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Abstract

Let S be the semigroup constructed from a one-sided shift of finite type. In this thesis, we will provide the construction of $H(S)$, the inverse hull of S, explore the properties of $H(S)$, and begin to characterize the structure of $H(S)$. We will also focus on a kind of one-sided shift of finite type, Markov shifts, and prove an invariant on isomorphic inverse hulls of Markov shifts.

Contents

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Chapter 1

Introduction

C*-algebras sit at the intersection of functional analysis, abstract algebra, and quantum physics, and thus research in this field is of much intellectual and practical interest. One can think of C*-algebras as Banach algebras with extra structure. Due to the complexity of this structure, researchers are extraordinarily interested in the classification of C*-algebras, where they derive C*-algebras from simpler objects, and then use these objects to determine if two C*-algebras are isomorphic. A class of C*-algebras that has been of special interest are the graph C*-algebras, which are built from directed graphs generated by partial isometries (Raeburn, 2005).

This thesis is concerned with the fact that the algebraic structure of these partial isometries is a semigroup, and specifically, the graph C*-algebra class is a subset of the C*-algebras that can be built from inverse hulls of one-sided shifts (Starling, 2016). By studying the structure of inverse hulls of one-sided shifts, specifically one-sided shifts of finite type, this thesis will allow for a better understanding of the structure of graph C*-algebras and contribute to the discourse of C*-algebra classification.

This thesis directly extends our research on the inverse hulls of Markov shifts, where Markov shifts are a specific kind of one-sided shift of finite type (Beaupré et al., 2021). In that paper we fully characterized the inverse hull of a Markov shift. In this thesis we first introduce the necessary background material in Chapter 2, then we prove properties of the inverse hull of a one-sided shift of finite type in Chapter 3, as well as use the characterization of the inverse hull of a Markov shift to determine an invariant of isomorphic inverse hulls of Markov shifts in Chapter 4. Lastly, in Chapter 5 we will begin a characterization of the inverse hull of a one-sided shift of finite type, as well as provide a discussion on future work.

Chapter 2

Background

2.1 Introduction to Languages

Within this thesis, we construct inverse semigroups out of *languages*, as defined in the field of symbolic dynamics. We define $\mathcal{A}$ to be a finite collection of symbols, called an *alphabet*. The elements of $\mathcal{A}$ are called *letters*. See D. Lind (1995) for standard literature on languages and symbolic dynamics.

Let a concatenation of elements in $\mathcal{A}$ be a *word*. For example, with the alphabet $\mathcal{A} = \{a, b, c\}$, $abcb$ is a word. A *subword* is a "section" of an existing word. Therefore, bc is a subword of $abcb$ (also $abcb$ is a subword of itself). The *length* of a word is the number of concatenated elements from $\mathcal{A}$ it contains. So $abcb$ is of length 4, which we notate as $|abcb| = 4$. We define $f(w)$ as the left most letter of w (the first letter of w), and $l(w)$ as the right most letter of w (the last letter of w). Thus, $f(abcb) = a$ and $l(abcb) = b$.

Remark 2.1.1. Note that two words w and s are equal in L if, an only if, $w = x_1 \ldots x_N$ and $s = y_1 \ldots y_M$, for $x_i, y_i \in \mathcal{A}$ where $N = M$ and $x_i = y_i$ for $1 \leq i \leq N$. Thus if $wv = su$, either $w = st$, $w = s$, or $wt = s$ for some word t.

We notate the set of all words of length $M \in \mathbb{N}$, with alphabet $\mathcal{A}$, as $\mathcal{A}^M$.

With some alphabet $\mathcal{A}$, let L be a collection of words with finite non-zero length. The set L is called a *language*.

Below are a couple examples of languages.

Example 2.1.2. Let L be the language associated with the alphabet

$$\mathcal{A} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

with the rule:

- For $x_i \in \mathcal{A}$, $w = x_1 \ldots x_N \in L$ if, and only if, $x_1 \neq 0$.

Fun note: this one way to describe the set of natural numbers.

Example 2.1.3. Let L be the language associated with the alphabet $\mathcal{A} = \{0, 1\}$ and the rule:

- $w \in L$ if, and only if, w does not contain the subword 11.

2.1.1 Languages associated with One-Sided Shifts

In the study of symbolic dynamics *shift spaces* are the fundamental building blocks.

Definition 2.1.4. The language L is a *language associated to a shift* if, and only if, for all $w \in L$,

1. every subword of w belongs to L
2. there are words $u, v \in L$ such that $uwv \in L$.

Definition 2.1.5. A *shift* is the set of all possible bi-infinite words,

$$\ldots x_{-2}x_{-1}x_0x_1x_2 \ldots$$

such that all finite subwords of these infinite words is an element of the language associated to the shift.

Within this thesis we will be working with a generalization of shifts, call *one-sided shifts*.

Definition 2.1.6. The language L is a *language associated to a one-sided shift* if, and only if, for all $w \in L$,

1. every subword of w belongs to L
2. there is a word $v \in L$ such that $wv \in L$.

note the difference between Definition 2.1.4 and Definition 2.1.6 is that there does not have to be a $u \in L$ such that $uw \in L$ for languages associated to one-sided shift. All languages associated to shifts are also trivially languages associated to one-sided shifts.

Definition 2.1.7. A *one-sided shift* is the set of all possible one-directional infinite words,

$$x_0 x_1 x_2 \ldots$$

such that all finite subwords of these infinite words is an element of the associated language.

If we consider Example 2.1.2 and Example 2.1.3, we could ask whether the described languages are languages associated to a one-sided shift.

In Example 2.1.2, the word $101 \in L$, however $01 \notin L$, as the left most letter in the word 01 is 0. Thus not all subwords of 101 are in L, which implies that The language described in Example 2.1.2 is not a language associated to a one-sided shift.

In Example 2.1.3, for all $w \in L$, a subword of w, h, will also not contain the subword 11, thus $h \in L$. if $w \in L$, $w0$ will also not contain the subword 11. Thus $w0 \in L$, which means that for all words $w \in L$ there does exist a $v \in L$, namely 0, such that $wv \in L$. Therefore, the language described in Example 2.1.3 is a languages associated to a one-sided shift.

2.1.2 Languages associated with One-Sided Shifts of Finite Type

For this thesis we will focus our attention on exploring the structure of languages associated with one-sided shifts of finite type. Shifts of finite type are often seen as the "simplest" shift spaces.

A language, L, can be uniquely described by its alphabet, $\mathcal{A}$, and its set of forbidden words $\mathcal{F}$. A set of *forbidden words*, $\mathcal{F}$, is a set of finite words, such that an infinite word, s, is not in the corresponding shift if, and only if, s contains a subword in $\mathcal{F}$.

Lemma 2.1.8. *Let $\mathcal{F}$ be a set of forbidden words for some one-sided shift, S, with language L. Then $w \notin L$ if, and only if, w contains a subword $u \in \mathcal{F}$ or there exists an $N \in \mathbb{N}$ such that for all $t \in \mathcal{A}^N$, there exists a subword v of wt such that $v \in \mathcal{F}$.*
(*assuming w is a finite concatenation of letters in $\mathcal{A}$*).

Proof. Let $\mathcal{F}$ be a set of forbidden words for some shift, S. Let L_1 be the language such that $w \notin L_1$ if, and only if, w contains a subword $u \in \mathcal{F}$ or there exists an $N \in \mathbb{N}$ such that for all $t \in \mathcal{A}^N$, there exists a subword v of wt such that $v \in \mathcal{F}$. Let L_2 be the language associated with S. Thus L_2 is the set of all finite subwords of $s \in S$. Our goal is to show that $L_1 = L_2$.

Let us assume for the sake of contradiction that there exists a $w \notin L_2$ such that no subword of w is in $\mathcal{F}$ and for all $N \in \mathbb{N}$ there exits a $t \in \mathcal{A}^N$, such that there is no subword of wt in $\mathcal{F}$. Given that $N \in \mathbb{N}$ was arbitrary, there must exist an $s \in S$ such that w is a subword of s, by the definition of $\mathcal{F}$. However this is a contradiction that $w \notin L_2$. Thus, $L_1 \subseteq L_2$.

If $w \notin L_1$ either a subword of w is in $\mathcal{F}$ (which would imply that w is not a subword of an element in S, which in turn would imply that $w \notin L_2$), or there exists an $N \in \mathbb{N}$ such that for all $t \in \mathcal{A}^N$, there exists a subword v of wt such that $v \in \mathcal{F}$. If that is the case, then w is not a subword of any $s \in S$, as the length of s is not finite. Therefore $w \notin L_2$, meaning $L_2 \subseteq L_1$.

Thus we have shown that $L_1 = L_2$. □

Note there is not a unique form of $\mathcal{F}$. For example if $\mathcal{A} = \{a, b, c\}$ and $\mathcal{F} = \{aa\}$, then we could also clam that $\mathcal{F} = \{aaa, aab, aac\}$.

Definition 2.1.9. A language L is a *language associated to a one-sided shift of finite type* if:

1. L satisfies the conditions of Definition 2.1.6 (L is a language associated to a one-sided shift)
2. there is a form of $\mathcal{F}$ that has finite cardinally.

A language, L, associated to a one-sided shift of finite type is *M-step* if there is a form of $\mathcal{F}$ such that all elements of $\mathcal{F}$ have length $M + 1$. Note that the corresponding shift of an M-step language is also called M-step. In practice this means if L is M-step, then for all $w \in \mathcal{A}^M$ where $sw, wt \in L$, $swt \in L$, as there is no $M + 1$ length subword of swt that is not a subword of sw or wt. This assertion can actually be proven as an if, and only if, statement.

Lemma 2.1.10. *Let L be a language associated to a one-sided shift. The language L is M-step if, and only if, for all $w \in \mathcal{A}^M$ where $sw, wt \in L$, $swt \in L$.*

Proof. As argued previously, if L is M-step and $w \in \mathcal{A}^M$ where $sw, wt \in L$ then $swt \in L$, as there is no $M + 1$ length subword of swt that is not a subword of sw or wt.

If there exits a language associated to a one-sided shift, L, such that for all $w \in \mathcal{A}^M$ where $sw, wt \in L$, $swt \in L$, we can consider the set $\mathcal{F} = \{w \in \mathcal{A}^{M+1} : w \notin L\}$. Using Lemma 2.1.8 we will show that $\mathcal{F}$ is a set of forbidden words associated to L. Let us assume, for the sake of contradiction, that there exists a $v \notin L$ such that v does not contain a subword of $\mathcal{F}$ and there does not exist an $N \in \mathbb{N}$ such that for all $t \in \mathcal{A}^N$ there exists a subword w of vt such that $w \in \mathcal{F}$.

If $|v| > M$ then $v = w_1 \dots w_n u$ where $|w_i| = M$ and $|u| \leq M$ If v does not contain a subword of $\mathcal{F}$ then $l(w_i)w_{i+1} \in L$ for all $1 \leq i \leq n$ which means that $w_l \dots w_n \in L$. Using similar logic we can show that $w_1 \dots w_n u \in L$, which is a contradiction.

Now let us consider if $|v| \leq M$. Given that there does not exist an $N \in \mathbb{N}$ such that for all $t \in \mathcal{A}^N$ there exists a subword of vt in $\mathcal{F}$, there exists an word $vt \in \mathcal{A}^{M+1}$ such that vt does not contain a subword in $\mathcal{F}$. Therefore $vt \notin \mathcal{F}$ which by definition of $\mathcal{F}$ means that $vt \in L$. However, as L is a language associated to a one-sided shift, $v \in L$, a contradiction. Therefore by Lemma 2.1.8, $\mathcal{F}$ is a set of forbidden words associated to L, meaning L is M-step.

We have proven that a language, L, associated to a one-sided shift is M-step if, and only if, for all $w \in \mathcal{A}^M$ where $sw, wt \in L$, $swt \in L$. □

If we reconsider Example 2.1.3, we can note that $\mathcal{F} = \{11\}$. Thus L is a language associated with a 1-step one-sided shift.

The following lemma shows that all languages associated to one-sided shifts of finite type are M-step, for some $M \in \mathbb{N}$.

Lemma 2.1.11. *Let L be a language associated to a one-sided shift of finite type. Then L is M-step for some $M \in \mathbb{N}$.*

Proof. Let L be a language associated to a one-sided shift of finite type. As $\mathcal{F}$ has finite cardinally, we can select $w \in \mathcal{F}$ such that w has the maximum length of any word in $\mathcal{F}$. Let us notate length of w as $M + 1 \in \mathbb{N}$. We can construct an new set $\mathcal{F}^*$, where $v \in \mathcal{F}^*$ if, and only if, $v = ct$ where $c \in \mathcal{F}$ and $t \in \mathcal{A}^{M-1-|c|}$. So $|v| = M + 1$, which means $\mathcal{F}^*$ is a finite set of words, all of length $M + 1$. We will show that the language associated with $\mathcal{F}$ is the same language associated with $\mathcal{F}^*$.

Let L^* be the language associated with $\mathcal{F}^*$. By Lemma 2.1.8, if $w \in L$, then w does not contain a subword $u \in \mathcal{F}$ and there does not exists an $N \in \mathbb{N}$ such that for all $t \in \mathcal{A}^N$, there exists a subword of wt in $\mathcal{F}$. Recall that by construction, for all $k \in \mathcal{F}^*$, there exists a $v \in \mathcal{F}$ such that v is a subword of k. Thus w can not have a subword in $\mathcal{F}^*$.

So, the only way for $w \notin L^*$ is if there exists a $K \in \mathbb{N}$ such that for all $t \in \mathcal{A}^K$, there exists a subword v of wt such that $v \in \mathcal{F}^*$. For the sake of contradiction, let us assume that there is such a $K \in \mathbb{N}$. Thus, as all words in $\mathcal{F}^*$ have a subword in $\mathcal{F}$, we know that for all words wt, where $|t| = K$, there exists a subword j in wt such that $j \in \mathcal{F}$. However, by Lemma 2.1.8, this implies that $w \notin L$, another contradiction. Therefore, we have shown $L \subseteq L^*$.

If $w \notin L$ then for all $t \in \mathcal{A}^{M+1-|w|}$, wt contains a subword $v \in \mathcal{F}$, as we defined the longest word in $\mathcal{F}$ to be of length $M+1$. Note, it is possible that $v = w$. Thus for each wt, this means that for all $u \in \mathcal{A}^{M+1-|v|}$, wtu contains a subword vj where $j \in \mathcal{A}^{M+1-|v|}$. By definition $vj \in \mathcal{F}^*$. So by Lemma 2.1.8 $w \notin L^*$, meaning $L^* \subseteq L$.

Therefore $L^* = L$. □

Sometimes, if M and $\mathcal{A}$ are small enough to be manageable, it is useful to describe L using a *transition array* which has dimensions $(|\mathcal{A}|, |\mathcal{A}|, |\mathcal{A}^{M-1}|)$, or, in other words, using a collection of $|\mathcal{A}^{M-1}|$, $|\mathcal{A}| \times |\mathcal{A}|$ matrices. Each $|\mathcal{A}|X|\mathcal{A}|$ matrix has it's rows and columns labeled with the letters of $\mathcal{A}$ and the matrices are labeled with a unique $M-1$ length word in the alphabet $\mathcal{A}$. Therefore, all words of length $M+1$ has a unique position within the transition array designate by the first letter of the word being the row label, the second letter being the column label, and the next $M-1$ letters being denoted by the label of the matrix. If said $M+1$ length word is in $\mathcal{F}$, that entry of the array will be 0, and if the word is in L, the entry will be 1.

Example 2.1.12. If we consider the language described by the set of forbidden words, $\mathcal{F} = \{bba, aba, aab\}$, the corresponding transition array is,

$$\overset{a}{\begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \end{array}} \quad \overset{b}{\begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array}}$$

Note, not all arrays of dimension $(|\mathcal{A}|, |\mathcal{A}|, |\mathcal{A}^{M-1}|)$ with entries in $\{1, 0\}$ correspond to a language associated to a one-sided shift of finite type. The following array is not a valid transition array, because it makes the assertion that $aab \in L$ but there is no letter that can be left appended to by ab, given that $aba, abb \in \mathcal{F}$. Thus aab is not infinitely right extendable.

$$\overset{a}{\begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \end{array}} \quad \overset{b}{\begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array}}$$

Before we consider the inverse hulls of one-sided shifts of finite type, let us first explore some useful definitions and properties of semigoups.

2.2 Introduction to Semigroups

A *semigroup* is a set, S, together with a binary, closed, and associative operation, $\times$. For simplicity of notation, we will refer to $(S, \times)$ as S for the rest of this thesis and for $s, t \in S$ we will notate $s \times t$ as st. For $s \in S$, we define the set $sS = \{st : t \in S\}$. See Grillet (1995) for standard literature on semigroup theory.

A semigroup, S, has a *zero element* if there exists a $0 \in S$ such that for all $t \in S$, $0t = t0 = 0$. A semigroup is *0-left cancellative* if for $s, t, r \in S$, $0 \neq st = sr$ implies that $t = r$.

Within semigroups we say that for $s, r \in S$, s *divides* r if there exists some $u \in S$ such that $su = r$ or $s = r$.

Definition 2.2.1. An element $r \in S$ is a *least common multiple* of s and t when

1. $sS \cap tS = rS$
2. both s and t divide r.

A semigroup *admits least common multiples* if there exists a least common multiple for each pair of elements in the semigroup.

Definition 2.2.2. S is an *inverse semigroup* if S is a semigroup and for each $t \in S$ there exists a unique $t^* \in S$ such that

$$t = tt^*t \text{ and } t^* = t^*tt^*.$$

Note, all groups are inverse semigroups. For the entirety of this thesis "inverse" will refer to semigroup inverses.

In this thesis, we will consider heavily the idempotents of semigroups. The elements $e \in S$ satisfying $ee = e$ are called *idempotents*. The set of all idempotents in S is denoted by $E(S)$. Note, if S is an inverse semigroup, given $eee = e$, for $e \in E(S)$, and inverses are unique, $e = e^*$. We can describe $E(S)$ in set notation as follows,

Proposition 2.2.3. *Let S be an inverse semigroup. Then $E(S) = \{ss^* : s \in S\}$.*

Proof. First, let us show that $\{ss^* : s \in S\} \subseteq E(S)$. If $s \in S$, then $ss^*ss^* = ss^*$ as $ss^*s = s$. Thus $ss^* \in E(S)$.

Now we will show that $E(S) \subseteq \{ss^* : s \in S\}$. If $e \in E(S)$, then $e = ee = ee^*$, as $e = e^*$ for all idempotents. Thus $e \in \{ss^* : s \in S\}$.

Therefore, we have shown that $E(S) = \{ss^* : s \in S\}$. □

Note that for all $s \in S$, it is also the case that $s^* \in S$, so $s^*s \in E(S)$. For the majority of this thesis we will be studying inverse semigroups, so it is vital that we understand arithmetic and common relations within inverse semigroups.

2.2.1 Inverse Semigroups

The following propositions build intuition on inverse semigoup arithmetic regarding the behavior of idempotents and inverses.

Proposition 2.2.4. *Let S be an inverse semigroup. If $e, f \in E(S)$, then $ef, fe \in E(S)$ and $ef = fe$.*

Proof. Let us assume $e, f \in E(S)$. First we will show that $f(ef)^*e$ is an idempotent and $(ef)^* = f(ef)^*e$. We see that $f(ef)^*ef(ef)^*e = f(ef)^*e$ as $(ef)^*ef(ef)^* = (ef)^*$. Thus $f(ef)^*e$ is a idempotent. Then,

$$\begin{aligned} f(ef)^*e(ef)f(ef)^*e &= f(ef)^*ef(ef)^*e \\ &= f(ef)^*e \end{aligned}$$

And,

$$\begin{aligned} eff(ef)^*eef &= ef(ef)^*ef \\ &= ef \end{aligned}$$

Given the uniqueness of inverses, it must be the case that $(ef)^* = f(ef)^*e$, and thus $(ef)^* \in E(S)$. Therefore, $ef = (ef)^*$. Note this also implies that $fe \in E(S)$.

Now we will prove that $ef = fe$. Note that,

$$\begin{aligned} ef(fe)ef &= e(ff)(ee)f \\ &= efef \\ &= ef \end{aligned}$$

And,

$$\begin{aligned} fe(ef)fe &= f(ee)(ff)e \\ &= fefe \\ &= fe \end{aligned}$$

Thus by definition $ef = (fe)^*$. However, $fe \in E(S)$, so $ef = fe$. □

Note that as a result of Proposition 2.2.4 we know that $E(S)$ is closed under the operation of S, and thus $E(S)$ is a subsemigroup of S.

Just like for groups, the following proposition holds for inverse semigroups:

Proposition 2.2.5. *Let S be an inverse semigroup. if $s, t \in S$ then $s^*t^* = (ts)^*$.*

Proof. Given that inverses are unique in S, we can prove that $s^*t^* = (ts)^*$ by showing that both sides of this equation are the inverse of ts. By definition $(ts)^*$ is the inverse of ts. Recall that $ss^*, t^*t \in E(S)$ from Proposition 2.2.3. So by Proposition 2.2.4,

$$\begin{aligned} ts(s^*t^*)ts &= tss^*t^*ts \\ &= tt^*tss^*s \\ &= ts \end{aligned}$$

Similarly,

$$\begin{aligned} (s^*t^*)ts(s^*t^*) = s^*t^*tss^*t^* &= s^*ss^*t^*tt^* \\ &= s^*t^* \end{aligned}$$

Thus s^*t^* is the inverse of ts. However, by the uniqueness of inverses this implies that $s^*t^* = (ts)^*$. □

Proposition 2.2.6. *Let S be an inverse semigroup. Then for $s, t \in S$,*

1. $stt^* = st(st)^*s$
2. $t^*ts = s(ts)^*ts$.

Proof. The proof for (1) and (2) are nearly identical, so we will prove (1) and leave the proof of (2) to the reader. By definition $stt^* = ss^*stt^*$. Thus, by Proposition 2.2.3 and Proposition 2.2.4,

$$stt^* = ss^*stt^* = stt^*s^*s.$$

Finally, by Proposition 2.2.5, $stt^* = st(st)^*s$. □

Propositions 2.2.3 through 2.2.6 will be used without reference throughout the rest of this thesis.

Sometimes, it can be useful to consider orderings of elements in S.

Definition 2.2.7. Let S be an inverse semigroup. The *natural partial order* on S is defined such that $s \leq t$ if $s = te$ for $s, t \in S$ and $e \in E(S)$.

There are multiple useful ways to describe the natural partial order.

Proposition 2.2.8. *Let S be an inverse semigroup. For $s, t \in S$, the following are equivalent under the natural partial order,*

1. $s \leq t$
2. $s = ts^*s$
3. $s = ss^*t$

Proof. We start by showing that $s \leq t$ implies $s = ts^*s$. Suppose $s \leq t$. Then there exists an $e \in E(S)$ such that $s = te$. Notice,

$$ts^*s = t(te)^*te = te^*t^*te = tet^*te = tt^*tee = te = s$$

Therefore (1) implies (2). Now we will show that $s = ts^*s$ implies $s = ss^*t$. Suppose $s = ts^*s$. Therefore, $s^* = (ts^*s)^* = s^*st^*$. Notice,

$$ss^*t = ts^*ss^*t = ts^*t = ts^*st^*t = tt^*ts^*s = ts^*s = s$$

.

Thus (2) implies (3). Finally, we will show that $s = ss^*t$ implies $s \leq t$. Suppose $s = ss^*t$. Then $s = ss^*t = (s^*)^*s^*t = t(s^*t)^*s^*t$. As $(s^*t)^*s^*t \in E(S)$, it must be the case that $s = te$ for some $e \in E(S)$. Thus (3) implies (1).

Therefore, the three statements are equivalent. □

Corollary 2.2.9. *Let S be an inverse semigroup. For $s, t \in E(S)$, $s \leq t$ if, and only if, $s = ts = st$.*

Proof. By Proposition 2.2.8, if $s, t \in E(S)$ then $s \leq t$ if, and only if, $s = ts^*s = ss^*t$. Note that for $s, t \in E(S)$, $ts^*s = tss = ts$ and $ss^*t = sst = st$. Thus, for $s, t \in E(S)$, $s \leq t$ if and only if $s = ts = st$. □

Note that the natural partial order in an inverse semigroup S is, in fact, a partial order. Recall that a partial order is reflexive, asymmetric, and transitive. Given that $s = ss^*s$, where $s^*s \in E(S)$, it is true that $s \leq s$. Given Proposition 2.2.8, if $s, t \in S$ held the property that $s \leq t$ and $t \leq s$ then $s = ts^*s$ and $t = st^*t$. Thus,

$$t = st^*t = ts^*st^*t = tt^*ts^*s = ts^*s = s.$$

Lastly, we will show that if $s = te$ and $t = ui$ for s,t,$u \in S$ and e, $i \in E(S)$, then $s = uj$ for some $j \in E(S)$. We find that $s = te = uie$ and $ie \in E(S)$. So, natural partial order is, in fact, a partial order.

Within this thesis we will primarily consider the natural partial order on $E(S)$. Under the natural partial order, $E(S)$ forms a meet semilattice.

Definition 2.2.10. A *meet semilattice* is a partial order on a set such that every finite subset, K, has a *meet*, denoted $\wedge K$. The *meet* is an element such that $\wedge K \leq k$ for all $k \in K$, and $a \leq \wedge K$ for $a \leq k$ for all $k \in K$.

As $E(S)$ is an inverse semigroup, we have already shown that $E(S)$ forms a partial order under the natural partial order. Let us consider $\{e_1, \ldots, e_n\} \subseteq E(S)$. Then $e_1 e_2 \ldots e_n \leq e_i$ for any $1 \leq i \leq n$, as

$$e_1 e_2 \ldots e_n = e_1 e_2 \ldots e_{i-1} e_i e_i e_{i+1} \ldots e_n = e_1 e_2 \ldots e_n e_i.$$

Also, for any $a \leq e_i$ for all $1 \leq i \leq n$, where $a \in E(S)$, $a = ae_i$ by Corollary 2.2.9. Thus, as a is an idempotent $a = ae_1 ae_2 \ldots ae_n = ae_1 \ldots e_n$, which implies that $a \leq e_1 \ldots e_n$. Therefore, $e_1 \ldots e_n$ is the meet of the set $\{e_1, \ldots, e_n\} \subseteq E(S)$, meaning under natural partial order, $E(S)$ forms a meet semilattice.

There are a number useful equivalence relations defined on an inverse semigroup which are called the *Green's relations*.

Definition 2.2.11. Let S be an inverse semigroup. For $s, t \in S$ we say that:

1. $s \mathcal{L} t$ if, and only if, $s^*s = t^*t$,
2. $s \mathcal{R} t$ if, and only if, $ss^* = tt^*$,
3. $s \mathcal{H} t$ if, and only if, $s^*s = t^*t$ and $ss^* = tt^*$,
4. $s \mathcal{D} t$ if, and only if, there exists a $r, k \in S$ such that $s \mathcal{L} r$, $r^* \mathcal{R} t$, $s \mathcal{R} b$, and $b^* \mathcal{L} t$. If $s, t \in E(S)$ then $s \mathcal{D} t$ if, and only if, there exists a $r \in S$ such that $s \mathcal{L} r$ and $t \mathcal{R} r$. We denote the set of elements that are $\mathcal{D}$-related to an element $s \in S$ as $\mathcal{D}_s$.

