(w)$ and $r(v) > r(w)$, we see that $r(u \otimes v) > r(u \otimes w)$. Therefore, we see that $\beta = 1_u \otimes \gamma$ decreases rank whenever γ is a forward α-arrow that also decreases rank.

Finally, let $\beta = \gamma \otimes 1_u$ where $\gamma : v \to w$ is a forward α arrow for which the statement is already true. Then we may write $\beta : v \otimes u \to w \otimes u$ Now observe that

$$r(v \otimes u) = r(v) + r(u) + \mathcal{L}(u) - 1$$

while

$$r(w \otimes u) = r(w) + r(u) + \mathcal{L}(u) - 1.$$

Since $\gamma : v \to w$ decreases rank, we see that $r(v) > r(w)$ and therefore $r(v \otimes u) > r(w \otimes u)$, as desired.

This completes the proof by induction, so that the statement is true for all forward α-arrows.

■

Thus what we have on our hands is the following. We know that the rank of word w is zero if and only if $w = w^{(n)}$. Further, we know that applying α-arrows to a pure binary word will decrease its rank. In other words, shifting the parentheses of a pure binary word w brings w "closer" to $w^{(n)}$ (whose parentheses are all on the left). Therefore, the rank of a pure binary word gives us a measure for how far a binary word w is away from $w^{(n)}$.

The following lemma demonstrates our interest in the word $w^{(n)}$.

Proposition 3.3.14. Let w be a pure binary word of length n. If $w \neq w^{(n)}$, then there exists a finite sequence of forward α-arrows from w to $w^{(n)}$.

Proof: We first show that for every pure binary word $w \neq w^{(n)}$ there exists a forward α-arrow β with domain w. We prove this statement by induction on length.

Observe the result is immediate for $n = 1, 2$. Suppose the result is true for binary words with length less than $n \geq 3$. Let w be a pure binary word with length n. Then $w = u \otimes v$, with u, v other pure binary words. We now consider two cases for u and v.

(1) The first case is when $\mathcal{L}(v) = 1$, so that $v = x_1$. As $w \neq w^{(n)}$ we know that $u \neq w^{(n-1)}$, and since u has length less than w, we see that by induction there exists a forward α-arrow $\beta : u \to u'$. Using β, we can construct the forward α-arrow

$$\beta \otimes 1_{x_1} : u \otimes x_1 \to u' \otimes x_1.$$

Hence $\beta \otimes 1_{x_1}$ is our desired forward α-arrow with domain w.

(2) The second case is when $\mathcal{L}(v) > 1$. In this case we may write $w = u \otimes (r \otimes s)$. A natural choice for a forward α-arrow in this case is simply

$$\alpha_{u,v,s} : u \otimes (r \otimes s) \to (u \otimes r) \otimes s$$

so that this case is also satisfied.

As we see, in all cases for $w \neq w^{(n)}$, we can find a forward α-arrow with domain w. As α-arrows decrease rank, and $r(w) = 0$ if and only if $w^{(n)}$, this guarantees a sequence of α-arrows from w to $w^{(n)}$, which is what we set out to show.

■

The previous proposition has an immediate, useful corollary. It will be used as one of the building blocks for the next section.

Corollary 3.3.15. Every morphism in $\mathcal{W}_{\mathrm{P}}$ can be expressed as a finite composition of α-arrows.

Proof: Let v, w be arbitrary pure binary words. Denote $\varphi_{v,w} : v \to w$ to be the unique morphism from v to w. By Proposition 3.3.14 there exists chains of forward α-arrows whose composite we denote as $\Gamma_1 : v \to w^{(n)}, \Gamma_2 : w \to w^{(n)}$. Our situation is pictured below.

$$\begin{tikzcd}
v \arrow[rr, "\varphi_{v,w}"] \arrow[dr, "\Gamma_1"'] & & w \\
& w^{(n)} \arrow[ur, "\Gamma_2^{-1}"'] &
\end{tikzcd}$$

However, $\mathcal{W}_{\mathrm{P}}$ is a thin category, so parallel morphisms must be equal. Therefore

$$\varphi_{v,w} = \Gamma_2^{-1} \circ \Gamma_1.$$

Hence $\varphi_{v,w}$ is a composition of α-arrows. As $\varphi_{v,w}$ was arbitrary, we see that every morphism in $\mathcal{W}_{\mathrm{P}}$ is a finite composition of α-arrows.

■

What this corollary says is that every morphism in $\mathcal{W}_{\mathrm{P}}$ can be expressed as a composite of forward and backward α-arrows. However, we emphasize that there can be many different ways to represent a morphism in $\mathcal{W}_{\mathrm{P}}$ via α-arrows. This will be an issue which we discuss later in the next section.

3.4 Step Three: Coherence for $A^{\otimes n}$ in α

Using our results from the previous section, we are almost ready to take our first major step in the proof of Mac Lane's Coherence Theorem. Before we do so, we need to introduce terminology to even state the theorem which we will prove in this section. Towards that goal we introduce a few more definitions.

Definition 3.4.1. Let $(\mathcal{M}, \alpha, \lambda, \rho, I, \otimes)$ be a monoidal category. For an object A of $\mathcal{M}$, we define the **proxy map** of A to be a partial functor

$$(-)_A : \mathcal{W}_{\mathrm{P}} \to \mathcal{M}$$

as follows. Note by partial functor, we mean a functor defined on all objects of $\mathcal{W}_{\mathrm{P}}$, but only a *subset* of all morphisms of $\mathcal{W}_{\mathrm{P}}$.

Objects. We define the action on objects recursively as follows.

- We set $(x_1)_A = A$.
- For a binary word $w = u \otimes v$, we define

$$(w)_A = (u \otimes v)_A = (u)_A \otimes (v)_A$$

Morphisms. We define the partial functor only on α-arrows. We do this recursively as follows.

- For $\alpha_{u,v,w}$ with u, v, w as pure binary words, we set:

$$(\alpha_{u,v,w})_A = \alpha_{(u)_A,(v)_A,(w)_A}$$
$$(\alpha^{-1}_{u,v,w})_A = \alpha^{-1}_{(u)_A,(v)_A,(w)_A}$$

- For $1_u \otimes \beta$ and $\beta \otimes 1_u$ with β an α-arrow, we set:

$$(1_u \otimes \beta)_A = 1_{(u)_A} \otimes (\beta)_A$$
$$(\beta \otimes 1_u)_A = (\beta)_A \otimes 1_{(u)_A}$$

We now introduce the theorem of the section. This theorem is the first major step in the proof of the coherence theorem, and the rest of this section will be dedicated to proving it.

Theorem 3.4.2 (Coherence in α.)**.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. For every object A, there exists a unique functor $\Phi_A : \mathcal{W}_P \to \mathcal{M}$ which restricts to the proxy map $(-)_A$ on objects and α-arrows of $\mathcal{W}_\mathrm{P}$.

We address the question the reader most likely has in mind right now: Why did we only define the proxy map on α-arrows? Why not define it on all of the morphisms of $\mathcal{W}_\mathrm{P}$ to get a functor to begin with? We did this to avoid a potential well-definedness issue, which we now elaborate on.

Let us attempt to naturally extend the proxy map to a functor. With Corollary 3.3.15, it is clear how to proceed on defining $(-)_A$ on general morphisms.

Let $\gamma : v \to w$ be any morphism in $\mathcal{W}_\mathrm{P}$. By Corollary 3.3.15, there exist forward and backward α-arrows $\gamma_1, \ldots, \gamma_n$ such that

$$\gamma = \gamma_n \circ \cdots \circ \gamma_1.$$

Since the proxy map is in fact defined on α-arrows, and since functors preserve composition, we are required to define

$$(\gamma)_A = (\gamma_n)_A \circ \cdots \circ (\gamma_1)_A.$$

However, we need to be careful. Suppose that we can also express γ as the finite composition of α-morphisms $\delta_1, \ldots, \delta_m$.

$$\gamma = \delta_m \circ \cdots \circ \delta_1.$$

While $\gamma_n \circ \cdots \circ \gamma_1 = \delta_m \circ \cdots \circ \delta_1$ because $\mathcal{W}_\mathrm{P}$ is a thin category, and therefore parallel morphisms are equal, we have no idea if

$$(\gamma_n)_A \circ \cdots \circ (\gamma_1)_A = (\delta_m)_A \circ \cdots \circ (\delta_1)_A$$

is true in $\mathcal{M}$. That is, we do not know if equivalent morphisms in $\mathcal{W}_\mathrm{P}$ are mapped to equal morphisms under the proxy map. Our issue is one of well-definedness.

This issue is similar to one which arises in group theory. When one attempts to define a group homomorphism on a quotient group, they must understand that there are different, equivalent ways to represent an element. In this situation they must make sure that the equivalent elements are mapped to the same target in the codomain.

Example 3.4.3. To illustrate our point, we include a concrete example of our problem which also demonstrates its nontriviality. For notational convenience, we suppress the instances of the monoidal product $\otimes$. Let

$$\gamma : x_1((x_1x_1)(x_1x_1)) \to ((x_1(x_1x_1))x_1)x_1.$$

Then we have many possible ways of expressing γ in terms of our α-arrows. Some potential ways we could express γ are displayed below in purple, blue, or orange.

$$
\begin{tikzcd}
 & & & x_1((x_1x_1)(x_1x_1)) \arrow[dll, shift right, "1_{x_1}\otimes\alpha_{x_1x_1,x_1,x_1}"'] \arrow[dll, shift left] \arrow[drr, "\alpha_{x_1,x_1x_1,x_1x_1}"] \arrow[dddr, "\gamma"'] & & & \\
 & x_1(((x_1x_1)x_1)x_1) \arrow[dl, "(1_{x_1}\otimes\alpha^{-1}_{x_1,x_1,x_1})\otimes 1_{x_1}"'] \arrow[ddr, "\alpha_{x_1,x_1(x_1x_1),x_1}"] & & & & (x_1(x_1x_1))(x_1x_1) \arrow[ddl, "\alpha_{x_1(x_1x_1),x_1,x_1}"'] \arrow[dr, "\alpha_{x_1,x_1,x_1}\otimes(1_{x_1}\otimes 1_{x_1})"] & \\
x_1((x_1(x_1x_1))x_1) \arrow[ddr, "\alpha_{x_1,x_1x_1x_1,x_1}"'] & & & & & & ((x_1x_1)x_1)(x_1x_1) \arrow[ddl, "\alpha_{(x_1x_1)x_1,x_1,x_1}"] \\
 & & (x_1((x_1x_1)x_1))x_1 \arrow[rr, "\alpha_{x_1,x_1x_1,x_1}\otimes 1_{x_1}"] & & ((x_1(x_1x_1))x_1)x_1 & & \\
 & (x_1(x_1(x_1x_1)))x_1 \arrow[ur, "(1_{x_1}\otimes\alpha_{x_1,x_1,x_1})\otimes 1_{x_1}"'] \arrow[drr, "\alpha_{x_1,x_1,x_1x_1}\otimes 1_{x_1}"'] & & & & (((x_1x_1)x_1)x_1)x_1 \arrow[ul, "(\alpha^{-1}_{x_1,x_1,x_1}\otimes 1_{x_1})\otimes 1_{x_1}"] & \\
 & & & ((x_1x_1)(x_1x_1))x_1 \arrow[urr, "\alpha_{x_1x_1,x_1,x_1}\otimes 1_{x_1}"'] & & &
\end{tikzcd}
$$

As this is a thin category, we know that the composition of these paths are equal in $\mathcal{W}_{\mathrm{P}}$. However, we now have many ways to define γ under the proxy map $(-)_A$. We could write

$$
\begin{aligned}
(\gamma)_A &= ((\alpha^{-1}_{x_1,x_1,x_1} \otimes 1_{x_1}) \otimes 1_{x_1})_A \circ \cdots \circ (1_{x_1} \otimes \alpha_{x_1x_1,x_1,x_1})_A \\
&= (\alpha^{-1}_{A,A,A} \otimes 1_A) \otimes 1_A \circ \cdots \circ 1_A \otimes \alpha_{AA,A,A}
\end{aligned}
$$

or

$$
\begin{aligned}
(\gamma)_A &= (\alpha_{x_1,x_1x_1,x_1} \otimes 1_{x_1})_A \circ \cdots \circ (1_{x_1} \otimes \alpha_{x_1x_1,x_1,x_1})_A \\
&= \alpha_{A,AA,A} \otimes 1_A \circ \cdots \circ 1_A \otimes \alpha_{AA,A,A}
\end{aligned}
$$

or

$$
\begin{aligned}
(\gamma)_A &= (\alpha_{x_1(x_1x_1),x_1,x_1})_A \circ (\alpha_{x_1,x_1x_1,x_1x_1})_A \\
&= \alpha_{A(AA),A,A} \circ \alpha_{A,AA,AA}
\end{aligned}
$$

But as morphisms in $\mathcal{M}$, we don't know if these compositions in $\mathcal{M}$, displayed below, are all equal.

$$\begin{tikzcd}
& & A((AA)(AA)) \arrow[dl, blue, shift left, "1_A\otimes\alpha_{AA,A,A}"'] \arrow[dl, purple, shift right] \arrow[dr, orange, "\alpha_{A,AA,AA}"] \arrow[dddr, thick, "(\gamma)_A"'] & & \\
& A(((AA)A)A) \arrow[dl, purple, "(1_A\otimes\alpha^{-1}_{A,A,A})\otimes 1_A"'] \arrow[dd, blue, "\alpha_{A,A(AA),A}"] & & (A(AA))(AA) \arrow[dd, orange, "\alpha_{A(AA),A,A}"'] \arrow[dr, "\alpha_{A,A,A}\otimes(1_A\otimes 1_A)"] & \\
A((A(AA))A) \arrow[ddr, purple, "\alpha_{A,AAA,A}"'] & & & & ((AA)A)(AA) \arrow[ddl, "\alpha_{(AA)A,A,A}"] \\
& (A((AA)A))A \arrow[rr, blue, "\alpha_{A,AA,A}\otimes 1_A"] & & ((A(AA))A)A & \\
& (A(A(AA)))A \arrow[u, "(1_A\otimes\alpha_{A,A,A})\otimes 1_A"'] \arrow[dr, purple, "\alpha_{A,A,AA}\otimes 1_A"'] & & (((AA)A)A)A \arrow[u, purple, "(\alpha^{-1}_{A,A,A}\otimes 1_A)\otimes 1_A"] & \\
& & ((AA)(AA))A \arrow[ur, purple, "\alpha_{AA,A,A}\otimes 1_A"'] & &
\end{tikzcd}$$

Hence we need to show that the purple, blue, and orange compositions are equal in $\mathcal{M}$. While we could perform tedious diagram chases to show that they are equal in $\mathcal{M}$, that would only address three of the many possible ways to express γ. It also would not take care of the case for much larger binary words! Hence, this problem is very nontrivial in general; we need higher level techniques to get what we want.

Therefore, to define a functor in the first place, we need to prove the following fact.

Proposition 3.4.4. Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category, and let A be an object of $\mathcal{M}$. Let v, w be binary words of the same length. If $\beta_1, \ldots, \beta_k$ and $\gamma_1, \ldots, \gamma_\ell$ are α-arrows with

$$\beta_k \circ \cdots \circ \beta_1, \gamma_\ell \circ \cdots \circ \gamma_1 : v \to w$$

then $(\beta_k)_A \circ \cdots \circ (\beta_1)_A = (\gamma_\ell)_A \circ \cdots \circ (\gamma_1)_A$ in $\mathcal{M}$.

To prove this proposition, we will see that it actually suffices to prove the the special case with $w = w^{(n)}$ and with $\beta_1, \ldots, \beta_k$ and $\gamma_1, \ldots, \gamma_\ell$ all *forward* α-arrows. That is, it suffices to prove the following proposition.

