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Exploring Winning Strategies for the Game of Cycles

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Abstract

This report details my adventures exploring the Game of Cycles in search of winning strategies. I started by studying combinatorial game theory with hopes to use the Sprague-Grundy Theorem and the structure of Nimbers to gain insight for the Game of Cycles. In the second semester, I pivoted to studying specific types of boards instead. In this thesis I show that variations of the mirror-reverse strategy developed by Alvarado et al. (2020) in the original Game of Cycles paper can be used to win on additional game boards with special structure, such as lollipops, steering wheel locks, and 3-spoke trees. Additionally I propose a new "working conjecture" that the player with the winning strategy is always determined by the parity of the number of markable edges on the board at the start of the game.

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Acknowledgments

I would like to thank many many people for helping me through what seemed like a daunting process, but ended up being a really rewarding thesis experience. Thanks to my housemates and friends from afar for always cheering me on. Special shoutout to Hannah (who has asked that Chapter 4 be dedicated to her) for helping me with just about every single stage of the thesis work, especially the editing, and thanks to Chris for playing endless games on Zoom with me and for working on the code. Thanks to Max for being the thesis buddy who is there to text in the wee hours of the morning, even though we probably should not have been awake. And thanks to Prof. Su for not only showing me this really great game, but also that exploring math is cool, unexpected, and fun!

Chapter 1

Introduction

> There should be no such thing as boring mathematics.
>
> Edsger W. Dijkstra

Welcome to my thesis, where I will be sharing some of the math I have been working on this year. (Shoutout to Zoom whiteboard for being my trusty "chalkboard" in the age of Covid-19.) I have been exploring the Game of Cycles, a game played by directing edges on planar graphs, from Francis Su's book, *Mathematics for Human Flourishing* [Su (2021)]. The rules for the game are actually quite simple, and will be taught in Chapter 2. On the other hand, how to confidently always win the game is not always so straightforward. This was one of the main motivations for my thesis work. *The Game of Cycles* paper by Alvarado et al. (2020) established important groundwork (especially the mirror-reverse strategy) on which I based my own exploration. I started off playing a bunch of games with both Prof. Su and my friends in order to understand how to play, and eventually started looking for patterns and developing intuition with guidance and inspiration from the Alvarado et al. (2020) paper. Hopefully, Chapter 2 will do the same for you as the first several weeks of losing against Prof. Su did for me (minus the bruised ego).

After this preliminary exploration, I spent time drawing out exhaustive game trees (iterating through every possible sequence of moves), in search of insights that might lead to new kinds of winning strategies. However, this quickly become too tedious as the boards got more complicated and had more edges.

At the same time, I learned about combinatorial game theory from the textbook *Winning Ways for Your Mathematical Plays* [Berlekamp et al. (2001)]. I tried to focus on learning about impartial combinatorial games (to be defined in Chapter 3), since the Game of Cycles is one. While there isn't much existing framework for thinking about winning strategies for the Game of Cycles, there is much more machinery established for understanding combinatorial games. Thus, trying to understand combinatorial game theory and applying it to the Game of Cycles seemed like a good idea. In Chapter 3, I explain the Sprague-Grundy Theorem and the structure of Nimbers, and then show how to apply these ideas to the Game of Cycles. Unfortunately, this added structure did not lead to any revolutionary insights about winning strategies like I had hoped, although this seems like an avenue that definitely could and should be revisited and poked at in the future.

The world of game theory does provide an immediately useful tool called Zermelo's Theorem which tells us that for every finite, two-player, perfect information game, either Player 1 or Player 2 must have a winning strategy [Zermelo (1912)]. The Game of Cycles is a finite two player perfect information game, so this applies! But do we know *which* player has the winning strategy? An open question from Alvarado et al. (2020) followed from an observation that "All of the theorems we have proved so far have shown, for various classes of boards, that if the number of edges in the board is odd, Player 1 has a winning strategy, and otherwise Player 2 has a winning strategy. Is there a board that does not follow this pattern?" I show that an edge with an endpoint with a leaf (a vertex of degree 1) will never be markable, and the presence of this kind of "rogue" edge breaks the pattern observed by Alvarado et al. (2020). Chapter 4 is a discussion on the impact of these "rogue" edges, and the impact they can have on who wins what games. (Spoiler alert: Boards that are just cycles or cycles with lots of rogue edges will have a predetermined winner before any moves are made!)