If S satisfies the property that $s \mathcal{H} t$ implies that $s = t$, then S is said to be *combinatorial*.

Proposition 2.2.12. *Let S be an inverse semigroup. Then S is combinatorial provided $s^*s = ss^*$ implies s is an idempotent for $s \in S$.*

Proof. Let us assume that $s^*s = ss^*$ implies s is an idempotent for $s \in S$. Now let us consider $s, t \in S$ such that $s \mathcal{H} t$. Then $s^*s = t^*t$ and $ss^* = tt^*$. Note that,

$$st^*(st^*)^* = st^*ts^* = ss^*ss^* = ss^*$$

and,

$$(st^*)^*st^* = ts^*st^* = tt^*tt^* = tt^*.$$

As $ss^* = tt^*$, $st^*(st^*)^* = (st^*)^*st^*$, which implies that $st^* \in E(S)$. Thus, $st^* = ss^* = tt^*$. Notice,

$$st^*s = ss^*s = s,$$

and,

$$t^*st^* = t^*tt^* = t^*.$$

By the uniqueness of inverses, it must be the case that $s = t$. Therefore, S is combinatorial.

□

2.2.2 Example Inverse Semigroup: Partial Bijections

A common and useful example of an inverse semigroup is the partial bijections. Recall in group theory we can construct the symmetric group, which is the set of bijections of any set to itself, with the operation of function composition. The semigroup equivalent to the symmetric group is the partial bijections.

Definition 2.2.13. The *partial bijections on a set X*, denoted $I(X)$, is the set of bijections $g : A \rightarrow B$ such that $A, B \subseteq X$, together with the operation of function composition.

Let us consider $g \in I(X)$ such that $g : A \rightarrow B$. Then we denote the domain of g to be $D(g) = A$, and similarly $R(g) = B$. Note symmetric group on X is the subset of $I(X)$ such that A and B are equal to X. The map with empty domain is denoted 0.

All $g \in I(X)$, where $g : A \rightarrow B$, have an inverse, g^{-1}, such that $g^{-1} : B \rightarrow A$ and $g^{-1}(g(a)) = a$ for all $a \in A$. Note that these are inverses in the semigroup sense. Given that $g^{-1}g : A \rightarrow A$ where $g^{-1}g(a) = a$ for all $a \in A$, $gg^{-1}g : A \rightarrow B$ where $gg^{-1}g(a) = g(a)$, so $gg^{-1}g = g$. Similarly we can show that $gg^{-1} : B \rightarrow B$ where $gg^{-1}(b) = b$ for all $b \in B$, and $g^{-1}gg^{-1} = g^{-1}$. Thus, all $g \in I(X)$ has an inverse, in the semigroup sense, $g^{-1} \in I(X)$.

Given that $I(X)$ has the operation of function composition, if $g, f \in I(X)$, and $g : A \rightarrow B$ and $f : C \rightarrow D$, then fg is the bijection of $g^{-1}(B \cap C)$ onto $f(B \cap C)$ such that $fg(x) = f(g(x))$ for $x \in g^{-1}(B \cap C)$.

Let us denote $1_A \in I(X)$ as the mapping $1_A : A \rightarrow A$ where $1_A(a) = a$ for all $a \in A$. Thus the set $\{1_A : A \subseteq X\}$ is the set of identity mappings in $I(X)$.

Proposition 2.2.14. *For any set X, $E(I(X)) = \{1_A : A \subseteq X\}$.*

Proof. We have shown that for $g \in I(X)$ such that $g : A \to B$, $g^{-1}g = 1_A$ and $gg^{-1} = 1_B$. Thus $E(I(X)) \subseteq \{1_A : A \subseteq X\}$ by Proposition 2.2.3. Let us consider an arbitrary $C \in X$. Then there exists an $g \in I(X)$ such that $g : C \to C$. So $g^{-1}g = 1_C$. Thus, $\{1_A : A \subseteq X\} \subseteq E(I(X))$ by Proposition 2.2.3. Therefore, $E(I(X)) = \{1_A : A \subseteq X\}$. □

Proposition 2.2.15. *For any set X, $1_A \leq 1_B$ if, and only if, $A \subseteq B$, for $1_A, 1_B \in E(I(X))$.*

Proof. Let us consider $1_A, 1_B \in E(I(X))$ such that $1_A \leq 1_B$. Then $1_A = 1_B 1_A$ by Corollary 2.2.9. We know that $1_B 1_A$ has domain $1_A^{-1}(A \cap B) = 1_A(A \cap B) = A \cap B$, as $1_A^{-1} = 1_A$. But as $1_A = 1_B 1_A$, it must be the case that $A = A \cap B$. Thus $A \subseteq B$.

Now let us consider $1_A, 1_B \in E(I(X))$ such that $A \subseteq B$. Again we know that $1_B 1_A$ has domain $1_A^{-1}(A \cap B) = A \cap B$. However, $A \subseteq B$ so $A \cap B = A$. Thus, $1_B 1_A$ has domain A where $1_B 1_A(a) = 1_B(a) = a$ for all $a \in A$. So, $1_A = 1_B 1_A$, which implies that $1_A \leq 1_B$ by Corollary 2.2.9.

Thus $1_A \leq 1_B$ if, and only if, $A \subseteq B$ for $1_A, 1_B \in E(I(X))$. □

Due to the unique properties of partial bijections, the Green's relations have an intuitive interpretation beyond the definition,

Proposition 2.2.16. *For some set X and $g, f \in I(X)$, we say that*:

1. *$g \ \mathcal{L} \ f$ if, and only if, $D(g) = D(f)$,*
2. *$g \ \mathcal{R} \ f$ if, and only if, $R(g) = R(f)$,*
3. *$g \ \mathcal{H} \ f$ if, and only if, $D(g) = D(f)$ and $R(g) = R(f)$,*
4. *if $g, f \in E(I(X))$ then $g \ \mathcal{D} \ f$ if, and only if, there exists a $r \in I(X)$ such that $D(g) = D(r)$ and $R(r) = R(f)$.*

Proof. Recall that by definition for $s, t \in S$ we say that:

1. $s \ \mathcal{L} \ t$ if, and only if, $s^*s = t^*t$,
2. $s \ \mathcal{R} \ t$ if, and only if, $ss^* = tt^*$,
3. $s \ \mathcal{H} \ t$ if, and only if, $s^*s = t^*t$ and $ss^* = tt^*$,
4. if $s, t \in E(S)$ then $s \ \mathcal{D} \ t$ if, and only if, there exists a $r \in S$ such that $s \ \mathcal{L} \ r$ and $r \ \mathcal{R} \ t$.

Recall that for $g \in I(X)$, $g^{-1}g = 1_{D(g)}$. Thus $g^{-1}g = f^{-1}f$ if, and only if, $1_{D(g)} = 1_{D(f)}$, which is only true if $D(g) = D(f)$. Thus $g \mathcal{L} f$ if, and only if, $D(g) = D(f)$. Similarly, as $gg^{-1} = 1_{R(g)}$, $g \mathcal{R} f$ if, and only if, $R(g) = R(f)$. The properties for $\mathcal{H}$ and $\mathcal{D}$ follow directly from the two above. □

The next section will define a subsemigroup of interest in $I(X)$, the inverse hull, which we will study in relation to a one-sided shift of finite type later in this thesis.

2.2.3 Inverse Hulls

All these useful properties of inverse semigroups do not necessarily hold for arbitrary semigroups, and thus it can be of interest to "build" an inverse semigroup from an arbitrary semigroup, using partial bijections.

Let us consider a 0-left cancellative semigroup, S. Then for all $s \in S$, we can consider the mapping $\theta_s \in I(S)$ such that $D(\theta_s) = \{t \in S : st \neq 0\}$, where $\theta_s(t) = st$ for $t \in D(\theta_s)$. Thus, $R(\theta_s) = \{st : t \in S \text{ and } st \neq 0\}$. We know that θ_s is a bijection because it is surjective by construction, and if $\theta_s(x) = \theta_s(w)$ for $x, w \in D(\theta_s)$, then $sx = sw \neq 0$, meaning $x = w$ by 0-left cancellativity.

We can also consider the mapping $\theta_s^{-1} \in I(S)$ such that $D(\theta_s^{-1}) = \{st : t \in S \text{ and } st \neq 0\}$ where $\theta_s^{-1}(st) = t$ for $st \in D(\theta_s^{-1})$. So, $R(\theta_s^{-1}) = \{t \in S : st \neq 0\}$. Note that θ_0 is the empty map.

Definition 2.2.17. Let S be a 0-left cancellative semigroup. The *inverse hull of* S, $H(S)$, is the inverse semigroup generated by the all mappings $\theta_s \in I(S)$ such that $s \in S$: ie $H(S) = \langle \theta_s, \theta_s^{-1} : s \in S \rangle$.

Note that as $H(S) \subseteq I(S)$, Propositions 2.2.14, 2.2.15 and 2.2.16 can be extended to the inverse hull of S.

Proposition 2.2.18. *Let S be a 0-left cancellative semigroup. For $s, t \in S$,*

$$\theta_s \theta_t = \theta_{st}$$

Proof. We know that $D(\theta_{st}) = \{k \in S : stk \neq 0\}$ and we know that $D(\theta_s\theta_t) = \theta_t^{-1}(D(\theta_s) \cap R(\theta_t)) = \theta_t^{-1}(\{tk \neq 0 : k \in S \text{ and } stk \neq 0\}) = \{k \in S : stk \neq 0\}$. Thus, $D(\theta_{st}) = D(\theta_s\theta_t)$. If $k \in D(\theta_{st})$ then,

$$\theta_s\theta_t(k) = \theta_s(tk) = stk = \theta_{st}(k).$$

Therefore, we have shown that $\theta_s\theta_t = \theta_{st}$. □

Corollary 2.2.19. *Let S be a 0-left cancellative semigroup. For $s, t \in S$,*

$$\theta_s^{-1}\theta_t^{-1} = \theta_{ts}^{-1}$$

Proof. Given that $\theta_s^{-1}\theta_t^{-1} = (\theta_t\theta_s)^{-1} = \theta_{ts}^{-1}$ by Proposition 2.2.18, $\theta_s^{-1}\theta_t^{-1} = \theta_{ts}^{-1}$. □

The inverse hull of S is a natural extension of S as an inverse semigroup because $H(S) \cong S$ if S is an inverse semigroup. If S is an inverse semigroup, for all $s \in S$, there exists an $s^* \in S$, where s^* is the inverse of s in S. From Proposition 2.2.18, $\theta_{s^*}\theta_s\theta_{s^*} = \theta_{s^*ss^*} = \theta_{s^*}$. Similarly $\theta_s\theta_{s^*}\theta_s = \theta_{ss^*s} = \theta_s$. Thus by the uniqueness of inverses in inverse semigroups, $\theta_s^{-1} = \theta_{s^*}$. So, $H(S) = \langle\theta_s : s \in S\rangle$, and for $s, t \in S$, $\theta_s\theta_t = \theta_{st}$. So actually, $H(S) = \{\theta_s : s \in S\}$.

We can define the mapping $\Psi : S \to H(S)$ such that $\Psi(s) = \theta_s$ for all $s \in S$, which is clearly bijective, as $H(S) = \{\theta_s : s \in S\}$. Also for $s, t \in S$, $\Psi(s)\Psi(t) = \theta_s\theta_t = \theta_{st} = \Psi(st)$. Thus, Ψ is an isomorphic, and $H(S) \cong S$. Therefore, the inverse hull of S is a natural extension of S as an inverse semigroup, and it allows us to study the structure of S using all the useful inverse semigroup properties.

Let S^1 be the semigroup S together with a formal 1 such that for all $s \in S$, $1s = s1 = s$, thus $\theta_1\theta_s = \theta_s\theta_1 = \theta_s$. Note, further in the thesis it will also be convent to consider $L^1 = L \cup \{1\}$, where 1 is this same formal 1, and L is a language associated to a one-sided shift of finite type.

In Exel and Steinberg (2018), the authors prove the following useful theorem. Note that we have adjusted the notation of the theorem to fit this thesis.

Theorem 2.2.20. *Let S be a 0-left cancellative semigroup admitting least common multiples. Then,*

$$H(S) = \{\theta_s\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}\theta_w^{-1} : s, w \in S^1 \text{ and } x_i \in S \text{ for } 1 \leq i \leq n\}$$

Proof. See Proof of Theorem 7.11 in Exel and Steinberg (2018). □

Thus we have a general form for elements of $H(S)$ as long as S is a 0-left cancellative semigroup admitting least common multiples. Note that we have not defined a normal form for elements of $H(S)$, as $\theta_s = \theta_s\theta_s^{-1}\theta_s\theta_1 = \theta_s\theta_s^{-1}\theta_s\theta_s^{-1}\theta_s\theta_1$. There exists a normal form of these elements, see Exel and Steinberg (2018), but for this thesis, the general form described in Theorem 2.2.20 is sufficient.

We have defined and explored all the necessary background in semigroups. The next section introduces the subject of this thesis, the inverse hulls of one-sided shifts of finite type. The inverse hull is defined on a semigroup, and thus we must construct a semigroup from a one-sided shift of finite type, to determine its inverse hull.

2.3 Construction of the Inverse Hull of A Shift

For any language L associated to a shift we can construct a semigroup $S = L \cup \{0\}$. We will show that S is a semigroup under the concatenation operation.

Proposition 2.3.1. *Let $S = L \cup \{0\}$, where L is a language associated with a shift. Then, S is a semigroup under the concatenation operation,*

$$w * u = \begin{cases} wu & \text{if } wu \in L \\ 0 & \text{otherwise} \end{cases}$$

Proof. $*$ is closed and binary by construction. for arbitrary $w, u, c \in S$,

$$(w * u) * c = \begin{cases} (wu) * c & \text{if } wu \in L \\ 0 & \text{otherwise} \end{cases} = \begin{cases} wuc & \text{if } wuc \in L \\ 0 & \text{otherwise} \end{cases}$$

$$w * (u * c) = \begin{cases} w * (uc) & \text{if } uc \in L \\ 0 & \text{otherwise} \end{cases} = \begin{cases} wuc & \text{if } wuc \in L \\ 0 & \text{otherwise} \end{cases}$$

Thus $*$ is associative. Therefore, S is a semigroup. □

Now let us consider the language associated to a one-sided shift of finite type.

Lemma 2.3.2. *Let $S = L \cup \{0\}$, where L is a language associated to a one-sided shift of finite type. Then S is 0-left cancellative and admits least common multiples.*

Proof. Recall a semigroup, G, is 0-left cancellative if $s, t, r \in G$ and $0 \neq st = sr$ implies that $t = r$. Let $S = L \cup \{0\}$, where L is a language associated to a one-sided shift of finite type. Let us consider $s, t, r \in S$ and $0 \neq st = sr$. Thus, $st, sr \in L$. Note that as L is associated to a shift it must be the case that $s, t, r \in L$ as well. Thus, by Remark 2.1.1, it must be the case that $t = r = e_1 \dots e_N$ for $e_i \in \mathcal{A}$. Thus S is 0-left cancellative.

Recall that a semigroup, G, admits least common multiples if there exists a least common multiple for each pair of elements of G, where $r \in G$ is a least common multiple for s and t when,

1. $sG \cap tG = rG$
2. both s and t divide r

We say that s divides r if there exists some $u \in G$ such that $su = r$ or $s = r$. Again let $S = L \cup \{0\}$, where L is a language associated to a one-sided shift of finite type. for some $s \in L$, $sS = \{sw : w \in L, sw \neq 0\} \cup \{0\}$. Note that if $s = tv$ for some $v \in L^1$ then $sS \cap tS = sS$ given that $\{sw : w \in L, sw \neq 0\} \subseteq \{tw : w \in L, tw \neq 0\}$. Also, $s = s$ and $s = tv$ where $v \in L^1$. Therefore, s is the least common multiple of s and t if $s = tv$. Note that it quickly follows that if $sv = t$, then t is the least common multiple of s and t, and if $s = t$ then t is again the least common multiple of s and t. If $t \neq s$ and there does not exist a $v \in L$ such that $s = tv$ or $sv = t$, then there does not exist a $k \in L$ such that $k = sw = tp$ for $w, p \in L$. Thus $sS \cap tS = 0 = 0S$. Also, $s0 = 0$ and $t0 = 0$, so 0 is the least common multiple of s and t. Therefore, we have shown that S admits least common multiples □

Given that $S = L \cup \{0\}$ is 0-left cancellative, by Lemma 2.3.2, we can consider $H(S) = \langle \theta_s, \theta_s^{-1} : s \in S \rangle$. We call $H(S)$ the *inverse hull of the one sided shift of finite type that is associated with L*. Given that all one sided shifts of finite type has an associated language, all one sided shifts of finite type has an associated inverse hull.

Recall that if L is a language associated to a one-sided shift of finite type, then L is M-step for some $M \in \mathbb{N}$, by Lemma 2.1.11.

Lemma 2.3.3. *Let $S = L \cup \{0\}$, where L is a language associated to a M-step one-sided shift. Then for any $w \in L$, such that $w = uv$, for $u, v \in L$, and $|v| = M$, it is the case that $\theta_w^{-1}\theta_w = \theta_v^{-1}\theta_v$.*

Proof. Let us consider $w \in L$ such that $w = uv$, for $u, v \in L$, and $|v| = M$. We know that $\theta_w^{-1}\theta_w$ is the identity mapping on the domain $\{t : wt \in L\}$. Similarly $\theta_v^{-1}\theta_v$ is the identity mapping on the domain $\{t : vt \in L\}$. Note that for all $t \in \{t : vt \in L\}$, $uvt \in L$, by Lemma 2.1.10. Therefore, $\{t : vt \in L\} \subseteq \{t : wt \in L\}$. Note also that for $t \in \{t : wt \in L\}$, $wt \neq 0$, thus $uvt \neq 0$, which implies that $vt \neq 0$. Therefore, $\{t : wt \in L\} \subseteq \{t : vt \in L\}$. So, $\{t : vt \in L\} = \{t : wt \in L\}$, meaning that $\theta_w^{-1}\theta_w = \theta_v^{-1}\theta_v$. □

We are now prepared to define a general form for elements in the inverse hulls of M-step one-sided shifts.

Theorem 2.3.4. *Let $S = L \cup \{0\}$, where L is a language associated to a M-step one-sided shift. Then,*

$$H(S) = \{\theta_s \theta_{x_1}^{-1} \theta_{x_1} \dots \theta_{x_n}^{-1} \theta_{x_n} \theta_w^{-1} : s, w \in S^1, x_i \in S, |x_i| \leq M \textit{ for } 1 \leq i \leq n\}.$$

Proof. By Lemma 2.3.2 we know that S is 0-left cancellative and admits least common multiples. Thus by Theorem 2.2.20,

$$H(S) = \{\theta_s \theta_{x_1}^{-1} \theta_{x_1} \dots \theta_{x_n}^{-1} \theta_{x_n} \theta_w^{-1} : s, w \in S^1 \text{ and } x_i \in S \text{ for } 1 \leq i \leq n\}.$$

Note that from Lemma 2.3.3, if there exists an x_i such that $|x_i| > M$, then there exists a $v_i \in L$ such that $|v_i| = M$ and $\theta_{x_i}^{-1} \theta_{x_i} = \theta_{v_i}^{-1} \theta_{v_i}$. Thus, without loss of generality, we can assume that $|x_i| \leq M$. Therefore, we have shown our desired result,

$$H(S) = \{\theta_s \theta_{x_1}^{-1} \theta_{x_1} \dots \theta_{x_n}^{-1} \theta_{x_n} \theta_w^{-1} : s, w \in S^1, x_i \in S, |x_i| \leq M \text{ for } 1 \leq i \leq n\}.$$

□

We can also provide a general form from elements of $E(H(S))$ specifically.

Corollary 2.3.5. *Let $S = L \cup \{0\}$, where L is a language associated to a M-step one-sided shift. Then,*

$$E(H(S)) = \{\theta_s \theta_{x_1}^{-1} \theta_{x_1} \dots \theta_{x_n}^{-1} \theta_{x_n} \theta_s^{-1} : s \in S^1, x_i \in S, |x_i| \leq M \textit{ for } 1 \leq i \leq n\}.$$

Proof. By Proposition 2.2.3, $E(H(S)) = \{ss^* : s \in H(S)\}$. Also, by Theorem 2.3.4, a general element in $H(S)$ is of the form $\theta_s \theta_{x_1}^{-1} \theta_{x_1} \dots \theta_{x_n}^{-1} \theta_{x_n} \theta_w^{-1}$. Thus as,

$$\theta_s \theta_{x_1}^{-1} \theta_{x_1} \dots \theta_{x_n}^{-1} \theta_{x_n} \theta_w^{-1} (\theta_s \theta_{x_1}^{-1} \theta_{x_1} \dots \theta_{x_n}^{-1} \theta_{x_n} \theta_w^{-1})^{-1} = \theta_s \theta_{x_1}^{-1} \theta_{x_1} \dots \theta_w^{-1} \theta_w \theta_s^{-1}$$

the idempotents in $H(S)$ are of the form $\theta_s \theta_{x_1}^{-1} \theta_{x_1} \dots \theta_{x_n}^{-1} \theta_{x_n} \theta_s^{-1}$. □

The structure of $E(H(S))$ is incredibly rich, and will be explored in the next chapter.

Chapter 3

Properties of the Inverse Hull of a One-sided M-step Shift

3.1 The Elements of E(H(S))

We showed in the last chapter that for $S = L \cup \{0\}$ associated to an M-step one-sided shift,

$E(H(S)) = \{\theta_s \theta_{x_1}^{-1} \theta_{x_1} \dots \theta_{x_n}^{-1} \theta_{x_n} \theta_s^{-1} : s \in S^1, x_i \in S, |x_i| \leq M \text{ for } 1 \leq i \leq n\}.$

Given that the inverse hull of a shift is a subsemigroup of $I(S)$, we can utilize the properties of partial bijections explored in the Subsection 2.2.2 when considering the inverse hull of a shift. Recall from Proposition 2.2.14, $E(H(S))$ is a set of identity maps. Thus, it is often convenient to think of the elements of $E(H(S))$ as uniquely described by their domains, especially because the natural partial order is determined by the domains of elements in $E(H(S))$, by Proposition 2.2.15. Therefore, it is impotent to build an intuition about what the domains of idempotents are.

We know that if $s \in L$ then $\theta_s : D(\theta_s) \to R(\theta_s)$ where $D(\theta_s) = \{w \in L : sw \in L\}$ and $R(\theta_s) = sL = \{sw \neq 0 : w \in L\}$.

Recall that if L is M-step, $w \in \mathcal{A}^M$ and $sw, wt \in L$, then $swt \in L$, by Lemma 2.1.10. Thus for $w \in \mathcal{A}^M$ and $sw \in L$, $wL \subseteq D(\theta_s)$, which means that $D(\theta_s) = \{w : sw \in L \text{ and } |w| \leq M\} \cup \{wL : sw \in L \text{ and } |w| = M\}$.

Given that $\theta_s^{-1}\theta_s = 1_{D(\theta_s)}$ and $\theta_s\theta_s^{-1} = 1_{R(\theta_s)}$ we know that,

$$\begin{aligned} D(\theta_s^{-1}\theta_s) &= D(1_{D(\theta_s)}) \\ &= \{w : sw \in sL\} \\ &= \{w : sw \in L \text{ and } |w| \leq M\} \cup \{wL : sw \in L \text{ and } |w| = M\}. \end{aligned}$$

And,

$$\begin{aligned} D(\theta_s \theta_s^{-1}) &= D(1_{R(\theta_s)}) \\ &= \{sw \neq 0 : w \in L\} \\ &= sL. \end{aligned}$$

Note that if $|s| = M$ and $vs \in L$ for some $v \in L$, then by Lemma 2.3.3, $\theta_s^{-1}\theta_s = \theta_{vs}^{-1}\theta_{vs}$. So, $D(\theta_s^{-1}\theta_s) = D(\theta_{vs}^{-1}\theta_{vs})$.

Recall that if g, f are partial bijections, and $g : A \to B$ and $f : C \to D$, then fg is the bijection of $g^{-1}(B \cap C)$ onto $f(B \cap C)$, such that $fg(x) = f(g(x))$ for $x \in g^{-1}(B \cap C)$. Given that $\theta_{x_i}^{-1}\theta_{x_i}$ are identity maps, $D(\theta_{x_1}^{-1}\theta_{x_1} \dots \theta_{x_n}^{-1}\theta_{x_n}) = D(\theta_{x_1}^{-1}\theta_{x_1}) \cap \dots \cap D(\theta_{x_n}^{-1}\theta_{x_n})$. More generally we find that,

Proposition 3.1.1. *Let $H(S)$ be the inverse hull of an M-step one-sided shift. If we consider an arbitrary element $\theta_s\theta_{x_1}^{-1}\theta_{x_1} \dots \theta_{x_n}^{-1}\theta_{x_n}\theta_s^{-1} \in E(H(S))$,*

$$D(\theta_s\theta_{x_1}^{-1}\theta_{x_1} \dots \theta_{x_n}^{-1}\theta_{x_n}\theta_s^{-1}) = \{sw \in L : x_i w, w \in L \,\forall\, i \in \{1, \dots n\}\}.$$

Now we have a multitude of different ways to think about the domains of elements in $E(H(S))$. We will use this intuition to prove properties of the inverse hull. For example, this construction of the domains for elements in $E(H(S))$ helps us prove that the inverse hull of a one-sided shift of finite type is combinatorial.

Proposition 3.1.2. *Let $H(S)$ be the inverse hull of an M-step one-sided shift. $H(S)$ is combinatorial.*

Proof. Let $\gamma = \theta_s\theta_{x_1}^{-1}\theta_{x_1} \dots \theta_{x_n}^{-1}\theta_{x_n}\theta_w^{-1} \neq 0$ and suppose that $\gamma\gamma^* = \gamma^*\gamma$. We know that $D(\gamma^*\gamma) = \{wu \in L : x_i u, su, u \in L \,\forall\, i \in \{1, \dots n\}\}$ and $D(\gamma\gamma^*) = \{sv \in L : x_i v, wv, v \in L \,\forall\, i \in \{1, \dots n\}\}$.

Since $\gamma\gamma^* = \gamma^*\gamma \neq 0$, there must exist $wu = sv \neq 0$ for some $u, v \in L$. Then $w = sy$ or $s = wy$ for some $y \in L^1$. First suppose that $w = sy$. As $\gamma^*\gamma \neq 0$, there exists $a \in \mathcal{A}$, such that $sa \in L$ and $x_i a \in L$ for all i. Therefore, $sa \in D(\gamma\gamma^*)$ and $sa \in D(\gamma^*\gamma)$. Thus, $sa = wz \neq 0$ for some $z \in L$. By 0-left cancellativity, $s = wz'$ for some $z' \in L^1$. Since $w = sy$ for some $y \in L^1$ and $s = wz'$ for some $z' \in L^1$, we conclude that $s = w$. A symmetric argument works in the case that $s = wy$.