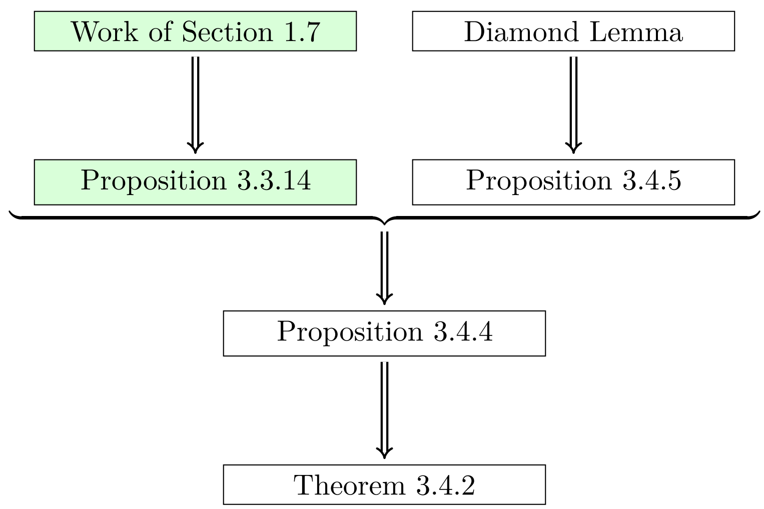
Proposition 3.4.5. Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category, and let A be an object of $\mathcal{M}$. Let w be a pure binary word of length n. If $\beta_1, \dots, \beta_k$ and $\gamma_1, \dots, \gamma_\ell$ are forward α-arrows with

$$\beta_k \circ \cdots \circ \beta_1, \gamma_\ell \circ \cdots \circ \gamma_1 : w \to w^{(n)}$$

in $\mathcal{W}_{\mathrm{P}}$, then $(\beta_k)_A \circ \cdots \circ (\beta_1)_A = (\gamma_\ell)_A \circ \cdots \circ (\gamma_1)_A$ in $\mathcal{M}$.

To prove this it will suffice to prove the Diamond Lemma (stated below). It will turn out the bulk of the overall proof toward our theorem will be spent on the Diamond Lemma. At the risk of downplaying its importance, we leave the proof of the Diamond Lemma to the end since it is very tedious and involved, and we do not want to disrupt the flow of the current discussion.

We summarize our plan on how to prove Theorem 3.4.2. The uncolored boxes, and the implications between them, are what is left to do.



Lemma 3.4.6 (Diamond Lemma)**.** Let w be a pure binary word and suppose β_1, β_2 are two forward α-arrows as below.

$$\begin{tikzcd} & w \arrow[dl, "\beta_1"'] \arrow[dr, "\beta_2"] & \\ w_1 & & w_2 \end{tikzcd}$$

There exists a pure binary word z and two $\gamma_1 : w_1 \to z, \gamma_2 : w_2 \to z$, with γ_1, γ_2 a composition of forward α-arrows, such that for any monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ the diagram below is commutative in $\mathcal{M}$.

$$\begin{tikzcd}
 & (w)_A \arrow[dl, "(\beta_1)_A"'] \arrow[dr, "(\beta_2)_A"] & \\
(w_1)_A \arrow[dr, "(\gamma_1)_A"'] & & (w_2)_A \arrow[dl, "(\gamma_2)_A"] \\
 & (z)_A &
\end{tikzcd}$$

Since the above lemma is an existence result, we emphasize this fact by coloring the arrows, which we are asserting to exist, Green. This is a practice we will continue.

As promised, we now prove Proposition 3.4.5 using the Diamond lemma. We restate the statement of the proposition for the reader's convenience.

Proposition 3.4.5. Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category, and let A be an object of $\mathcal{M}$. Let w be a pure binary word of length n. If $\beta_1, \ldots, \beta_k$ and $\gamma_1, \ldots, \gamma_\ell$ are forward α-arrows with

$$\beta_k \circ \cdots \circ \beta_1, \gamma_\ell \circ \cdots \circ \gamma_1 : w \to w^{(n)}$$

in $\mathcal{W}_{\mathrm{P}}$, then $(\beta_k)_A \circ \cdots \circ (\beta_1)_A = (\gamma_\ell)_A \circ \cdots \circ (\gamma_1)_A$ in $\mathcal{M}$.

Proof: To prove the desired statement, we proceed by induction on the rank of a pure binary word w. In what follows we write we will write $w = u \otimes v$ since $\mathcal{L}(w) \geq 3$.

For our base case let w be a word of rank 0. Then by Proposition 3.3.12 we see that $w = w^{(n)}$ so that this statement is trivial.

Next suppose the statement is true for all words with rank at most k where $k \geq 0$. Let w be a pure binary word of rank $k+1$. We want to show that the diagram in $\mathcal{M}$

$$\begin{tikzcd}
 & (w)_A \arrow[dl, "(\beta_1)_A"'] \arrow[dr, "(\gamma_1)_A"] & \\
(u_1)_A \arrow[dr, "(\beta_k)_A\circ\cdots\circ(\beta_2)_A"'] & & (v_1)_A \arrow[dl, "(\gamma_\ell)_A\circ\cdots\circ(\gamma_2)_A"] \\
 & (w^{(n)})_A &
\end{tikzcd}$$

is commutative. By the Diamond Lemma 3.4.6, there exists exist a pure binary word z and two composites of forward α-arrows β' and γ' such that the diagram below is commutative in $\mathcal{M}$.

$$\begin{tikzcd}
 & (w)_A \arrow[dl, "(\beta_1)_A"'] \arrow[dr, "(\gamma_1)_A"] & \\
(u_1)_A \arrow[dr, "(\beta')_A"'] & & (v_1)_A \arrow[dl, "(\gamma')_A"] \\
 & (z)_A &
\end{tikzcd}$$

Let $\Gamma_z : z \to w^{(n)}$ by any composition of forward α-arrows from z to $w^{(n)}$; at least one must exist by Proposition 3.3.14. We can now combine our two diagrams in $\mathcal{M}$ to obtain the diagram below.

$$\begin{tikzcd}
 & (w)_A \arrow[dl, "(\beta_1)_A"'] \arrow[dr, "(\gamma_1)_A"] & \\
(u_1)_A \arrow[dr, "(\beta')_A"] \arrow[ddr, "(\beta_k)_A \circ \cdots \circ (\beta_2)_A"'] & & (v_1)_A \arrow[dl, "(\gamma')_A"'] \arrow[ddl, "(\gamma_\ell) \circ \cdots \circ (\gamma_2)_A"] \\
 & (z)_A \arrow[d, "(\Gamma_z)_A"] & \\
 & w^{(n)} &
\end{tikzcd}$$

By Lemma 3.3.13, we know that forward α-arrows decrease rank, so that $r(u_1) < r(w)$ and $r(v_1) > r(w)$. Hence we invoke our induction hypothesis to conclude that both the lower left and lower right triangles commute in $\mathcal{M}$. As the original upper diamond already commutes via the Diamond Lemma, we see that the entire diagram is commutative. Therefore we have that

$$(\beta_k)_A \circ \cdots \circ (\beta_1)_A = (\gamma_\ell)_A \circ \cdots \circ (\gamma_1)_A$$

in $\mathcal{M}$. This completes our induction and hence the proof.

■

As promised, we use the above proposition to prove Proposition 3.4.4.

Proposition 3.4.4. Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category, and let A be an object of $\mathcal{M}$. Let v, w be binary words of the same length. If $\beta_1, \ldots, \beta_k$ and $\gamma_1, \ldots, \gamma_\ell$ are α-arrows with

$$\beta_k \circ \cdots \circ \beta_1, \gamma_\ell \circ \cdots \circ \gamma_1 : v \to w$$

then $(\beta_k)_A \circ \cdots \circ (\beta_1)_A = (\gamma_\ell)_A \circ \cdots \circ (\gamma_1)_A$ in $\mathcal{M}$.

Proof: We begin by denoting the domain and codomain of the α-arrows to make our discussion clear. Let $u_0, \ldots, u_k, t_0, \ldots, t_\ell$ be the pure binary words such that $u_0 = t_0 = v$, $v_k = u_\ell = w$ and

$$\begin{aligned} \beta_i &: u_{i-1} \to u_i, \quad i = 1, 2, \ldots, k \\ \gamma_j &: t_{j-1} \to t_j, \quad j = 1, 2, \ldots, \ell \end{aligned}$$

Note that each morphism may either be forward or backward. With this notation we can picture our parallel α-arrows in $\mathcal{W}_{\mathrm{P}}$ as below.

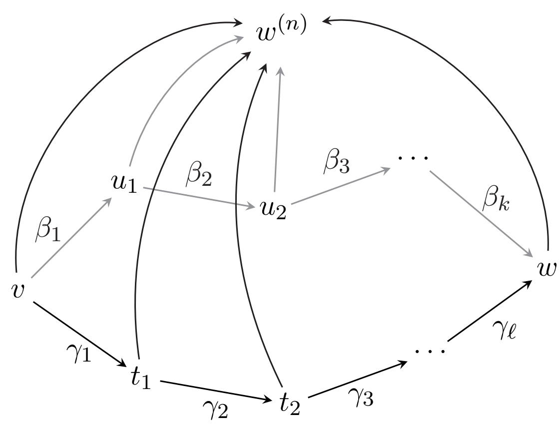
$$\begin{tikzcd}
 & u_1 \arrow[r, "\beta_2"] & u_2 \arrow[r, "\beta_3"] & \cdots \arrow[dr, "\beta_k"] & \\
v \arrow[ur, "\beta_1"] \arrow[dr, "\gamma_1"'] & & & & w \\
 & t_1 \arrow[r, "\gamma_2"'] & t_2 \arrow[r, "\gamma_3"'] & \cdots \arrow[ur, "\gamma_\ell"'] &
\end{tikzcd}$$

Now consider the image of this diagram in $\mathcal{M}$, which we do not yet know to be commutative.

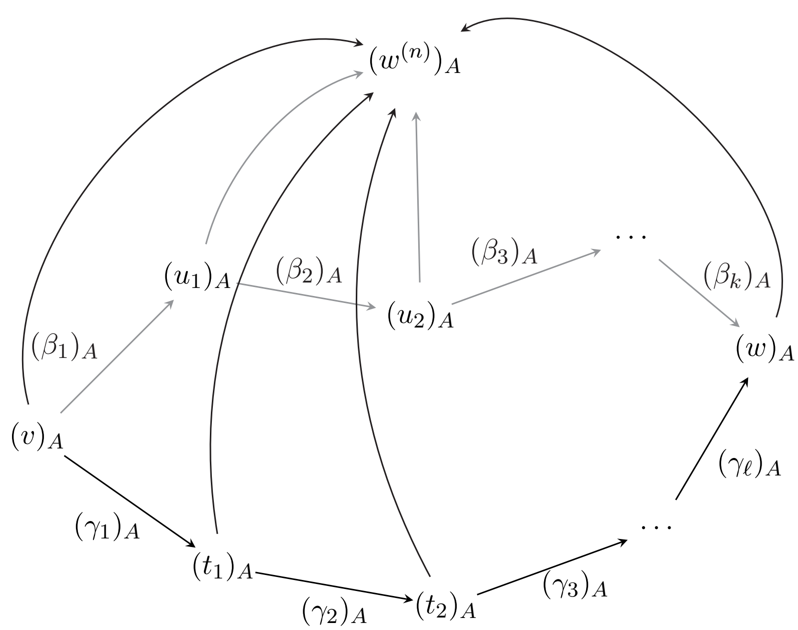
$$\begin{tikzcd}
 & (u_1)_A \arrow[r, "(\beta_2)_A"] & (u_2)_A \arrow[r, "(\beta_3)_A"] & \cdots \arrow[dr, "(\beta_k)_A"] & \\
(v)_A \arrow[ur, "(\beta_1)_A"] \arrow[dr, "(\gamma_1)_A"'] & & & & (w)_A \\
 & (t_1)_A \arrow[r, "(\gamma_2)_A"'] & (t_2)_A \arrow[r, "(\gamma_3)_A"'] & \cdots \arrow[ur, "(\gamma_\ell)_A"'] &
\end{tikzcd}$$

Our goal is to show that this diagram in $\mathcal{M}$ is in fact commutative. This will then show our desired equality.

By Proposition 3.3.14, we can connect each pure binary word u_i and t_i to the terminal word $w^{(n)}$ with forward α-arrows $\Gamma_{u_i} : u_i \to w^{(n)}$ and $\Gamma_{t_i} : t_i \to w^{(n)}$. If we add these to our diagram (and suppress the notation on the Γ's), it becomes



whose image under the proxy map in $\mathcal{M}$ is



Thus the diagram has become a cone, with apex $w^{(n)}$, which is sliced by the triangles. The base of this cone is the original diagram. We now show that each triangle is commutative.

Note that each triangle is of two possible forms: it either consists of β_i or γ_i. Without loss of generality, consider a triangle with an instance of β_i, as below.

$$\begin{tikzcd}
(u_{i-1})_A \arrow[rr, "(\beta_i)_A"] \arrow[dr, "(\Gamma_{u_{i-1}})_A"'] & & (u_i)_A \arrow[dl, "(\Gamma_{u_i})_A"] \\
& (w^{(n)})_A &
\end{tikzcd}$$

Now if β_i is a forward α-arrow, observe that by Proposition 3.4.5 it is a commutative diagram in $\mathcal{M}$.

On the other hand, suppose β_i is a backward α-arrow. Then β_i^{-1} is a forward α-arrow. Then we may rewrite the triangle as

$$\begin{tikzcd}
(u_{i-1})_A \arrow[dr, "(\Gamma_{u_{i-1}})_A"'] & & (u_i)_A \arrow[ll, "(\beta_i^{-1})_A"'] \arrow[dl, "(\Gamma_{u_i})_A"] \\
& (w^{(n)})_A &
\end{tikzcd}$$

so that it now consists entirely of forward α-arrows. This then allows us to apply Proposition 3.4.5 to guarantee that it is a commutative diagram in $\mathcal{M}$. Thus, what we have shown is that each triangle in the above diagram is commutative in $\mathcal{M}$. This literally means that for each i,

$$(\Gamma_{u_i})_A \circ (\beta_i)_A = (\Gamma_{u_{i-1}})_A \implies (\beta_i)_A = (\Gamma_{u_i})_A^{-1} \circ (\Gamma_{u_{i-1}})_A$$
$$(\Gamma_{t_i})_A \circ (\gamma_i)_A = (\Gamma_{t_{i-1}})_A \implies (\gamma_i)_A = (\Gamma_{t_i})_A^{-1} \circ (\Gamma_{t_{i-1}})_A$$

Therefore, we see that $(\beta_k)_A \circ \cdots \circ (\beta_1)_A$ can be written as

$$\Big((\Gamma_{u_k})_A^{-1} \circ (\Gamma_{u_{k-1}})_A\Big) \circ \Big((\Gamma_{u_{k-1}})_A^{-1} \circ (\Gamma_{u_{k-2}})_A\Big) \circ \cdots \circ \Big((\Gamma_{u_1})_A^{-1} \circ (\Gamma_{u_0})_A\Big)$$

which is a "telescoping" composition that reduces to

$$(\Gamma_{u_k})_A^{-1} \circ (\Gamma_{u_0})_A.$$

Similarly, we can expression $(\gamma_\ell)_A \circ \cdots \circ (\gamma_1)_A$ as

$$\Big((\Gamma_{t_\ell})_A^{-1} \circ (\Gamma_{t_{\ell-1}})_A\Big) \circ \Big((\Gamma_{t_{\ell-1}})_A^{-1} \circ (\Gamma_{t_{\ell-2}})_A\Big) \circ \cdots \circ \Big((\Gamma_{t_1})_A^{-1} \circ (\Gamma_{t_0})_A\Big)$$

which also reduces to

$$(\Gamma_{t_\ell})_A^{-1} \circ (\Gamma_{t_0})_A.$$

However, $u_k = t_\ell$ and $u_0 = t_0$, so that

$$(\Gamma_{u_k})_A^{-1} \circ (\Gamma_{u_0})_A = (\Gamma_{t_\ell})_A^{-1} \circ (\Gamma_{t_0})_A \implies (\beta_k)_A \circ \cdots \circ (\beta_1)_A = (\beta_k)_A \circ \cdots \circ (\beta_1)_A$$

Thus we have that our original diagram in $\mathcal{M}$

$$\begin{tikzcd}
& (u_1)_A \arrow[r, "(\beta_2)_A"] & (u_2)_A \arrow[r, "(\beta_3)_A"] & \cdots \arrow[dr, "(\beta_n)_A"] & \\
(v)_A \arrow[ur, "(\beta_1)_A"] \arrow[dr, "(\gamma_1)_A"'] & & & & (w)_A \\
& (t_1)_A \arrow[r, "(\gamma_2)_A"'] & (t_2)_A \arrow[r, "(\gamma_3)_A"'] & \cdots \arrow[ur, "(\gamma_m)_A"'] &
\end{tikzcd}$$

is commutative. Therefore we have that parallel sequences of α-arrows are equal in $\mathcal{M}$, as desired.