In the spring semester, I started studying specific classes of graphs, and looked for winning strategies. I was especially curious to explore trees (graphs with no cycles), and graphs with vertices of degree 3. Chapter 5 is where a lot of the hot action is, as it contains new results about winning strategies for new classes of boards including 3-Spoke Trees, Lollipops, and Steering-Wheel-Lock shapes, as well as a generalization that allows us to glue together multiple components into one board that we know how to win. All of these games use an adapted version of the mirror-reverse strategy originally shown in the Alvarado et al. (2020) paper.

Even after spending a year with the Game of Cycles, there is still so much left to learn and understand. For example, there are boards as simple as the "Two-Story House" (seen in Figure 2.4) for which we still do not know a good winning strategy! Whether this sounds intriguing and exciting, or simply appeals to your secret desire to beat your friends at a game over and over again, read on!

Chapter 2

Background

> Mathematics is a game played according to certain simple rules with meaningless marks on paper.
>
> David Hilbert

The Game of Cycles was introduced in Francis Su's book *Mathematics for Human Flourishing*[Su (2021)]. The rules are simple, but the winning strategies do not always seem apparent. Let's first learn how to play, and then see if we can figure out how to win.

2.1 How to Play the Game of Cycles

The Game of Cycles is a two-player game played on an undirected planar graph, and we'll call this graph our game board. Basically, we play on a bunch of vertices and edges. For now, we'll assume that the graph is connected, so every vertex is reachable via some path from every other vertex. Figure 2.1 shows an example of one such board, but the possibilities are really endless!

Let's say you and I wanted to play a game. On each of our turns, we mark one edge with an arrow to give it a direction. Once an edge has been marked, no one can mark it again. Aside from that, the only rule is that we cannot mark an edge such that it will create either a source (a vertex whose edges all point away from it), or a sink (a vertex whose edges all point

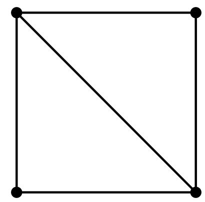

Figure 2.1 An example of a planar graph with 4 vertices and 5 edges. This is one possible board we can use to play the Game of Cycles.

towards it). We will call this restriction the Sink-Source Rule. An example of a source and a sink is shown in Figure 2.2.

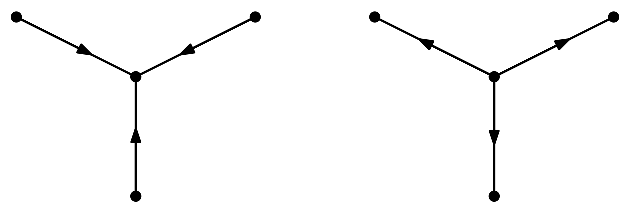

Figure 2.2 For a vertex of degree 3, we cannot have all 3 edges pointing towards that vertex (this is a sink!), and we also can't have all 3 edges away from it (this is a source!).

You and I keep taking turns, until one of us wins. How does one of us win? One of us wins by either completing a cycle cell on the graph, or by playing the last playable edge. Notice that every game ends with a winner - we cannot end in a tie. A cycle cell is a directed cycle of edges that contains no other cycles. Figure 2.3 shows an example and non-examples of this idea.

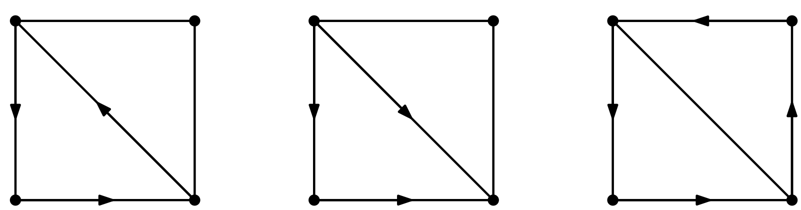

Figure 2.3 Left: This *is* a cycle cell.
Center: This *is not* a cycle cell because the edges are not all directed in the same direction along the cell.
Right: This *is not* a cycle cell because the cycle contains two cycles within it.