Therefore γ is idempotent, which by Proposition 2.2.12, proves $H(S)$ is combinatorial.

□

Next we are going to define subsets of $E(H(S))$ that are of special interest moving forward.

3.1.1 Subsets of E(H(S))

For any $N \geq 1$, we define the set $\mathcal{O}(N) = \{\theta_s\theta_s^{-1} : s \in L \text{ and } |s| = N\}$, which we can describe in words as *the range idempotents of words of length* N, as $D(\theta_s\theta_s^{-1}) = R(\theta_s)$. Then *the domain idempotents*, $\theta_s^{-1}\theta_s$ for $s \in L$, and their multiplications make up the set: $\mathcal{T} = \{(\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n} \neq 0 : x_i \in L \text{ and } n \geq 1\}$.

Note that due to the construction $\mathcal{T}$, $\mathcal{T} \cup \{0\}$ is a closed under the operation of $H(S)$.

Proposition 3.1.3. *Let us consider the inverse hull of a one-sided shift of finite type, $H(S)$. Then the set $\mathcal{T} \cup \{0\} = \{(\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n} \neq 0 : x_i \in L$ and $n \geq 1\} \cup \{0\}$ is closed under the operation of $H(S)$.*

Proof. Let us consider $\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}, \theta_{y_1}^{-1}\theta_{y_1}\ldots\theta_{y_m}^{-1}\theta_{y_m} \in \mathcal{T}$. Then

$$\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}\theta_{y_1}^{-1}\theta_{y_1}\ldots\theta_{y_m}^{-1}\theta_{y_m} \in \mathcal{T}$$

or $\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}\theta_{y_1}^{-1}\theta_{y_1}\ldots\theta_{y_m}^{-1}\theta_{y_m} = 0$. Either way,

$$\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}\theta_{y_1}^{-1}\theta_{y_1}\ldots\theta_{y_m}^{-1}\theta_{y_m} \in \mathcal{T} \cup \{0\}.$$

Hence, $\mathcal{T} \cup \{0\}$ is a closed under the operation of $H(S)$. □

Now, let us consider the set $\mathcal{O}(N)$ in greater detail. Let define a set $X \subseteq S$, for some semigroup S, to be *mutually orthogonal* if for all $\alpha, \beta \in X$, such that $\alpha \neq \beta$, $\alpha\beta = 0$. The set $\mathcal{O}(N)$ is mutually orthogonal.

Proposition 3.1.4. *Let us consider the inverse hull of a one-sided shift of finite type, $H(S)$. The set $\mathcal{O}(N) = \{\theta_s\theta_s^{-1} : s \in L$ and $|s| = N\}$ is mutually orthogonal for any $N \geq 1$.*

Proof. Let us consider $\theta_s\theta_s^{-1}$ and $\theta_w\theta_w^{-1}$ in $\mathcal{O}(N)$. We know that $D(\theta_s\theta_s^{-1}) = sL$ and $D(\theta_w\theta_w^{-1}) = wL$. If it were the case that $D(\theta_s\theta_s^{-1}) \cap D(\theta_w\theta_w^{-1}) \neq \emptyset$, then there must equal a t in sL and wL. Therefore, $t = su = wv$ for $u, v \in L^1$. Note, as $|w| = |s| = N$, by Remark 2.1.1, it must be the case that $s = w$. Thus, $\mathcal{O}(N)$ is mutually orthogonal for any $N \geq 1$. □

A keen eyed reader might notice that for a set $X \subseteq E(H(S))$ being mutually orthogonal is equivalent to saying that there exists no nonzero $\gamma \in E(H(S))$ such that $\gamma \leq \alpha, \beta$, for $\alpha, \beta \in X$ and $\alpha \neq \beta$.

It is natural to frame mutually orthogonality in terms of the natural partial order of $E(H(S))$, because the natural partial order is excessively rich in information. The next section derives properties of $H(S)$ using the natural partial order.

3.2 The Natural Partial Order of E(H(S))

Recall that the natural partial order is defined such that for $\alpha, \beta \in E(H(S))$, $\alpha \leq \beta$ if, and only if, $\alpha\beta = \alpha$. By Proposition 2.2.15, $\alpha\beta = \alpha$ if, and only if, $D(\alpha) \subseteq D(\beta)$ for elements $\alpha, \beta \in E(H(S))$. It often becomes useful to consider the elements above, or below, some $\alpha \in E(H(S))$, in the natural partial order on $E(H(S))$.

Definition 3.2.1. Let $H(S)$ be the inverse hull of a one-sided shift of finite type. For $\alpha \in E(H(S))$, the *up-set of* α is the set, $\alpha^{\uparrow} = \{\beta \in E(H(S)) : \beta \geq \alpha\}$, and the *down-set of* α is the set, $\alpha^{\downarrow} = \{\beta \in E(H(S)) : \beta \leq \alpha\}$. We will also define $X^{\uparrow}$ and $X^{\downarrow}$ for $X \subseteq E(H(S))$, where,

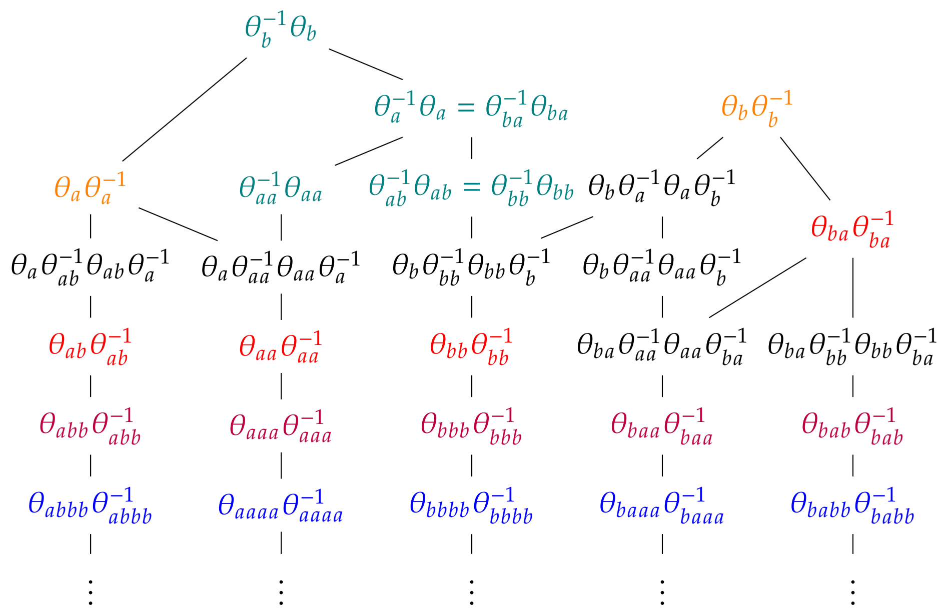
$$X^{\uparrow} = \bigcup_{x \in X} x^{\uparrow} \text{ and } X^{\downarrow} = \bigcup_{x \in X} x^{\downarrow}.$$

Constructing semilattice examples of $E(H(S))$ is useful to start building intuition for the properties of $H(S)$.

Example 3.2.2. Let us consider the language L that is described by the forbidden word set $\mathcal{F} = \{bba, aba, aab\}$, or the corresponding transition array,

$$\begin{array}{cc} a & b \\ \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \end{array} & \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array} \end{array}$$

We can construct the semilattice of $E(H(S))$ below. the set $\mathcal{T}$ is denoted in teal, and the sets $\mathcal{O}(1) - \mathcal{O}(4)$ are denoted in orange, red, plum, and blue, respectively.



You might notice some relations between the colored sets. For example, it appears that the elements of $\mathcal{T}$ are the only elements not below an element of $\mathcal{O}(1)$. Also, in the above example, every element $e \in \mathcal{T}^{\downarrow} - \mathcal{T}$ has a unique $t \in \mathcal{T}$ such that $e < t$ and there does not exist any $g \in \mathcal{T}$ such that $e < g < t$. Lastly, every element of $\mathcal{O}(N+1)$ seems to be below an element of $\mathcal{O}(N)$. It turns out these properties hold for every inverse hull of a one-sided shift of finite type.

Proposition 3.2.3. *Let us consider the inverse hull of a one-sided shift of finite type, $H(S)$. Then $\mathcal{T}$ equals the complement of $\mathcal{O}(1)^{\downarrow}$ in $E(H(S))$, denoted $(\mathcal{O}(1)^{\downarrow})^c$.*

Proof. Consider $\theta_s\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}\theta_s^{-1} \in (\mathcal{O}(1)^{\downarrow})^c$. For the sake of contradiction, let us assume that $\theta_s\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}\theta_s^{-1} \notin \mathcal{T}$. Then, $s = a_1\ldots a_P$ for $a_i \in \mathcal{A}$ and $P \geq 1$.

$$\begin{aligned}
\theta_s\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}\theta_s^{-1} &= \theta_s\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}\theta_{a_P}^{-1}\ldots\theta_{a_1}^{-1} \\
&= \theta_s\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}\theta_{a_P}^{-1}\ldots\theta_{a_1}^{-1}\theta_{a_1}\theta_{a_1}^{-1} \\
&= \theta_s\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}\theta_s^{-1}\theta_{a_1}\theta_{a_1}^{-1}
\end{aligned}$$

Therefore $\theta_{a_1}\theta_{a_1}^{-1} \geq \theta_s\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}\theta_s^{-1}$, which is a contradiction, as $\theta_{a_1}\theta_{a_1}^{-1} \in \mathcal{O}(1)$. Thus, $(\mathcal{O}(1)^{\downarrow})^c \subseteq \mathcal{T}$.

Let us consider some arbitrary $\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n} \in \mathcal{T}$. Note that if $\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}$ is less than or equal to some $\theta_a\theta_a^{-1} \in \mathcal{O}(1)$, then it must be the case that $D(\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}) \subseteq aL$, as $D(\theta_a\theta_a^{-1}) = aL$. For the sake of contradiction, let us assume that there exists such an $\theta_a\theta_a^{-1}$.

As $\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}$ is nonzero, there must exist an $aw \in aL$ such that $w \in L$ and $aw \in D(\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n})$. Note, if $x_iaw \in L$ for all $1 \leq i \leq n$, then $x_ia \in L$. So, $a \in D(\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n})$. However, $a \notin aL$, so $D(\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}) \nsubseteq D(\theta_a\theta_a^{-1})$, which implies that $\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n} \nleq \theta_a\theta_a^{-1}$, a contradiction. Therefore we know that $\mathcal{T} \subseteq (\mathcal{O}(1)^{\downarrow})^c$.

Hence we have proven our desired result, $\mathcal{T} = (\mathcal{O}(1)^{\downarrow})^c$. □

An interesting result of Proposition 3.2.3 is that elements of $\mathcal{T}$ have unique $\mathcal{D}$-classes.

Proposition 3.2.4. *Let $H(S)$ be the inverse hull of a one-sided shift of finite type. Fix $e, t \in \mathcal{T}$. If $e \mathcal{D} t$ then $e = t$. Also, for all nonzero $f \in E(H(S))$, $f \mathcal{D} t$ for some $t \in \mathcal{T}$.*

Proof. Consider an arbitrary nonzero $\gamma = \theta_s\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}\theta_w^{-1}$ in $H(S)$ and suppose that $\gamma\gamma^*, \gamma^*\gamma \in \mathcal{T}$. Given that,

$$\begin{aligned}\gamma\gamma^* &= \theta_s\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}\theta_w^{-1}\theta_w\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}\theta_s^{-1}\\ &= \theta_s\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}\theta_w^{-1}\theta_w\theta_s^{-1},\end{aligned}$$

as $\gamma\gamma^* \in \mathcal{T}$, $s = 1$, by Proposition 3.2.3. Similarly, as $\gamma^*\gamma \in \mathcal{T}$, $w = 1$. Therefore, $\gamma\gamma^* = \gamma^*\gamma = \theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}$. Now suppose that $e, t \in \mathcal{T}$ with $e \mathcal{D} t$. Then, by Definition 2.2.11, there exists $\gamma \in H(S)$ such that $\gamma\gamma^* = e$ and $\gamma^*\gamma = t$. By the above argument, $e = t$.

We know that for any nonzero $f \in E(H(S))$, $f = \theta_w t\theta_w^{-1}$ for some $w \in L^1$ and $t \in \mathcal{T}$. Thus, $\theta_w t \neq 0$ and,

$$\theta_w t(\theta_w t)^{-1} = \theta_w tt^{-1}\theta_w^{-1} = f$$

and,

$$(\theta_w t)^{-1}\theta_w t = t^{-1}\theta_w^{-1}\theta_w t = t\theta_w^*\theta_w.$$

So, $f\mathcal{D}t\theta_w^*\theta_w$ where $t\theta_w^*\theta_w \in \mathcal{T}$. Therefore, any nonzero $f \in E(H(S))$ is $\mathcal{D}$-related to some element of $\mathcal{T}$. □

Now let us prove our second observation from Example 3.2.2.

Proposition 3.2.5. *Let us consider the inverse hull of a one-sided shift of finite type, $H(S)$. For nonzero $e \in \mathcal{T}^{\downarrow} - \mathcal{T}$ there exists a unique $t \in \mathcal{T}$ such that $e < t$ where there does not exist any $g \in \mathcal{T}$ such that $e < g < t$.*

Proof. Let us imagine, for the sake of contradiction, there exists $t \neq k$ such that $t, k \in \mathcal{T}$, $e < t$ and $e < k$ for some $e \in \mathcal{O}(1)^{\downarrow}$, and there exists no $g \in \mathcal{T}$ such that $e < g < t$ or $e < g < k$. Given that $t \neq k$, $D(t) \neq D(k)$, however, $D(e) \subseteq D(t) \cap D(k)$, so $D(tk) \neq \emptyset$. Therefore $tk \neq 0$, and $e \leq tk$. Recall that $\mathcal{T}$ is closed by Proposition 3.1.3, so $tk \in \mathcal{T}$. If $tk < t$ or $tk < k$ then there exists a $g \in \mathcal{T}$ such that $e < g < t$ or $e < g < k$, a contradiction. Thus, $tk = t = k$, which is another contradiction. Therefore, for all nonzero $e \in \mathcal{T}^{\downarrow} - \mathcal{T}$, there exists a unique $t \in \mathcal{T}$ such that $e < t$ and there exists no $g \in \mathcal{T}$ such that $e < g < t$. □

We noted previously that in Example 3.2.2, it appeared as though every element of $\mathcal{O}(N+1)$ was below an element of $\mathcal{O}(N)$, with is a property that we can formalize.

Definition 3.2.6. Let us say that two sets A and B hold the property that A *is less than* B, $A < B$, over the natural partial order if,

1. for all $a \in A$ there exists a $b \in B$ such that $a < b$
2. for all $b \in B$ there exists a $a \in A$ such that $a < b$
3. there exists no $b \in B$ such that there exists an $a \in A$ where $b \leq a$.

Proposition 3.2.7. *Let us consider the inverse hull of a one-sided shift of finite type, $H(S)$. Then $\mathcal{O}(N+1) < \mathcal{O}(N)$.*

Proof. Let us consider $\theta_w \theta_w^{-1} \in \mathcal{O}(N+1)$. Then, $w = va$ where $v \in L$, $a \in \mathcal{A}$ and $|v| = N$. Therefore,

$$\begin{aligned} \theta_w \theta_w^{-1} &= \theta_w \theta_a^{-1} \theta_v^{-1} \\ &= \theta_w \theta_a^{-1} \theta_v^{-1} \theta_v \theta_v^{-1} \\ &= \theta_w \theta_w^{-1} \theta_v \theta_v^{-1}. \end{aligned}$$

So, $\theta_w \theta_w^{-1} \leq \theta_v \theta_v^{-1}$ where $\theta_v \theta_v^{-1} \in \mathcal{O}(N)$. We know that $D(\theta_v \theta_v^{-1}) = vL$ and $w = va \in vL$. However, $D(\theta_w \theta_w^{-1}) = wL$, and $w \notin wL$, therefore $\theta_w \theta_w^{-1} < \theta_v \theta_v^{-1}$.

If we consider $\theta_v \theta_v^{-1} \in \mathcal{O}(N)$, there must exist a word $w = va \in vL$ such that $a \in \mathcal{A}$, as L is language associated to a one-sided shift. Given that $w = va$, all $ws \in L$ hold the property that $ws = vas$, so $wL \subseteq vL$. Therefore, $\theta_w \theta_w^{-1} \leq \theta_v \theta_v^{-1}$, where $\theta_w \theta_w^{-1} \in \mathcal{O}(N+1)$. Again, given that $w \notin wL$ but $w \in vL$, it must be the case that $\theta_w \theta_w^{-1} < \theta_v \theta_v^{-1}$.

Lastly, for the sake of contradiction, let us assume there exists a $\theta_v\theta_v^{-1} \in \mathcal{O}(N)$ such that $\theta_w\theta_w^{-1} \geq \theta_v\theta_v^{-1}$ for some $\theta_w\theta_w^{-1} \in \mathcal{O}(N+1)$. We have already shown that there exists an $\theta_s\theta_s^{-1} < \theta_v\theta_v^{-1}$ where $\theta_s\theta_s^{-1} \in \mathcal{O}(N+1)$. Therefore, by transitivity, $\theta_s\theta_s^{-1} < \theta_w\theta_w^{-1}$, which is a contradiction by Proposition 3.1.4. Therefore, there can not exist a $\theta_v\theta_v^{-1} \in \mathcal{O}(N)$ such that $\theta_w\theta_w^{-1} \geq \theta_v\theta_v^{-1}$ for some $\theta_w\theta_w^{-1} \in \mathcal{O}(N+1)$.

Hence, we have proven our desired result, $\mathcal{O}(N+1) < \mathcal{O}(N)$. □

Corollary 3.2.8. *Let us consider the inverse hull of a one-sided shift of finite type, $H(S)$, and $N, P \in \mathbb{N}$ such that $N < P$. Then $\mathcal{O}(P) < \mathcal{O}(N)$.*

Proof. Let us consider sets A, B and C, such that $A < B$ and $B < C$. Then for all $a \in A$ there exists a $b \in B$, such that there is a $c \in C$, where $a < b < c$, by transitivity. Similarly, for all $c \in C$ there exists a $b \in B$, such that there is an $a \in A$, where $a < b < c$, by transitivity. Lastly, if there existed a $c \in C$ such that $c \leq a$ for some $a \in A$, then as there exists a $b \in B$, where $b < c \leq a$, however, that contradicts the assumption that $A < B$. Therefore there must exist no such $c \in C$, and we have shown that $A < C$ if $A < B$ and $B < C$.

Thus, given that $\mathcal{O}(N+1) < \mathcal{O}(N)$ by Proposition 3.2.7, $\mathcal{O}(P) < \mathcal{O}(N)$ for $N < P$. □

Above we showed that the sets $\mathcal{O}(N)$, for $N \in \mathbb{N}$, have specific relationships in the natural partial order on $E(H(S))$. In the next proposition, we consider the relationship of individual elements in the natural partial order on $E(H(S))$.

Proposition 3.2.9. *Let us consider the inverse hull of a one-sided shift of finite type, $H(S)$. If $0 \neq \theta_s t\theta_s^{-1} \leq \theta_w e\theta_w^{-1}$, for $t, e \in \mathcal{T}$ and $s, w \in L$, then $s = wu$ for some $u \in L^1$.*

Proof. Recall that if $\theta_s t\theta_s^{-1} \leq \theta_w e\theta_w^{-1}$, by Proposition 2.2.15, $D(\theta_s e\theta_s^{-1}) \subset D(\theta_w t\theta_w^{-1})$. Also, as $\theta_w e\theta_w^{-1} \leq \theta_w\theta_w^{-1}$, $D(\theta_w t\theta_w^{-1}) \subseteq wL$. Similarly, it must be the case that $D(\theta_s e\theta_s^{-1}) \subseteq sL$. Thus, for all $sa \in D(\theta_s e\theta_s^{-1})$ where $a \in \mathcal{A}$, $sa = wv$ for $v \in L$. Therefore, $v = ua$ for $u \in L^1$, and $s = wu$. □

Corollary 3.2.10. *Let us consider the inverse hull of a one-sided shift of finite type, $H(S)$. If $\theta_s t\theta_s^{-1} = \theta_w e\theta_w^{-1} \neq 0$, for $t, e \in \mathcal{T}$ and $s, w \in L$, then $s = w$ and $t\theta_s^{-1}\theta_s = e\theta_s^{-1}\theta_s$.*

Proof. by Proposition 3.2.9 we know that $s = wu$ and $w = sv$ for $u, v \in L^1$. Therefore, $u = v = 1$ and $s = w$.

Given that $s = w$, $\theta_s t\theta_s^{-1} = \theta_s e\theta_s^{-1} \neq 0$. Thus,

$$\theta_s^{-1}\theta_s t\theta_s^{-1}\theta_s = \theta_s^{-1}\theta_s e\theta_s^{-1}\theta_s.$$

Therefore, $t\theta_s^{-1}\theta_s = e\theta_s^{-1}\theta_s$. □

Now let us consider two propositions that explore properties that the inverse hull of a one-sided shift of finite type has, specifically due to being M-step.

Proposition 3.2.11. *Let us consider the inverse hull of a one-sided M-step shift, $H(S)$. Then for $s, w \in L$, where $|w| = M$, and $sw \neq 0$, $\theta_w\theta_w^{-1} \leq \theta_s^{-1}\theta_s$.*

Proof. Consider $s, w \in L$, where $|w| = M$, and $sw \neq 0$. Then by Lemma 2.1.10, for any $wu \in wL$, $swu \in L$. Therefore, $wL \subseteq D(\theta_s^{-1}\theta_s)$ and $\theta_w\theta_w^{-1} \leq \theta_s^{-1}\theta_s$. □

Proposition 3.2.12. *Let us consider the inverse hull of a one-sided M-step shift, $H(S)$. For all $s \in L$, $\theta_s^{-1}\theta_s$ is comparable to some $\theta_w\theta_w^{-1}$, for $|w| \leq M$.*

Proof. Let us consider $s \in L$. Then, given that L is associated to a one-sided shift, there exists a $w \in L$ such that $|w| = M$ and $sw \in L$. By Lemma 2.1.10, for any $wu \in L$, $swu \in L$. Therefore, $wL \subseteq D(\theta_s^{-1}\theta_s)$, and $\theta_w\theta_w^{-1} \leq \theta_s^{-1}\theta_s$. □

Given that we defined $\mathcal{A}$ to be a finite set, it follows that $\mathcal{O}(N)$ is a finite set for any $N \in \mathbb{N}$, as $\mathcal{O}(N) \subseteq \mathcal{A}^N$. We can prove that other sets of interest are also finite, given the finiteness of $\mathcal{A}$.

Proposition 3.2.13. *For any inverse hull of a one-sided M-step shift, $H(S)$, the following sets are finite,*

1. *$\mathcal{O}(N)$ for any $N \in \mathbb{N}$*
2. *$\mathcal{T}$*
3. *$e^{\uparrow}$ for all $e \in E(H(S))$.*

Proof. Recall that for any one-sided shift, $\mathcal{A}$ is defined to be a finite set. Therefore, as $\mathcal{T} = \{(\theta_{x_1}^{-1}\theta_{x_1} \ldots \theta_{x_n}^{-1}\theta_{x_n} \neq 0 : x_i \in L, |x_i| \leq M \text{ and } n \geq 1\}$, n is finite, and $\mathcal{O}(N)$ is finite for all $N \in \mathbb{N}$, $\mathcal{T}$ is also a finite set.

From Proposition 3.2.9, we know that for an arbitrary $e = \theta_s t\theta_s^{-1} \in \mathcal{O}(1)^{\downarrow}$, where $s \in L$ and $t \in \mathcal{T}$, all elements in $e^{\uparrow} \cap \mathcal{O}(1)^{\downarrow}$ must be of the form $\theta_w k\theta_w^{-1}$, where $s = wr$ for some $w \in L$, $r \in L^1$, and $k \in \mathcal{T}$. Given that $|s|$ and $|\mathcal{T}|$ are finite, there is only a finite number of such $\theta_w k\theta_w^{-1}$. Therefore, $e^{\uparrow} \cap \mathcal{O}(1)^{\downarrow}$ is a finite set. In addition, by Proposition 3.2.3, $T = (\mathcal{O}(1)^{\Downarrow})^c$. Thus, again due to the finiteness of $\mathcal{T}$, $e^{\uparrow}$ is a finite set.

For an arbitrary $e \in \mathcal{T}$, $e^{\uparrow} \subseteq \mathcal{T}$, which in turn means that $e^{\uparrow}$ must also be finite.

□

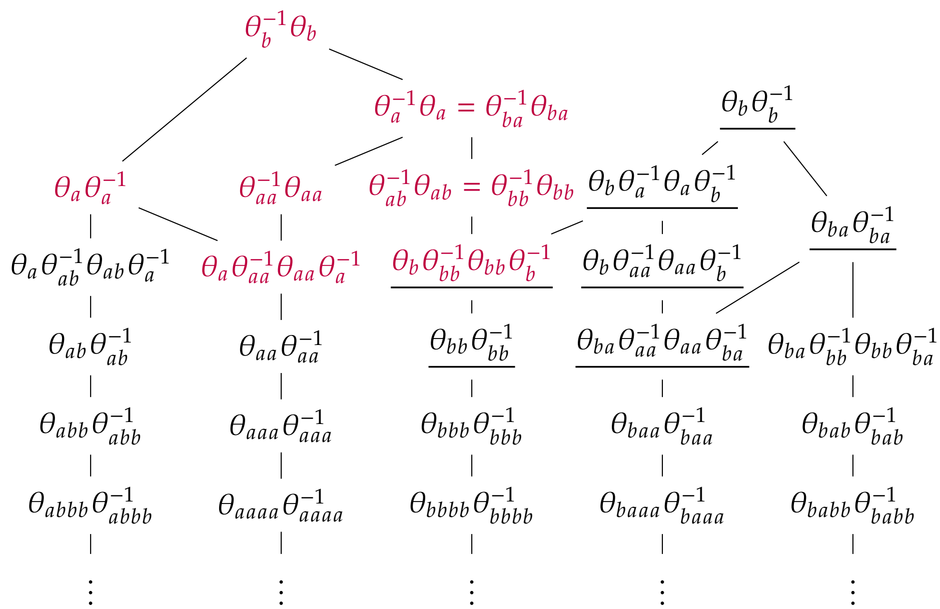
In the following section, a different color coding of the semilattice from Example 3.2.2 will illuminate even more extremely useful structure, and lead us to be able to fully describe the semilattice of $E(H(S))$ using a finite subset of $E(H(S))$.

3.3 Shapes

Let us consider the semilattice of a specific idempotent set, $E(H(S))$, where we color-code the elements of $E(H(S))$ based on their $\mathcal{D}$-classes. What is all the information we could glean from this color-coded semilattice? We would know the set of elements that make up $E(H(S))$. We would know the $\mathcal{D}$-classes associated with each element of $E(H(S))$, and we would know the natural partial order of $E(H(S))$.