∎

Finally, we use all of our previous work to prove Theorem 3.4.2. In this case, the proof is simply the definition of our desired functor. We state the theorem here for the reader's convenience.

Theorem 3.4.2 (Associator Coherence.)**.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. For every object A, there exists a unique functor $\Phi_A : \mathcal{W}_P \to \mathcal{M}$ which agrees with the proxy map $(-)_A$ on the objects and α-arrows.

To define this functor, we will (in this order) define the functor on (1) object, (2) α-arrows, (3) general morphisms of $\mathcal{W}_\mathrm{P}$, and then finally show that our definition preserves composition.

Objects. For a pure binary word w, we define $\Phi_A(w) = (w)_A$.

Morphisms. (1) If β is an α-arrow, we define $\Phi_A(\beta) = (\beta)_A$.

(2) Now we define our functor on a general morphism $v \to w$ in $\mathcal{W}_{\mathrm{P}}$. For convenience denote this as $\varphi_{v,w} : v \to w$.

We know by Corollary 3.3.15 that there exist finitely many forward and backward α-arrows $\gamma_1, \dots, \gamma_k$ such that

$$\varphi_{v,w} = \gamma_k \circ \cdots \circ \gamma_1.$$

Therefore, define

$$\Phi_A(\varphi_{v,w}) = \Phi(\gamma_k \circ \cdots \circ \gamma_1) = (\gamma_k)_A \circ \cdots \circ (\gamma_1)_A.$$

By Proposition 3.4.4, we see that this definition is well-defined.

Note that this definition allows the functor to also be well-defined on identities, i.e., in all instances, $\Phi_A(1_u) = 1_{u_A}$.

We now show that this definition of our functor behaves under composition. Let $\varphi_{u,v} : u \to v$ and $\varphi_{v,w} : v \to w$ be morphisms in $\mathcal{W}_{\mathrm{P}}$. Then there exist sequences of α-arrows $\beta_1, \dots, \beta_k$ and $\gamma_1, \dots, \gamma_\ell$ such that

$$\varphi_{u,v} = \beta_k \circ \cdots \circ \beta_1 \qquad \varphi_{v,w} = \gamma_\ell \circ \cdots \circ \gamma_1.$$

Then we can write

$$\begin{aligned} \Phi(\varphi_{v,w} \circ \varphi_{u,v}) &= \Phi(\gamma_\ell \circ \cdots \circ \gamma_1 \circ \beta_k \circ \cdots \circ \beta_1) \\ &= (\gamma_\ell)_A \circ \cdots \circ (\gamma_1)_A \circ (\beta_k)_A \circ \cdots \circ (\beta_1)_A \\ &= \Phi(\gamma_\ell \circ \cdots \circ \gamma_1) \circ \Phi(\beta_k \circ \cdots \circ \beta_1) \\ &= \Phi(\varphi_{v,w}) \circ \Phi(\varphi_{u,v}) \end{aligned}$$

Hence we see that our definition on morphisms behaves appropriately on composition, so that Φ is in fact a functor.

We conclude this section by proving the Diamond Lemma, which we have now seen to play a critical role in this proof.

Lemma 3.4.6 (Diamond Lemma)**.** Let w be a pure binary word and suppose β_1, β_2 are two forward α-arrows as below.

$$\begin{tikzcd} & w \arrow[dl, "\beta_1"'] \arrow[dr, "\beta_2"] & \\ w_1 & & w_2 \end{tikzcd}$$

There exists a pure binary word z and two $\gamma_1 : w_1 \to z, \gamma_2 : w_2 \to z$, with γ_1, γ_2 a composition of forward α-arrows, such that for any monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ the diagram below is commutative in $\mathcal{M}$.

$$
\begin{tikzcd}
 & (w)_A \arrow[dl, "(\beta_1)_A"'] \arrow[dr, "(\beta_2)_A"] & \\
(w_1)_A \arrow[dr, "(\gamma_1)_A"'] & & (w_2)_A \arrow[dl, "(\gamma_2)_A"] \\
 & (z)_A &
\end{tikzcd}
$$

is commutative.

As we said before, the above lemma is an existence result, so we emphasize this fact by coloring the arrows, which we are asserting to exist, Green.

Proof: We will prove this using induction on the length of $w = u \otimes v$. Therefore, throughout the proof, suppose the result is already true for all words of length less than that of w.

We proceed in a case-by-case basis, exhausting the possible forms of β_1 and β_2. For our purposes, we will express $w = u \otimes v$. Whenever $\mathcal{L}(v) > 1$, we write $v = s \otimes t$.

Let β_1, β_2 be forward α-arrows. Then β_1 could be of the forms

$$\alpha_{u,s,t} \qquad 1_u \otimes \gamma_1 \qquad \gamma_1 \otimes 1_v$$

and β_2 could be of the forms

$$\alpha_{u,s,t} \qquad 1_u \otimes \gamma_2 \qquad \gamma_2 \otimes 1_v.$$

with γ_1, γ_2 already forward α-arrows. Therefore, our cases for β_1, β_2, displayed in tuples, are listed in the table below.

(β_1, β_2)	$\alpha_{u,s,t}$	$1_u \otimes \gamma_2$	$\gamma_2 \otimes 1_v$
$\alpha_{u,s,t}$	$(\alpha_{u,s,t}, \alpha_{u,s,t})$	$(\alpha_{u,s,t}, 1_u \otimes \gamma_2)$	$(\alpha_{u,s,t}, \gamma_2 \otimes 1_v)$
$1_u \otimes \gamma_1$	$(1_u \otimes \gamma_1, \alpha_{u,s,t})$	$(1_u \otimes \gamma_1, 1_u \otimes \gamma_2)$	$(1_u \otimes \gamma_1, \gamma_2 \otimes 1_v)$
$\gamma_1 \otimes 1_v$	$(\gamma_1 \otimes 1_v, \alpha_{u,s,t})$	$(\gamma_1 \otimes 1_v, 1_u \otimes \gamma_2)$	$(\gamma_1 \otimes 1_v, \gamma_2 \otimes 1_v)$

While there are 9 cases displayed above, we have pointed out via color the pairs of cases which are logically equivalent to each other due to the

symmetry of our problem. Therefore, we actually have 6 cases to check We now proceed to the proof.
Case 1: $(\alpha_{u,s,t}, \alpha_{u,s,t})$.
In this case, we have that $\beta_1 = \beta_2$, for which the statement is trivially true.

Case 2: $(\gamma_1 \otimes 1_v, 1_u \otimes \gamma_2)$
Suppose $\beta_1 = \gamma_1 \otimes 1_v$ and $\beta_2 = 1_u \otimes \gamma_2$. Here, $\gamma_1 : u \to u'$ and $\gamma_2 : v \to v'$ for some pure binary words u', v'. Then we get the diagram

$$\begin{tikzcd} & u \otimes v \arrow[dl, "\gamma_1 \otimes 1_v"'] \arrow[dr, "1_u \otimes \gamma_2"] & \\ u' \otimes v \arrow[dr, "1_{u'} \otimes \gamma_2"'] & & u \otimes v' \arrow[dl, "\gamma_1 \otimes 1_{v'}"] \\ & u' \otimes v' & \end{tikzcd}$$

which commutes by the bifunctoriality of $\otimes$.

Case 3: $(\gamma_1 \otimes 1_v, \gamma_2 \otimes 1_v)$
Suppose $\beta_1 = \gamma_1 \otimes 1_v$ and $\beta_2 = \gamma_2 \otimes 1_v$ with $\gamma_1 : u \to u_1$ and $\gamma_2 : u \to u_2$ both forward α-arrows. Then in this case we have the triangle below in $\mathcal{M}$.

$$\begin{tikzcd} & (u \otimes v)_A \arrow[dl, "(\gamma_1)_A \otimes 1_{(v)_A}"'] \arrow[dr, "(\gamma_2)_A \otimes 1_{(v)_A}"] & \\ (u_1 \otimes v)_A & & (u_2 \otimes v)_A \end{tikzcd}$$

Note that the above diagram is the image of diagram

$$\begin{tikzcd} & (u)_A \arrow[dl, "(\gamma_1)_A"'] \arrow[dr, "(\gamma_2)_A"] & \\ (u_1)_A & & (u_2)_A \end{tikzcd}$$

under the functor $(-) \otimes (v)_A$. As $\mathcal{L}(u) < \mathcal{L}(u \otimes v)$, we know by our induction hypothesis that there exists a pure binary word z and a pair of composite, forward α-arrows $\sigma_1 : u_1 \to z$ and $\sigma_2 : u_2 \to z$ such that the diagram below commutes in $\mathcal{M}$.

$$\begin{tikzcd} & (u)_A \arrow[dl, "(\gamma_1)_A"'] \arrow[dr, "(\gamma_2)_A"] & \\ (u_1)_A \arrow[dr, "(\sigma_1)_A"'] & & (u_2)_A \arrow[dl, "(\sigma_2)_A"] \\ & (z)_A & \end{tikzcd}$$

Therefore we can apply the functor $(-) \otimes (v)_A$ on the above diagram to obtain the commutative diagram below

$$\begin{tikzcd} & (u)_A \otimes (v)_A \arrow[dl, "(\beta_2)_A \otimes (1_v)_A"'] \arrow[dr, "(\gamma_1)_A \otimes (1_v)_A"] & \\ (u_1)_A \otimes (v)_A \arrow[dr, "(\sigma_1)_A \otimes (1_v)_A"'] & & (u_2)_A \otimes (v)_A \arrow[dl, "(\sigma_2) \otimes (1_v)_A"] \\ & (z)_A \otimes (v)_A & \end{tikzcd}$$

which proves this case.

Case 4: $(1_u \otimes \gamma_1, 1_u \otimes \gamma_2)$
The next case is when $\beta_1 = 1_u \otimes \gamma_1$ and $\beta_2 = 1_u \otimes \gamma_2$ with $\gamma_1 : v \to v_1$ and $\gamma_2 : v \to v_2$. However, this can be proved in a similar manner as the previous case using the induction hypothesis and the functor $(u)_A \otimes (-)$.
Case 5: $(\alpha_{u,s,t}, \gamma_2 \otimes 1_v)$
Let $\beta_1 = \alpha_{u,s,t}$, so that $w = u \otimes (s \otimes t)$. Let $\beta_2 = \gamma_2 \otimes 1_v = \gamma_2 \otimes 1_{s \otimes t}$ with $\gamma_2 : u \to u'$ a forward α-arrow. Then we will have the diagram in $\mathcal{M}$

$$\begin{tikzcd} & (u \otimes (s \otimes t))_A \arrow[dl, "(\alpha_{u,s,t})_A"'] \arrow[dr, "(\gamma_2 \otimes (1_s \otimes 1_t))_A"] & \\ ((u \otimes s) \otimes t)_A \arrow[dr, "((\gamma_2 \otimes 1_s) \otimes 1_t)_A"'] & & (u' \otimes (s \otimes t))_A \arrow[dl, "(\alpha_{u',s,t})_A"] \\ & ((u' \otimes s) \otimes t)_A & \end{tikzcd}$$

which commutes in $\mathcal{M}$ by naturality of α.

Case 6: $(\alpha_{u,s,t}, 1_u \otimes \gamma_2)$
Let $\beta_1 = \alpha_{u,s,t}$, $\beta_2 = 1_u \otimes \gamma$ with γ a forward α-arrow with domain $s \otimes t$. By the recursive definition of a forward α-arrow, we have three possible cases for γ.

Case 6.1: $\gamma = 1_s \otimes \gamma'$
With $\gamma = 1_s \otimes \gamma'$ with $\gamma' : t \to t'$ already a forward α-arrow, we have the diagram in $\mathcal{M}$

$$\begin{tikzcd}
 & (u \otimes (s \otimes t))_A \arrow[dl, "(\alpha_{u,s,t})_A"'] \arrow[dr, "(1_u \otimes (1_s \otimes \gamma))_A"] & \\
((u \otimes s) \otimes t)_A \arrow[dr, "((1_u \otimes 1_s) \otimes \gamma)_A"'] & & (u \otimes (s \otimes t'))_A \arrow[dl, "(\alpha_{u,s,t'})_A"] \\
 & ((u \otimes s) \otimes t')_A &
\end{tikzcd}$$

which commutes in $\mathcal{M}$ by naturality of α.

Case 6.2: $\gamma = \gamma' \otimes 1_t$
If $\gamma = \gamma' \otimes 1_t$ with $\gamma' : s \to s'$ already a forward α-arrow, we can create the diagram

$$\begin{tikzcd}
 & (u \otimes (s \otimes t))_A \arrow[dl, "(\alpha_{u,s,t})_A"'] \arrow[dr, "(1_u \otimes (\gamma' \otimes 1_t))_A"] & \\
((u \otimes s) \otimes t)_A \arrow[dr, "((1_u \otimes \gamma') \otimes 1_t)_A"'] & & (u \otimes (s' \otimes t))_A \arrow[dl, "(\alpha_{u,s',t})_A"] \\
 & ((u \otimes s') \otimes t)_A &
\end{tikzcd}$$

which also commutes in $\mathcal{M}$ by naturality of α.
Case 6.3: $\gamma = \alpha_{s,p,q}$
The third case for γ is when $\gamma = \alpha_{s,p,q}$. In this case, we express $w = u \otimes (s \otimes (p \otimes q))$. We can then construct the diagram

$$\begin{tikzcd}
 & (u \otimes (s \otimes (p \otimes q)))_A \arrow[dl, "(\alpha_{u,s,p\otimes q})_A"'] \arrow[dr, "(1_u \otimes \alpha_{s,p,q})_A"] & \\
((u \otimes s) \otimes (p \otimes q))_A \arrow[d, "(\alpha_{u\otimes s,p,q})_A"'] & & (u \otimes ((s \otimes p) \otimes q))_A \arrow[d, "(\alpha_{u,s\otimes p,q})_A"] \\
(((u \otimes s) \otimes p) \otimes q)_A & & ((u \otimes (s \otimes p)) \otimes q)_A \arrow[ll, "(\alpha_{u,s,p}\otimes 1_q)_A"]
\end{tikzcd}$$

which is always commutative in $\mathcal{M}$. In this case, the word $((u \otimes s) \otimes p) \otimes q$ acts as our vertex z which completes the diagram.

As we have exhausted all possible cases, we see that the statement is true for pure binary words of rank $k + 1$ if it is true for all pure binary words with rank at most k. By induction, the statement is true for all binary words of any rank, so that we have proved the theorem.

■

3.5 Step Four: Binary Words

So far we have established a unique functor $\Phi_A : \mathcal{W}_{\mathrm{P}} \to \mathcal{M}$ for each object A of any given monoidal category $\mathcal{M}$, and this functor grants us coherence in the associators between iterated monoidal products of a single object. We now consider such monoidal products with the identity I as well, so that we may say something about coherence with regard to the unitors λ and ρ in a general monoidal category. Towards that goal, we now consider binary words (not just pure binary words) and introduce some definitions.

Recall that $\mathcal{L}$ calculates the length of a binary word, or more informally, the number of x_1's in a binary word. We now introduce a dual quantity which instead counts the number of x_0

Definition 3.5.1. Let w be a binary word. Define the **identity length** of w, denoted $\mathcal{E}$, recursively as follows.

- $\mathcal{E}(x_0) = 1$ and $\mathcal{E}(x_1) = 0$.
- $\mathcal{E}(u \otimes v) = \mathcal{E}(u) + \mathcal{E}(v)$.

Similarly to how $\mathcal{L}(-)$ counts the number of x_1's in a binary word, $\mathcal{E}(-)$ counts the number of x_0's in a binary word.

Next, we introduce the following concept that will later on be key to our proof of Mac Lane's Coherence Theorem.

Definition 3.5.2. Let w be a binary word. We define the **clean word** derived from w, denoted $\overline{w}$, recursively as follows.