There are some very interesting things that we can observe pretty quickly. First, the Sink-Source Rule will sometimes leave edges *unmarkable*, meaning both of the two possible ways we could mark that edge will violate the Sink-Source Rule. Since we both play by the rules, hopefully, it will be left unmarked and unmarkable for the rest of the game. For the figures in this thesis, unmarkable edges will be indicated with a double hash mark as shown by edges ab and de (as seen in Figure 2.4). You are welcome to do this when you play as well, just as a reminder to you and your opponent that that edge is out of commission for the rest of the game. Marking unmarkable edges with the double hash mark is not a move a player makes, and does not change game play in any way, it is simply a tool I found helpful for spotting patterns and gaining intuition, so I thought it might help you too!

Unmarkable edges mean that the game can end without any cycle cells being formed as well as without all of the edges being played (as seen in Figure 2.4). The board shown in Figure 2.4 is actually not very well understood, and we will call it the "Two-Story House" board (as named in the Alvarado et al. (2020) paper).

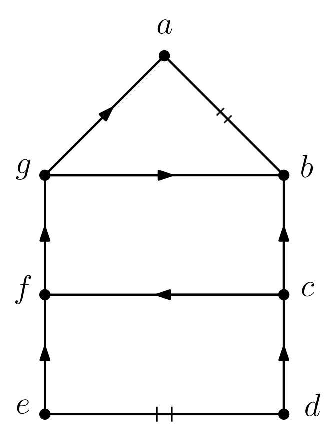

Figure 2.4 Here is an example where at first glance it seems like the game isn't over yet! No one has made a cycle cell, and there are still unmarked edges left on the board. However, we see that both ab and de are both actually unmarkable as a result of the Sink-Source Rule, so the game is in fact over. There were 7 moves made, so Player 1 must have played last markable edge and thereby won the game.

2.1.1 Example Games

Let's look at a couple of example games just to solidify our understanding of game mechanics, and gain a bit more intuition about this whole Sink-Source

Rule and unmarkable edges situation. We will examine two iterations of the Game of Cycles played on the same board, and see one game where Player 1 wins, and one game where Player 2 wins.

To understand some notation, when just describing an edge in a graph, we will name it by its two endpoints, and when we name a cycle cell we will name it by the vertices along the cycle. So in Figure 2.4, edge ab (or equivalently ba) is unmarkable, and the bottommost rectangle $cdef$ (or equivalently any of $defc$, $efcd$, $fced$, $cfed$, $fedc$, $edcf$, or $dcfe$) is not a cycle cell.

On the other hand, when we are describing a move a player makes, the direction of the marking matters. If a move labels an edge between vertices u and v with the arrow pointing away from v and towards u, we say the player played uv, and vice versa if the arrow points away from u and towards v, then they played vu. So the moves that were made in Figure 2.4 were ga, gb, fg, cb, ef, and dc.

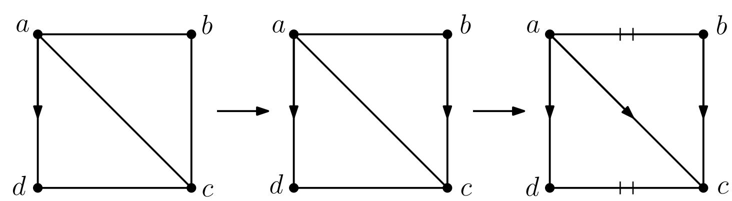

Figure 2.5 Example Game #1

In our first example game, shown in Figure 2.5, Player 1 plays ad, and then Player 2 plays bc. Player 1 uses their turn play ac, and the game is over. Why? If Player 2 tried to play ab, they'd create a source at vertex a. If Player 2 tried to play ba, they'd create a source at vertex b. Player 2 is a law-abiding game player, and doesn't want to violate the Sink-Source Rule, and so Player 2 cannot mark the edge ab - in other words, ab is *unmarkable*. We can use analogous reasoning to see that the edge cd is also unmarkable. There are no markable edges left to play, so Player 1 has won! Congratulations to Player 1!