In theory, these three very fundamental properties of $H(S)$ would make the semilattice of $E(H(S))$ an extremely useful tool. However, finding the semilattice for $E(H(S))$ is quite difficult to do. We have learned that there is the set $\mathcal{T}$ and there are the sets $\mathcal{O}(N)$ where $\mathcal{O}(P) < \mathcal{O}(N)$ for $N < P$. However, there are many other elements of $E(H(S))$ that do not fall into these nice well described sets, so determining the $\mathcal{D}$-classes of these elements, or their relation in the natural partial order, becomes very tedious computationally. In addition, $E(H(S))$ is an infinite set, so we will never be able to fully compute the entire semilattice. Thus can make it difficult when we are trying to discover the properties of $E(H(S))$, because it is possible that the hypothesised properties break farther down the semilattice then was computed. Therefore, it would be extraordinarily helpful to be able to compute the natural partial order of, and $\mathcal{D}$-classes on, a finite set of elements in $E(H(S))$, and then extrapolate the entirety of the rest of the color-coded semilattice for $E(H(S))$ from that small amount of computation.

It is with this goal in mind, that we turn our attention back to Example 3.2.2, with the set of forbidden word, $\mathcal{F} = \{bba, aba, aab\}$. We will color a set of elements plum, and we will underline the elements of another set.



A keen reader might start to see some interesting connections between the plum elements and the underlined elements. For example, there are the same number of underlined elements as plum elements. In fact, the underlined elements seem to have the same general "shape" as the plum elements, with one element at the top ($\theta_b\theta_b^{-1}$ and $\theta_b^{-1}\theta_b$ respectively), a diamond shape in the middle, and ending in three chains of elements. Potentially surprisingly, the underlined elements are equivalent to the set $\{\theta_b e\theta_b^{-1} : e \text{ is plum}\}$. Clearly, the plum and underlined elements are deeply related, and this begs the question, "what exactly are these plum elements?".

The plum elements consist of $\mathcal{T}$ and the elements directly beneath $\mathcal{T}$. Also, note that the maximal element of the plum set, $\theta_b^{-1}\theta_b$, and the maximal element of the underlined set, $\theta_b\theta_b^{-1}$, are deeply related. Thus, it is sensible to consider the plum elements as the intersection of $\theta_b^{-1}\theta_b^{\downarrow}$ with $\mathcal{T}$ union the elements directly beneath $\mathcal{T}$.

Let us formalize the set $\mathcal{T}$ union the elements directly beneath $\mathcal{T}$.

Definition 3.3.1. Let us consider the inverse hull of a one-sided shift of finite type, $H(S)$. We define $\mathcal{T}^+ = \mathcal{T} \cup \{e \in \mathcal{T}^{\downarrow} - \mathcal{T} : e \lessdot t \text{ for some } t \in \mathcal{T}\}$. For $e \in \mathcal{T}^+$ such that $e \notin \mathcal{T}$, e is *directly beneath* $\mathcal{T}$. Used standardly, $e \lessdot t$ is defined such that $e < t$ and there exists no $q \in E(H(S))$ such that $e < q < t$.

Note there exits no infinite chains of elements in $\mathcal{T}^{\downarrow} - \mathcal{T}$, such that for all $f \in \mathcal{T}^{\downarrow} - \mathcal{T}$, where $f < t$ for some $t \in \mathcal{T}$, there exists a $q \in \mathcal{T}^{\downarrow} - \mathcal{T}$ such that

$f < q < t$. This assertion is a direct result of Proposition 3.2.13 part 3. Also, it is clear that for all $e \in \mathcal{O}(1)$, such that $e \in \mathcal{T}^{\downarrow}$, $e \in \mathcal{T}^{+}$.

One extremely important property of $\mathcal{T}^{+}$ is that it is a finite set.

Proposition 3.3.2. *Let us consider the inverse hull of a one-sided M-step shift, $H(S)$. Then the set $\mathcal{T}^{+}$ is finite.*

Proof. Recall that for $\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n} \in \mathcal{T}$, $D(\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}) = \{w \in L : x_i w \in L, \forall i \in \{1,\ldots,n\}\}$. By Proposition 3.2.11, if $x_i w \in L$, for $|w| = M$, $\theta_w\theta_w^{-1} \leq \theta_{x_i}^{-1}\theta_{x_i}$. Thus, for any $w \in D(\theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n})$, where $|w| = M$, $\theta_w\theta_w^{-1} \leq \theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}$.

For the sake of contradiction, let us assume that $\mathcal{T}^{+}$ is infinite. Recall that by Proposition 3.2.13, $\mathcal{T}$ is a finite set. Thus if $\mathcal{T}^{+}$ is an infinite set, there would have to exist an infinite number of nonzero elements $\theta_s t\theta_s^{-1} \in \mathcal{T}^{+} - \mathcal{T}$, where $s \in L$ and $t \in \mathcal{T}$. Given that $t \in \mathcal{T}$, and again, $\mathcal{T}$ is finite, there must be an infinite number of $s \in L$ such that $\theta_s t\theta_s^{-1} \in \mathcal{T}^{+} - \mathcal{T}$, where $t \in \mathcal{T}$.

Given that there must be an infinite number of $s \in L$ such that $\theta_s t\theta_s^{-1} \in \mathcal{T}^{+} - \mathcal{T}$, where $t \in \mathcal{T}$, there must exist such an s, where $|s| > M$. Therefore, $s = wu$, for $w, u \in L$ and $|w| = M$. Recall that $\theta_s t\theta_s^{-1} \subseteq sL$, thus there exists an $sv \in sL$ such that $sv \in D(\theta_s t\theta_s^{-1})$. Therefore, for all $e \in \mathcal{T}$, such that $\theta_s t\theta_s^{-1} < e$, $w \in D(e)$. Thus, $\theta_s t\theta_s^{-1} < \theta_w\theta_w^{-1} < e$ for all $e > \theta_s t\theta_s^{-1}$, where $e \in T$, which a contradiction to the assumption that $\theta_s t\theta_s^{-1} \in \mathcal{T}^{+} - \mathcal{T}$.

Hence, we have shown that $\mathcal{T}^{+}$ is a finite set.

□

With the formalization of $\mathcal{T}^{+}$, we can define the plum elements in our example to be the set $\theta_b^{-1}\theta_b^{\downarrow} \cap \mathcal{T}^{+}$ and the underlined elements to be the set $\{\theta_b e\theta_b^{-1} : e \in \theta_b^{-1}\theta_b^{\downarrow} \cap \mathcal{T}^{+}\}$. More generally, we provide the following definition.

Definition 3.3.3. Let us consider the inverse hull of a one-sided shift of finite type, $H(S)$. For any $w \in L$ we define,

1. $\mathcal{T}_w = \theta_w^{-1}\theta_w^{\downarrow} \cap \mathcal{T}^{+}$. The set $\mathcal{T}_w$ is called the *preshape of* w.
2. $\mathcal{S}_w = \{\theta_w e\theta_w^{-1} : e \in \mathcal{T}_w\}$. The set $\mathcal{S}_w$ is called the *shape of* w.

It is interesting to note that for $\theta_w^{-1}\theta_w = \theta_v^{-1}\theta_v$, where $w = st$, $v = sr$, and $|s| = M$, for some $t, r \in L^1$, the preshapes of w and v are identical to $\mathcal{T}_s$. The following proposition shows that every nonzero element of $E(H(S))$ is an element of some $\mathcal{S}_w$ or $\mathcal{T}_w$, and for all words, $s \in L$, such that $|s| > 1$, $\theta_s\theta_s^{-1}$ is in a shape of a word that does not equal s.

Proposition 3.3.4. *Let us consider the inverse hull of a one-sided shift of finite type, $H(S)$. For all $e \in \mathcal{T}^+$, $e \in \mathcal{T}_w$ for some $w \in L$. For all nonzero $e \in O(1)^{\downarrow}$, $e \in \mathcal{S}_w$ for some $w \in L$. For all words $wa \in L$ where $a \in \mathcal{A}$ and $w \in L$, $\theta_{wa}\theta_{wa}^{-1} \in \mathcal{S}_w$.*

Proof. Let us consider an arbitrary $t \in \mathcal{T}$. We know that $t = \theta_{x_1}^{-1}\theta_{x_1} \dots \theta_{x_n}^{-1}\theta_{x_n}$, for $x_i \in L$. Thus as $\theta_{x_1}^{-1}\theta_{x_1} \dots \theta_{x_n}^{-1}\theta_{x_n} = \theta_{x_1}^{-1}\theta_{x_1}\theta_{x_1}^{-1}\theta_{x_1} \dots \theta_{x_n}^{-1}\theta_{x_n}$, it must be the case that $t \leq \theta_{x_1}^{-1}\theta_{x_1}$. Therefore, $t \in \mathcal{T}_{x_1}$. Given that for all $e \in \mathcal{T}^+ - \mathcal{T}$, it must be the case that $e < t$, for some $t \in \mathcal{T}$, every $e \in \mathcal{T}^+ - \mathcal{T}$ is also in some $\mathcal{T}_w$, for $w \in L$.

Now let us consider some nonzero $e \in O(1)^{\downarrow}$. Therefore, $e = \theta_w t \theta_w^{-1}$ for some $t \in \mathcal{T}$ and $w \in L$. If $t\theta_w^{-1}\theta_w = 0$, then $e = \theta_w t \theta_w^{-1} = \theta_w t \theta_w^{-1}\theta_w\theta_w^{-1} = 0$, which is a contradiction that e is nonzero. If $t\theta_w^{-1}\theta_w \neq 0$, then $t\theta_w^{-1}\theta_w \in \mathcal{T}^+$ and $t\theta_w^{-1}\theta_w \leq \theta_w^{-1}\theta_w$. Therefore, $t\theta_w^{-1}\theta_w \in \mathcal{T}_w$. Given that $e = \theta_w t\theta_w^{-1} = \theta_w t\theta_w^{-1}\theta_w\theta_w^{-1}$, by definition, $e \in \mathcal{S}_w$.

Lastly, let us consider some word $wa \in L$, where $a \in \mathcal{A}$ and $w \in L$. Recall that if $wa \in L$, there exists a $wau \in L$ for some $u \in L$. Therefore, $\theta_a\theta_a^{-1}\theta_w^{-1}\theta_w \neq 0$. Note that $\theta_a\theta_a^{-1}\theta_w^{-1}\theta_w = \theta_a\theta_{wa}^{-1}\theta_{wa}\theta_a^{-1} < \theta_w^{-1}\theta_w$. Thus, there must exist an $e \in \mathcal{T}_w - \mathcal{T}$ such that $e \geq \theta_a\theta_{wa}^{-1}\theta_{wa}\theta_a^{-1}$. As $O(1)$ is mutually orthogonal, by Proposition 3.1.4, $e = \theta_a t \theta_a^{-1}$ for some $t \in \mathcal{T}$. However, given that $e \in \mathcal{T}_w$,

$$e = \theta_w^{-1}\theta_w\theta_a t\theta_a^{-1} = \theta_a t\theta_{wa}^{-1}\theta_{wa}\theta_a^{-1}.$$

As, $\theta_a t\theta_{wa}^{-1}\theta_{wa}\theta_a^{-1} \leq \theta_a\theta_{wa}^{-1}\theta_{wa}\theta_a^{-1}$, it must be the case that $e = \theta_a\theta_{wa}^{-1}\theta_{wa}\theta_a^{-1}$. By definition, $\theta_w e\theta_w^{-1} \in \mathcal{S}_w$ and,

$$\theta_w e\theta_w^{-1} = \theta_w\theta_a\theta_{wa}^{-1}\theta_{wa}\theta_a^{-1}\theta_w^{-1} = \theta_{wa}\theta_{wa}^{-1}\theta_{wa}\theta_{wa}^{-1} = \theta_{wa}\theta_{wa}^{-1}.$$

Thus, $\theta_{wa}\theta_{wa}^{-1} \in \mathcal{S}_w$

□

Now that we know that every element is in a shape or a preshape, it becomes an interesting question to ask if there is any useful mappings between $\mathcal{T}_w$ and $\mathcal{S}_w$, with the hopes of fully understanding the color-coded semilattice of $E(H(S))$ using only $\mathcal{T}^+$.

Definition 3.3.5. Let us consider the inverse hull of a one-sided shift of finite type, $H(S)$. For $w \in L$, we define the following mapping $\Delta_w : \mathcal{T}_w \to \mathcal{S}_w$, such that $\Delta_w(e) = \theta_w e\theta_w^{-1}$, for all $e \in \mathcal{T}_w$.

In the next two propositions, we will show that Δ_w respects $\mathcal{D}$-classes and is a bijection.

Proposition 3.3.6. *Let us consider the inverse hull of a one-sided shift of finite type, $H(S)$. The mapping Δ_w, for $w \in L$, respects $\mathcal{D}$-classes.*

Proof. For any nonzero $e \in \mathcal{T}_w$, it must be the case that $\theta_w e \neq 0$. We find that

$$\theta_w e(\theta_w e)^{-1} = \theta_w e e^{-1}\theta_w^{-1} = \theta_w e\theta_w^{-1}$$

and,

$$(\theta_w e)^{-1}\theta_w e = e\theta_w^{-1}\theta_w e = ee = e.$$

Therefore, $\theta_w e\theta_w^{-1}\mathcal{D}e$, which means that Δ_w respects $\mathcal{D}$-classes. □

Proposition 3.3.7. *Let us consider the inverse hull of a one-sided shift of finite type, $H(S)$. The mapping Δ_w, for $w \in L$, is a bijection.*

Proof. Let us consider the mapping $\Delta_w : \mathcal{T}_w \to \mathcal{S}_w$, where $\Delta_w(e) = \theta_w e\theta_w^{-1}$. Recall that $\mathcal{S}_w = \{\theta_w e\theta_w^{-1} : e \in \mathcal{T}_w\}$. Thus, Δ_w is surjective.

Let us assume that there exists elements $e, k \in \mathcal{T}_w$, such that $\theta_w e\theta_w^{-1} = \theta_w k\theta_w^{-1}$. Then,

$$\theta_w^{-1}\theta_w e\theta_w^{-1}\theta_w = \theta_w^{-1}\theta_w k\theta_w^{-1}\theta_w.$$

As, $e, k \in \mathcal{T}_w$, $e = e\theta_w^{-1}\theta_w$ and $k = k\theta_w^{-1}\theta_w$. Thus, $e = k$, meaning that Δ_w is injective. □

In the next proposition, we will show that Δ_w respects the natural partial order of $E(H(S))$.

Proposition 3.3.8. *Let us consider the inverse hull of a one-sided shift of finite type, $H(S)$. The mapping Δ_w, for $w \in L$, respects the natural partial order of $E(H(S))$.*

Proof. Let us consider nonzero elements $e, t \in \mathcal{T}_w$. If $e < t$ then,

$$\theta_w e\theta_w^{-1}\theta_w t\theta_w^{-1} = \theta_w et\theta_w^{-1} = \theta_w e\theta_w^{-1}.$$

Therefore $\theta_w e\theta_w^{-1} \leq \theta_w t\theta_w^{-1}$. Given that Δ_w is a bijection, by Proposition 3.3.7, $\theta_w e\theta_w^{-1} < \theta_w t\theta_w^{-1}$.

Now let us assume that eIt, meaning that $e \nleq t$ and $e \ngeq t$. For the sake of contradiction, we will assume that $\theta_w e\theta_w^{-1}$ and $\theta_w t\theta_w^{-1}$ are not incomparable. Without loss of generality, let us assume that $\theta_w e\theta_w^{-1} \leq \theta_w t\theta_w^{-1}$. Therefore, $\theta_w e\theta_w^{-1} = \theta_w et\theta_w^{-1}$. So, it must be the case that,

$$\theta_w^{-1}\theta_w e\theta_w^{-1}\theta_w = \theta_w^{-1}\theta_w et\theta_w^{-1}\theta_w.$$

As $e, t \in \mathcal{T}_w$, $e = e\theta_w^{-1}\theta_w$ and $et = et\theta_w^{-1}\theta_w$. Therefore, $e = et$, which means that $e < t$, a contradiction that eIt.

Hence, we have shown that the mapping Δ_w respects the natural partial order of $E(H(S))$. □

So we know the set of mappings Δ_w, for $w \in L$, allow shapes and preshapes to completely describe the elements of $E(H(S))$ and their $\mathcal{D}$-classes. We have also shown that shapes and preshapes respect the natural partial order, so the last thing that we must prove is that shapes and preshapes completely describe the natural partial order of $E(H(S))$, to satisfy our goal of extrapolating the entirety of the color-coded semilattice for $E(H(S))$, using a finite set, namely $\mathcal{T}^+$.

Lemma 3.3.9. *Let us consider the inverse hull of a one-sided shift of finite type, $H(S)$. For any $e \in \mathcal{O}(1)^{\downarrow}$, there exists a shape, $\mathcal{S}_w$ such that $e, k \in \mathcal{S}_w$ for all $k \lessdot e$.*

Proof. Let $e = \theta_w t\theta_w^{-1} \neq 0$ where $t \in \mathcal{T}$ and $w \in L$. Then $e \in \mathcal{S}_w$, given that $t\theta_w^{-1}\theta_w \in \theta_w^{-1}\theta_w^{\downarrow} \cap \mathcal{T}$ and $\theta_w t\theta_w^{-1} = \theta_w t\theta_w^{-1}\theta_w\theta_w^{-1}$. For the sake of notational simplicity, let $t = t\theta_w^{-1}\theta_w$.

For the sake of contradiction, let us assume that there exists an $k \lessdot e$ such that $k \notin \mathcal{S}_w$.

First, let us assume that $k \neq 0$. We know that $k = \theta_s f\theta_s^{-1}$, for some $f \in \mathcal{T}$ and $s \in L$. By Proposition 3.2.9, we know that $s = wu$ for $u \in L^1$. So, $k = \theta_w\theta_u f\theta_u^{-1}\theta_w^{-1}$. Given that $k < e$,

$$k = \theta_w t\theta_w^{-1}\theta_w\theta_u f\theta_u^{-1}\theta_w^{-1} = \theta_w t\theta_u f\theta_u^{-1}\theta_w^{-1}.$$

Given $k \notin \mathcal{S}_w$, there must be a $q \in E(H(S))$, such that $t\theta_u f\theta_u^{-1} < q < t$, else $t\theta_u f\theta_u^{-1} \lessdot t$. Therefore, $t\theta_u f\theta_u^{-1} < q < t < \theta_w\theta_w^{-1}$ and

$$D(t\theta_u f\theta_u^{-1}) \subsetneq D(q) \subsetneq D(t) \subsetneq D(\theta_w\theta_w^{-1}).$$

Thus, for all x in $D(t\theta_u f\theta_u^{-1})$, $D(q)$, or $D(t)$, it is the case that $wx \in L$. So, $D(\theta_w t\theta_u f\theta_u^{-1}\theta_w^{-1}) = \{wx : x \in D(t\theta_u f\theta_u^{-1})\}$, $D(\theta_w q\theta_w^{-1}) = \{wx : x \in D(q)\}$, and $D(\theta_w t\theta_w^{-1}) = \{wx : x \in D(t)\}$. Therefore,

$$D(\theta_w t\theta_u f\theta_u^{-1}\theta_w^{-1}) \subsetneq D(\theta_w q\theta_w^{-1}) \subsetneq D(\theta_w t\theta_w^{-1}).$$

So,

$$(\theta_w t\theta_u f\theta_u^{-1}\theta_w^{-1} = k) < \theta_w q\theta_w^{-1} < (\theta_w t\theta_w^{-1} = e).$$

Hence, it is not the case that $k \lessdot e$, which is a contradiction.

Similarly if $k \neq 0$, there must there must be a q such that $0 < q < t$, and the same argument from above holds, where it must be the case that $0 < \theta_w q \theta_w^{-1} < \theta_w t \theta_w^{-1}$. □

Now, given Proposition 3.2.13 part 3, we know that $e^{\uparrow}$ is finite for all $e \in E(H(S))$, therefore if $e < f$, for $e, f \in \mathcal{O}(1)^{\downarrow}$, there are a finite set of elements of $E(H(S))$, $\{k_1, \ldots k_n\}$, such that $e \lessdot k_1 \lessdot \cdots \lessdot k_n \lessdot f$: ie there is a finite saturated chain between e and f. From Lemma 3.3.9, we know that there exists words w_0 through w_n such that $e, k_1 \in \mathcal{S}_{w_0}$, $k_1, k_2 \in \mathcal{S}_{w_1}$, and so on. Therefore, Lemma 3.3.9 and Proposition 3.3.8 imply that,

Proposition 3.3.10. *for elements $e, f \in E(H(S))$, $e < f$ if, and only if, there exists a finite set of shapes, $\{\mathcal{S}_{w_0}, \ldots, \mathcal{S}_{w_n}\}$, such that $e, k_1 \in \mathcal{S}_{w_0}$, $k_1, k_2 \in \mathcal{S}_{w_1}$, ..., $k_n, f \in \mathcal{S}_{w_n}$, where $e \lessdot k_1 \lessdot \cdots \lessdot k_n \lessdot f$.*

We have proven that using shapes we are able to extrapolate the entirety of the color-coded semilattice for $E(H(S))$, computing the natural partial order of, and $\mathcal{D}$-classes on, the finite set $\mathcal{T}^+ \in E(H(S))$.

Though shapes serve their motivating purpose, allowing the color-coded semilattice of $E(H(S))$ to be fully described using a finite subset of $E(H(S))$, they can also serve as a useful algebraic tool in the proofs for properties of $H(S)$.

One useful feature of shapes is the following:

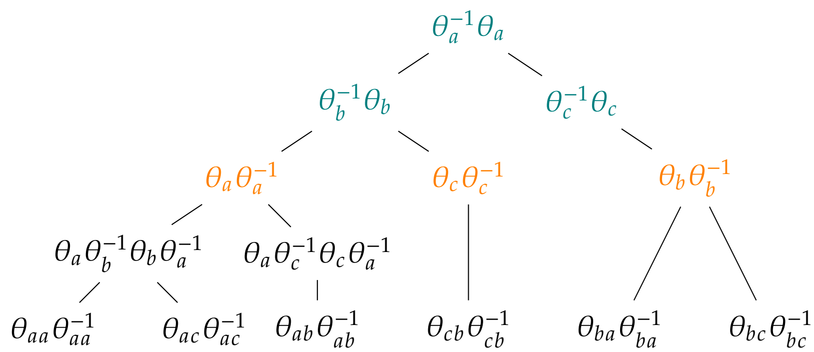
Proposition 3.3.11. *Let $H(S)$ be the inverse hull of a one-sided shift of finite type. For all $e \;\mathcal{D}\; t$, where $t \in \mathcal{T}$ and $e \in \mathcal{O}(1)^{\downarrow}$, there exists a $w \in L$ such that $\Delta_w(t) = e$.*

Proof. Recall that for nonzero $e \in \mathcal{O}(1)^{\downarrow}$, $e = \theta_w t \theta_w^{-1}$ for $t \in \mathcal{T}$ and $w \in L$. Thus, $\theta_w^{-1}\theta_w t \neq 0$ and $\theta_w^{-1}\theta_w t \in \mathcal{T}_w \cap \mathcal{T}$. Therefore, $\Delta_w(\theta_w^{-1}\theta_w t) = \theta_w \theta_w^{-1} \theta_w t \theta_w = e$. Also, by Proposition 3.3.6, $\theta_w^{-1}\theta_w t \mathcal{D} e$. Thus, by Proposition 3.2.4, for all $e \;\mathcal{D}\; t$, where $t \in \mathcal{T}$ and $e \in \mathcal{O}(1)^{\downarrow}$, there exists a $w \in L$ such that $\Delta_w(t) = e$. □

3.3.1 Shapes And One-sided 1-step Shifts

In Chapter 4, we will employ shapes to prove a result regarding one-sided 1-step shifts. Due to the unique structure of the inverse hulls of one-sided 1-step shifts, shapes have a few extra properties in this setting. Note, another name for one-sided 1-step shifts is Markov shift.

Example 3.3.12. Let us consider the Markov shift associated to the set of forbidden words, $\mathcal{F} = \{bb, ca, cc\}$. Below we have the set $\mathcal{O}(1)$ denoted in orange, and $\mathcal{T}$ denoted in teal.



We see that in Example 3.3.12, $\mathcal{T} = \mathcal{O}(1)^{\uparrow} - \mathcal{O}(1)$. In the article Beaupré et al. (2021), we prove that for Markov shifts, every element of $E(H(S))$ is comparable to some element of $\mathcal{O}(1)$ and $\mathcal{T} = \mathcal{O}(1)^{\uparrow} - \mathcal{O}(1)$. This is because for a 1-step shift, if $wa \in L$, for $a \in \mathcal{A}$, then for any $as \in L$, $was \in L$, by Lemma 2.1.10. So, $wa \in L$ if, an only if, $\theta_a\theta_a^{-1} < \theta_w^{-1}\theta_w$. Therefore, for any $t \in \mathcal{T}$, such that $t = \theta_{x_1}^{-1}\theta_{x_1}\ldots\theta_{x_n}^{-1}\theta_{x_n}$,

$$D(t) = \{a \in \mathcal{A} : x_i a \in L, \forall 1 \leq i \leq n\} \cup \{aL : a \in \mathcal{A} \text{ and } x_i a \in L, \forall 1 \leq i \leq n\}.$$

Thus, $t \in \mathcal{T}$ if, an only if, there exits an $e \in \mathcal{O}(1)$ such that $e < t$.

Proposition 3.3.13. *Let $H(S)$ be the inverse hull of a Markov shift. Then, $\mathcal{T} = \mathcal{O}(1)^{\uparrow} - \mathcal{O}(1)$.*

Also, in Example 3.3.12, we see that $\mathcal{T}^{+} = T \cup (\mathcal{O}(1) \cap \mathcal{T}^{\downarrow})$, which turns out to be true for all Markov shifts.

Proposition 3.3.14. *Let $H(S)$ be the inverse hull of a Markov shift. Then $\mathcal{T}^{+} = \mathcal{T} \cup (\mathcal{O}(1) \cap \mathcal{T}^{\downarrow})$.*

Proof. If $\theta_a\theta_a^{-1}t \neq 0$ for $a \in \mathcal{A}$ and $t \in \mathcal{T}$, then there exists an $as \in L$ such that $as \in D(t)$. Thus $aL \subseteq D(t)$ and $\theta_a\theta_a^{-1} < t$. Therefore, $\theta_a\theta_a^{-1}t = \theta_a\theta_a^{-1}$. As $\mathcal{O}(1) \cup 0$ and $\mathcal{T} \cup 0$ are closed sets, $\mathcal{O}(1)^{\uparrow} \cup \{0\}$ is a closed set for Markov shifts.

For $e \in \mathcal{O}(1)^{\downarrow}$ such that $e \lessdot t$ for some $t \in \mathcal{T}$, $e \leq t\theta_a\theta_a^{-1}$ for some $a \in \mathcal{A}$. However, as $t\theta_a\theta_a^{-1} = \theta_a\theta_a^{-1} < t$, $e = \theta_a\theta_a^{-1}$.