- We set $\overline{x_1} = x_1$.
- If $\mathcal{L}(w) = 0$ (i.e., it has no instance of x_1) then $\overline{w} = x_0$.
- Let u, v be binary words with $\mathcal{L}(u) = 0$ and $\mathcal{L}(v) > 0$. Then

$$\overline{u \otimes v} = \overline{v \otimes u} = \overline{v}$$

- Let u, v be binary words with $\mathcal{L}(u), \mathcal{L}(v) > 0$. Then $\overline{u \otimes v} = \overline{u} \otimes \overline{v}$.

Note that for a pure binary word w, we have that $\overline{w} = w$. Informally, the clean word of a binary word of nonzero length is simply the pure binary word obtained by removing all instances of the identity from its expression. In the case for a binary word with zero length, we naturally define the clean word to be x_0 .

Example 3.5.3. We offer some examples of clean words obtained from binary words.

Word	Clean Word
$x_0 \otimes (x_0 \otimes x_0)$	x_0
$x_0 \otimes (x_1 \otimes x_0)$	x_1
$(x_1 \otimes x_0) \otimes x_1$	$x_1 \otimes x_1$
$((x_1 \otimes x_0) \otimes x_0) \otimes x_1$	$x_1 \otimes x_1$
$(x_1 \otimes x_0) \otimes ((x_1 \otimes x_0) \otimes x_1)$	$x_1 \otimes (x_1 \otimes x_1)$

The above example also shows that two different binary words can have the same clean word.

Definition 3.5.4 (Monoidal Arrows)**.** A **forward monoidal** arrow of $\mathcal{W}$ is defined recursively as follows.

- For any triple of binary words u, v, w, the morphisms

$$\begin{aligned} \alpha_{u,v,w} &: u \otimes (v \otimes w) \xrightarrow{\sim} (u \otimes v) \otimes w \\ \lambda_u &: x_0 \otimes u \xrightarrow{\sim} u \\ \rho_u &: u \otimes x_0 \xrightarrow{\sim} u \end{aligned}$$

  are, respectively, **forward** α-, λ-, and ρ-**arrows**. They are collectively defined to be **forward monoidal arrows**.

- For any binary word u and forward monoidal arrow μ, the morphisms

$$1_u \otimes \mu \qquad \mu \otimes 1_u$$

  are forward monoidal arrows.

Finally, we say a **backward monoidal arrow** is the inverse of a forward monoidal arrow.

We also establish the following terminology to distinguish our α-arrows from our λ and ρ arrows.

Definition 3.5.5. A **forward unitor arrow** is either a forward λ-arrow or a forward ρ-arrow. Similarly, a **backward unitor arrow** is the inverse of a forward unitor arrow.

As we have already seen forward α-arrows, we provide examples of forward and backward λ, ρ-arrows.

Example 3.5.6. Below we have a forward and backward λ-arrow.

$$\begin{tikzcd} x_1 \otimes ((x_0 \otimes x_1) \otimes x_1) \arrow[d, "1_{x_1} \otimes (\lambda_{x_1} \otimes 1_{x_1})"] \\ x_1 \otimes (x_1 \otimes x_1) \end{tikzcd} \qquad \begin{tikzcd} (x_1 \otimes x_1) \otimes x_1 \arrow[d, "\lambda^{-1}_{(x_1 \otimes x_1) \otimes x_1}"] \\ x_0 \otimes ((x_1 \otimes x_1) \otimes x_1) \end{tikzcd}$$

We also have forward and backward ρ-arrows below.

$$\begin{tikzcd} (x_1 \otimes x_0) \otimes x_1 \arrow[d, "\rho_{x_1} \otimes 1_{x_1}"] \\ x_1 \otimes x_1 \end{tikzcd} \qquad \begin{tikzcd} x_1 \otimes (x_1 \otimes x_1) \arrow[d, "1_{x_1} \otimes \rho^{-1}_{x_1 \otimes x_1}"] \\ x_1 \otimes ((x_1 \otimes x_1) \otimes x_0) \end{tikzcd}$$

We now move onto proving some important lemmas regarding monoidal arrows that we will use for the coherence theorem.

The first three are quick, but have particular importance.

Lemma 3.5.7. Let w be a binary word, $w \neq x_0$. Then $\mathcal{E}(w) = 0$ if and only if $w = \overline{w}$.

Note that $w = x_0$ is the only case for which the above proposition is not true, since $x_0 = \overline{x_0}$ but $\mathcal{E}(x_0) \neq 0$. Hence, our reasoning for excluding it (and it is not a case we will need to concern ourselves with).

Proof: Suppose $\mathcal{E}(w) = 0$, and let us prove the forward direction by induction on the length of the word. Let us write $w = u \otimes v$, suppose that the statement is true for all pure binary words with length less than w. Observe that

$$w = u \otimes v = \overline{u} \otimes \overline{v} = \overline{u \otimes v} = \overline{w}.$$

where we used the induction hypothesis on u, v which have smaller length than w. Thus we see that $w = \overline{w}$.

Conversely, suppose $\overline{w} = w$, $w \neq x_0$, and suppose the statement is true for binary words with length less than w. Write $w = u \otimes v$. By the definition of a clean word, the only way we can have $\overline{w} = w$ is if u, v are binary words with nonzero length. Therefore, if $\overline{w} = w$ we see that

$$\overline{u} \otimes \overline{v} = u \otimes v.$$

Since u, v have smaller length than w, we may use the induction hypothesis to conclude that $\mathcal{E}(u) = \mathcal{E}(v) = 0$. Hence, $\mathcal{E}(w) = 0$, as desired.

■

Lemma 3.5.8. Let w be a binary word. Suppose $\iota : w \to w'$ is a forward unitor arrow. Then $\mathcal{E}(w') = \mathcal{E}(w) - 1$.

In other words, any unitor arrow always takes away exactly one identity.

Proof: We prove this by examining the possible cases for ι. Write $w = u \otimes v$. As ι is a forward unitor arrow, it has four possible forms.

(1) Suppose $\iota = \lambda_v : x_0 \otimes v \to v$. As

$$\mathcal{E}(v) = \mathcal{E}(v) + \mathcal{E}(x_0) - 1 = \mathcal{E}(v \otimes x_0) - 1$$

we see that the statement is satisfied in this case.

(2) If $\iota = \rho_u : u \otimes x_0 \to u$, we can use a similar argument as in (1) to prove the statement.

(3) Suppose $\iota = 1_u \otimes \kappa : u \otimes v \to u \otimes v'$ where $\kappa : v \to v'$ is a forward unitor arrow for which the statement is already true. Then $\mathcal{E}(v') = \mathcal{E}(v) - 1$. Hence,

$$\mathcal{E}(u \otimes v') = \mathcal{E}(u \otimes v) - 1.$$

Therefore the statement is satisfied for $1_u \otimes \kappa$ if it is true for κ.

(4) If $\iota = \kappa \otimes 1_v : u \otimes v \to u' \otimes v$ where κ is a forward unitor for which the statement is already true, then we may prove this case by following a similar argument as in (3).

As we have examined all cases, we may conclude that for every forward unitor $\iota : w \to w'$, we have that $\mathcal{E}(w') = \mathcal{E}(w) - 1$ as desired.

■

Lemma 3.5.9. Let $\iota : w \to w'$ be a forward unitor arrow. Then $\overline{w} = \overline{w}'$.

In other words, unitor arrows do not alter the particular format of a clean word.

Proof: First, observe that the result is trivial if $\mathcal{L}(w) = \mathcal{L}(w') = 0$. Therefore, let $w = u \otimes v$ be such a binary word with $\mathcal{E}(w) > 0$. Suppose the statement is true for binary words v such that $\mathcal{E}(v) < \mathcal{E}(w)$. Let $\iota : w \to w'$ be a forward unitor arrow. By the recursive definition of ι, our forward unitor arrow has four possible forms.

(1) Suppose $\iota = \lambda_v : x_0 \otimes v \to v$. However, note that $\overline{x_0 \otimes v} = \overline{v}$, so that this case is true.

(2) If $\iota = \rho_u : u \otimes x_0 \to u$, then this case may be proven in a similar manner as case (1).

(3) Suppose $\iota = 1_u \otimes \kappa : u \otimes v \to u \otimes v'$ where κ is a forward unitor arrow for which the result is already true. Since $\mathcal{L}(u \otimes v) < 0$, we have a few subcases.

Suppose $\mathcal{L}(v) > 0$. Then by our assumption on κ, $\overline{v} = \overline{v}'$. Therefore, if $\mathcal{L}(u) = 0$, we see that

$$\overline{u \otimes v} = \overline{v} = \overline{v}' = \overline{u \otimes v}'$$

which satisfies this case. If instead $\mathcal{L}(u) > 0$, then

$$\overline{u \otimes v} = \overline{u} \otimes \overline{v} = \overline{u} \otimes \overline{v}' = \overline{u \otimes v}'$$

which again satisfies the case.

Finally, suppose $\mathcal{L}(v) = 0$. Then $\overline{u \otimes v} = \overline{u} = \overline{u \otimes v}'$.

In all cases we see that $\overline{u \otimes v} = \overline{u \otimes v}'$ as desired.

(3) Our third case if when $\iota = \kappa \otimes 1_v : u \otimes v \to u' \otimes v$ with κ a forward unitor for which the result is already true. However, this case can be proved similarly as in case (2).

In all instances, we see that for a forward unitor arrow $\iota : w \to w'$, we have that $\overline{w} = \overline{w}'$, as desired.

■

The following lemma is an important existence result that will be used in the next proposition.

Lemma 3.5.10. Let w be a binary word with $\mathcal{E}(w) > 0$. Then there exists a forward unitor with domain w.

Proof: We prove this by induction on the total length of a binary word $\mathcal{L}(w) + \mathcal{E}(w)$. Thus, let $w = u \otimes v$ be a binary word with $\mathcal{E}(w) > 0$ and suppose the statement is true for all binary words z with

$$\mathcal{L}(z) + \mathcal{E}(z) < \mathcal{L}(w) + \mathcal{E}(w).$$

Then we have a few cases for w.

(1) Suppose $u = x_0$. Then we take the forward unitor $\lambda_v : x_0 \otimes v \to v$.

(2) Suppose $v = x_0$. We may similarly take $\rho_u : u \otimes x_0 \to u$, so that this case is satisfied.

(3) Suppose $u, v \neq x_0$. Since $\mathcal{E}(w) > 1$, either $\mathcal{E}(u)$ or $\mathcal{E}(v) > 0$. Without loss of generality, suppose $\mathcal{E}(u) > 0$. Since

$$\mathcal{L}(u) + \mathcal{E}(u) = \mathcal{L}(u) + \mathcal{E}(u)$$

we may apply our induction hypothesis to conclude that there exists a forward unitor $\iota : u \to u'$ with domain u. Hence, the morphism

$$\iota \otimes 1_v : u \otimes v \to u' \otimes v$$

is a forward unitor with domain $u \otimes v = w$.

As we have evaluated all cases, we see that the statement is true for all binary words as desired.

■

The previous four lemmas now give rise to the following proposition.

Proposition 3.5.11. Let w be a binary word with $\mathcal{E}(w) = \ell$. Then there exists a composable sequence of ℓ-many forward unitor arrows $\iota_\ell, \cdots, \iota_1$ as below:

$$\iota_\ell \circ \cdots \circ \iota_1 : w \to w'.$$

Moreover, for every such chain, we have that $w' = \overline{w}$.

Proof: To prove existence of such a chain for every binary word with nonzero identity length, we may proceed by induction. Let w be a binary word with $\mathcal{E}(w) > 0$, and suppose that such a chain exists for binary words v with $\mathcal{E}(v) < \mathcal{E}(w)$. Then by Lemma 2.5.10, there exists a forward unitor $\iota : w \to w'$. By Lemma 2.5.8, $\mathcal{E}(w') = \mathcal{E}(w) - 1$, so by our induction hypothesis, there exists a chain of forward unitor arrows

$$\iota_{\ell-1} \circ \cdots \circ \iota_1 : w' \to \overline{w}'.$$

Hence, $\iota \circ \iota_{\ell-1} \circ \cdots \circ \iota_1 : w \to \overline{w}$ is a forward chain of unitors with initial domain w, which proves existence.

To prove that $w' = \overline{w}$, denote the domain and codomain of our unitors $\iota_i : w_{i-1} \to w_i$, so that $w_0 = w$. By Lemma 2.5.9, for each i we have that $\overline{w_{i-1}} = \overline{w_i}$. Hence $\overline{w} = \overline{w_\ell}$. By Lemma 2.5.8, we have that $\mathcal{E}(w_i) = \mathcal{E}(w_{i-1}) - 1$. Therefore,

$$\mathcal{E}(w_\ell) = \mathcal{E}(w) - \ell = 0.$$

However, by Lemma 2.5.7, we see that this implies $w_\ell = \overline{w_\ell} = \overline{w}$. Hence we see that

$$\iota_\ell \circ \cdots \circ \iota : w \to \overline{w}$$

as desired.

■

The previous proposition immediately implies the next.

Proposition 3.5.12. Let w be a binary word with $\mathcal{L}(w) > 0$. Then there exists a sequence of forward monoidal arrows from w to $w^{(n)}$.

Proof: By Lemma 3.5.10, we have a sequence of forward unitor arrows from w to $\overline{w}$.

$$\mu_k \circ \cdots \circ \mu_1 : w \to \overline{w}$$

Since $\overline{w}$ is a pure binary word, we can then use Proposition 3.3.14 to guarantee a sequence of forward α-arrows from $\overline{w}$ to $w^{(n)}$.

$$\beta_\ell \circ \cdots \circ \beta_1 : \overline{w} \to w^{(n)}$$

Composing these morphisms then gives us our desired monoidal arrow:

$$\beta_\ell \circ \cdots \circ \beta_1 \circ \mu_k \circ \cdots \circ \mu_1 : w \to w^{(n)}$$

so that such a sequence of forward monoidal arrows exists.

■

And the previous proposition gives us the following corollary.

Corollary 3.5.13. Every morphism in $\mathcal{W}$ can be expressed as a composition of a sequence of forward and backward monoidal arrows.

Proof: The proof is the same exact proof as that of Corollary 3.3.15. We use the previous proposition with the fact that $\mathcal{W}$ is a thin category to conclude this.

■

3.6 Step Five: Coherence for $A^{\otimes n}$ for ρ, λ

In this section, we extend the work we've completed with the associators to now include the unitors. We will obtain a theorem similar to Theorem 3.4.2. To even state the theorem, we need to introduce a new definition.

Definition 3.6.1. Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. For each object A in $\mathcal{M}$, we define the **general proxy map** of A to be the partial functor $(-)_A : \mathcal{W} \to \mathcal{M}$ defined as follows.

Objects We define the general proxy map on objects recursively.

- We set $(x_0)_A = I$ and $(x_1)_A = A$
- For a binary word $w = u \otimes v$ we set:

$$(w)_A = (u \otimes v)_A = (u)_A \otimes (v)_A$$

Morphisms We define the partial functor only on α-, λ-, and ρ-arrows. This is also done recursively.