This next example game, shown in Figure 2.6, is played on the same board, but has a totally outcome. Player 1 starts the same way, by playing ad. Now Player 2 plays ab. Player 1 now decides to play ca. Player 2 spies their chance to win, and plays bc, completing a cycle cell abc. Player 2 wins! Yay!

Let's back it up and look a bit more closely at what happened in this game. When Player 1 marked ca, they made what we call a *death move*. In other words, Player 1 played the second to last edge in a cycle cell, allowing Player 2 to win on their next move. As you might guess, you *don't* want to

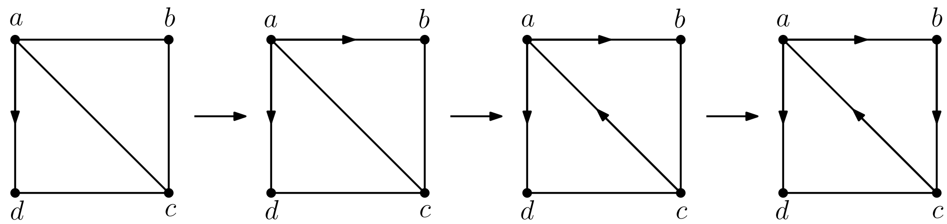

Figure 2.6 Example Game #2

play a death move if you can avoid it. Could Player 1 have avoided this? It turns out that Player 2 made a very smart first move, and the only moves available to Player 1 after that point were all death moves. Very cool, Player 2.

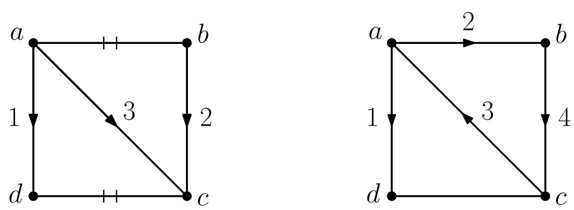

Figure 2.7 Left: New notation used to illustrate Example Game #1 (Figure 2.5). Right: New notation used to illustrate Example Game #2 (Figure 2.6).

From here on out, we will change the notation we use to draw out games. Instead of redrawing the board each time a new move is made, we will instead put all moves made in a game on one board. The moves will be labeled with numbers to show the order in which they were made. Figure 2.7 shows the examples from Figure 2.5 and Figure 2.6 drawn with the new notation.

2.2 How to *Win* the Game of Cycles on a Cycle

Now that we know how to play the game, there is a much more fun question that we can ask: How do we *win* the game? This is the big question of the semester. There are some boards that we already know how to play and win from work done by Alvarado et al. (2020), and I will share those here.

First, let me teach you some cool things about a game played on an n-cycle, or a cycle with n vertices. We call this board C_n. Pay attention, because here's the secret to winning.

We start with a cool fact.

Lemma 2.1. *At the end of a game on a line of edges, there is a direction change if and only if there is an unmarkable edge.*

Proof. Figure 2.8 provides a visualization of this idea.

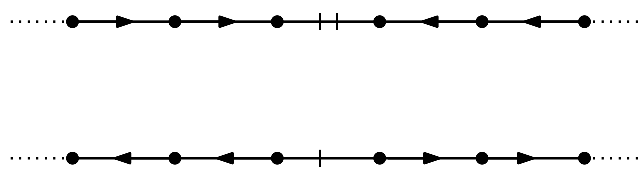

Figure 2.8 An unmarkable edge occurs only when a chain of arrows changes direction.

First we see if there is a direction change, there must be an unmarkable edge (an edge left unmarked). Suppose for the sake of contradiction that we are traveling along the line in some direction and there is a direction change with no unmarked edge. The vertex at which this change occurred must either be a sink or a source, which would have violated the Sink-Source Rule of the Game of Cycles. This is not possible, thus a contradiction.