Hence, $\mathcal{T}^{+} = \mathcal{T} \cup (\mathcal{O}(1) \cap \mathcal{T}^{\downarrow})$. □

For Markov shifts, this new description of $\mathcal{T}^{+}$ reveals more shape structure.

Proposition 3.3.15. *Let $H(S)$ be the inverse hull of a Markov shift. For all $w \in L$, $\mathcal{S}_w = \theta_w\theta_w^{-1\downarrow} \cap O(|w|+1)^{\uparrow}$.*

Proof. By definition $\mathcal{S}_w = \{\theta_w e \theta_w^{-1} : e \in \mathcal{T}_w\}$ where $\mathcal{T}_w = \theta_w^{-1}\theta_w^{\downarrow} \cap \mathcal{T}^+$.

First we will show that $\theta_w\theta_w^{-1\downarrow} \cap O(|w|+1)^{\uparrow} \subseteq \mathcal{S}_w$. If $e \in \theta_w\theta_w^{-1\downarrow} \cap O(|w|+1)^{\uparrow}$, then $e = \theta_w r \theta_w^{-1}$ for $r = \theta_s t \theta_s^{-1}$, where $s \in L^1$ and $t \in \mathcal{T}$, by Proposition 3.2.9. Also, it must be the case that $|s| \leq 1$. If $|s| = 1$, then $e = \theta_{ws}\theta_{ws}^{-1}$, given that $e \in O(|w|+1)^{\uparrow}$. Therefore, $e = \theta_w r \theta_w^{-1}$ for $r \in \mathcal{T}^+$, by Proposition 3.3.14. As we can write $e = \theta_w\theta_w^{-1}\theta_w r \theta_w^{-1}$, and $\theta_w^{-1}\theta_w r \in \mathcal{T}^+$ by closure of $\mathcal{T}^+$, $e \in \mathcal{S}_w$.

Now we will show that $\mathcal{S}_w \subseteq \theta_w\theta_w^{-1\downarrow} \cap O(|w|+1)^{\uparrow}$. Consider $e \in \mathcal{S}_w$. Thusn $e = \theta_w r \theta_w^{-1}$ such that $r \in \mathcal{T}_w$. If $r \in \mathcal{T}_w - \mathcal{T}$ then, by Proposition 3.3.14, $r \in O(1)$. Thus, $e \in O(|w|+1)$. So, given that shapes respect the natural partial order of $E(H(S))$, by Proposition 3.3.8, for any $e \in \mathcal{S}_w$, $e \in O(|w|+1)^{\uparrow}$. Also, $\theta_w\theta_w^{-1}\theta_w\theta_w^{-1} = \theta_w\theta_w^{-1}$, hence, also by Proposition 3.3.8, for any $e \in \mathcal{S}_w$, $e \in \theta_w\theta_w^{-1\downarrow}$. So, we showed that $\mathcal{S}_w \subseteq \theta_w\theta_w^{-1\downarrow} \cap O(|w|+1)^{\uparrow}$.

Therefore, we proved our desired result that $\mathcal{S}_w = \theta_w\theta_w^{-1\downarrow} \cap O(|w|+1)^{\uparrow}$. □

Proposition 3.3.16. *Let $H(S)$ be the inverse hull of a Markov shift. Let us consider nonzero $x \in O(1)^{\downarrow} - O(1)$. If there exists $\theta_w\theta_w^{-1} \neq x$ and $\theta_s\theta_s^{-1} \neq x$, such that $x \in \mathcal{S}_w$ and $x \in \mathcal{S}_s$, then $\theta_s\theta_s^{-1} = \theta_w\theta_w^{-1}$*

Proof. By Proposition 3.3.15, given $x \in \mathcal{S}_w$, we know that $x \in (\theta_w\theta_w^{-1\downarrow} - \theta_w\theta_w^{-1}) \cap O(|w|+1)$, and similarly we know that $x \in (\theta_s\theta_s^{-1\downarrow} - \theta_s\theta_s^{-1}) \cap O(|s|+1)^{\uparrow}$. Note that $x \in (\theta_w\theta_w^{-1\downarrow} - \theta_w\theta_w^{-1}) \cap O(|w|+1)^{\uparrow} \subseteq (O(|w|)^{\downarrow} - O(|w|)) \cap O(|w|+1)^{\uparrow}$ and $x \in (\theta_s\theta_s^{-1\downarrow} - \theta_s\theta_s^{-1}) \cap O(|s|+1)^{\uparrow} \subseteq (O(|s|)^{\downarrow} - O(|s|)) \cap O(|s|+1)^{\uparrow}$. Thus, $|s| = |w|$, by Corollary 3.2.8. Also, as $x \in \theta_s\theta_s^{-1\downarrow}$ and $x \in \theta_w\theta_w^{-1\downarrow}$, $x = \theta_s\theta_s^{-1}\theta_w\theta_w^{-1}x \neq 0$. Therefore, by Proposition 3.1.4, $s = w$. □

Using these propositions, our understanding of shapes developed in Section 3.3, and the paper Beaupré et al. (2021), we will prove a major result in the next chapter. Note that due to the prevalence of shapes in the proofs of Chapter 4, many of the particular propositions from Section 3.3 will be used without reference, in the interest of clarity.

Chapter 4

Isomorphic Inverse Hulls of Markov Shifts

In this chapter, we will focus our attention on the inverse hulls of Markov shifts. The inverse hulls of Markov shifts was the primary focus of our former research with David Milan, Anthony Dickson, and Christin Sum (Beaupré et al. (2021)). Using a set of properties, which will be fully explained in Section 4.1, we characterized the inverse hulls of Markov shifts. With this characterization we were able to show that two non-isomorphic Markov shifts can result in isomorphic inverse hulls. Even more strongly, we showed that non-conjugate Markov shifts can result in isomorphic inverse hulls, while conjugate Markov shifts can have non-isomorphic inverse hulls (Beaupré et al. (2021), section 5). This implies that the common shift invariants that are studied in symbolic dynamics with regards to shift conjugacy (ex, entropy), are not capturing the shifts' structural similarities or differences that are captured by considering the inverse hulls of said shifts.

Given that we have already shown in this thesis the rich structural information of the inverse hulls of one-sided shifts of finite type, the above results in our past research beg the question, "what shifts invariant exist with respect to shifts that have isomorphic inverse hulls?". In Beaupré et al. (2021), we conjectured that "isomorphic inverse hulls of Markov shifts must have associated alphabets that are the same size." In this chapter, we will prove this conjecture.

First, we will introduce the main results of Beaupré et al. (2021), which we will use to prove our conjecture. Note that for Markov shifts, we will denote $\mathcal{O}(1)$ as $\mathcal{O}$ in this chapter, due to the uniquely important properties of $\mathcal{O}(1)$ when studying Markov shifts. Also in this chapter, as $\mathcal{T} = \mathcal{O}^{\uparrow} - \mathcal{O}$ for Markov shifts, we will refer to $\mathcal{T}$ as $\mathcal{O}^{\uparrow} - \mathcal{O}$.

4.1 Former Research on the Inverse Hulls of Markov Shifts

In Beaupré et al. (2021), we prove the following characterization of the inverse hull of a Markov shift.

Theorem 4.1.1. *Let H be an inverse semigroup with 0. Then H is isomorphic to the inverse hull of a Markov subshift if, and only if,*

1. *H is combinatorial*
2. *there is a set $\mathcal{O}$, of nonzero idempotents in H, satisfying* $(O1) - (O5)$
3. *the language L associated with $\mathcal{O}$ generates H.*

We define the properties $(O1) - (O5)$ to be:

(O1) the elements of $\mathcal{O}$ are mutually orthogonal

(O2) every idempotent in H is comparable to some element of $\mathcal{O}$

(O3) both $\mathcal{O}^{\uparrow} \cup \{0\}$ and $(\mathcal{O}^{\uparrow} - \mathcal{O}) \cup \{0\}$ are closed under multiplication

(O4) elements of $\mathcal{O}^{\uparrow} - \mathcal{O}$ are uniquely determined by the set of idempotents in $\mathcal{O}$ that they lie above

(O5) for each $e \in \mathcal{O}$, the $\mathcal{D}$-class of e contains at most one element of $\mathcal{O}^{\uparrow} - \mathcal{O}$.

The *language associated with $\mathcal{O}$, L*, is defined as follows:

$$\mathcal{A} = \{a \in H : a^*a \in \mathcal{O}^{\uparrow} - \mathcal{O} \text{ and } aa^* \in \mathcal{O}\}, \text{ and}$$
$$L = \{a_1 a_2 \ldots a_n \neq 0 : n \in \mathbb{N}, a_i \in A\}.$$

See Beaupré et al. (2021) for a proof of theorem 4.1.1. Note that due to the special properties of Markov shifts, we can provide a few new propositions, which increase the ease of operating elements in $H(S)$.

Proposition 4.1.2. *Let $H(S)$ be the inverse hull of a Markov shift. then for $s, w \in L$,*

$$\theta_s^{-1}\theta_w = \begin{cases} \theta_{w'} & \text{if } w = sw' \text{ for some } w' \in L, \\ \theta_{s'}^{-1} & \text{if } s = ws' \text{ for some } s' \in L, \\ \theta_{l(s)}^{-1}\theta_{l(s)} & \text{if } s = w, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

and,

$$\theta_s^{-1}\theta_s\theta_w = \begin{cases} \theta_w & \text{if } sw \in L \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First we will prove the case when $w = sw'$ for some $w' \in L$. If $w = sw'$, then $\theta_s^{-1}\theta_w = \theta_s^{-1}\theta_s\theta_{w'}$. Given that $H(S)$ is the inverse hull of a Markov shift, for all $w'u \in L$, $sw'u \in L$, by Lemma 2.1.10. Therefore, $D(\theta_s^{-1}\theta_s\theta_{w'}) = D(\theta_{w'})$, and $\theta_s^{-1}\theta_w = \theta_{w'}$.

The proof that $\theta_s^{-1}\theta_w = \theta_{s'}^{-1}$ if $s = ws'$, for some $s' \in L$, is nearly identical to the above argument, so we leave it as an exercise to the reader.

If $s = w$, then $\theta_s^{-1}\theta_w = \theta_s^{-1}\theta_s$. By Lemma 2.3.3, it must be the case that $\theta_s^{-1}\theta_s = \theta_{l(s)}^{-1}\theta_{l(s)}$. Thus $\theta_s^{-1}\theta_w = \theta_{l(s)}^{-1}\theta_{l(s)}$.

Lastly, if we consider $\theta_s^{-1}\theta_s\theta_w \neq 0$, then $\theta_s^{-1}\theta_s\theta_w = \theta_s^{-1}\theta_{sw}$. However, $\theta_s^{-1}\theta_{sw}$ falls into the first case above (where $w = sw'$ for some $w' \in L$), which means that $\theta_s^{-1}\theta_s\theta_w = \theta_w$. □

4.2 Proof Markov Shift Conjecture

In this section, we will prove the conjecture that isomorphic inverse hulls of Markov shifts must have the same sized alphabets. Throughout this section we will assume that we have two isomorphic inverse hulls of Markov Shifts, $H(S_1)$ and $H(S_2)$, with corresponding $\mathcal{O}$ sets, $\mathcal{O}_1$ and $\mathcal{O}_2$. Given that $H(S_1)$ and $H(S_2)$ are isomorphic, there exists a unique isomorphism $\Gamma : H(S_2) \to H(S_1)$ that respects the natural partial order of elements, element operations, and $\mathcal{D}$-classes.

Our proof will utilize the fact that $|\mathcal{O}| = |\mathcal{A}|$, as we will prove that it must be the case that $|\mathcal{O}_1| = |\mathcal{O}_2|$. Even more strongly, we will show that $|\mathcal{O}_1 \cap \mathcal{D}_t| = |\Gamma(\mathcal{O}_2) \cap \mathcal{D}_t|$ for any $t \in \mathcal{O}_1^{\uparrow} - \mathcal{O}_1$. As Γ is an isomorphism that respects $\mathcal{D}$-classes, $|\Gamma(\mathcal{O}_2) \cap \mathcal{D}_t| = |\mathcal{O}_2 \cap \mathcal{D}_{\Gamma^{-1}(t)}|$. Given Proposition 3.2.4, every element of $E(H(S))$ is $\mathcal{D}$-related to a unique $t \in \mathcal{T}$. Therefore, proving that $|\mathcal{O}_1 \cap \mathcal{D}_t| = |\Gamma(\mathcal{O}_2) \cap \mathcal{D}_t| = |\mathcal{O}_2 \cap \mathcal{D}_{\Gamma^{-1}(t)}|$ for all $t \in \mathcal{O}_1^{\uparrow} - \mathcal{O}_1$, would prove that $|\mathcal{O}_1| = |\mathcal{O}_2|$.

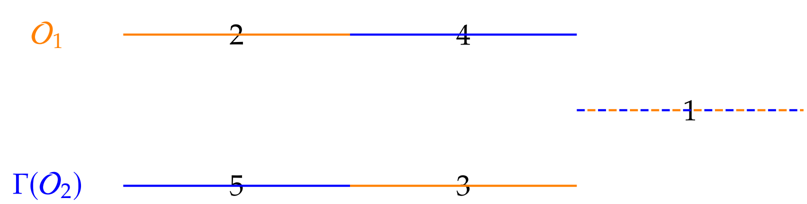
Throughout this chapter, again and again, we will have to understand the relationship that different $e \in E(H(S_1))$ have with $\mathcal{O}_1$ and $\Gamma(\mathcal{O}_2)$. For example, e could be an element of $\mathcal{O}_1^{\uparrow}$, while also being an element of $\Gamma(\mathcal{O}_2^{\downarrow}) - \Gamma(\mathcal{O}_2)$. Sometimes it is helpful to visualize these relationships, when we start looking at many elements of $E(H(S_1))$.

By Property (O2) of Theorem 4.1.1, we know that every element of $\Gamma(\mathcal{O}_2)$ is comparable to some element of $\mathcal{O}_1$ and vice-versa. Also, as $\mathcal{O}(N)$ is mutually orthogonal, for all $N \in \mathbb{N}$, by Proposition 3.1.4, if $e \in \mathcal{O}_1$ and $e > x$ for some $x \in \Gamma(\mathcal{O}_2)$, then there can not be an $f \in \Gamma(\mathcal{O}_2)$ such that $f \geq e$ or an $g \in \mathcal{O}_1$ such that $g \leq x$. In other words,

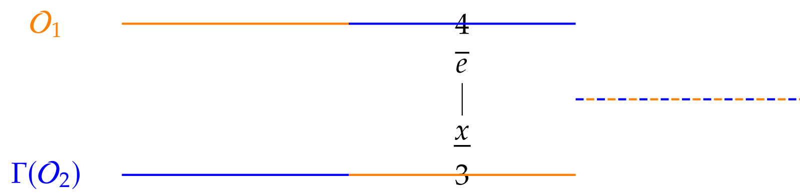
Remark 4.2.1. We can partition the set $\mathcal{O}_1 \cup \Gamma(\mathcal{O}_2)$ into five subsets:

1. $\mathcal{O}_1 \cap \Gamma(\mathcal{O}_2) = \{e : e \in \mathcal{O}_1 \text{ and } e \in \Gamma(\mathcal{O}_2)\}$
2. $\mathcal{O}_1 \cap (\Gamma(\mathcal{O}_2)^{\uparrow} - \Gamma(\mathcal{O}_2)) = \{e : e \in \mathcal{O}_1 \text{ and } e > x \text{ for some } x \in \Gamma(\mathcal{O}_2)\}$
3. $\mathcal{O}_1 \cap (\Gamma(\mathcal{O}_2)^{\downarrow} - \Gamma(\mathcal{O}_2)) = \{e : e \in \mathcal{O}_1 \text{ and } e < x \text{ for some } x \in \Gamma(\mathcal{O}_2)\}$
4. $(\mathcal{O}_1^{\uparrow} - \mathcal{O}_1) \cap \Gamma(\mathcal{O}_2) = \{e : e \in \Gamma(\mathcal{O}_2) \text{ and } e > x \text{ for some } x \in \mathcal{O}_1\}$
5. $(\mathcal{O}_1^{\downarrow} - \mathcal{O}_1) \cap \Gamma(\mathcal{O}_2) = \{e : e \in \Gamma(\mathcal{O}_2) \text{ and } x < e \text{ for some } x \in \mathcal{O}_1\}$

In the interest of visualization, we can consider the below line diagram, where the five colored lines represent the five subsets of $\mathcal{O}_1 \cup \Gamma(\mathcal{O}_2)$ (as described in Remark 4.2.1), and are labeled accordingly:



We can visualize the relationships of elements of $E(H(S_1))$ with $\mathcal{O}_1$ and $\Gamma(\mathcal{O}_2)$, and each other, using the above diagram. For example, let us consider $x \in \mathcal{O}_1^{\uparrow} \cap (\Gamma(\mathcal{O}_2^{\downarrow}) - \Gamma(\mathcal{O}_2))$, $e \in (\mathcal{O}_1^{\uparrow} - \mathcal{O}_1) \cap \Gamma(\mathcal{O}_2^{\downarrow})$, and $x < e$:



The placement of x and e, between lines 4 and 3, indicates that they are both within the set $\Gamma(O_2^{\downarrow}) \cap O_1^{\uparrow}$. Also, the underline under x indicates that $x \in O_1^{\uparrow}$ (ie could be an element of O_1 or $O_1^{\uparrow} - O_1$), and the lack of overline on x indicates that $x \in \Gamma(O_2^{\downarrow}) - \Gamma(O_2)$ (ie x is not an element of $\Gamma(O_2)$). Additionally, the line between x and e is solid as $x < e$, while it would be dotted if $x \leq e$.

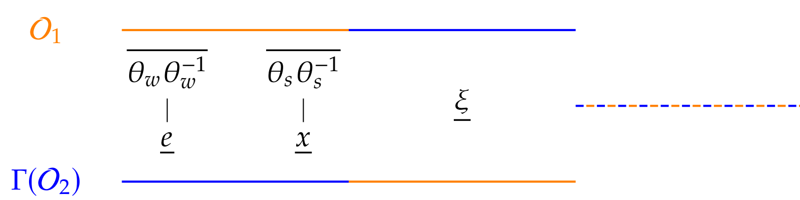
Periodically, we will use this visualization method to help our explanation of scenarios that arise in the following proofs. We encourage our readers to draw this visualization for scenarios depicted in the following proof, that we do not provide the complete visualisation for, and update the visualization as the proof introduces more elements of $E(H(S))$ and their relationships to $O_1 \cup \Gamma(O_2)$.

In the next subsection, we will prove the Diamond Lemma, which is vital in our larger proof.

4.2.1 The Diamond Lemma

In this lemma we will prove the following statement: If $\xi \in O_1^{\uparrow} \cap (\Gamma(O_2)^{\downarrow} - \Gamma(O_2))$, with $\Delta_w(\xi) = e$, and $\Delta_s(\xi) = x$, for some $e, x \in (O_1^{\downarrow} - O_1) \cap \Gamma(O_2)^{\uparrow}$, with $\theta_w \theta_w^{-1} \neq e$ and $\theta_s \theta_s^{-1} \neq x$, then $e = x$.

We can visualize these lemma conditions as follows:

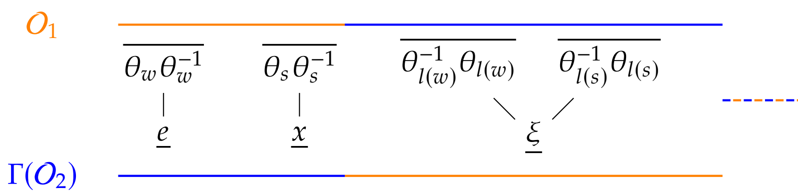


In the Diamond Lemma we will periodically update our visualization, while in the following main proof we will leave that largely to the readers, if they find it helpful for understanding.

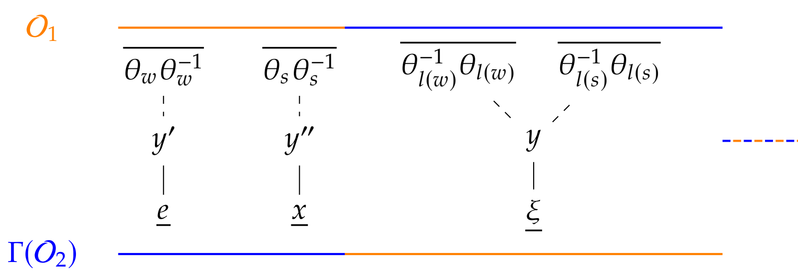
Lemma 4.2.2 (Diamond Lemma)**.** *If $\xi \in O_1^{\uparrow} \cap (\Gamma(O_2)^{\downarrow} - \Gamma(O_2))$, with $\Delta_w(\xi) = e$, and $\Delta_s(\xi) = x$, for some $e, x \in (O_1^{\downarrow} - O_1) \cap \Gamma(O_2)^{\uparrow}$, with $\theta_w \theta_w^{-1} \neq e$ and $\theta_s \theta_s^{-1} \neq x$, then $e = x$.*

Proof. Let us imagine that there exists an $\xi \in \mathcal{O}_1^{\uparrow} \cap (\Gamma(\mathcal{O}_2)^{\downarrow} - \Gamma(\mathcal{O}_2))$, with $\Delta_w(\xi) = e$, and $\Delta_s(\xi) = x$, for some $e, x \in (\mathcal{O}_1^{\downarrow} - \mathcal{O}_1) \cap \Gamma(\mathcal{O}_2)^{\uparrow}$, with $\theta_w\theta_w^{-1} \neq e$ and $\theta_s\theta_s^{-1} \neq x$. For the sake of contradiction, let us assume $x \neq e$.

Given $\Delta_s(\xi) = x$ and $\Delta_w(\xi) = e$, and $x \neq \theta_s\theta_s^{-1}$ and $e \neq \theta_w\theta_w^{-1}$, we know that $\theta_{l(w)}^{-1}\theta_{l(w)}, \theta_{l(s)}^{-1}\theta_{l(s)} > \xi$. Given that $e, x \in \Gamma(\mathcal{O}_2^{\uparrow})$, $\theta_w\theta_w^{-1}, \theta_s\theta_s^{-1} \in \Gamma(\mathcal{O}_2^{\uparrow}) - \Gamma(\mathcal{O}_2)$. Thus by Proposition 3.2.4, and the fact that $\theta_{l(w)}^{-1}\theta_{l(w)}\mathcal{D}\theta_w\theta_w^{-1}$ and $\theta_{l(s)}^{-1}\theta_{l(s)}\mathcal{D}\theta_s\theta_s^{-1}$, it must be the case that $\theta_{l(w)}^{-1}\theta_{l(w)}, \theta_{l(s)}^{-1}\theta_{l(s)} \in \Gamma(\mathcal{O}_2^{\downarrow})$.



Note that if there exists a y such that $\theta_{l(w)}^{-1}\theta_{l(w)} \geq y > \xi$ and $\theta_{l(s)}^{-1}\theta_{l(s)} \geq y > \xi$, there would exist $y', y'' \in \mathcal{D}_y$ such that $\Delta_w(y) = y', \Delta_s(y) = y''$. We will show that no such y can exist.



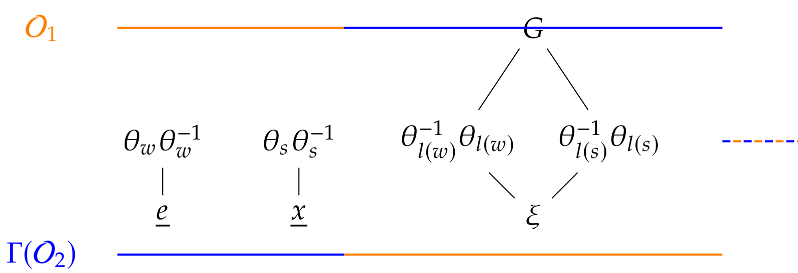
Thus $y' > e, y'' > x$, meaning that $y', y'' \in \Gamma(\mathcal{O}_2^{\uparrow}) - \Gamma(\mathcal{O}_2)$. By Proposition 3.2.4, $y' = y''$. Thus, if $y' \neq \theta_w\theta_w^{-1}$ and $y'' \neq \theta_s\theta_s^{-1}$, as $y' \in \theta_w\theta_w^{-1\downarrow} \cap \mathcal{O}_1(|w|+1)^{\uparrow}$, and $y'' \in \theta_s\theta_s^{-1\downarrow} \cap \mathcal{O}_1(|s|+1)^{\uparrow}$, it must be the case that $\theta_w\theta_w^{-1} = \theta_s\theta_s^{-1}$, by Proposition 3.3.16. Note that if $y' = \theta_w\theta_w^{-1}$ and $y'' = \theta_s\theta_s^{-1}$, it is trivially true that $\theta_w\theta_w^{-1} = \theta_s\theta_s^{-1}$.

Lastly, without loss of generality, we will consider if $y' \neq \theta_w\theta_w^{-1}$ but $y'' = \theta_s\theta_s^{-1}$. By Proposition 3.3.15, it must be the case that $s = wa$ for some $a \in \mathcal{A}$. However, that means that there exists no $e < y'$ such that $e \in \mathcal{S}_w$ (also by Proposition 3.3.15), which is a contradiction. Thus it must be the case that $\theta_w\theta_w^{-1} = \theta_s\theta_s^{-1}$.

As $\Delta_w(\xi) = e$ and $\Delta_s(\xi) = x$, it must be true that $x = e$, which is a contradiction to our assumption that $e \neq x$. Thus there exists no such y. Note also that this must imply that $\theta_{l(w)}^{-1}\theta_{l(w)} \not> \theta_{l(s)}^{-1}\theta_{l(s)}$, and $\theta_{l(w)}^{-1}\theta_{l(w)} \not< \theta_{l(s)}^{-1}\theta_{l(s)}$. In addition, by Proposition 3.2.5, the fact that no such y exists implies that $\xi \in \mathcal{O}_1^{\uparrow} - \mathcal{O}_1$.

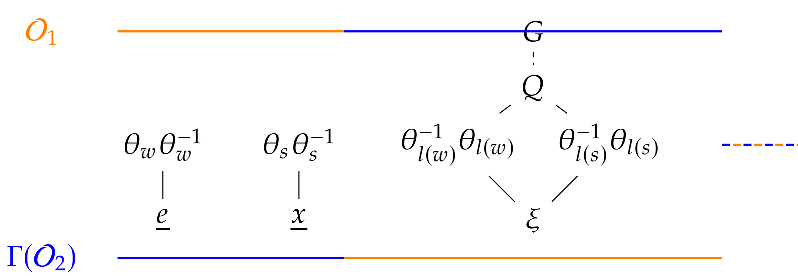
Given that $\theta_{l(w)}^{-1}\theta_{l(w)}, \theta_{l(s)}^{-1}\theta_{l(s)} \in \Gamma(\mathcal{O}_2^{\downarrow})$, there must exist an $G \geq \theta_{l(w)}^{-1}\theta_{l(w)}$ and an $B \geq \theta_{l(s)}^{-1}\theta_{l(s)}$ such that $G, B \in \Gamma(\mathcal{O}_2)$. Given that $\theta_{l(w)}^{-1}\theta_{l(w)}, \theta_{l(s)}^{-1}\theta_{l(s)} > \xi$, it must be the case that $G > \xi$ and $B > \xi$. Thus, $GB \neq 0$. By the mutual orthogonality of $\Gamma(\mathcal{O}_2)$, $G = B$. As $\theta_{l(w)}^{-1}\theta_{l(w)} \not> \theta_{l(s)}^{-1}\theta_{l(s)}$ and $\theta_{l(w)}^{-1}\theta_{l(w)} \not< \theta_{l(s)}^{-1}\theta_{l(s)}$, it must be true that $G > \theta_{l(w)}^{-1}\theta_{l(w)}, \theta_{l(s)}^{-1}\theta_{l(s)}$, assuming that $\theta_{l(w)}^{-1}\theta_{l(w)} \neq \theta_{l(s)}^{-1}\theta_{l(s)}$.