- For binary words u, v, w, we set:

$$\begin{aligned} (\alpha_{u,v,w})_A &= \alpha_{(u_A,v_A,w_A)} : u_A \otimes (v_A \otimes w_A) \xrightarrow{\sim} (u_A \otimes v_A) \otimes w_A \\ (\lambda_u)_A &= \lambda_{u_A} : I \otimes u_A \xrightarrow{\sim} u_A \\ (\rho_u)_A &= \rho_{u_A} : u_A \otimes I \xrightarrow{\sim} u_A \end{aligned}$$

- For a more general α, λ, or ρ-arrow of the form $1_u \otimes \beta$ or $\beta \otimes 1_u$ we set:

$$\begin{aligned} (1_u \otimes \beta)_A &= 1_{u_A} \otimes (\beta)_A \\ (\beta \otimes 1_u)_A &= (\beta)_A \otimes 1_{u_A} \end{aligned}$$

Before concluding this definition, we note that there is some potential ambiguity in our definition on the unitors. This is because sometimes a forward unitor arrow in $\mathcal{W}$ can be expressed in two ways. The reader may check that all possible cases for ambiguity are the three cases below.

$$\begin{tikzcd} x_0 \otimes x_0 \arrow[d, shift right, "\rho_{x_0}"'] \arrow[d, shift left, "\lambda_{x_0}"] \\ x_0 \end{tikzcd} \qquad \begin{tikzcd} x_0 \otimes (x_0 \otimes v) \arrow[d, shift right, "1_{x_0} \otimes \lambda_v"'] \arrow[d, shift left, "\lambda_{(x_0 \otimes v)}"] \\ x_0 \otimes v \end{tikzcd} \qquad \begin{tikzcd} (u \otimes x_0) \otimes x_0 \arrow[d, shift right, "\rho_u \otimes 1_{x_0}"'] \arrow[d, shift left, "\rho_{(u \otimes x_0)}"] \\ u \otimes x_0 \end{tikzcd}$$

As parallel morphisms in $\mathcal{W}$, they are equal. Therefore, in order for our definition to be well-defined, we need that the corresponding pairs of morphisms

$$\begin{tikzcd} I \otimes I \arrow[d, shift right, "\rho_I"'] \arrow[d, shift left, "\lambda_I"] \\ I \end{tikzcd} \qquad \begin{tikzcd} I \otimes (I \otimes (v)_A) \arrow[d, shift right, "1_I \otimes \lambda_{(v)_A}"'] \arrow[d, shift left, "\lambda_{(I \otimes (v)_A)}"] \\ I \otimes (v)_A \end{tikzcd} \qquad \begin{tikzcd} ((u)_A \otimes I) \otimes I \arrow[d, shift right, "\rho_{(u)_A} \otimes 1_I"'] \arrow[d, shift left, "\rho_{((u)_A \otimes I)}"] \\ (u)_A \otimes I \end{tikzcd}$$

to be equal in $\mathcal{M}$. One can show that these morphisms are equal in $\mathcal{M}$ using the unitor diagrams 2.2, 2.3, and 2.4.

Regarding our notation, note that we are recycling the same notation from the proxy map to the general proxy map. This is because the only difference between the two is that the general proxy map is simply an extension of the proxy map which is now defined on identity elements x_0 and unitors.

The goal of this section is to prove the following theorem, which can be thought of as an extension of Theorem 3.4.2.

Theorem 3.6.2 (Coherence in Unitors)**.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. For each object A, there exists a unique strict monoidal functor $\Delta_A : \mathcal{W} \to \mathcal{M}$ which agrees with the general proxy map on objects and monoidal morphisms.

The above theorem is implied by Proposition 3.6.3 (stated below), in the same way that Theorem 3.4.2 followed from Proposition 3.4.4.

Proposition 3.6.3. Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category, and consider two binary words v, w. Let $\mu_1, \dots, \mu_k$ and $\eta_1, \dots, \eta_\ell$ be monoidal arrows with:

$$\mu_k \circ \cdots \circ \mu_1,\ \eta_\ell \circ \cdots \circ \eta_1 : v \to w$$

Then $(\mu_k)_A \circ \cdots \circ (\mu_1)_A = (\eta_\ell)_A \circ \cdots \circ (\eta_1)_A$ in $\mathcal{M}$.

The above proposition is implied by Proposition 3.6.4 (stated below), in the same way that Proposition 3.4.4 followed from Proposition 3.4.5

Proposition 3.6.4. Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category, and consider a binary word w. Let $\mu_1, \dots, \mu_k$ and $\eta_1, \dots, \eta_\ell$ be forward monoidal arrows with:

$$\mu_k \circ \cdots \circ \mu_1,\ \eta_\ell \circ \cdots \circ \eta_1 : w \to w^{(n)}$$

Then $(\mu_k)_A \circ \cdots \circ (\mu_1)_A = (\eta_\ell)_A \circ \cdots \circ (\eta_1)_A$ in $\mathcal{M}$.

Once we have the above proposition, we can prove Proposition 3.6.3, and hence our desired theorem, using the same technique as in in the Proof of Proposition 3.4.4.

We briefly recall such techniques: We consider two parallel chains of monoidal arrows. We then connect each object in the chain to $w^{(n)}$ with a chain of forward monoidal arrow (recall that a chain must exist for each object). We then have a bunch of adjacent triangles with apex $w^{(n)}$ and we can conclude via the Proposition 3.6.4 that each such triangle commutes. We then conclude that the original two parallel chains form a commutative diagram in $\mathcal{M}$. Thus, our two chains have the same composite in $\mathcal{M}$. This then proves Proposition 3.6.3, which then grants us Theorem 3.6.2.

As our goal has been reduced to proving Proposition 3.6.4, we prove this proposition using the following two results.

The first result is the following proposition.

Proposition 3.6.5 (Arrow Reorganization)**.** Let $\mu_1, \dots, \mu_k$ be composable forward monoidal arrows with ℓ-many unitor arrows. Then there exist composable forward unitor arrows $\eta_1, \dots, \eta_\ell$ and forward α-arrows $\eta_{\ell+1}, \dots \eta_m$ such that, for any monoidal category $\mathcal{M}$ with object A, we have that

$$(\mu_k)_A \circ \cdots \circ (\mu_1)_A = \overbrace{(\eta_m)_A \circ \cdots \circ (\eta_{\ell+1})_A}^{\text{Forward } \alpha' s} \circ \overbrace{(\eta_\ell)_A \circ \cdots \circ (\eta_1)_A}^{\text{Unitors in front}}$$

in $\mathcal{M}$.

The above proposition basically states that monoidal arrows can be reorganized in a particular way with all of the unitors in the front. The second result that we need in order to prove Proposition 3.6.4 is the following proposition.

Proposition 3.6.6 (Unitor-Chain Equivalence)**.** Let w be a binary word with nonzero length and with $\mathcal{E}(w) = k$. Suppose $\mu_1, \dots, \mu_k$ and $\eta_1, \dots, \eta_k$ are a composable sequence of forward unitor arrows:

$$\mu_k \circ \cdots \circ \mu_1, \ \eta_k \circ \cdots \circ \eta_1 : w \to \overline{w}$$

Then $(\mu_k)_A \circ \cdots \circ (\mu_1)_A = (\eta_k)_A \circ \cdots \circ (\eta_1)_A$ in $\mathcal{M}$.

For the sake of organization, we will assume the validity of these two results now so that we may prove 3.6.4 We will then prove these two results in the next section.

Proof of Proposition 3.6.4

Let

$$\mu_{n_1} \circ \cdots \circ \mu_1,\ \eta_{n_2} \circ \cdots \circ \eta_1 : w \to w^{(n)}$$

be any two composites of forward monoidal arrows from w to $w^{(n)}$. Since $\mathcal{E}(w) = k$ and $\mathcal{E}(w^{(n)}) = 0$, we know by Lemma 3.5.8 that there are exactly k-many forward unitors in each expression. We can then use Proposition 3.6.5 to find forward unitor arrows $\gamma_1, \ldots \gamma_k, \delta_1, \ldots, \delta_k$ and forward α-arrows $\gamma_{k+1}, \ldots, \gamma_{m_1},\ \delta_{k+1}, \ldots, \delta_{m_2}$ such that:

$$(\mu_{n_1})_A \circ \cdots \circ (\mu_1)_A = \overbrace{(\gamma_{m_1})_A \circ \cdots \circ (\gamma_{k+1})_A}^{\text{Forward } \alpha' s} \circ \overbrace{(\gamma_k)_A \circ \cdots \circ (\gamma_1)_A}^{\text{Unitors in front}}$$

$$(\eta_{n_2})_A \circ \cdots \circ (\eta_1)_A = \overbrace{(\delta_{m_2})_A \circ \cdots \circ (\delta_{k+1})_A}^{\text{Forward } \alpha' s} \circ \overbrace{(\delta_k)_A \circ \cdots \circ (\delta_1)_A}^{\text{Unitors in front}}$$

By Proposition 3.5.11, we know that the domain of the composition of our unitors is $\overline{w}$:

$$\gamma_k \circ \cdots \circ \gamma_1,\ \delta_k \circ \cdots \circ \delta_1 : w \to \overline{w}$$

Diagramatically, our situation is displayed below.

$$\begin{tikzcd}
 & (w)_A \arrow[dl, "(\gamma_1)_A"'] \arrow[dr, "(\delta_1)_A"] & \\
(r_1)_A \arrow[d, "(\gamma_2)_A"'] & & (s_1)_A \arrow[d, "(\delta_2)_A"] \\
\vdots \arrow[dr, "(\gamma_k)_A"'] & & \vdots \arrow[dl, "(\delta_k)_A"] \\
 & (\overline{w})_A \arrow[dl, "(\gamma_{k+1})_A"'] \arrow[dr, "(\delta_{k+1})_A"] & \\
(r_{k+1})_A \arrow[d, "(\gamma_{k+2})_A"'] & & (s_{k+1})_A \arrow[d, "(\delta_{k+2})_A"] \\
\vdots \arrow[dr, "(\gamma_{m_1})_A"'] & & \vdots \arrow[dl, "(\delta_{m_2})_A"] \\
 & \left(w^{(n)}\right)_A &
\end{tikzcd}$$

By Proposition 3.7.3, the upper half of this diagram (above $(\overline{w})_A$) must commute. By Proposition 3.4.4, the bottom half of this diagram (below $(\overline{w})_A$), which consists entirely of forward α-arrows, must commute. Therefore, the entire diagram commutes, and this completes the proof.

■

3.7 Step Six: Arrow Reorganization and Unitor Chain Equivalence

We now discuss what it takes to prove the Arrow Reorganization and Unitor-Chain Equivalence results.

To prove the Arrow Reorganization result, it suffices to prove a special case which is precisely stated in the following lemma.

Lemma 3.7.1 (Associator-Unitor Swap.)**.** Let $\mu : w \to w_1$ be a forward α-arrow and let $\iota : w_1 \to w_2$ be a forward unitor arrow. Then either one of the following two situations must occur.

- There exists a binary word z, a forward unitor arrow $\iota' : w \to z$ and a forward α-arrow $\mu' : z \to w_2$ such that, for any monoidal category $\mathcal{M}$, the diagram below commutes.

$$\begin{tikzcd} & (w)_A \arrow[dl, "(\mu)_A"'] \arrow[dr, "(\iota')_A"] & \\ (w_1)_A \arrow[dr, "(\iota)_A"'] & & (z)_A \arrow[dl, "(\mu')_A"] \\ & (w_2)_A & \end{tikzcd}$$

- There exists a forward unitor arrow $\iota' : w \to w_2$ such that, for any monoidal category $\mathcal{M}$, the diagram below commutes.

$$\begin{tikzcd} & (w)_A \arrow[dl, "(\mu)_A"'] \arrow[dr, "(\iota')_A"] & \\ (w_1)_A \arrow[rr, "(\iota)_A"'] & & (w_2)_A \end{tikzcd}$$

As before, the above lemma is an existence result, so we emphasize this fact by coloring the arrows that we are asserting to exist Green.

Assuming the above lemma, we prove the Arrow Reorganization Proposition.

Proof of Arrow Reorganization (Proposition 3.6.5). We summarize rather than introducing too much notation, since the proof strategy is rather simple. Consider a sequence of monoidal arrows $\mu_1, \dots, \mu_k$. Suppose μ_j is a unitor arrow. If μ_{j-1} is an α-arrow, we perform an associator-unitor swap, obtaining a new chain whose composite is the same in $\mathcal{M}$. If not, we leave it alone and check the other unitor arrows.

We perform this reorganization, swapping associator arrows and unitor arrows one at a time, until we have a sequence of morphisms in which no unitor arrow is preceded by an α-arrow (and hence all unitors begin at the front of our chain). The repeated application of the Associator-Unitor swap guarantees that the composite of this new chain is equal to the composite of our original chain.

■

We now understand how to prove the Arrow Reorganization Proposition: it relies critically on the Associator-Unitor Swap. As we now understand how the Associator-Unitor swap is used, we offer its proof.

Proof of Associator-Unitor Swap (Lemma 3.7.1). We prove this using a case-by-case basis. For our proof, we write $w = u \otimes v$. Whenever $\mathcal{L}(v) > 1$, we write $w = u \otimes (s \otimes t)$. If $\mathcal{L}(t) > 1$, we will write $w = u \otimes (s \otimes (p \otimes q))$.

Since μ is a forward α-arrow, it could be of the forms

$$\alpha \quad 1_u \otimes \eta_1 \quad \eta_1 \otimes 1_v$$

with η_1 a forward α-arrow. Since ι is a forward unitor arrow, it could be of the forms

$$\lambda_v \quad \rho_u \quad 1_u \otimes \eta_2 \quad \eta_2 \otimes 1_v$$

with η_2 either a forward unitor arrow. We display our table below, this time coloring the entries in order to group together similar cases.

(μ, ι)	$1_u \otimes \eta_2$	$\eta_2 \otimes 1_v$	λ_v	ρ_u
α	$(\alpha_{u,s,t}, 1_u \otimes \eta_2)$	$(\alpha_{u,s,t}, \eta_2 \otimes 1_v)$	$(\alpha_{u,s,t}, \lambda_v)$	$(\alpha_{u,s,t}, \rho_u)$
$1_u \otimes \eta_1$	$(1_u \otimes \eta_1, 1_u \otimes \eta_2)$	$(1_u \otimes \eta_1, \eta_2 \otimes 1_v)$	$(1_u \otimes \eta_1, \lambda_v)$	$(1_u \otimes \eta_1, \rho_u)$
$\eta_1 \otimes 1_v$	$(\eta_1 \otimes 1_v, 1_u \otimes \eta_2)$	$(\eta_1 \otimes 1_v, \eta_2 \otimes 1_v)$	$(\eta_1 \otimes 1_v, \lambda_v)$	$(\eta_1 \otimes 1_v, \rho_u)$

Case 1: $(\alpha_{u,s,t}, 1_{u \otimes s} \otimes \eta_2)$
First consider $\mu = \alpha_{u,s,t} : u \otimes (s \otimes t) \to (u \otimes s) \otimes t$ and $\iota = 1_{u \otimes s} \otimes \eta_2$ with

$\eta_2 : t \to t'$ either a forward λ or ρ arrow. We select the forward unitor arrow $1_{u_A} \otimes (1_{s_A} \otimes (\eta_2)_A)$ and the forward α-arrow α_{u_A, s_A, t'_A} to obtain the diagram

$$\begin{tikzcd}
& u_A \otimes (s_A \otimes t_A) \arrow[dl, "\alpha_{u,s,t}"'] \arrow[dr, "1_{u_A} \otimes (1_{s_A} \otimes (\eta_2)_A)"] & \\
(u_A \otimes s_A) \otimes t_A \arrow[dr, "(1_{u_A} \otimes 1_{s_A}) \otimes (\eta_2)_A"'] & & u_A \otimes (s_A \otimes t'_A) \arrow[dl, "\alpha_{u_A, s_A, t'_A}"] \\
& (u_A \otimes s_A) \otimes t'_A &
\end{tikzcd}$$

which commutes by naturality of α.
Case 2: $(\alpha_{u,s,t}, \eta_2 \otimes 1_t)$.
In this case, $\mu = \alpha_{u,s,t} : u \otimes (s \otimes t) \to (u \otimes s) \otimes t$, while $\iota = \eta_2 \otimes 1_t$. Hence, η_2 must act on $(u \otimes s)$. With that said, η_2 must be of the form

$$\lambda_s \qquad \rho_u \qquad \tau \otimes 1_s \qquad 1_u \otimes \sigma$$

with $\tau : u \to u'$ and $\sigma : s \to s'$ either forward λ or ρ arrows. Thus we check each of these cases are satisfied.
Case 2.1: $\eta_2 = \lambda_{s_A}$
In this case, $u = I$. We can construct a triangular diagram by appending $\lambda_{s_A \otimes t_A} : I \otimes (s_A \otimes t_A) \to s_A \otimes t_A$ as below.

$$\begin{tikzcd}
& I \otimes (s_A \otimes t_A) \arrow[dl, "\alpha_{I, s_A, t_A}"'] \arrow[dr, "\lambda_{s_A \otimes t_A}"] & \\
(I \otimes s_A) \otimes t_A \arrow[rr, "\lambda_{s_A} \otimes 1_{t_A}"'] & & s_A \otimes t_A
\end{tikzcd}$$

which commutes in $\mathcal{M}$ by Proposition 2.1.2.
Case 2.2: $\eta_2 = \rho_u$
In this case, $s_A = I$. We can append the morphism $1_{u_A} \otimes \lambda_{t_A} : u_A \otimes (I \otimes t_A) \to u_A \otimes t_A$ to create a triangular diagram as below.

$$\begin{tikzcd}
& u_A \otimes (I \otimes t_A) \arrow[dl, "\alpha_{u_A, I, t_A}"'] \arrow[dr, "1_{u_A} \otimes \lambda_{t_A}"] & \\
(u_A \otimes I) \otimes t_A \arrow[rr, "\rho_{u_A} \otimes 1_{t_A}"'] & & u_A \otimes t_A
\end{tikzcd}$$

The above diagram is guaranteed to commute by unitor-axiom (Diagram 2.2) in any monoidal category $\mathcal{M}$.