Now we see that if there is an unmarkable edge, there must be a direction change. Suppose again for the sake of contradiction that we are traveling along the line in some direction and there is an unmarked edge that we claim is unmarkable, but no direction change. Then this unmarked edge can be marked in the same direction as the two adjacent edges on the line, thus a contradiction.

This proves that when marking edges along line, there is a direction change if and only if there is an unmarkable edge, as desired. □

This allows us to prove the following lemma about cycles.

Lemma 2.2. *If a C_n board has no markable edges, the number of unmarkable edges must be even.*

Proof. The idea for this proof relies on a few key points. First, we see that if there are no markable edges left, then there cannot be two adjacent unmarkable edges. Otherwise one of these edges actually would be markable, which is a contradiction. We also see that adjacent marked edges must be pointing in the same direction along the boundary of the cycle, otherwise we'd have an illegal source or sink. So, in the end we have chains of marked edges pointing in the same direction around the cycle, and each unmarkable edge separates two chains pointing opposite directions around the cycle by Lemma 2.1. If we pick one marked edge and trace around the cycle in one direction, we know that the number of times the chains change direction must be even, because we have to end up facing the same direction when we return to the original edge. (A visualization of this idea is shown in Figure 2.9.) If the number of times the chains change direction must be even, this shows the number of unmarkable edges must also be even, as desired. □

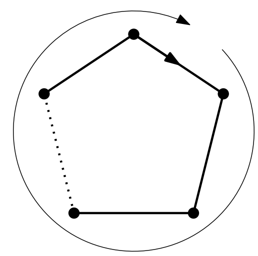

Figure 2.9 Pick any marked edge and trace around the cycle clockwise. We see that the total number of direction changes must be even once we cycle back to the starting edge.

Theorem 2.1. *The play on a C_n board is entirely determined by parity. If n is odd, Player 1 wins. If n is even, Player 2 wins.*

Proof. We know that the game ends when a player completes a cycle cell (which on an n-cycle board takes n moves), or when a player has played the last markable edge. By Lemma 2.2, the number of unmarkable edges left at the end of a game must be even, so we know that the parity of the edges played until the game is over will always be the same as the parity of n. Thus, the winner of the game is determined entirely by parity as desired. □

That's it! Now you know how to win if you and a friend are playing on a board that is an n-cycle. If n is odd, "bravely" volunteer to go first, and if n is even, "graciously" allow your friend to start the game off. Then, just make sure everyone plays by the rules, and you're guaranteed a win!

2.3 The Mirror Reverse Strategy

Previous work done by Alvarado et al. (2020) explored boards with 180° rotational symmetry as well as boards with reflective symmetry across any axis. They proposed and proved that by following a "mirror-reverse" strategy, the player with the winning strategy could always win in certain instances of these symmetric boards. How on earth does this work and what does it mean to "mirror-reverse"?

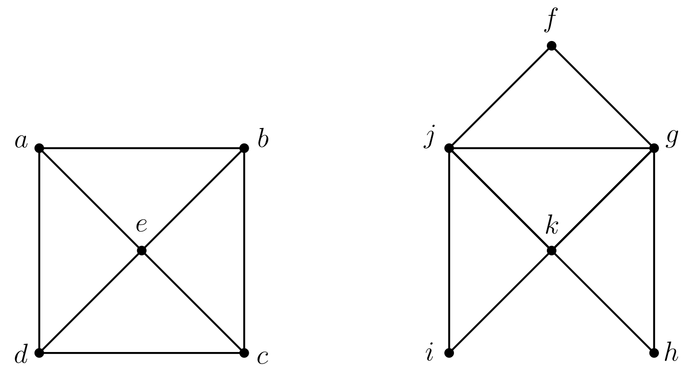

Figure 2.10 Left: A board with both 180° rotational symmetry and reflective symmetry (across vertical, horizontal, and diagonal axes).
Right: A board with reflective symmetry across a vertical axis of symmetry.
Figures appeared originally in Alvarado et al. (2020).