Note that if $\theta_{l(w)}^{-1}\theta_{l(w)} = \theta_{l(s)}^{-1}\theta_{l(s)}$, then $\theta_w\theta_w^{-1} \mathcal{D} \theta_s\theta_s^{-1}$, which by Proposition 3.2.4, implies that $\theta_w\theta_w^{-1} = \theta_s\theta_s^{-1}$ and $x = \Delta_s(\xi) = \Delta_w(\xi) = e$, a contradiction. Therefore we can assume that $\theta_{l(w)}^{-1}\theta_{l(w)} \neq \theta_{l(s)}^{-1}\theta_{l(s)}$.



It is at this point that the visualization explains the name of this lemma.

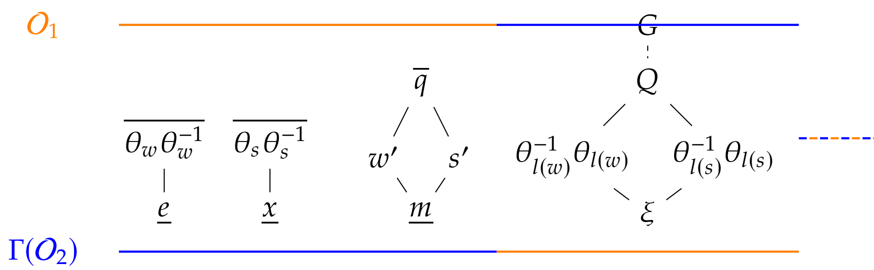
We know by Proposition 3.2.13, $\mathcal{O}_1^{\uparrow} - \mathcal{O}_1$ must be finite. Thus all sequences in $\mathcal{O}_1^{\uparrow} - \mathcal{O}_1$ must terminate, meaning that there exists a Q, such that $G \geq Q > \theta_{l(w)}^{-1}\theta_{l(w)}, \theta_{l(s)}^{-1}\theta_{l(s)}$ and there does not exist a y for which $Q > y > \theta_{l(w)}^{-1}\theta_{l(w)}, \theta_{l(s)}^{-1}\theta_{l(s)}$.



As $Q \in \Gamma(O_2^{\downarrow})$, there exists an $q \in \Gamma(O_2^{\uparrow}) - \Gamma(O_2)$ where, by Proposition 3.3.11, $\Delta_v(Q) = q$ for some $v \in L_1$. Such a q must exist given there is an element of $\Gamma(O_2^{\uparrow}) - \Gamma(O_2)$ that is $\mathcal{D}$-related to Q.

Thus, as $\theta_{l(w)}^{-1}\theta_{l(w)}, \theta_{l(s)}^{-1}\theta_{l(s)}, \xi \in \mathcal{T}_v$, there must exist $w', s', m \in E(H(S_1))$, such that $\Delta_v(\theta_{l(w)}^{-1}\theta_{l(w)}) = w'$, $\Delta_v(\theta_{l(s)}^{-1}\theta_{l(s)}) = s'$, and $\Delta_v(\xi) = m$.

Given that there does not exist a y, where $Q > y > \theta_{l(w)}^{-1}\theta_{l(w)}, \theta_{l(s)}^{-1}\theta_{l(s)}$, $w', s' \in \Gamma(O_2^{\uparrow})$. As $s', w' > \xi$, $s'w' \neq 0$. thus s' and w' can not both be in $\Gamma(O_2)$, due to mutual orthogonality of $\Gamma(O_2)$. Also, one element of $\{s', w'\}$ can not be in $\Gamma(O_2)$ while the other be in $\Gamma(O_2^{\uparrow}) - \Gamma(O_2)$, due to the closure of $\Gamma(O_2^{\uparrow}) - \Gamma(O_2)$ under multiplication. Thus $w', s' \in \Gamma(O_2^{\uparrow}) - \Gamma(O_2)$.



However we also know that $\theta_w\theta_w^{-1}, \theta_s\theta_s^{-1} \in \Gamma(O_2^{\uparrow}) - \Gamma(O_2)$, thus due to Proposition 3.2.4, $w' = \theta_w\theta_w^{-1}$ and $s' = \theta_s\theta_s^{-1}$. This means that $\Delta_w(\xi) = m$, and $\Delta_s(\xi) = m$. Recall that $\Delta_w(\xi) = e$, and $\Delta_s(\xi) = x$. Thus $x = m = e$, a contradiction that $x \neq e$. We have proven our desired result. □

Now that we have the Diamond Lemma, we are prepared to prove that $|O_1| = |O_2|$.

4.2.2 Theorem Preliminary Discussion

Let us consider a mapping $\Psi : (\Gamma(O_2) - O_1) \cap \mathcal{D}_\mu \to (O_1 - \Gamma(O_2)) \cap \mathcal{D}_\mu$, where $\mu \in O_1^{\uparrow} - O_1$. Later we will precisely define $\Psi(e)$, but for now it is important to note that if we show that Ψ is injective for an arbitrary $\mu \in O_1^{\uparrow} - O_1$, it must be the case that $|O_1 \cap \mathcal{D}_\mu| \geq |\Gamma(O_2) \cap \mathcal{D}_\mu|$. Given that O_1 and O_2 are arbitrarily indexed we can equivalently show $|O_1 \cap \mathcal{D}_\mu| \leq |\Gamma(O_2) \cap \mathcal{D}_\mu|$. Thus, proving that Ψ is injective will prove that $|O_1 \cap \mathcal{D}_\mu| = |\Gamma(O_2) \cap \mathcal{D}_\mu|$. As all $\mathcal{D}$ classes contain an element of $O_1^{\uparrow} - O_1$, showing Ψ is injective will prove $|O_1| = |O_2|$.

Let us consider an $e \in (\Gamma(\mathcal{O}_2) - \mathcal{O}_1) \cap \mathcal{D}_\mu$. Note that if $(\Gamma(\mathcal{O}_2) - \mathcal{O}_1) \cap \mathcal{D}_\mu$ is empty, Ψ is trivially a one-to-one mapping. Thus, we will assume that $(\Gamma(\mathcal{O}_2) - \mathcal{O}_1) \cap \mathcal{D}_\mu$ is not empty.

In this subsection we are going to exhaust all possible mutually-exclusive scenarios of the relations between $e \in (\Gamma(\mathcal{O}_2) - \mathcal{O}_1) \cap \mathcal{D}_\mu$ and other associated elements of H, $\mathcal{O}_1$, and $\Gamma(\mathcal{O}_2)$. In each case we attempt to identify an image for e in Ψ, or show that the scenario leads to a contradiction. Finally, we will describe $\Psi(e)$ with regards to all viable scenarios.

Given that $e \in (\Gamma(\mathcal{O}_2) - \mathcal{O}_1) \cap \mathcal{D}_\mu$, e is not in $\mathcal{O}_1$. Thus either,

1. $e \in \mathcal{O}_1^{\downarrow} - \mathcal{O}_1$. If this is the case, there must exist a $\theta_w \theta_w^{-1} \in \mathcal{O}_1^{\downarrow}$ where $\theta_w \theta_w^{-1} \neq e$ and $\Delta_w(e') = e$ for some $e' \in \mathcal{T}_w$. Note that Δ_w is the shape map of w, defined in relation to $\mathcal{O}_1$. Recall that e' must be in $\mathcal{O}_1^{\uparrow}$, thus either,
   (a) $e' \in \mathcal{O}_1$.
   (b) $e' \in \mathcal{O}_1^{\uparrow} - \mathcal{O}_1$. Let us now consider the unique $\gamma \in \Gamma(\mathcal{O}_2)^{\uparrow} - \Gamma(\mathcal{O}_2) \cap \mathcal{D}_\mu$. It is trivial that either,
      i. $\underline{\gamma \in \mathcal{O}_1^{\uparrow} - \mathcal{O}_1.}$
      ii. $\gamma \in \mathcal{O}_1$.
      iii. $\gamma \in \mathcal{O}_1^{\downarrow} - \mathcal{O}_1$. If this is the case, there must exist a $\theta_z \theta_z^{-1} \in \mathcal{O}_1^{\downarrow}$, where $\theta_z \theta_z^{-1} \neq \gamma$ and $\Delta_z(\gamma') = \gamma$, for some $\gamma' \in \mathcal{T}_z$. Again, Δ_z is the shape map of z, defined in relation to $\mathcal{O}_1$. Recall that γ' must be in $\mathcal{O}_1^{\uparrow}$, Thus either,
         A. $\underline{\gamma' \in \mathcal{O}_1^{\uparrow} - \mathcal{O}_1.}$
         B. $\gamma' \in \mathcal{O}_1$.

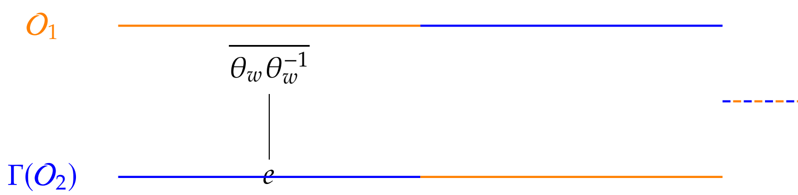
2. $e \in \mathcal{O}_1^{\uparrow} - \mathcal{O}_1$. Let us again consider the unique $\gamma \in \Gamma(\mathcal{O}_2)^{\uparrow} - \Gamma(\mathcal{O}_2) \cap \mathcal{D}_\mu$. It is trivial that either,
   (a) $\underline{\gamma \in \mathcal{O}_1^{\uparrow} - \mathcal{O}_1.}$
   (b) $\gamma \in \mathcal{O}_1$.
   (c) $\gamma \in \mathcal{O}_1^{\downarrow} - \mathcal{O}_1$. If this is the case, there must exist a $\theta_z \theta_z^{-1} \in \mathcal{O}_1^{\downarrow}$, such that $\theta_z \theta_z^{-1} \neq \gamma$ and $\Delta_z(\gamma') = \gamma$, for some $\gamma' \in \mathcal{T}_z$. Note, Δ_z is with regards to $\mathcal{O}_1$. Recall γ' must be in $\mathcal{O}_1^{\uparrow}$, thus either,
      i. $\underline{\gamma' \in \mathcal{O}_1^{\uparrow} - \mathcal{O}_1.}$

ii. $\gamma' \in \mathcal{O}_1$.

We have underlined the scenarios that lead to a contradiction, and we will prove these contradictions now. Note also that all scenarios above that were not underlined describe an element in $\mathcal{O}_1 \cap \mathcal{D}_\mu$. We will prove these elements are in $(\mathcal{O}_1 - \Gamma(\mathcal{O}_2)) \cap \mathcal{D}_\mu$. The proofs follow in order of appearance.

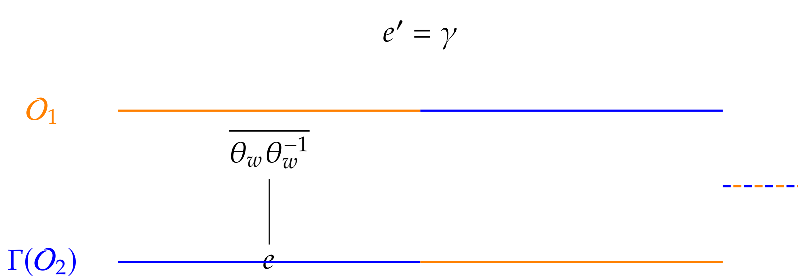
For all applicable scenarios it is assumed that $e \in (\Gamma(\mathcal{O}_2) - \mathcal{O}_1) \cap \mathcal{D}_\mu$ and $\gamma \in \Gamma(\mathcal{O}_2)^{\uparrow} - \Gamma(\mathcal{O}_2) \cap \mathcal{D}_\mu$. Also recall that by Proposition 3.3.6, for all words $v \in L$ and $t \in \mathcal{T}_v$, $t\mathcal{D}\Delta_v(t)$.

1a) Let us consider the situation where $e \in \mathcal{O}_1^{\downarrow} - \mathcal{O}_1$ and $e' \in \mathcal{O}_1$, for $\Delta_w(e') = e$. Note $\theta_w\theta_w^{-1} \neq e$. We do not yet know the relation of e' to $\Gamma(\mathcal{O}_2)$, so we can not yet place it on the visualization.



Note as $e \in \mathcal{S}_w$, it must be the case that $\theta_{l(w)}^{-1}\theta_{l(w)} > e'$. Also as $\theta_w\theta_w^{-1} > e$ and $e \in \Gamma(\mathcal{O}_2)$, $\theta_w\theta_w^{-1} \in \Gamma(\mathcal{O}_2)^{\uparrow} - \Gamma(\mathcal{O}_2)$. Thus by Proposition 3.2.4, $\theta_{l(w)}^{-1}\theta_{l(w)} \notin \Gamma(\mathcal{O}_2)^{\uparrow} - \Gamma(\mathcal{O}_2)$, meaning $e' \in \Gamma(\mathcal{O}_2)^{\downarrow} - \Gamma(\mathcal{O}_2)$. Therefore, as $e' \in \mathcal{O}_1 \cap \mathcal{D}_\mu$, it is true that $e' \in (\mathcal{O}_1 - \Gamma(\mathcal{O}_2)) \cap \mathcal{D}_\mu$.

1bi) We are going to assume for the sake of contradiction that $e \in \mathcal{O}_1^{\downarrow} - \mathcal{O}_1$, $e' \in \mathcal{O}_1^{\uparrow} - \mathcal{O}_1$ where $\Delta_w(e') = e$, and $\gamma \in \mathcal{O}_1^{\uparrow} - \mathcal{O}_1$ where $\gamma \in (\Gamma(\mathcal{O}_2)^{\uparrow} - \Gamma(\mathcal{O}_2)) \cap \mathcal{D}_\mu$. Note $\theta_w\theta_w^{-1} \neq e$. Given Proposition 3.2.4, $e' = \mu = \gamma$.



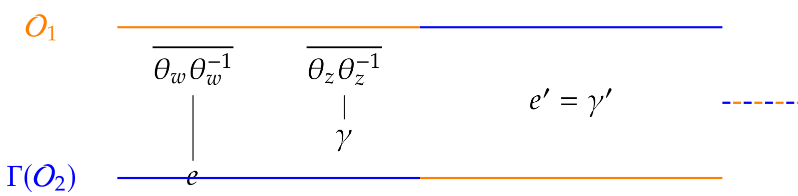
Note as $e \in \mathcal{S}_w$, it must be the case that $\theta_{l(w)}^{-1}\theta_{l(w)} > e'$. Also as $\theta_w\theta_w^{-1} > e$, and $e \in \Gamma(\mathcal{O}_2)$, $\theta_w\theta_w^{-1} \in \Gamma(\mathcal{O}_2)^{\uparrow} - \Gamma(\mathcal{O}_2)$. Thus by Proposition 3.2.4, $\theta_{l(w)}^{-1}\theta_{l(w)} \in$

$\Gamma(O_2)^{\downarrow}$, meaning that $e' \in \Gamma(O_2)^{\downarrow} - \Gamma(O_2)$. However, this is a contradiction because $\gamma \in \Gamma(O_2)^{\uparrow} - \Gamma(O_2)$.

1bii) Now let us consider the situation where $e \in O_1^{\downarrow} - O_1$, $e' \in O_1^{\uparrow} - O_1$ for $\Delta_w(e') = e$, and $\gamma \in O_1$. As $\gamma \in (\Gamma(O_2)^{\uparrow} - \Gamma(O_2)) \cap \mathcal{D}_{\mu}$, clearly $\gamma \in (O_1 - \Gamma(O_2)) \cap \mathcal{D}_{\mu}$.

1biiiA) We are going to assume, for the sake of contradiction, that $e \in O_1^{\downarrow} - O_1$, $e' \in O_1^{\uparrow} - O_1$ where $\Delta_w(e') = e$, $\gamma \in O_1^{\downarrow} - O_1$ and $\gamma' \in O_1^{\uparrow} - O_1$ where $\Delta_z(\gamma') = \gamma$. Note $\theta_w\theta_w^{-1} \neq e$ and $\theta_z\theta_z^{-1} \neq \gamma$. Given Proposition 3.2.4, $e' = \mu = \gamma'$.

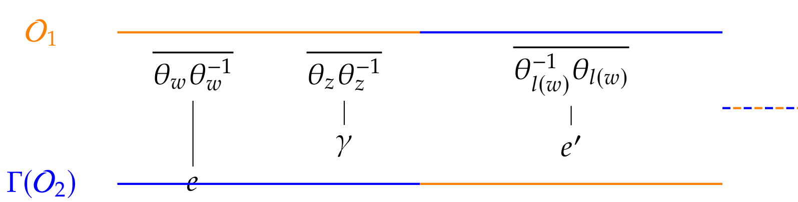
Also, it must be the case that $\theta_{l(w)}^{-1}\theta_{l(w)} > e'$, and $\theta_w\theta_w^{-1} \in \Gamma(O_2)^{\uparrow} - \Gamma(O_2)$. Given $\theta_{l(w)}^{-1}\theta_{l(w)}\mathcal{D}\theta_w\theta_w^{-1}$, by Proposition 3.2.4, $\theta_{l(w)}^{-1}\theta_{l(w)} \in \Gamma(O_2)^{\downarrow}$. Therefore $e' \in \Gamma(O_2)^{\downarrow} - \Gamma(O_2)$.



Given $\Delta_w(e') = e$ and $\Delta_z(\gamma') = \gamma$, $\Delta_w(\mu) = e$ and $\Delta_z(\mu) = \gamma$. Also, $e, \gamma \in \Gamma(O_2)^{\uparrow} \cap (O_1^{\downarrow} - O_1)$, $e \neq \theta_w\theta_w^{-1}$, and $\gamma \neq \theta_z\theta_z^{-1}$. Thus, by the Diamond Lemma, $e = \gamma$. This is a contradiction, as $e \in \Gamma(O_2)$ and $\gamma \in \Gamma(O_2)^{\uparrow} - \Gamma(O_2)$.

1biiiB) Now let us consider the situation where $e \in O_1^{\downarrow} - O_1$ and $e' \in O_1^{\uparrow} - O_1$ for $\Delta_w(e') = e$, and $\gamma \in O_1^{\downarrow} - O_1$ and $\gamma' \in O_1$ for $\Delta_z(\gamma') = \gamma$. Note, $\theta_w\theta_w^{-1} \neq e$ and $\theta_z\theta_z^{-1} \neq \gamma$.

Also note, this means that $\theta_{l(w)}^{-1}\theta_{l(w)} > e'$, and $\theta_w\theta_w^{-1} \in \Gamma(O_2)^{\uparrow} - \Gamma(O_2)$. Given $\theta_{l(w)}^{-1}\theta_{l(w)}\mathcal{D}\theta_w\theta_w^{-1}$, by Proposition 3.2.4, $\theta_{l(w)}^{-1}\theta_{l(w)} \in \Gamma(O_2)^{\downarrow}$. Therefore, $e' \in \Gamma(O_2)^{\downarrow} - \Gamma(O_2)$. We do not yet know the relation of γ' to $\Gamma(O_2)$, so we can not yet place it on the visualization.

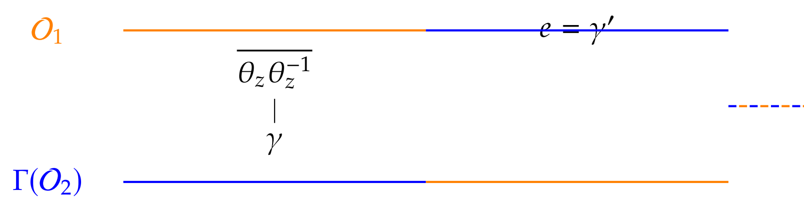


Given that $\gamma \in \Gamma(O_2)^{\uparrow} - \Gamma(O_2)$, we know that $\theta_z\theta_z^{-1} \in \Gamma(O_2)^{\uparrow} - \Gamma(O_2)$, where $\theta_z\theta_z^{-1} > \gamma$ and $\theta_{l(z)}^{-1}\theta_{l(z)} > \gamma'$. Due to Proposition 3.2.4, $\theta_{l(z)}^{-1}\theta_{l(z)} \in \Gamma(O_2)^{\downarrow}$, and $\gamma' \in \Gamma(O_2)^{\downarrow} - \Gamma(O_2)$. Therefore, $\gamma' \in (O_1 - \Gamma(O_2)) \cap \mathcal{D}_\mu$.

2a) Assume, for the sake of contradiction, that $e \in O_1^{\uparrow} - O_1$, and $\gamma \in O_1^{\uparrow} - O_1$. Given Proposition 3.2.4, $e = \mu = \gamma$. This is a contradiction, as $e \in \Gamma(O_2)$ and $\gamma \in \Gamma(O_2)^{\uparrow} - \Gamma(O_2)$.

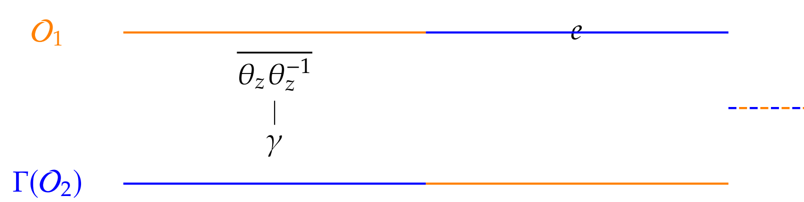
2b) Now let us consider the situation where $e \in O_1^{\uparrow} - O_1$ and $\gamma \in O_1$. Recall, $\gamma \in (\Gamma(O_2)^{\uparrow} - \Gamma(O_2)) \cap \mathcal{D}_\mu$. Thus $\gamma \in (O_1 - \Gamma(O_2)) \cap \mathcal{D}_\mu$.

2ci) Assume, for the sake of contradiction, that $e \in O_1^{\uparrow} - O_1$, $\gamma \in O_1^{\downarrow} - O_1$, and $\gamma' \in O_1^{\uparrow} - O_1$, where $\Delta_z(\gamma') = \gamma$ for $\theta_z\theta_z^{-1} \neq \gamma$. By Proposition 3.2.4, $e = \gamma'$.



We know that $\theta_z\theta_z^{-1} > \gamma$ and $\theta_{l(z)}^{-1}\theta_{l(z)} > \gamma'$. As $\gamma \in \Gamma(O_2)^{\uparrow} - \Gamma(O_2)$, $\theta_z\theta_z^{-1} \in \Gamma(O_2)^{\uparrow} - \Gamma(O_2)$. By Proposition 3.2.4, $\theta_{l(z)}^{-1}\theta_{l(z)} \in \Gamma(O_2)^{\downarrow}$ and $\gamma' \in \Gamma(O_2)^{\downarrow} - \Gamma(O_2)$. This is a contradiction, as $e = \gamma'$ and $e \in \Gamma(O_2)$.

2cii) Lastly, let us consider the case where $e \in O_1^{\uparrow} - O_1$, $\gamma \in O_1^{\downarrow} - O_1$, and $\gamma' \in O_1$ where $\Delta_z(\gamma') = \gamma$. Note, $\theta_z\theta_z^{-1} \neq \gamma$. We do not yet know the relation of γ' to $\Gamma(O_2)$, so we can not yet place it on the visualization.



We know that $\theta_z\theta_z^{-1} > \gamma$, and $\theta_{l(z)}^{-1}\theta_{l(z)} > \gamma'$. As $\gamma \in \Gamma(O_2)^{\uparrow} - \Gamma(O_2)$, $\theta_z\theta_z^{-1} \in \Gamma(O_2)^{\uparrow} - \Gamma(O_2)$. By Proposition 3.2.4, $\theta_{l(z)}^{-1}\theta_{l(z)} \in \Gamma(O_2)^{\downarrow}$. Thus, $\gamma' \in \Gamma(O_2)^{\downarrow} - \Gamma(O_2)$. So, $\gamma' \in (O_1 - \Gamma(O_2)) \cap \mathcal{D}_\mu$.

Now we are ready to define $\Psi(e)$, for any $e \in (\Gamma(O_2) - O_1) \cap \mathcal{D}_\mu$. We have proved that for all of the scenarios above that do not lead to contradictions, there exists an element, $y \in (O_1 - \Gamma(O_2)) \cap \mathcal{D}_\mu$. We will suggest the following mapping, $\Psi : (\Gamma(O_2) - O_1) \cap \mathcal{D}_\mu \rightarrow (O_1 - \Gamma(O_2)) \cap \mathcal{D}_\mu$, where $\Psi(e) = y$. Therefore,

1. If $e \in O_1^{\downarrow} - O_1$ and $\Delta_w(e') = e$ where $\theta_w\theta_w^{-1} \neq e$,
   - a. if $e' \in O_1$: $\Psi(e) = e'$
   - b. if $e' \in O_1^{\uparrow} - O_1$, consider $\gamma \in (\Gamma(O_2)^{\uparrow} - \Gamma(O_2)) \cap \mathcal{D}_\mu$.
     - ii. if $\gamma \in O_1$: $\Psi(e) = \gamma$
     - iiiB. if $\gamma \in O_1^{\downarrow} - O_1$, then there exists $\theta_z\theta_z^{-1} \in O_1^{\downarrow}$, where $\theta_z\theta_z^{-1} \neq \gamma$ and $\Delta_z(\gamma') = \gamma$: $\Psi(e) = \gamma'$
2. if $e \in O_1^{\uparrow} - O_1$, consider $\gamma \in (\Gamma(O_2)^{\uparrow} - \Gamma(O_2)) \cap \mathcal{D}_\mu$.
   - b. if $\gamma \in O_1$: $\Psi(e) = \gamma$
   - cii. if $\gamma \in O_1^{\downarrow} - O_1$, then there exists $\theta_z\theta_z^{-1} \in O_1^{\downarrow}$, where $\theta_z\theta_z^{-1} \neq \gamma$ and $\Delta_z(\gamma') = \gamma$: $\Psi(e) = \gamma'$

We have already proven that this is an exhaustive list of the mutually-exclusive possible scenarios, and that for all such scenarios, $\Psi(e) \in (O_1 - \Gamma(O_2)) \cap \mathcal{D}_\mu$. Note, for consistency, we labeled the scenarios equivalently to previously, when we included contradictory scenarios.

Also note that e', γ', and γ must be unique, the first two due to Proposition 3.3.7 and Proposition 3.3.16, and the latter being due to Proposition 3.2.4. As a result, Ψ is well defined.