Case 2.3: $\eta_2 = \tau \otimes 1_s$

In this case, $\eta_2 = \tau \otimes 1_s$ with τ a forward λ or ρ-arrow. We can first apply the forward arrow $\tau \otimes (1_{s_A} \otimes 1_{t_A})$ followed by $\alpha_{u'_A, s_A, t_A}$ to obtain the diagram

$$
\begin{tikzcd}
 & u_A \otimes (s_A \otimes t_A) \arrow[dl, "\alpha_{u_A,s_A,t_A}"'] \arrow[dr, "\tau\otimes(1_{s_A}\otimes 1_{t_A})"] & \\
(u_A \otimes s_A) \otimes t_A \arrow[dr, "(\tau\otimes 1_{s_A})\otimes 1_{t_A}"'] & & u'_A \otimes (s_A \otimes t_A) \arrow[dl, "\alpha_{u'_A,s_A,t_A}"] \\
 & (u'_A \otimes s_A) \otimes t_A &
\end{tikzcd}
$$

which commutes by naturality of α.

Case 2.4: $\eta_2 = 1_u \otimes \sigma$. This case is nearly identical to the previous, creating a desired diagram which commutes by naturality of α.

This proves all of our cases for when $\mu = \alpha_{u_A, s_A, t_A}$ and $\iota = (\eta_2)_A \otimes 1_{t_A}$, and so we move onto our other cases.

Case 3: $(\alpha_{u,s,t}, \lambda_t)$

This case cannot happen, since we cannot apply $\lambda : x_0 \otimes t \to x_0$ after $\alpha_{u,s,t} : u \otimes (s \otimes t) \to (u \otimes s) \otimes t$ as $u \otimes s \neq x_0$ for any binary words u, s.

Case 4: $(\alpha_{u,s,t}, \rho_{u \otimes s})$

In this case, we'll have that $\mu = \alpha_{u_A, s_A, t_A}$ and $\iota = \rho_{u_A \otimes s_A}$. This implies that $t_A = I$. We can then append the forward ρ-arrow $1_{u_A} \otimes \rho_{s_A}$ to obtain the diagram

$$
\begin{tikzcd}
 & u_A \otimes (s_A \otimes I) \arrow[dl, "\alpha_{u_A,s_A,I}"'] \arrow[dr, "1_{u_A}\otimes\rho_{s_A}"] & \\
(u_A \otimes s_A) \otimes I \arrow[rr, "\rho_{u_A\otimes s_A}"'] & & u_A \otimes s_A
\end{tikzcd}
$$

which we know commutes due to Proposition 2.1.2.

Case 5: $(1_u \otimes \eta_1, 1_u \otimes \eta_2)$. In this case $\mu = 1_{u_A} \otimes (\eta_1)_A$ and $\iota = 1_{u_A} \otimes (\eta_2)_A$ with η_1 a forward α-arrow and η_2 either a forward λ or ρ-arrow. We can prove this case by induction.

Suppose the statement is true for word of length less than n, and let $w = u \otimes v$ be a binary word of length n. Then we have the diagram on the left

$$\begin{tikzcd}
& u_A \otimes v_A \arrow[dl, "1_{u_A}\otimes(\eta_1)_A"'] \\
u_A \otimes v'_A \arrow[dr, "1_u\otimes(\eta_2)_A"'] & \\
& u_A \otimes v''_A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& v_A \arrow[dl, "(\eta_1)_A"'] \\
v'_A \arrow[dr, "(\eta_2)_A"'] & \\
& v''_A
\end{tikzcd}$$

which is the image of the diagram on the right under the functor $u_A \otimes (-)$. By induction, there exists either a binary word z, and a forward λ or ρ arrow $\eta' : v_A \to z$ and a forward α-arrow $\eta'' : z \to v''_A$ such that the diagram below commutes in $\mathcal{M}$.

$$\begin{tikzcd}
& v_A \arrow[dl, "(\eta_1)_A"'] \arrow[dr, "(\eta')_A"] & \\
v'_A \arrow[dr, "(\eta_2)_A"'] & & z \arrow[dl, "(\eta'')_A"] \\
& v''_A &
\end{tikzcd}$$

We can then take the image of this under the functor $u_A \otimes (-)$ to obtain the commutative diagram below.

$$\begin{tikzcd}
& u_A \otimes v_A \arrow[dl, "1_{u_A}\otimes(\eta_1)_A"'] \arrow[dr, "1_{u_A}\otimes(\eta')_A"] & \\
u_A \otimes v'_A \arrow[dr, "1_{u_A}\otimes(\eta_2)_A"'] & & u_A \otimes z \arrow[dl, "1_{u_A}\otimes(\eta'')_A"] \\
& u_A \otimes v''_A &
\end{tikzcd}$$

As $1_{u_A} \otimes (\eta')_A$ is a forward λ or ρ arrow since $(\eta')_A$ is, and since $1_{u_A} \otimes (\eta'')_A$ is a forward α-arrow since $(\eta'')_A$ is, we have that the case must be true for all words by induction.

Case 6: $(1_u \otimes \eta_1, \eta_2 \otimes 1_{v'})$
In this case, $\mu = 1_{u_A} \otimes (\eta_1)_A$ with $\eta_1 : v \to v'$ a forward α-arrow, and $\iota = (\eta_2)_A \otimes 1_{v'}$ with $\eta_2 : u \to u'$ either a forward λ or ρ arrow. We can use the forward λ or ρ arrow $(\eta_2)_A \otimes 1_{v_A}$ followed by the α-arrow $1_{u'_A} \otimes (\eta_1)_A$ to obtain the diagram below.

$$\begin{tikzcd}
 & u_A \otimes v_A \arrow[dl, "1_{u_A}\otimes(\eta_1)_A"'] \arrow[dr, "(\eta_2)_A\otimes 1_{v_A}"] & \\
u_A \otimes v'_A \arrow[dr, "(\eta_2)_A\otimes 1_{v'_A}"'] & & u' \otimes v \arrow[dl, "1_{u'_A}\otimes(\eta_1)_A"] \\
 & u'_A \otimes v'_A &
\end{tikzcd}$$

The above diagram commutes by functoriality of $\otimes$, completing this case.
Case 7: $(1_u \otimes \eta_1, \lambda_{v'})$
In this case we'll have $\mu = 1_u \otimes \eta_1$ with η_1 a forward α-arrow and $\iota = \lambda_{v'}$. This then implies that $u = I$. We can then append the λ-arrow λ_{v_A} followed by the α-arrow $(\eta_1)_A : v_A \to v'_A$ to obtain the diagram

$$\begin{tikzcd}
 & I \otimes v_A \arrow[dl, "1_I\otimes(\eta_1)_A"'] \arrow[dr, "\lambda_{v_A}"] & \\
I \otimes v'_A \arrow[dr, "\lambda_{v'_A}"'] & & v_A \arrow[dl, "(\eta_1)_A"] \\
 & v'_A &
\end{tikzcd}$$

which commutes by naturality of λ.
Case 8: $(1_u \otimes \eta_1, \rho_u)$
This case cannot happen, since to apply ρ_u after $1_u \otimes \eta_1$ implies that the codomain of η_1 is x_0, which is not possible if η_1 is an α-morphism.
Case 9: $(\eta_1 \otimes 1_v, 1_u \otimes \eta_2)$
Equivalent to **Case 5**.
Case 10: $(\eta_1 \otimes 1_v, \eta_2 \otimes 1_v)$
Equivalent to **Case 6**.
Case 11:$(\eta_1 \otimes 1_v, \lambda_v)$
This case cannot happen, since to apply λ_v after $\eta_1 \otimes 1_v$ implies that the

codomain of η_1 is x_0, which is not possible for an α-arrow.
Case 12: $(\eta_1 \otimes 1_v, \rho_u)$
In this case, we have that $\mu = (\eta_1)_A \otimes 1_{v_A}$ and $\eta_2 = \rho_{u_A}$. This implies that $v_A = I$. We can then append the forward ρ arrow ρ_{u_A} followed by the forward α-arrow $(\eta_1)_A$ to the diagram to obtain

$$\begin{tikzcd}
& u_A \otimes I \arrow[dl, "(\eta_1)_A \otimes 1_{v_A}"'] \arrow[dr, "\rho_{u_A}"] & \\
u'_A \otimes I \arrow[dr, "rho_{u'_A}"'] & & u_A \arrow[dl, "(\eta_1)_A"] \\
& u'_A &
\end{tikzcd}$$

which commutes by naturality of ρ.

This proves all the cases, which completes the proof.

■

Thus we have proven the Associator-Unitor Swap. Our final task is to prove the Unitor-Chain Equivalence. To do so, it suffices to prove the following lemma.

Lemma 3.7.2. (Unitor Diamond Lemma.) Let w be a binary word, and μ_1, μ_2 a pair of forward unitor arrows as below.

$$\begin{tikzcd}
& w \arrow[dl, "\mu_1"'] \arrow[dr, "\mu_2"] & \\
w_1 & & w_2
\end{tikzcd}$$

There there exists a binary word z and a pair of forward unitor arrows $\eta_1 : w_1 \to z$, $\eta_2 : w_2 \to z$ such that for any monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$, the diagram below is commutative in $\mathcal{M}$.

$$\begin{tikzcd} & (w)_A \arrow[dl, "(\mu_1)_A"'] \arrow[dr, "(\mu_2)_A"] & \\ (w_1)_A \arrow[dr, "(\eta_1)_A"'] & & (w_2)_A \arrow[dl, "(\eta_2)_A"] \\ & (z)_A & \end{tikzcd}$$

As before, we color the arrows which we are asserting to exist Green.

Proof: To prove this, we do a case-by-case basis again. In general, we will write $w = u \otimes v$, and if $\mathcal{L}(v) > 1$, we write $w = u \otimes (s \otimes t)$.

Now since μ_1, μ_2 are forward unitor arrows, μ_1 could be of the form

$$1_u \otimes \eta_1 \qquad \eta_1 \otimes 1_v \qquad \lambda_v \qquad \rho_u$$

while μ_2 could be of the form

$$1_u \otimes \eta_2 \qquad \eta_2 \otimes 1_v \qquad \lambda_v \qquad \rho_u$$

with η_1, η_2 both forward unitor arrows. Therefore, our possible cases are as follows. We could have $\mu_1 = \mu_2$. Or, we could have any of the cases below. The paired-coloring indicates logically equivalent cases due to the symmetry of our problem.

(β_1, β_2)	$1_u \otimes \eta_2$	$\eta_2 \otimes 1_v$	λ_v	ρ_u
$1_u \otimes \eta_1$	$(1_u \otimes \eta_1, 1_u \otimes \eta_2)$	$(1_u \otimes \eta_1, \eta_2 \otimes 1_v)$	$(1_u \otimes \eta_1, \lambda_v)$	$(1_u \otimes \eta_1, \rho_u)$
$\eta_1 \otimes 1_v$	$(\eta_1 \otimes 1_v, 1_u \otimes \eta_2)$	$(\eta_1 \otimes 1_v, \eta_2 \otimes 1_v)$	$(\eta_1 \otimes 1_v, \lambda_v)$	$(\eta_1 \otimes 1_v, \rho_u)$
λ_v	$(\lambda_v, 1_u \otimes \eta_2)$	$(\lambda_v, \eta_2 \otimes 1_v)$	(λ_v, λ_v)	(λ_v, ρ_u)
ρ_u	$(\rho_u, 1_u \otimes \eta_2)$	$(\rho_u, \eta_2 \otimes 1_v)$	(ρ_u, λ_v)	(ρ_u, ρ_u)

Since we've already implemented this case-by-case proof strategy several times, we will point out the cases which we've seen before, and take care of the cases that are new.

Case 1: $(1_u \otimes \eta_1, 1_u \otimes \eta_2)$ This case can be proven by induction on total length $\mathcal{L}(w) + \mathcal{E}(w)$, using a similar argument as in Case 3 of Lemma 3.4.6.

Case 2: $(1_u \otimes \eta_1, \eta_2 \otimes 1_v)$ This case can be proven via functoriality, in a similar manner as Case 2 of Lemma 3.4.6.

Case 3: $(1_u \otimes \eta_1, \lambda_v)$.
With $\mu_1 = 1_u \otimes \eta_1$ and $\mu_2 = \lambda_v$, denote $\eta_1 : v \to v'$. In this case, we can use the morphisms $\lambda_{(v')_A}$ and η_1 to obtain the diagram

$$\begin{tikzcd} & I \otimes (v)_A \arrow[dl, "1_I \otimes \eta_1"'] \arrow[dr, "\lambda_{(v)_A}"] & \\ I \otimes (v')_A \arrow[dr, "\lambda_{(v')_A}"'] & & v \arrow[dl, "\eta_1"] \\ & v' & \end{tikzcd}$$

which commutes by naturality of λ.
Case 5: $(1_u \otimes \eta_1, \rho_u)$.
With $\mu_1 = 1_u \otimes \eta_1, \mu_2 = \rho_u$, note that the only choice for η_1 is $\eta_1 = 1_{x_0}$. However, there is no unitor arrow with domain x_0, so this does not result in a valid case for us to consider.
Case 6: $(\eta_1 \otimes 1_v, \lambda_v)$.
With $\mu_1 = \eta_1 \otimes 1_v, \mu_2 = \lambda_v$, note that the only choice for η_1 is again 1_{x_0}. Once again, there is no unitor arrow with domain x_0, so this is also not a valid case that we need to consider.
Case 7: $(\eta_1 \otimes 1_v, \rho_u)$.
With $\mu_1 = \eta_1 \otimes 1_v, \mu_2 = \rho_u$, we can use the morphisms $\rho_{(u')_A}$ and η_1 to obtain

$$\begin{tikzcd} & (u)_A \otimes I \arrow[dl, "\eta_1 \otimes 1_I"'] \arrow[dr, "\rho_{(u)_A}"] & \\ (u')_A \otimes I \arrow[dr, "\rho_{(u')_A}"'] & & (u)_A \arrow[dl, "\eta_1"] \\ & (u')_A & \end{tikzcd}$$

which commutes by naturality of ρ.

Case 8: (λ_v, λ_v). In this case, we see that $\mu_1 = \mu_2$, so that the statement is trivially satisfied in this case.

With all cases verified, we see that the statement must be true for all binary words, as desired.

■

We now show how this proves the Unitor-Chain Equivalence, which we restate for the readers convenience.

Proposition 3.7.3 (Unitor-Chain Equivalence)**.** Let w be a binary word with nonzero length and with $\mathcal{E}(w) = k$. Suppose $\mu_1, \ldots, \mu_k$ and $\eta_1, \ldots, \eta_k$ are forward unitors and that:

$$\mu_k \circ \cdots \circ \mu_1,\ \eta_k \circ \cdots \circ \eta_1 : w \to \overline{w}$$

Then $(\mu_k)_A \circ \cdots \circ (\mu_1)_A = (\eta_k)_A \circ \cdots \circ (\eta_1)_A$ in $\mathcal{M}$.