4.2.3 One-to-One, Step-by-Step

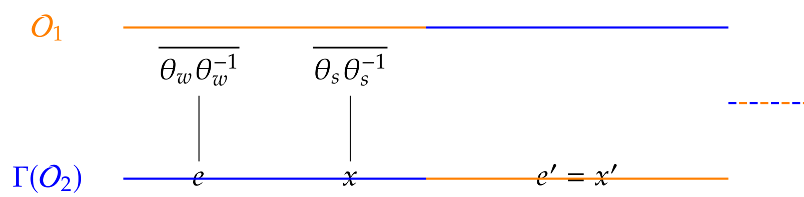
The rest of the proof follows a very consistent pattern, as we prove that Ψ is injective. Recall that if Ψ is injective, it must be the case that $|(O_1 - \Gamma(O_2)) \cap \mathcal{D}_\mu| = |(\Gamma(O_2) - O_1) \cap \mathcal{D}_\mu|$ and by extension, $|O_1| = |O_2|$. Therefore, after we prove that Ψ is injective, we will have proven our desired result.

The proof of the injectivity of Ψ is structured as follows: We assume that $e \in (\Gamma(O_2) - O_1) \cap \mathcal{D}_\mu$ falls under one of the five possible scenarios above. For the sake of contradiction we then assume that there exists an $x \in (\Gamma(O_2) - O_1) \cap \mathcal{D}_\mu$ such that $x \neq e$ and $\Psi(x) = \Psi(e)$. Lastly, we prove that x can not fall under any of the possible scenarios above, meaning that no such x exists.

Note that if we have already showed that there is not an x in scenario A, such that for e in scenario B, it is the case that $\Psi(x) = \Psi(e)$, we do not have to consider the situation where x is in scenario B and e is in scenario A, as the labeling of x and e are arbitrary.

1a) $e \in \mathcal{O}_1^{\downarrow} - \mathcal{O}_1$, and $e' \in \mathcal{O}_1$
We see that under these conditions $\Psi(e) = e'$. Let us imagine of the sake of contradiction, that there does exist an $x \neq e$, such that $x \in (\Gamma(\mathcal{O}_2) - \mathcal{O}_1) \cap \mathcal{D}_\mu$ and $\Psi(x) = \Psi(e)$. We will now consider all the properties that x could hold and show that each one leads to a contradiction.

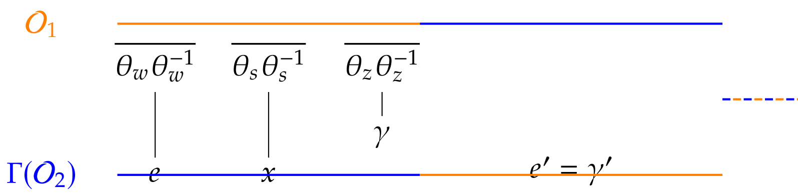
First let us assume that x holds the properties of 1a). Thus $x \in \mathcal{O}_1^{\downarrow} - \mathcal{O}_1$ where there exists an $\theta_s\theta_s^{-1} \in \mathcal{O}_1^{\downarrow}$, not equal to x, such that $\Delta_s(x') = x$ and $x' \in \mathcal{O}_1$. Given how we have defined Ψ, $\Psi(x) = x'$, thus $x' = e'$. Recall that in the last subsection, we showed that when e holds property 1a) it is the case that $e' \in \Gamma(\mathcal{O}_2)^{\downarrow} - \Gamma(\mathcal{O}_2)$.



By the Diamond Lemma, $e = x$, which is a contradiction of the assumption that $e \neq x$.

Now let us assume that x holds the properties of 1bii). Thus, $\Psi(x) = \gamma$ where $\gamma \in \mathcal{O}_1$ and $\gamma \in (\Gamma(\mathcal{O}_2)^{\uparrow} - \Gamma(\mathcal{O}_2)) \cap \mathcal{D}_\mu$. Thus $\gamma = e'$. Recall that in the last subsection we showed that when e holds property 1a) it is the case that $e' \in \Gamma(\mathcal{O}_2)^{\downarrow} - \Gamma(\mathcal{O}_2)$. Thus $e' \neq \gamma$, which contradicts the assumption that $\Psi(e) = \Psi(x)$.

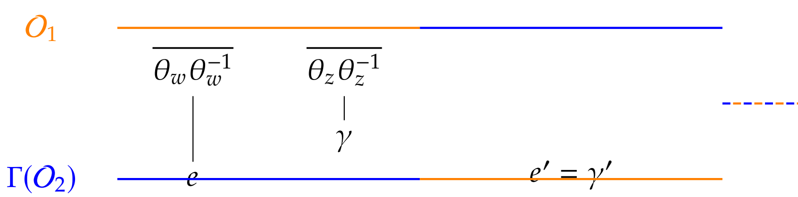
Let us consider if x holds the properties of 1biiiB). Thus $x \in \mathcal{O}_1^{\downarrow} - \mathcal{O}_1$ where there exists an $\theta_s\theta_s^{-1} \in \mathcal{O}_1^{\downarrow}$, not equal to x, such that $\Delta_s(x') = x$ and $x' \in \mathcal{O}_1^{\uparrow} - \mathcal{O}_1$. Ie, $x' = \mu$. In addition, $\gamma \in (\mathcal{O}_1^{\downarrow} - \mathcal{O}_1) \cap (\Gamma(\mathcal{O}_2)^{\uparrow} - \Gamma(\mathcal{O}_2)) \cap \mathcal{D}_\mu$, and there exists a $\theta_z\theta_z^{-1} \in \mathcal{O}_1^{\downarrow}$, such that $\theta_z\theta_z^{-1} \neq \gamma$, where $\Delta_z(\gamma') = \gamma$ and $\gamma' \in \mathcal{O}_1$. We find here that by the definition of Ψ, $\Psi(x) = \gamma'$. Given that $\Psi(e) = \Psi(x)$, $e' = \gamma'$. We do not yet know the relation of x' to $\Gamma(\mathcal{O}_2)$, so we can not yet place it on the visualization.



By the Diamond Lemma $e = \gamma$. This is a contradiction of the assumption that $e \in \Gamma(O_2)$ and $\gamma \in \Gamma(O_2)^{\uparrow} - \Gamma(O_2)$.

Let us assume that x holds the properties of 2b). Thus, $x \in O_1^{\uparrow} - O_1$ and $\gamma \in O_1 \cap (\Gamma(O_2)^{\uparrow} - \Gamma(O_2)) \cap \mathcal{D}_\mu$. By the definition of Ψ, $\Psi(x) = \gamma$. Given our assumption that $\Psi(x) = \Psi(e)$, it must be the case that $\gamma = e'$. Recall that in the last subsection, we showed that when e holds property 1a), it is the case that $e' \in \Gamma(O_2)^{\downarrow} - \Gamma(O_2)$. Thus $e' \neq \gamma$, which contradicts the assumption that $\Psi(e) = \Psi(x)$.

Finally, let us assume that x holds the properties of 2cii). Thus $x \in O_1^{\uparrow} - O_1$ and $\gamma \in (O_1^{\downarrow} - O_1) \cap (\Gamma(O_2)^{\uparrow} - \Gamma(O_2)) \cap \mathcal{D}_\mu$. In addition, there exists a $\theta_z\theta_z^{-1} \in O_1^{\downarrow}$, not equal to γ, such that $\Delta_z(\gamma') = \gamma$ and $\gamma' \in O_1$. By the definition of Ψ, $\Psi(x) = \gamma'$. Thus, by the assumption that $\Psi(x) = \Psi(e)$, $\gamma' = e'$.



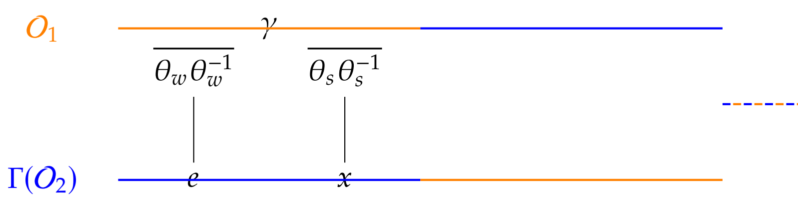
By the Diamond Lemma, $e = \gamma$. This is a contradiction of the assumption that $e \in \Gamma(O_2)$ and $\gamma \in \Gamma(O_2)^{\uparrow} - \Gamma(O_2)$.

We have shown that there there exists no situation where $\Psi(x) = \Psi(e)$, for $e \neq x$ when e satisfies the properties of 1a). Now let us assume that e satisfies the properties of 1bii).

1bii) $e \in O_1^{\downarrow} - O_1$, $e' \in O_1^{\uparrow} - O_1$, and $\gamma \in O_1$
We see that under these conditions $\Psi(e) = \gamma$. Let us imagine, for the sake of contradiction, there does exist an $x \neq e$, such that $x \in (\Gamma(O_2) - O_1) \cap \mathcal{D}_\mu$ and $\Psi(x) = \gamma$. Note that we have already shown that such an e can not hold the property that $\Psi(e) = \Psi(x)$ where x holds properties of 1a). We will

now consider the remaining scenarios that x could fall into, and show that each one leads to a contradiction. Note that we can immediately exclude x satisfying 1biiiB) and 2cii), as both assumes $\gamma \in \mathcal{O}_1^{\downarrow} - \mathcal{O}_1$, which is a contradiction of the assumption of $\gamma \in \mathcal{O}_1$.

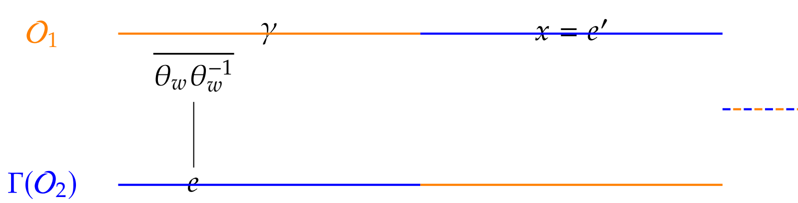
First let us assume that x holds the properties of 1bii). Thus $x \in \mathcal{O}_1^{\downarrow} - \mathcal{O}_1$, $x' \in \mathcal{O}_1^{\uparrow} - \mathcal{O}_1$, and $\gamma \in \mathcal{O}_1$. Given that $e', x' \in \mathcal{O}_1^{\uparrow} - \mathcal{O}_1$, by Proposition 3.2.4, $x' = e' = \mu$. Note that for both e and x, there exists an $\theta_w\theta_w^{-1} \neq e$ and $\theta_s\theta_s^{-1} \neq x$, such that $\theta_w\theta_w^{-1}, \theta_s\theta_s^{-1} \in \mathcal{O}^{\downarrow}$ where $\Delta_w(\mu) = e$ and $\Delta_s(\mu) = x$. We do not yet know the relation of μ to $\Gamma(\mathcal{O}_2)$, so we can not yet place it on the visualization.



Given that $\theta_w\theta_w^{-1} > e$, $\theta_{l(w)}^{-1}\theta_{l(w)} > e'$. Thus, as $e \in \Gamma(\mathcal{O}_2)$, $\theta_w\theta_w^{-1} \in \Gamma(\mathcal{O}_2)^{\uparrow} - \Gamma(\mathcal{O}_2)$ and, by the Proposition 3.2.4, $\theta_{l(w)}^{-1}\theta_{l(w)} \in \Gamma(\mathcal{O}_2)^{\downarrow}$. Thus $e' = x' \in \Gamma(\mathcal{O}_2)^{\downarrow} - \Gamma(\mathcal{O}_2)$.

By the Diamond lemma, we find that e=x, which contradicts our assumption that $e \neq x$.

Now Let us assume that x satisfies the properties for 2b). Thus, $x \in \mathcal{O}_1^{\uparrow} - \mathcal{O}_1$, $\gamma \in \mathcal{O}_1$, and $\Psi(x) = \gamma$. Also note that as $e' \in \mathcal{O}_1^{\uparrow} - \mathcal{O}_1$, by Proposition 3.2.4, $e' = x$.

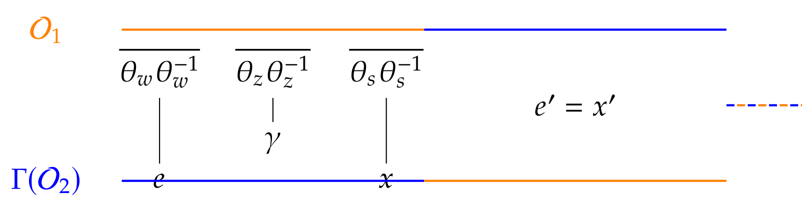


Given that $\theta_w\theta_w^{-1} > e$ and $\Delta_w(e') = e$, $\theta_{l(w)}^{-1}\theta_{l(w)} > e'$. However, given $e \in \Gamma(\mathcal{O}_2)$, $\theta_w\theta_w^{-1} \in \Gamma(\mathcal{O}_2)^{\uparrow} - \Gamma(\mathcal{O}_2)$. Therefore, $\theta_w\theta_w^{-1}, \theta_{l(w)}^{-1}\theta_{l(w)} \in \Gamma(\mathcal{O}_2)^{\uparrow} - \Gamma(\mathcal{O}_2)$, which is a contradiction, by Proposition 3.2.4.

We have shown that there there exists no situation where $\Psi(x) = \Psi(e)$, for $e \neq x$ when e satisfies the properties of 1bii). Now let us assume that e satisfies the properties of 1biiiB).

1biiiB) $e, \gamma \in O_1^{\downarrow} - O_1$, $e' \in O_1^{\uparrow} - O_1$, and $\gamma' \in O_1$
We see that under these conditions $\Psi(e) = \gamma'$. Let us imagine for the sake of contradiction there does exist an $x \neq e$, such that $x \in (\Gamma(O_2) - O_1) \cap \mathcal{D}_\mu$ and $\Psi(x) = \gamma'$. Note that we have already shown that there can not exist an x such that $\Psi(e) = \Psi(x)$, where x falls under the properties of 1a). We will now consider all the properties that x could hold, and show that each one leads to a contradiction. Note that we can immediately exclude x satisfying 1bii) and 2b), as both assumes $\gamma \in O_1$, which is a contradiction of the assumption of $\gamma \in O_1^{\downarrow} - O_1$.

First, let us assume that x holds the properties of 1biiiB). Thus $x \in O_1^{\downarrow} - O_1$, and there exists an $\theta_s\theta_s^{-1} > x$, such that $\Delta_s(x') = x$. Recall that $e \in O_1^{\downarrow} - O_1$, and there exists an $\theta_w\theta_w^{-1} > e$, such that $\Delta_w(e') = e$. Recall also, that $x', e' \in O_1^{\uparrow} - O_1$. Thus, by Proposition 3.2.4, $x' = \mu = e'$. In the last subsection we proved that when e is in scenario 1biiiB), $e' \in \Gamma(O_2)^{\downarrow} - \Gamma(O_2)$.



By the Diamond Lemma, $x = e$, which is a contradiction of our assumption that $x \neq e$.

Now let us assume that x holds the properties of 2cii). Thus $x \in O_1^{\uparrow} - O_1$, meaning $x = e'$. Recall that in the last subsection, we proved that when e is in scenario 1biiiB), $e' \in \Gamma(O_2)^{\downarrow} - \Gamma(O_2)$. This contradicts the assumption that $x \in \Gamma(O_2)$.

We have shown that there there exists no situation where $\Psi(x) = \Psi(e)$, for $e \neq x$ when e satisfies the properties of 1biiB). Now let us assume that e satisfies the properties of 2b).

2b) $e \in O_1^{\uparrow} - O_1$ and $\gamma \in O_1$
We see that under these conditions $\Psi(e) = \gamma$. Let us imagine, for the sake

of contradiction, there does exist an $x \neq e$, such that $x \in (\Gamma(O_2) - O_1) \cap \mathcal{D}_\mu$ and $\Psi(x) = \gamma$. Note that we have already shown that there can not be an $x \neq e$ where $\Psi(e) = \Psi(x)$ and x falls under the properties of 1a) or 1bii). If x satisfies the properties 1biiiB) or 2cii), there exists a contradiction to the assumption that $\gamma \in O_1$. We are left to consider x satisfying the properties of 2b). If x, satisfies the properties of 2b), $x \in O_1^{\uparrow} - O_1$, which means by Proposition 3.2.4, $e = x$. This contradicts the assumption that $e \neq x$.

We have shown that there there exists no situation $\Psi(x) = \Psi(e)$, for $e \neq x$ when e satisfies the properties of 2b). Finally, let us assume that e satisfies the properties of 2cii).

2cii) $e \in O_1^{\uparrow} - O_1$, $\gamma, \in O_1^{\downarrow} - O_1$, and $\gamma' \in \Gamma(O_2)$
We see that under these conditions $\Psi(e) = \gamma'$. Note that we have already shown that there can not be an $x \neq e$ where $\Psi(e) = \Psi(x)$ and x falls under the properties of 1a) or 1biiiB). If x satisfies the properties 1bii) or 2b). there exists a contradiction to the assumption that $\gamma \in O_1^{\downarrow} - O_1$. We are left to consider x satisfying the properties of 2cii). If x, satisfies the properties of 2cii), $x \in O_1^{\uparrow} - O_1$, which means by Proposition 3.2.4, $e = x$. This contradicts the assumption that $e \neq x$.

We have shown that there there exists no situation where $\Psi(x) = \Psi(e)$, for $e \neq x$ when e satisfies the properties of 2cii).

We have shown that for all possible scenarios of $e, x \in (\Gamma(O_2) - O_1) \cap \mathcal{D}_\mu$, if $\Psi(e) = \Psi(x)$, $e = x$. Therefore, $\Psi : (\Gamma(O_2) - O_1) \cap \mathcal{D}_\mu \to (O_1 - \Gamma(O_2)) \cap \mathcal{D}_\mu$ is an injective mapping. We have already shown that this implies that $|O_1 \cap \mathcal{D}_\mu| = |\Gamma(O_2) \cap \mathcal{D}_\mu|$ and $|O_1| = |O_2|$. Thus, we have proven our desired result:

Theorem 4.2.3. *Consider two inverse hulls of Markov shifts, $H(S_1)$ and $H(S_2)$, with corresponding alphabets $\mathcal{A}_1$ and $\mathcal{A}_2$. If $H(S_1) \cong H(S_2)$, then $|\mathcal{A}_1| = |\mathcal{A}_2|$.*

Chapter 5

Characterization of the Inverse Hull of a One-sided M-step Shift

In Chapter 4, we introduced our former research on the inverse hull of a Markov shift. In Theorem 4.1.1 we provide the characterization of the inverse hull of a Markov shift, from Beaupré et al. (2021). Using this characterization, we showed that non-conjugate Markov shifts can result in isomorphic inverse hulls, while conjugate Markov shifts can have non-isomorphic inverse hulls. In this thesis, we used the characterization to show that alphabet size is a shift invariant with respect to Markov shifts with isomorphic inverse hulls. With the usefulness of the characterization of the inverse hulls of Markov shifts, the clearest next step in studying the inverse hulls of one-sided M-step shifts, is to find a similar characterization.

This chapter explores the initial steps we have taken on the way to a characterization of the inverse hulls of one-sided M-step shifts. We intend to use this initial exploration in our continuing research to find this characterization.

When considering what a possible characterization might be for the inverse hulls of one-sided M-step shifts, it is useful to study the properties from the characterization of the inverse hull of a Markov shift, for two major reason. First of all, studying the properties of the characterization from Beaupré et al. (2021) can give us an indication of possible properties for our characterization if the inverse hull of a one-sided M-step shift still satisfies the properties. Secondly, when the inverse hull of a one-sided M-step shift does not satisfy the property, it is an indication of where the more complex structure of the inverse hull of an M-step shift deviates from that of a 1-step shift, and thus what structure we need to study more.

In the next section, we will consider the properties from the characterization in Beaupré et al. (2021), extended to the inverse hull of a one-sided M-step shift, as well as discuss the characteristics we want the properties to hold in our new characterization.

5.1 Possible Properties for the Characterization

Recall that if we let H be an inverse semigroup with 0, Theorem 4.1.1 states that H is isomorphic to the inverse hull of a Markov shift if, and only if,

1. H is combinatorial
2. there is a set $\mathcal{O}(1)$ of nonzero idempotents in H satisfying (O1)-(O5)
3. the language L associated with $\mathcal{O}(1)$ generates H.

We define the properties $(O1) - (O5)$ to be:

(O1) the elements of $\mathcal{O}(1)$ are mutually orthogonal

(O2) every idempotent in H is comparable to some element of $\mathcal{O}(1)$

(O3) both $\mathcal{O}(1)^{\uparrow} \cup \{0\}$ and $\mathcal{T} \cup \{0\}$ are closed under multiplication

(O4) elements of $\mathcal{T}$ are uniquely determined by the set of idempotents in $\mathcal{O}(1)$ that they lie above

(O5) for each $e \in \mathcal{O}(1)$, the $\mathcal{D}$-class of e contains at most one element of $\mathcal{T}$.

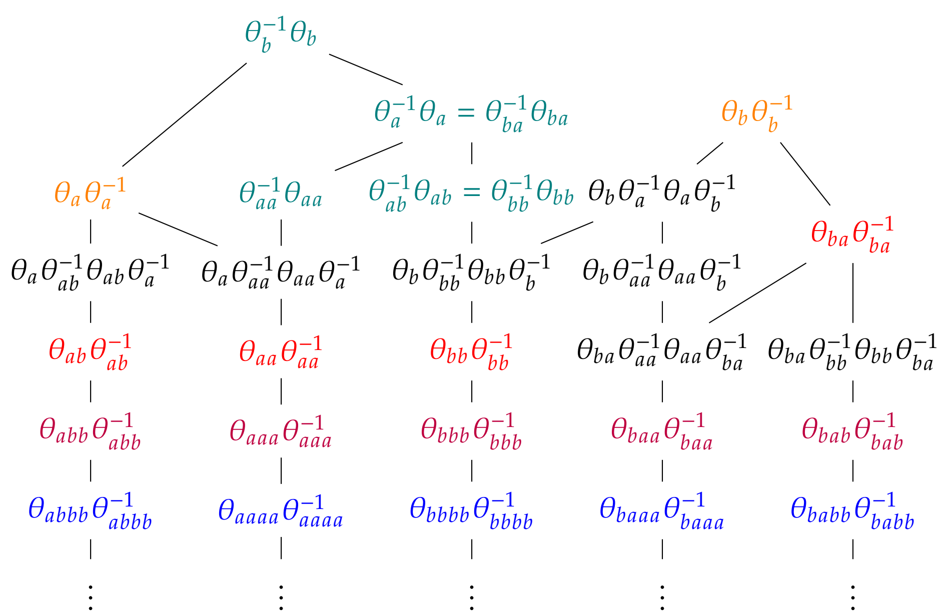
The *language associated with* $\mathcal{O}(1)$, L is defined by letting

$$\mathcal{A} = \{a \in H : a^*a \in \mathcal{T} \text{ and } aa^* \in \mathcal{O}(1)\}, \text{ and}$$
$$L = \{a_1 a_2 \dots a_n \neq 0 : n \in \mathbb{N}, a_i \in A\}.$$

If we let $H(S)$ be the inverse hull of a one-sided M-step shift, then clearly $H(S)$ will not satisfy all of these properties (unless $M = 1$). Note that in the above properties, we could identify them as relating to either $\mathcal{O}(1)$ or $\mathcal{O}(M)$, as $M = 1$ for Markov shifts. Therefore, when we consider the properties for general one-sided M-step shifts, we have to think critically about how we will extent them to situations where $M \neq 1$.

$H(S)$ by definition is an inverse semigroup with 0. By Proposition 3.1.2 we know that $H(S)$ is combinatorial. By Proposition 3.1.4, we know that $\mathcal{O}(N)$ is mutually orthogonal in $H(S)$, for all $N \in \mathbb{N}$, so $H(S)$ satisfies property (O1). Also $H(S)$ satisfies (O5), as every $\mathcal{D}$-class of $H(S)$ contains a unique element of $\mathcal{T}$, by Proposition 3.2.4.

If we consider Example 3.2.2 we see that it is not necessarily the case that $H(S)$ satisfies (O2). We provide the semilattice for Example 3.2.2, so our readers can quickly visually verify this assertion.



Clearly $\theta_{aa}^{-1}\theta_{aa}$ is not comparable to an element of $\mathcal{O}(1)$. Also note that $\theta_b\theta_{aa}^{-1}\theta_{aa}\theta_b^{-1}$ is not comparable to any elements of $\mathcal{O}(2) = \mathcal{O}(M)$. Therefore, $H(S)$ does not satisfy either interpretation of property (O2).

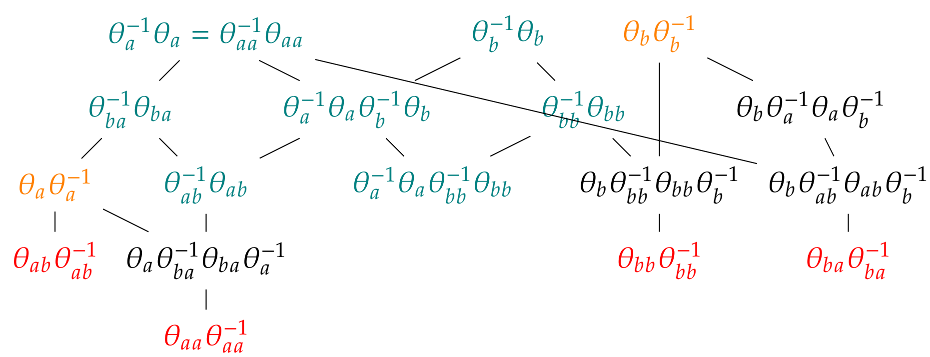
Though $H(S)$ satisfies half of property (O3) (namely $T \cap \{0\}$ is closed under multiplication, by Proposition 3.1.3), it is not necessarily the case that $\mathcal{O}(1)^{\uparrow} \cap \{0\}$ or $\mathcal{O}(M)^{\uparrow} \cap \{0\}$ are closed under multiplication, which is again illustrated in Example 3.2.2.

From Example 3.2.2, it might appear that $H(S)$ does satisfy (O4), as every element of $\mathcal{T}$ is uniquely determined by the set of idempotents in $\mathcal{O}(M)$ that it lies above. However, this is not universally true.

Example 5.1.1. Consider the shift with the transition array,

$$
\begin{array}{cc}
a & b \\
\begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array} &
\begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}
\end{array}
$$

and corresponding semilattice of the elements in $\mathcal{T} \cup O(M)^{\uparrow}$,



We see that $\theta_{ab}^{-1}\theta_{ab}^{\downarrow} \cap O(M) = \theta_a^{-1}\theta_a\theta_b^{-1}\theta_b^{\downarrow} \cap O(M)$, thus the elements of $\mathcal{T}$ are not uniquely determined by the set of idempotents in $O(M)$ that they lie above.

Now that we have an understanding of how we can, and can't, extend the properties from the characterization of Beaupré et al. (2021), we can begin compiling properties of the inverse hulls of one-sided M-step shifts that could work in our new characterization.