Proof: We prove this by induction on $\mathcal{E}(w)$. Suppose the result is true for binary words v with $\mathcal{E}(v) < \mathcal{E}(w)$, and consider two composable chains of forward unitors $\mu_1, \ldots, \mu_k, \eta_1, \ldots, \eta_k$ as described above. We seek to show that the diagram

$$\begin{tikzcd} & (w)_A \arrow[dl, "(\mu_1)_A"'] \arrow[dr, "(\eta_1)_A"] & \\ (u_1)_A \arrow[dr, "(\mu_k)_A \circ \cdots \circ (\mu_2)_A"'] & & (v_1)_A \arrow[dl, "(\eta_k)_A \circ \cdots \circ (\eta_2)_A"] \\ & (\overline{w})_A & \end{tikzcd}$$

is commutative in $\mathcal{M}$. By the Unitor Diamond Lemma, there exists a binary word z and two forward unitors $\iota_1 : u \to z$ and $\iota_2 : v \to z$ such that

$$\begin{tikzcd} & (w)_A \arrow[dl, "(\mu_1)_A"'] \arrow[dr, "(\eta_1)_A"] & \\ (u_1)_A \arrow[dr, "(\iota_1)_A"'] & & (v_1)_A \arrow[dl, "(\iota_2)_A"] \\ & (z)_A & \end{tikzcd}$$

is commutative in $\mathcal{M}$. Now, by Lemma 3.5.9, we have that $\overline{z} = \overline{w}$. By Lemma 3.5.8, $\mathcal{E}(z) = k - 2$. Hence, by Proposition 3.5.11, there exists a chain of forward unitors $\nu_1, \ldots, \nu_{k-2}$ such that $\nu_{k-2} \circ \cdots \circ \nu_1 : z \to \overline{w}$. Our situation is displayed below. For clarity, we suppress $\nu_{k-2} \circ \cdots \circ \nu_1 : z \to \overline{w}$ in the diagram below.

$$
\begin{tikzcd}
 & (w)_A \arrow[dl, "(\mu_1)_A"'] \arrow[dr, "(\eta_1)_A"] & \\
(u_1)_A \arrow[dr, "(\iota_1)_A"] \arrow[ddr, "(\mu_k)_A \circ \cdots \circ (\mu_2)_A"'] & & (v_1)_A \arrow[dl, "(\iota_2)_A"'] \arrow[ddl, "(\eta_k)_A \circ \cdots \circ (\eta_2)_A"] \\
 & (z)_A \arrow[d] & \\
 & (\overline{w})_A &
\end{tikzcd}
$$

By Lemma 3.5.8, we know that $\mathcal{E}(u_1), \mathcal{E}(v_1) < \mathcal{E}(w)$. Therefore, we may apply our induction hypothesis to conclude that the lower left and lower right triangles must commute. As the original upper square commutes by the Unitor Diamond Lemma, this implies that

$$(\mu_k)_A \circ \cdots \circ (\mu_1)_A = (\eta_k)_A \circ \cdots \circ (\eta_2)_A$$

as desired.

∎

At this point, we have formally filled in all of the potential gaps in the proof of Theorem 3.6.2. We have completed the hard work required to prove Mac Lane's Coherence Theorem. We will use the next section to see how our previous results immediately apply our desired coherence result.

3.8 Step Seven: Proving the Main Theorem

At this point we have proven coherence in associators and unitors, but only when considering iterated monoidal products of a single object. We have not yet achieved our desired result, which should say something about more general monoidal products with different objects in the expression. However, our previous work quickly implies our desired theorem. We first introduce a definition and perform a clever trick.

In what follows, we let $\mathbf{1}$ denote the terminal category whose sole object is denoted $\bullet$.

Definition 3.8.1. Let $(\mathcal{M}, \otimes, I)$ be a monoidal category. Define the **iterated functor category**[1] **of** $\mathcal{M}$, denoted as $\mathbf{It}(\mathcal{M})$, to be the category where:

Objects. Functors $F : \mathcal{M}^n \to \mathcal{M}$ for all $n = 0, 1, 2, \dots$ When $n = 0$, we let $\mathcal{M}^0 = \mathbf{1}$.

Morphisms. Natural transformations $\eta : F \to G$ between such functors.

We will give this category a monoidal structure. Towards that goal, we introduce the following bifunctor

$$\odot : \mathbf{It}(\mathcal{M}) \times \mathbf{It}(\mathcal{M}) \to \mathbf{It}(\mathcal{M})$$

whose behavior we describe on objects and morphisms as follows.

On objects. For two functors $F : \mathcal{M}^n \to \mathcal{M}$, $G : \mathcal{M}^m \to \mathcal{M}$, we define the functor $F \odot G : \mathcal{M}^{n+m} \to \mathcal{M}$ pointwise as

$$(F \odot G)(A_1, \dots, A_{n+m}) = F(A_1, \dots, A_n) \otimes G(A_{n+1}, \dots, A_{n+m})$$

where $\otimes$ is the monoidal product of $\mathcal{M}$.

On morphisms. Let $F_1, G_1 : \mathcal{M}^n \to \mathcal{M}$ and $F_2, G_2 : \mathcal{M}^m \to \mathcal{M}$. Given natural transformations

$$\eta : F_1 \to G_1 \qquad \mu : F_2 \to G_2$$

we define the natural transformation $\eta \odot \mu : F_1 \odot G_1 \to F_2 \odot G_2$ pointwise as

$$(\eta \odot \mu)_{(A_1, \dots, A_{n+m})} = (\eta)_{(A_1, \dots, A_n)} \otimes (\mu)_{(A_{n+1}, \dots, A_{n+m})}$$

[1] The notation of this category is due to Mac Lane, but he did not supply a name for this category. So I made one up. Today, this construction is known as an endomorphism operad.

The above bifunctor is what allows us to regard $\mathbf{It}(\mathcal{M})$ as a monoidal category. This is more precisely stated in the following lemma.

Lemma 3.8.2. Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. Then

$$(\mathbf{It}(\mathcal{M}), \odot, c, \boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\rho})$$

is a monoidal category where

- The monoidal product is the bifunctor $\odot : \mathbf{It}(\mathcal{M}) \times \mathbf{It}(\mathcal{M}) \to \mathbf{It}(\mathcal{M})$
- The identity object is the functor $c : \mathbf{1} \to \mathcal{M}$, where $c(\bullet) = I$
- For functors $F_j : \mathcal{M}^{i_j} \to \mathcal{M}$, $j = 1, 2, 3$, the associator

  $$\boldsymbol{\alpha}_{F_1,F_2,F_3} : F_1 \odot (F_2 \odot F_3) \to (F_1 \odot F_2) \odot F_3$$

  is the natural transformation defined pointwise for each $(A_1, \dots, A_{i_1+i_2+i_3}) \in \mathcal{M}^{(i_1+i_2+i_3)}$ as

  $$(\boldsymbol{\alpha}_{F_1,F_2,F_3})_{(A_1,\dots,A_{i_1+i_2+i_3})} = \alpha_{(F(A_1,\dots,A_{i_1}),F(A_{i_1+1},\dots A_{i_1+i_2}),F(A_{i_1+i_2+1},\dots,A_{i_1+i_2+i_3}))}$$

- For a functor $F : \mathcal{M}^n \to \mathcal{M}$, the left unitor $\boldsymbol{\lambda} : c \odot F \to F$ is the natural transformation defined pointwise for $(\bullet, A_1, \dots, A_n) \in \mathbf{1} \times \mathcal{M}^n$ as

  $$(\boldsymbol{\lambda}_F)_{(\bullet,A_1,\dots,A_n)} = \lambda_{F(A_1,\dots,A_n)}$$

  while the right unitor $\boldsymbol{\rho} : F \odot c \to F$ is the natural transformation defined similarly as

  $$(\boldsymbol{\rho}_F)_{(A_1,\dots,A_n,\bullet)} = \rho_{F(A_1,\dots,A_n)}$$

It is simple to check that these satisfy the axioms of a monoidal category. We now reach the final theorem.

Theorem 3.8.3 (Coherence Theorem for Monoidal Categories.)**.** For every monoidal category $\mathcal{M}$, there exists a unique, strict monoidal functor

$$\Phi_{\text{id}} : \mathcal{W} \to \mathbf{It}(\mathcal{M})$$

where $\Phi_{\text{id}}(x_1) = \text{id} : \mathcal{M} \to \mathcal{M}$.

Proof: As $(\mathbf{It}(\mathcal{M}), \odot, c)$ is a monoidal category by Lemma 3.8.2, the theorem follows by a simple application of Theorem 3.6.2 to this monoidal category.

■

A reader might be wondering: How does the above theorem grant us coherence? Let us first investigate the behavior of this functor.

Under the functor, the morphism in $\mathcal{W}$

$$x_1 \otimes (x_1 \otimes x_1) \xrightarrow{\alpha_{x_1,x_1,x_1}} (x_1 \otimes x_1) \otimes x_1$$

is mapped by Φ_{id} to the natural transformation between the functors in $\mathbf{It}(\mathcal{M})$

$$\mathrm{id} \odot (\mathrm{id} \odot \mathrm{id}) \xrightarrow{\alpha_{\mathrm{id},\mathrm{id},\mathrm{id}}} (\mathrm{id} \odot \mathrm{id}) \odot \mathrm{id}$$

and, as functors from $\mathcal{M}^3 \to \mathcal{M}$, we may substitute any A, B, C to obtain a natural isomorphism

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$$

in $\mathcal{M}$. Next, we know that functors preserve diagrams. Therefore, our commutative pentagon diagram in $\mathcal{W}$

$$\begin{tikzcd}
x_1 \otimes (x_1 \otimes (x_1 \otimes x_1)) \arrow[r] \arrow[d] & (x_1 \otimes x_1) \otimes (x_1 \otimes x_1) \arrow[r] & ((x_1 \otimes x_1) \otimes x_1) \otimes x_1 \\
x_1 \otimes ((x_1 \otimes x_1) \otimes x_1) \arrow[rr] & & (x_1 \otimes (x_1 \otimes x_1)) \otimes x_1 \arrow[u]
\end{tikzcd}$$

is mapped by Φ_{id} to a commutative diagram of natural transformations in $\mathbf{It}(\mathcal{M})$ between the functors below

$$\begin{tikzcd}
\mathrm{id} \odot (\mathrm{id} \odot (\mathrm{id} \odot \mathrm{id})) \arrow[r] \arrow[d] & (\mathrm{id} \odot \mathrm{id}) \odot (\mathrm{id} \odot \mathrm{id}) \arrow[r] & ((\mathrm{id} \odot \mathrm{id}) \odot \mathrm{id}) \odot \mathrm{id} \\
\mathrm{id} \odot ((\mathrm{id} \odot \mathrm{id}) \odot \mathrm{id}) \arrow[rr] & & (\mathrm{id} \odot (\mathrm{id} \odot) \mathrm{id}) \odot \mathrm{id} \arrow[u]
\end{tikzcd}$$

and as the above functors are of the form $\mathcal{M}^4 \to \mathcal{M}$, we may substitute any $A, B, C, D \in \mathcal{M}$ to obtain the commutative diagram

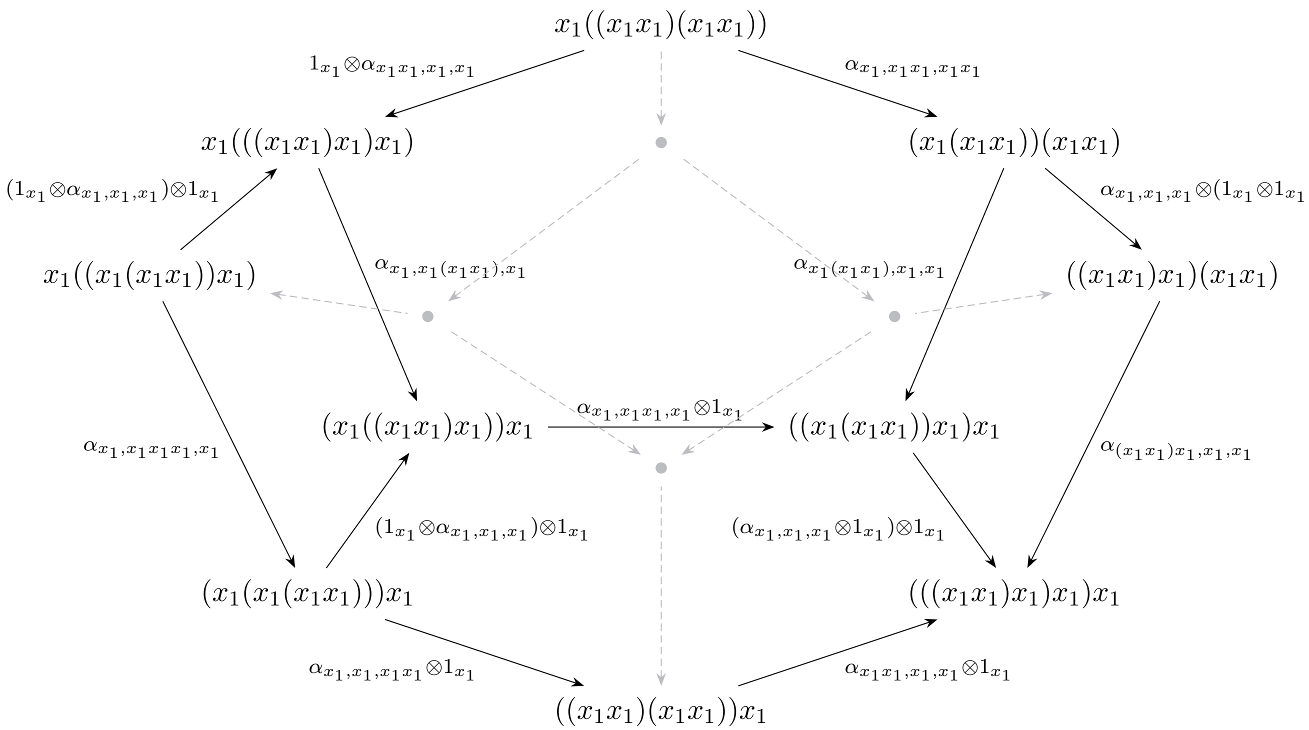
$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "\alpha_{A,B,C\otimes D}"] \arrow[d, "1_A \otimes \alpha_{B,C,D}"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha_{A\otimes B,C,D}"] & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha_{A,B\otimes C,D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha_{A,B,C} \otimes 1_D"']
\end{tikzcd}$$

in $\mathcal{M}$.

So far, our functor makes sense. Moreover, we already knew that the above pentagon commutes for all $A, B, C, D \in \mathcal{M}$. Thus, what about diagram **??**?

Again, functors preserve diagrams. Therefore, the commutative diagram in $\mathcal{W}$ (see next page) is mapped by Φ_{id} to the commutative diagram of natural transformations in $\mathbf{It}(\mathcal{M})$ between functors (see second page) and as functors from $\mathcal{M}^5 \to \mathcal{M}$, we may substitute any A, B, C, D, E to obtain the commutative diagram in $\mathcal{M}$ (on the third page).

Front. (Note that the product $\otimes$ in $\mathcal{W}$ has been suppressed).