Any viable extensions of the properties from the characterization of Markov shifts are possible. Namely,

1. H is a combinatorial inverse semigroup with 0
2. the elements of $O(1)$ and $O(M)$ are mutually orthogonal.
3. $\mathcal{T} \cup \{0\}$ is closed under multiplication
4. for each $e \in O(1)$ and/or $O(M)$, the $\mathcal{D}$-class of e contains at most one element of $\mathcal{T}$.

Given Chapter 3, we know a lot more properties of $H(S)$ than the four above. Therefore, it is a vital to consider what makes a good property in a characterization like this.

If we want the characterization to be useful, as the characterization in Beaupré et al. (2021) is, we need to find properties of $H(S)$ that are easily verifiable, so that it is viable to use the characterization to prove that specific semigroups are inverse hulls of one-sided M-step shifts. In practice, this means that properties like "the elements of $\mathcal{O}(1)$ are mutually orthogonal" are great properties (because $\mathcal{O}(1)$ is a finite set "near the top" of the semilattice), while properties like "the elements of $\mathcal{O}(N)$ are mutually orthogonal for all $N \in \mathbb{N}$" are not good properties (because there are an infinite number of $\mathcal{O}(N)$-sets, arbitrary "low" in the semilattice).

Also, like all characterizations, it is important to have as few (easily verifiable) properties as possible, as that would allow the characterization to be the least amount of work to use. With this in mind, it begs the question, what possible proof outlines are there to prove our characterization? With a better understanding of the proof outline, we will begin to seen what exact properties will we require, as we work to prove the sub-parts of the characterization proof.

With an eye towards the clearest inspiration, the characterization of the inverse hulls of Markov shifts, had four major parts. Firstly, we showed for that for any H, that had the appropriate properties, the elements of H had the general form: $sa_1^*a_1 \ldots a_n^*a_nw^*$ where $s, w \in L^1$ and $a_i \in \mathcal{A}$. Secondly, we showed that $L \cup \{0\}$ is isomorphic to the semigroup associated to a Markov Shift. Thirdly, we showed that H had a property which we called "right reductive relative to L" (see Beaupré et al. (2021), section 4.2). Lastly, these three parts lead to a relatively quick proof that H is isomorphic to the inverse hull of the Markov shift that $L \cup \{0\}$ is isomorphic to.

In the last section of this chapter, we will follow the structure of the characterization of the inverse hulls of Markov shifts, by defining a set of properties that allow us to finish parts one and two for our characterization. In the future, we intend to add to this list of properties to complete the characterization.

5.2 Beginning the Characterization

Let H be a combinatorial inverse semigroup with 0. Choose a set $\mathcal{O}(1) \subset E(H)$ and define the following sets,

1. $\mathcal{T} = (\mathcal{O}(1)^{\downarrow})^c \cap E(H)$
2. $\mathcal{A} = \{a \in H : a^*a \in \mathcal{T} \text{ and } aa^* \in \mathcal{O}(1)\}$

3. For $N \in \mathbb{N}$, $\mathcal{O}(N) = \{ww^* \in H : w = a_1 \dots a_N \text{ for } a_i \in \mathcal{A}\}$
4. $L = \{a_1 \dots a_N \neq 0 : N \in \mathbb{N} \text{ and } a_i \in \mathcal{A}\}$.

Note that for $a_1 \dots a_N \in L$ we denote $|a_1 \dots a_N| = N$. Require the chosen $\mathcal{O}(1)$ to have the following properties,

(T1) $\mathcal{O}(1)$ is mutually orthogonal

(T2) $\mathcal{T} \cup \{0\}$ is closed under multiplication in H

(T3) There exists an $M \in \mathbb{N}$, such that for $s, w \in L$, where $|w| = M$ and $sw \neq 0$, $ww^* \leq s^*s$

(T4) For all $s \in L$ such that $|s| \leq M$, $s^*s \in \mathcal{T}$, and s^*s is comparable to some ww^* for $|w| \leq M$

(T5) H is generated by L.

It is important to note that the inverse hull of a one-sided M-step shift, $H(S)$, satisfies properties (T1) though (T5). Notice, $H(S)$ satisfies (T1) by Proposition 3.1.4, (T2) by Proposition 3.1.3, (T3) by Proposition 3.2.11, (T4) by Proposition 3.2.12, and (T5) by construction. Also, $H(S)$ is combinatorial by Proposition 3.1.2 .

First we will prove three lemmas that will help us find a general form for elements of H.

Lemma 5.2.1. *Let H be a combinatorial inverse semigroup with 0, containing a subset $\mathcal{O}(1) \subset E(H)$ that satisfies properties (T1) through (T5). Then $\mathcal{O}(N)$ is mutually orthogonal for $N \in \mathbb{N}$, and for $w, s \in L$, where $|w| = |s|$, $w^*s \neq 0$ if and only if $w = s$.*

Proof. We will prove that $\mathcal{O}(N)$ is mutually orthogonal for $N \in \mathbb{N}$ using induction. By property (T1), we know that $\mathcal{O}(1)$ is mutually orthogonal. Now let us assume that $\mathcal{O}(N)$ is mutually orthogonal for some $N \in \mathbb{N}$. Consider $ww^*, ss^* \in \mathcal{O}(N+1)$, such that $ww^*ss^* \neq 0$. Then it must be the case that there is a representation of w and s such that $w = w'a$ and $s = s'b$, where $|w'| = |s'| = N$ and $a, b \in \mathcal{A}$. Let us consider, ww^*ss^*.

$$ww^*ss^* = w'aa^*w'^*s'bb^*s'^* = w'aa^*w'^*w'w'^*s's'^*s'bb^*s'^*.$$

Therefore as $ww^*ss^* \neq 0$, $w'w'^*s's'^* \neq 0$, which by the induction hypothesis would mean that $w' = s'$. So,

$$ww^*ss^* = w'aa^*w'^*s'bb^*s'^* = w'aa^*w'^*w'bb^*w'^* = w'w'^*w'aa^*bb^*w'^*.$$

Therefore as $ww^*ss^* \neq 0$, $aa^*bb^* \neq 0$, which by the base case would mean that $a = b$. So, it must be the case that $w = w'a = s'b = s$. Thus we have proved our desired result, that $\mathcal{O}(N)$ is mutually orthogonal for $N \in \mathbb{N}$.

Now if $w^*s \neq 0$ for $w, s \in L$ where $|w| = |s|$, then $w^*ww^*ss^*s \neq 0$. So $ww^*ss^* \neq 0$, which by the mutual orthogonality of $\mathcal{O}(N)$, means that $w = s$. □

Lemma 5.2.2. *Let H be a combinatorial inverse semigroup with 0, containing a subset $\mathcal{O}(1) \subset E(H)$ that satisfies properties $(T1)$ through $(T5)$. Then for $w, s \in L$ where $|w| = M$, $sw \neq 0$ if, and only if, $ww^* \leq s^*s$. Also for $w, s \in L$, where $|w| = M$ and $sw \neq 0$, $(sw)^*sw = w^*w$.*

Proof. Note that for any $w, s \in L$, if $ww^* \leq s^*s$, then $s^*sww^* \neq 0$ and $sw \neq 0$. By (T3) we know that for $w, s \in L$, where $|w| = M$, if $sw \neq 0$ then $ww^* \leq s^*s$.

If we consider $w, s \in L$, where $|w| = M$ and $sw \neq 0$, we know that $ww^* \leq s^*s$, so,

$$(sw)^*sw = w^*s^*sw = w^*ww^*s^*sw = w^*ww^*w = w^*w.$$

□

Lemma 5.2.3. *Let H be a combinatorial inverse semigroup with 0, containing a subset $\mathcal{O}(1) \subset E(H)$ that satisfies properties $(T1)$ through $(T5)$. Then for $v, u \in L$,*

$$u^*v = \begin{cases} w^*wk & \text{if } v = uk, u = sw \text{ where } |w| = M, \text{ or } w = u, \text{ and } k, w, s \in L \\ k^*w^*w & \text{if } u = vk, v = sw \text{ where } |w| = M, \text{ or } w = v, \text{ and } k, w, s \in L \\ w^*w & \text{if } u = v, v = sw \text{ where } |w| = M, \text{ or } w = v, \text{ and } w, s \in L \\ 0 & \text{otherwise} \end{cases}$$

Proof. Note that the first three cases are mutually-exclusive, as case one assumes that $|v| > |u|$, case two assumes that $|v| < |u|$, and case three assumes that $|v| = |u|$.

The proofs for cases one through three are nearly identical, so here we will prove case one, and leave the others to the reader. Let us assume that $v = uk$ and $u = sw$ where $|w| = M$, or $w = u$, and $k, w, s \in L$. Then $u^*v = (sw)^*(swk) = (sw)^*(sw)k$. Recall that by Lemma 5.2.2, $(sw)^*(sw) = w^*w$ if $|w| = M$. So, for ether case, $|w| = M$ or $w = u$, $u^*v = w^*wk$.

Now let us assume that $v \neq uk$, $u \neq vk$, and $u \neq v$. if $|u| = |v|$, then $u^*v = 0$, by Lemma 5.2.1. If $|u| > |v|$, then $u = wk$ where $|w| = |v|$. Then by Lemma 5.2.1, $u^*v = k^*w^*v \neq 0$ if, and only if, $w = v$, which contradicts our assumption that $u \neq vk$. Similarly, if $|u| < |v|$, then $v = wk$ where

$|w| = |u|$. Then by Lemma 5.2.1, $u^*v = u^*wk \neq 0$ if, and only if, $w = u$, which contradicts our assumption that $v \neq uk$. Therefore, if u and v do not hold the properties assumed in cases one through three, $u^*v = 0$. ◻

The following theorem proves all elements for H have a general form reminiscent of the general forms for elements in the inverse hulls of one-sided M-step shifts, as described in Theorem 2.3.4.

Theorem 5.2.4. *Let H be a combinatorial inverse semigroup with 0, containing a subset $O(1) \subset E(H)$ that satisfies properties $(T1)$ through $(T5)$. Then all nonzero elements of the inverse semigroup can be written in the form,*

$$sx_1^*x_1 \dots x_n^*x_nw^* \text{ such that } s, w \in L^1, x_i \in L, \text{ and } |x_i| \leq M.$$

Proof. We know that by property (T5), H is generated by L, so all nonzero elements of H can be written in one of the four following forms,

1. $s_1^*s_2 \dots s_{k-1}s_k^*$
2. $s_1s_2^* \dots s_{k-1}s_k^*$
3. $s_1^*s_2 \dots s_{k-1}^*s_k$
4. $s_1s_2^* \dots s_{k-1}^*s_k$

Where $k \in \mathbb{N}$ and $s_i \in L$. We will prove our desired result using induction on k. When $k = 1$, we can write $s = ss^*s1$ and $s^* = 1s^*ss^*$, which proves our base case.

Let us assume that there exists an $k \in \mathbb{N}$ such that for all nonzero $e \in H$, where e can be written as one of the four possibilities above, we can also write $e = sx_1^*x_1 \dots x_n^*x_nw^*$ such that $s, w \in L^1, x_i \in L,$ and $|x_i| \leq M$.

Now let us consider the $k+1$ case. Then e could be written as as one of the four length k possibilities above, g, right multiplied by either z or z^* for $z \in L$. By the induction hypothesis, $g = sx_1^*x_1 \dots x_n^*x_nw^*$ such that $s, w \in L^1, x_i \in L$, and $|x_i| \leq M$. Thus, $e = sx_1^*x_1 \dots x_n^*x_nw^*z$ or $e = sx_1^*x_1 \dots x_n^*x_nw^*z^*$. Note that if $e = sx_1^*x_1 \dots x_n^*x_nw^*z^*$, then $e = sx_1^*x_1 \dots x_n^*x_n(zw)^*$, which is in our desired general form.

If $e = sx_1^*x_1 \dots x_n^*x_nw^*z$ then, as $w^*z \neq 0$, we can use Lemma 5.2.3 to find that,

$$e = \begin{cases} sx_1^*x_1 \dots x_n^*x_nv^*vu & \text{if } z = wu, w = hv \text{ where } |v| = M \text{ or } v = w \\ sx_1^*x_1 \dots x_n^*x_nu^*v^*v & \text{if } w = zu, z = hv \text{ where } |v| = M \text{ or } v = z \\ sx_1^*x_1 \dots x_n^*x_nv^*v & \text{if } z = w, w = hv \text{ where } |v| = M \text{ or } v = w \end{cases}$$

The following arithmetic heavily relies on Proposition 2.2.6. If $z = wu$ and $w = hv$ where $|v| = M$, or $v = w$, then,

$$\begin{aligned} e &= sx_1^*x_1 \dots x_n^*x_n v^*vu \\ &= su(x_1u)^*(x_1u)\dots(x_nu)^*(x_nu)(vu)^*vu \\ &= su(x_1u)^*(x_1u)\dots(x_nu)^*(x_nu)(vu)^*vu1. \end{aligned}$$

By Lemma 5.2.2, we know that we can write $(x_iu)^*(x_iu)$ and $(vu)^*vu$ as y^*y, for some $y \in L$ such that $|y| \le M$.

If $w = zu$ and $z = hv$ where $|v| = M$, or $v = z$, then,

$$\begin{aligned} e &= sx_1^*x_1 \dots x_n^*x_n u^*v^*v \\ &= sx_1^*x_1 \dots x_n^*x_n (vu)^*(vu)u^*. \end{aligned}$$

By Lemma 5.2.2, we know that we can write $(vu)^*vu$ as y^*y, for some $y \in L$ such that $|y| \le M$.

Lastly if $z = w, w = hv$ where $|v| = M$, or $v = w$,

$$\begin{aligned} e &= x_1^*x_1 \dots x_n^*x_n v^*v \\ &= x_1^*x_1 \dots x_n^*x_n v^*v1. \end{aligned}$$

Therefore, for every possible case, e can be written in the desired general form. By induction we have proven that all nonzero elements of H can be written in the form,

$$sx_1^*x_1 \dots x_n^*x_n w^* \text{ such that } s, w \in L^1, x_i \in L, \text{ and } |x_i| \le M.$$

□

Corollary 5.2.5. *Let H be a combinatorial inverse semigroup with 0, containing a subset $O(1) \subset E(H)$ that satisfies properties $(T1)$ through $(T5)$. Then all nonzero elements of $E(H)$ can be written in the form,*

$$wx_1^*x_1 \dots x_n^*x_n w^* \textit{ such that } w \in L^1, x_i \in L, \textit{ and } |x_i| \le M.$$

Proof. Consider $e \in E(H)$. Then by Theorem 5.2.4, $e = sx_1^*x_1 \dots x_n^*x_n w^*$ such that $s, w \in L^1, x_i \in L,$ and $|x_i| \le M$. Thus, as $e = ee$ and $e = e^*$,

$$\begin{aligned} e &= sx_1^*x_1 \dots x_n^*x_n w^*(sx_1^*x_1 \dots x_n^*x_n w^*)^* \\ &= sx_1^*x_1 \dots x_n^*x_n w^*wx_1^*x_1 \dots x_n^*x_n s^* \end{aligned}$$

By Lemma 5.2.2 we know that we can write w^*w as y^*y, for $y \in L$ such that $|y| \leq M$. Therefore every nonzero $e \in E(H)$ can be written in the desired general form for elements of $E(H)$.

□

Now that we have a general form for elements of H, we can more explicitly consider the elements of $\mathcal{O}(1)$ and $\mathcal{T}$.

Proposition 5.2.6. *Let H be a combinatorial inverse semigroup with 0, containing a subset $\mathcal{O}(1) \subset E(H)$ that satisfies properties (T1) through (T5). Then $\mathcal{O}(1) = \{aa^* : a \in \mathcal{A}\}$ and $\mathcal{T} = \{x_1^*x_1 \dots x_n^*x_n \neq 0 : x_i \in L, |x_i| \leq M\}$.*

Proof. First let us prove that $\mathcal{O}(1) = \{aa^* : a \in \mathcal{A}\}$. By definition, $\{aa^* : a \in \mathcal{A}\} \subseteq \mathcal{O}(1)$. Consider $e \in \mathcal{O}(1)$. We know that $e = wx_1^*x_1 \dots x_n^*x_nw^*$ such that $w \in L^1, x_i \in L,$ and $|x_i| \leq M$, by Corollary 5.2.5. If $|w| \geq 1$ then,

$$e = wx_1^*x_1 \dots x_n^*x_nw^* \leq ww^* \leq f(w)f(w)^*.$$

As $f(w)f(w) \in \mathcal{O}(1)$, property (T1) requires that $e = f(w)f(w)^*$.

If $w = 1$ then $e = x_1^*x_1 \dots x_n^*x_n$. By (T4) we know that all $x_i^*x_i \in \mathcal{T}$. Given (T2), it must be the case that $e \in \mathcal{T}$. However, this is a contradiction because, by definition, $\mathcal{T} \cap \mathcal{O}(1) = \emptyset$. Thus, it can not be the case that $w = 1$.

Therefore, $\mathcal{O}(1) \subseteq \{aa^* : a \in \mathcal{A}\}$, implying that $\mathcal{O}(1) = \{aa^* : a \in \mathcal{A}\}$.

Now let us prove that $\mathcal{T} = \{x_1^*x_1 \dots x_n^*x_n \neq 0 : x_i \in L, |x_i| \leq M\}$. By (T4) we know that all $s^*s \in \mathcal{T}$ for all $s \in L$ such that $|s| \leq M$. Given (T2), it must be the case that $x_1^*x_1 \dots x_n^*x_n \in \mathcal{T}$, if $x_1^*x_1 \dots x_n^*x_n \neq 0$ and $|x_i| \leq M$. Thus, $\{x_1^*x_1 \dots x_n^*x_n \neq 0 : x_i \in L, |x_i| \leq M\} \subseteq \mathcal{T}$

Now let us consider $e \in \mathcal{T}$. We know that $e = wx_1^*x_1 \dots x_n^*x_nw^*$ such that $w \in L^1, x_i \in L,$ and $|x_i| \leq M$, by Corollary 5.2.5. If $|w| \geq 1$ then,

$$e = wx_1^*x_1 \dots x_n^*x_nw^* \leq ww^* \leq f(w)f(w)^*.$$

However that would mean that $e \notin (\mathcal{O}(1)^{\downarrow})^c$, which contradicts our assumption that $e \in \mathcal{T}$. Thus, $\mathcal{T} \subseteq \{x_1^*x_1 \dots x_n^*x_n \neq 0 : x_i \in L, |x_i| \leq M\}$, which implies that $\mathcal{T} = \{x_1^*x_1 \dots x_n^*x_n \neq 0 : x_i \in L, |x_i| \leq M\}$. □

Lastly we will show that the set L is isomorphic to a language associated to a one-sided M-step shift. To do this, we will first prove that words in L have a unique representation as the concatenation of elements in $\mathcal{A}$.

Lemma 5.2.7. *Let H be a combinatorial inverse semigroup with 0, containing a subset $\mathcal{O}(1) \subset E(H)$ that satisfies properties (T1) through (T5). Then if $a_1 \dots a_N = b_1 \dots b_K \in L$ where $b_i, a_j \in \mathcal{A}$, then $N = K$ and $a_i = b_i$ for all $1 \leq i \leq N$.*

Proof. Let us assume that $w = a_1 \dots a_N = b_1 \dots b_K \in L$. First we will show that if $N = K$ then $a_i = b_i$ using induction. If $N = K = 1$, then $w = a_1 = b_1$. Trivially, $a_i = b_i$ for all $1 \leq i \leq N$.

Now let us imagine that for some $Q \in \mathbb{N}$, where $Q = N = K$ and $w = a_1 \dots a_N = b_1 \dots b_K$, it is the case that $a_i = b_i$ for all $1 \leq i \leq Q$..

Now let is imagine that $N = K = Q+1$ where $w = a_1 \dots a_N = b_1 \dots b_K$. So, $w = a_1 \dots a_{Q+1} = b_1 \dots b_{Q+1}$. Let us define $w_1 = a_1 \dots a_Q$ and $w_2 = b_1 \dots b_Q$. Thus $w = w_1 a_{Q+1} = w_2 b_{Q+1}$. Then,

$$w^* w = (w_1 a_{Q+1})^* (w_2 b_{Q+1}) = a^*_{Q+1} w^*_1 w_2 b_{Q+1}.$$

By Lemma 5.2.1, $w_1 = w_2$. By the induction hypothesis, $a_i = b_i$ for all $1 \leq i \leq Q$.

Now let us consider $ww^* = ww^*ww^*$,

$$\begin{aligned} ww^*ww^* &= (w_1 a_{Q+1})(w_1 a_{Q+1})^* (w_2 b_{Q+1})(w_2 b_{Q+1})^* \\ &= w_1 a_{Q+1} a^*_{Q+1} w^*_1 w_2 b_{Q+1} b^*_{Q+1} w^*_2 \\ &= w_1 a_{Q+1} a^*_{Q+1} w^*_1 w_1 b_{Q+1} b^*_{Q+1} w^*_1 \\ &= w_1 w^*_1 w_1 a_{Q+1} a^*_{Q+1} b_{Q+1} b^*_{Q+1} w^*_1 \end{aligned}$$

Therefore, $a^*_{Q+1} b_{Q+1} \neq 0$, which by Lemma 5.2.1 means that $a_{Q+1} = b_{Q+1}$.

We have proven that if $w = a_1 \dots a_N = b_1 \dots b_K$, for $b_i, a_j \in \mathcal{A}$ and $N = K$, then $a_i = b_i$ for all $1 \leq i \leq N$.

Now we will prove by contradiction that it can not be the case that $w = a_1 \dots a_N = b_1 \dots b_K$ for $b_i, a_j \in \mathcal{A}$ and $N \neq K$.

Let us assume that $w = a_1 \dots a_N = b_1 \dots b_K$ for $b_i, a_j \in \mathcal{A}$ and $N \neq K$. Without loss of generality, let us assume that $N < K$. By Lemma 5.2.3 we find that,

$$w^* w = (a_1 \dots a_N)^* (b_1 \dots b_K) = u^* u (b_{N+1} \dots b_K),$$

where $u = (a_1 \dots a_N)$ if $N \leq M$, or $u = a_1 \dots a_M$ if $M < N$. Given this, we can see that,

$$\begin{aligned} w^*w &= w^*w(w^*w)^* \\ &= u^*u(b_{N+1}\dots b_K)(u^*u(b_{N+1}\dots b_K))^* \\ &= u^*u(b_{N+1}\dots b_K)(b_{N+1}\dots b_K)^*u^*u \\ &= u^*u(b_{N+1}\dots b_K)(b_{N+1}\dots b_K)^* \\ &= u^*u(b_{N+1}\dots b_K)(b_{N+1}\dots b_K)^*b_{N+1}b_{N+1}^* \end{aligned}$$

Therefore $w^*w \leq b_{N+1}b_{N+1}^*$, however this is a contradiction by Proposition 5.2.6, as $w^*w \in \mathcal{T}$ but $b_{N+1}b_{N+1}^* \in \mathcal{O}(1)$. Therefore, if $w = a_1 \dots a_N = b_1 \dots b_K \in L$, it must be the case that $N = K$, which we already showed means that $a_i = b_i$ for all $1 \leq i \leq N$. □

In light of Lemma 5.2.7, we can identify the operation of H, on L, as both the operation itself and concatenation of elements in $\mathcal{A}$. As L is a collection of words of finite non-zero length, $L \cup \{0\}$ is isomorphic to a semigroup associated to a language, under the operation in H. Now we will prove that the set L is isomorphic to a language associated to a one-sided M-step shift.

Theorem 5.2.8. *Let H be a combinatorial inverse semigroup with* 0*, containing a subset $\mathcal{O}(1) \subset E(H)$ that satisfies properties* (T1) *through* (T5)*. Then the set L is isomorphic to a language associated to a one-sided M-step shift.*

Proof. We have already shown that L is isomorphic to a language.

Recall that by Definition 2.1.6, L is a language associated to a one-sided shift if, and only if, for all $w \in L$, every subword of w belongs to L and there are words $v \in L$ such that $wv \in L$.

By Lemma 5.2.7, the operation in H acts as concatenation of elements in $\mathcal{A}$ on L. Thus, is we show that for all $w \in L$, every subword of w belongs to L and there are words $v \in L$ such that $wv \in L$, that will prove that L is isomorphic to a language associated to a one-sided shift.

For all $w \in L$ recall that $w = a_1 \dots a_N$, where $a_1 \dots a_N \neq 0$ in H. Thus, for any subword of w, $a_i \dots a_j$, where $a_1 \dots a_i \dots a_j \dots a_N \neq 0$, $a_i \dots a_j \neq 0$, which by definition means that $a_i \dots a_j$ is in the set L.

Now let us consider $w \in L$, and note that by Lemma 5.2.2, $w^*w = s^*s$ where either $w = ts$ for $|s| = M$ and $t \in L$, or $w = s$. By property (T4), we know that s^*s is comparable to some vv^*, for $v \in L$. So, $s^*svv^* \neq 0$, which implies that $w^*wvv^* \neq 0$. Therefore, $wv \neq 0$, and $wv \in L$. We have proven that that the set L is isomorphic to a language associated to a one-sided shift.

Now we must show that L is isomorphic to an M-step language. Recall that a language, L, is M-step, if $w \in \mathcal{A}^M$ and $sw, wt \in L$, implies that $swt \in L$, by Lemma 2.1.10. Let us assume that $w \in \mathcal{A}^M$ and $sw, wt \in L$. Given that $wt \in L$, $ww^*wtt^*t \neq 0$ in H, so $w^*wtt^* \neq 0$. Recall by Lemma 5.2.2, $(sw)^*(sw) = w^*w$. Thus $(sw)^*(sw)tt^* \neq 0$, which implies that $swt \neq 0$. So $swt \in L$, meaning L is isomorphic to a language associated to a one-sided M-step shift. □

We have now proven that when H is a combinatorial inverse semigroup with 0, containing a subset $\mathcal{O}(1) \subset E(H)$ that satisfies properties (T1) through (T5), the set L is isomorphic to a language associated to a one-sided M-step shift. Along with proving that elements of H have the desired general form, in Theorem 5.2.4, this thesis ends well on the way to a characterization of the inverse hull of a one-sided M-step shift. The next clear step in this characterization, is to consider how to extend the definition of "right reductive relative to L" provided in Beaupré et al. (2021) with reference to inverse hulls of Markov shifts, to the inverse hulls of one-sided M-step shifts.

In the future, we indent to finish this characterization of the inverse hull of a one-sided M-step shift, and use it to study invariants on one-sided M-step shifts with isomorphic inverse hulls, as we have with our characterization for the inverse hulls of Markov shifts.

Chapter 6

Conclusion

In this thesis, we constructed the inverse hull of a one-sided shift of finite type, $H(S)$ (Section 2.3). Then, in Chapter 3, we explored some properties of $H(S)$. Most notably, in Section 3.3, we proved the existence of shapes, which show that the ($\mathcal{D}$-class) color-coded semilattice of $E(H(S))$, under the natural partial order, is fully describable using only a finite subset of $E(H(S))$, namely $\mathcal{T}^+$. In Chapter 4, we proved Theorem 4.2.3, which was a conjecture from Beaupré et al. (2021). Lastly, in Chapter 5, we begin constructing a characterization of the inverse hull of a one-sided shift of finite type, and explored possible avenues for future work completing the characterization.

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