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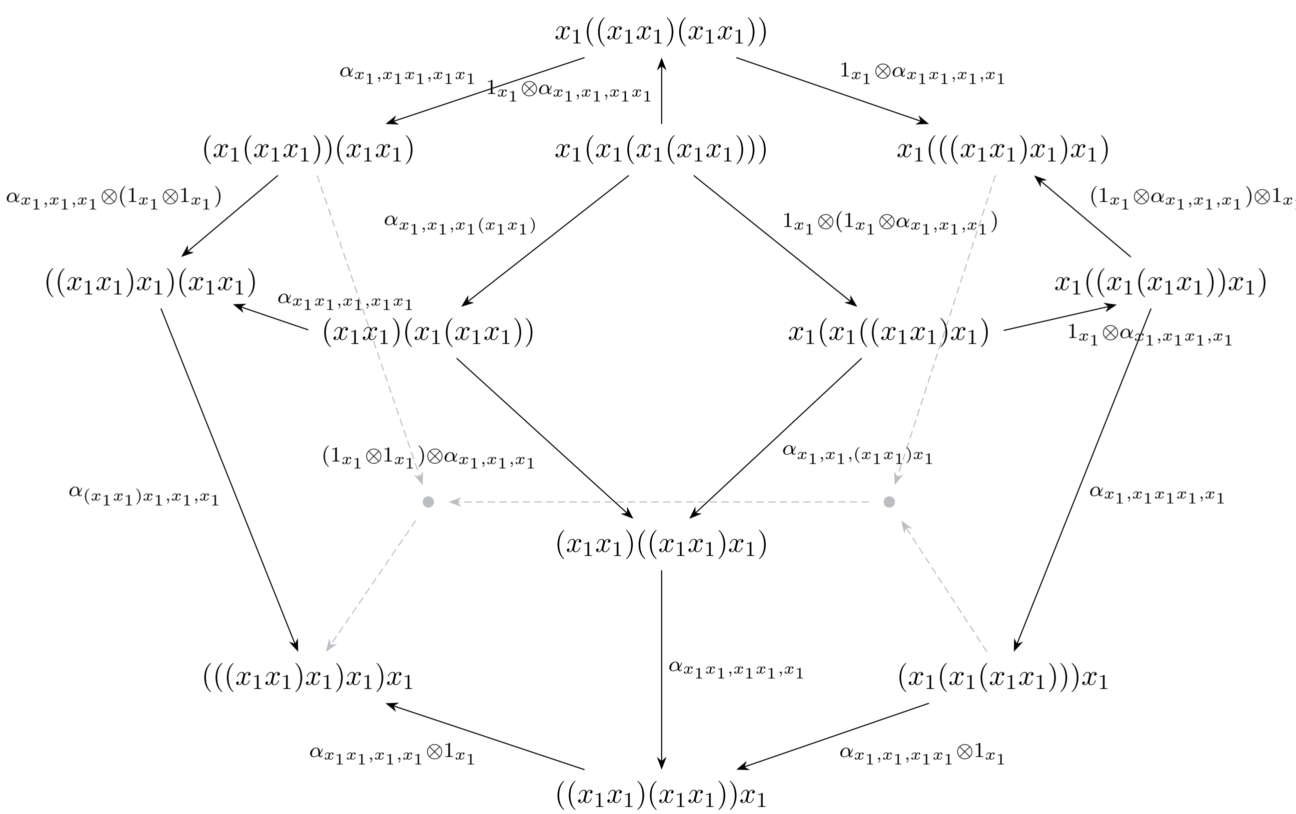
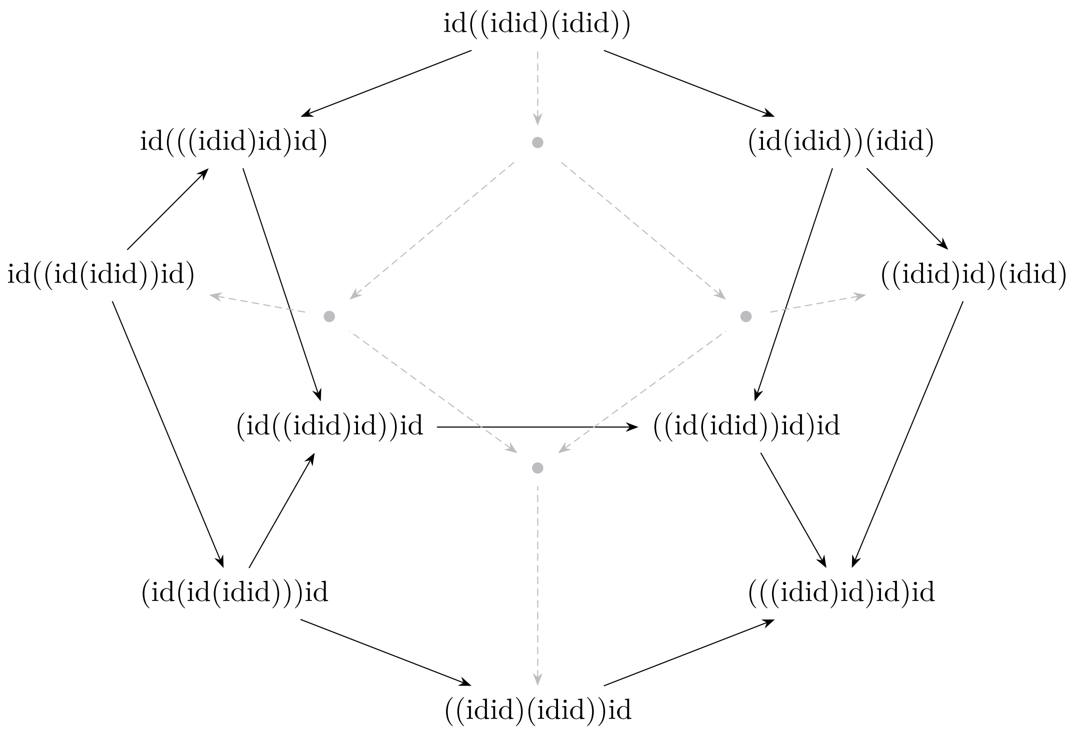


Front. (Note that the product $\odot$ and the associators in $\mathbf{It}(\mathcal{M})$ are suppressed.)



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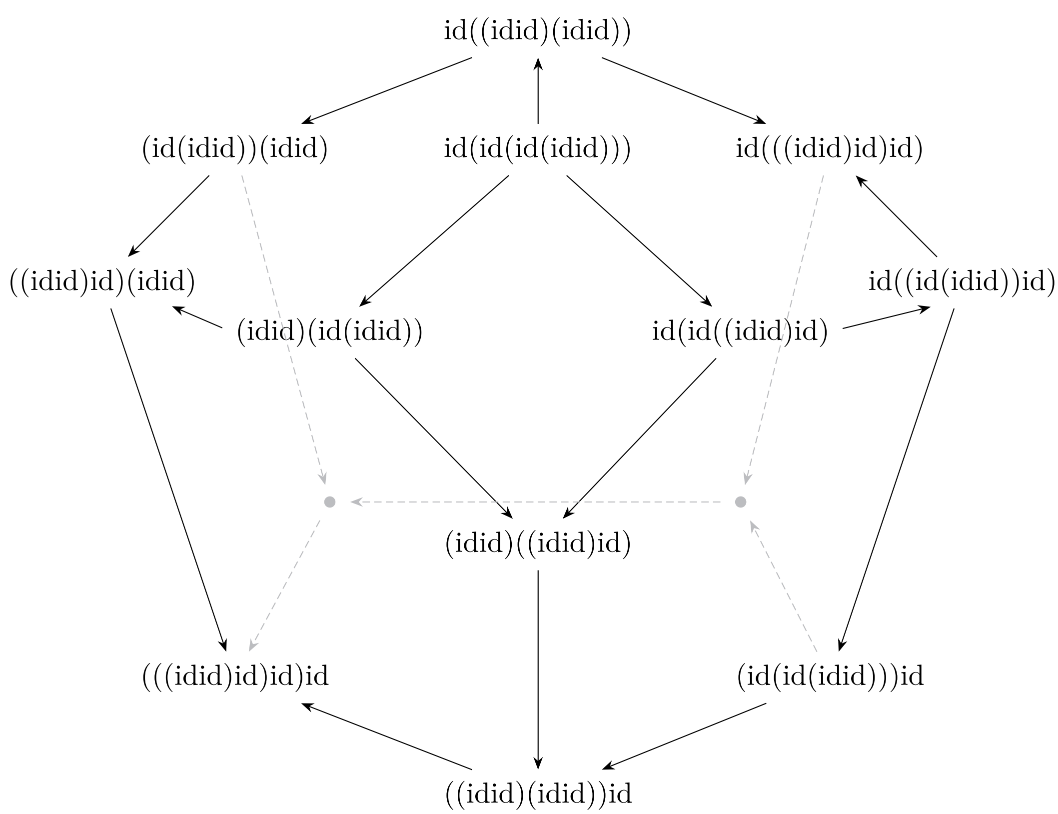
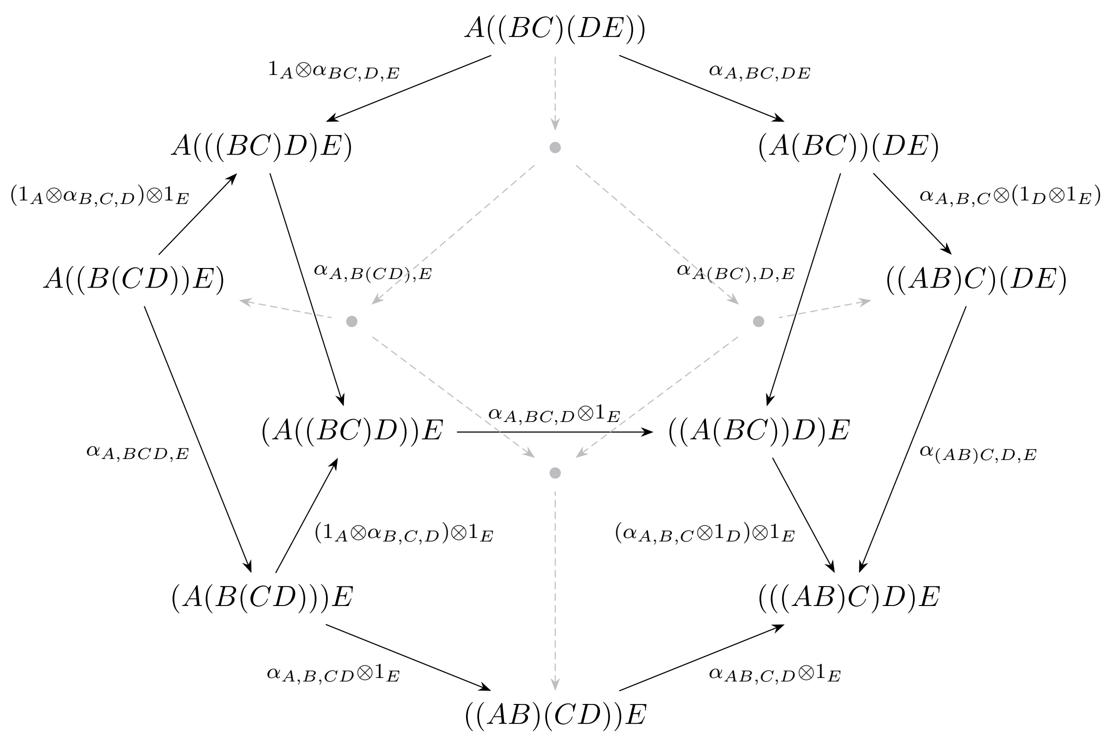


Front. (Note that the product $\otimes$ in $\mathcal{M}$ is suppressed.)



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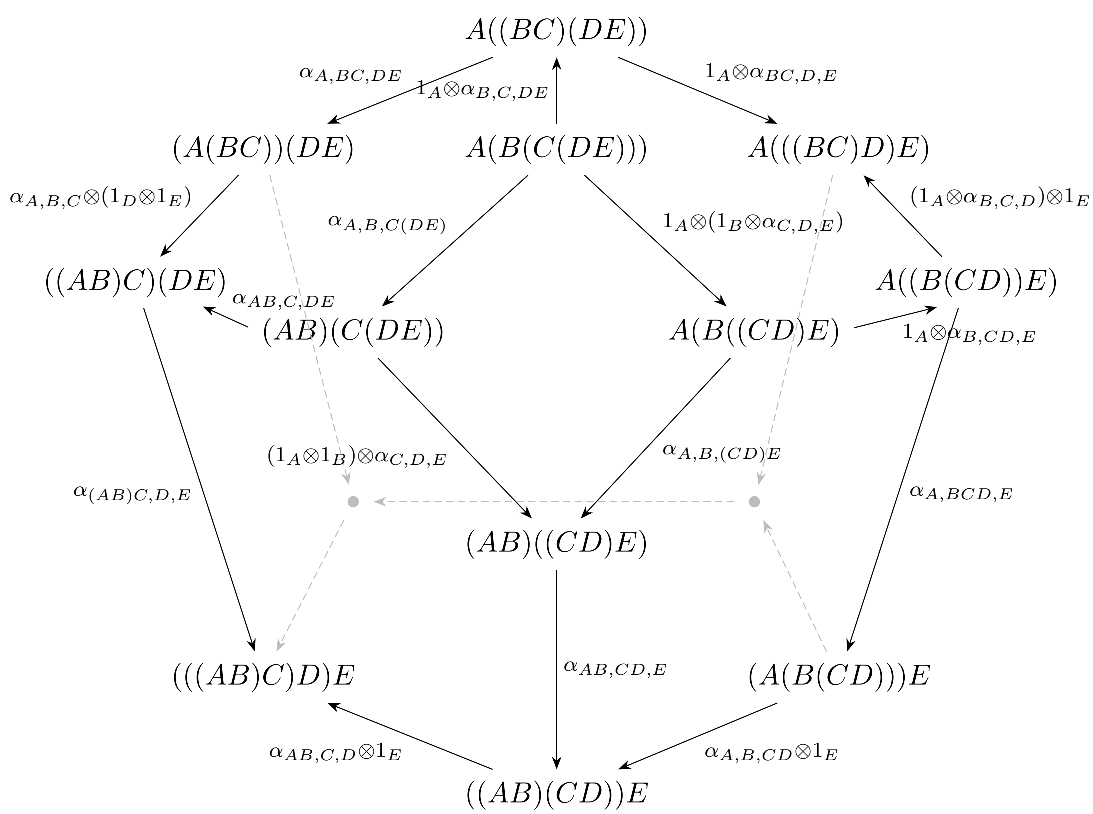

This process continues for every possible diagram in $\mathcal{W}$. Each diagram in $\mathcal{W}$ is mapped to a corresponding diagram in $\mathbf{It}(\mathcal{M})$ made up of identity functors, and with the identity functor, we are free to substitute whatever instance of $A, B, C, \dots \in M$ in it. The arrows between the identity functors are natural transformations which reduce to instances of α, ρ, λ in $\mathcal{M}$ upon substituting objects in the identity functor. What matters here is the functoriality of Φ_I. It guarantees that all the diagrams obtained as the image of Φ_{id} will commute.

This completes our work towards proving Mac Lane's Coherence Theorem.

3.9 Conclusion: What Does The Theorem *Actually* Say?

For myself, a significant point of confusion regarding Mac Lane's Coherence Theorem arose in understanding what it actually says. This is natural since the original writing of the theorem is somewhat vague and, with all due respect to Mac Lane, it is confusing in many places.

In writing this thesis, I think the reason for why it was written the way that it was is because (as I have seen) it takes many definitions and a lot of notation to clearly state what the theorem is saying. And in a pedagogical process, the limiting factors are *time* (from the author and reader) and *space* (in the author's body of text). If one does not have time or space, they will have to be less rigorous and hence use vague language (otherwise, they have to rush the audience). For example, in less formal presentations, mathematicians must be vague about their work since they cannot take the audience on a 5-year pure mathematics journey in order to bring them up to perfect speed. Thus, a limitation of time and space appropriately compels vague language.

We end this thesis by commenting on what the theorem does not say, as sometimes people walk away from Mac Lane's Coherence Theorem confused and think that it says:

- All diagrams built from instances of α, ρ, λ commute.

While the theorem does say a certain class of diagrams commute, it does not comment on all diagrams. Thus the issue with the above statement is that it includes diagrams which the theorem does not make comment on. The theorem only makes comment on diagrams which can by constructed via the functor in Theorem 3.8.3.

In fact, the above statement can get us in trouble. The canonical example is Isbell's counterexample, which I will not repeat since it is frequently mentioned enough in most discussions of monoidal categories.

What this theorem *does* say is instead more wordy:

- A binary word w of length n is actually a functor:

  $$\Phi_{\text{id}}(w) : \mathcal{M}^n \to \mathcal{M}$$

  For any other binary word v of length n, there is a unique morphism

  $$\psi_{v,w} : v \to w$$

in $\mathcal{W}$ (because it is a thin category). Moreover, $\psi_{v,w}$ corresponds to a *unique* natural transformation

$$\Phi_{\mathrm{id}}(\psi_{v,w}) : \Phi_{\mathrm{id}}(v) \to \Phi_{\mathrm{id}}(w)$$

between the functors corresponding to the binary words.

Now, by Corollary 3.5.13, $\psi_{v,w}$ is a composition of monoidal arrows in $\mathcal{W}$. Therefore, the components of the natural transformation $\Phi_{\mathrm{id}}(\psi_{v,w})$ are the familiar compositions made up of the associators, unitors, and their recursive instances in $\mathcal{M}$.

The uniqueness of this natural transformation is what guarantees coherence.

This certainly more wordy than some statement like "a certain class of diagrams commute", but it is correct.

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