If we have a board with 180° rotational symmetry, we can imagine pairing each vertex v with a *partner vertex* v', where v' is in the position of vertex v after a 180° rotation. Similarly, every edge e has a *partner edge* e', where e' is the edge e after 180° rotation.

Analogously we can do the same for a board with reflective symmetry about some axis. We can pair each vertex v with a *partner vertex* v', where v' is in the position of vertex v after reflection across the axis of symmetry. And again, every edge e has a *partner edge* e', where e' is the edge e after reflection across the axis of symmetry.

In the example shown on the left side of Figure 2.10, the vertex partner pairings under 180° rotational symmetry are:

- $a \leftrightarrow c$
- $b \leftrightarrow d$
- $e \leftrightarrow e$

The edge partner pairings are:

- $ab \leftrightarrow cd$
- $bc \leftrightarrow da$

- $be \leftrightarrow de$
- $ce \leftrightarrow ae$

In the example shown on the right side of Figure 2.10, the vertex partner pairings under reflective symmetry (with a vertical axis of symmetry) are:

- $f \leftrightarrow f$
- $g \leftrightarrow j$
- $h \leftrightarrow i$
- $k \leftrightarrow k$

The edge partner pairings are:

- $fg \leftrightarrow fj$
- $gj \leftrightarrow jg$
- $gh \leftrightarrow ji$
- $gk \leftrightarrow jk$
- $hk \leftrightarrow ik$

Let's assume Player A and Player B have a game where this mysterious "mirror-reverse" strategy supposedly works. If it is supposed that Player A has the winning strategy, then Player A should:

1. If possibly, complete a cycle cell! Victory!
2. Otherwise, "mirror" and "reverse" the move Player B made. If Player B played by directing an edge from vertex i to vertex j, then Player A should direct an edge from vertex j' to vertex i', where j' is the partner vertex for j and i' is the partner vertex for i.

Curious... There are just so many questions. When can we use this mirror-reverse strategy? Who can use it? Why does this work?

For starters, let us define some terms in order to understand what kinds of boards this strategy can work for!

Definition 2.1. The *trivial-symmetry* assigns each vertex with itself as its partner vertex and each edge with itself as its partner edge.

Definition 2.2. A board has *involutive symmetry*, τ, if there is a non-trivial symmetry that is its own inverse. We call τ an *involution*. τ assigns a unique partner vertex and parter edge to each vertex and edge on the board.

So, 180° rotational symmetry and reflective symmetry are both involutive symmetries.

Definition 2.3. Given some involution τ, we say that a vertex, edge, or cell is *self-involutive* if the involution assigns that vertex, edge, or cell to itself.

For example, in the right side example of Figure 2.10, f and k are both self-involutive vertices, and jg is a self-involutive edge. Furthermore, fgj and jgk are both self-involutive cells.

We can see that an involution might map an edge to itself, but not necessarily map the vertices that are the endpoints of that edge to themselves. In the left side example in Figure 2.10, vertex e is self-involutive. None of the edges or cells are self-involutive. In the right side example of Figure 2.10, the edge jg is a self-involutive, but the vertices j and g do not map to themselves, rather they map to each other. Similarly, an involution might assign a cell to itself, but not assign any vertices or edges on that cell to themselves. (Imagine a 4-cycle, C_4 under 180° rotational symmetry for an example of this.)

Definition 2.4. Given some involution τ, we say that a cell is *part-involutive* if the cell itself is not self-involutive, but at least one edge is assigned a partner edge also in the cell. We also call a cell *nowhere-involutive* if every edge in the cell is assigned a partner edge outside of the cell.

In the right side example of Figure 2.10, the cell ijk maps to the cell ghk, and every edge in the cell ijk is assigned to a partner edge outside ijk. Thus ijk is a nowhere-involutive cell.

This framework allows us to state the following theorem and corollaries about using the mirror reverse strategy.

Theorem 2.2. *Let G be a board with an involution such that each cell is either self-involutive or nowhere-involutive. If there is no self-involutive edge, then Player* 2 *has a winning strategy. If there is exactly one self-involutive edge whose vertices are not fixed by the involution, then Player* 1 *has a winning strategy.*

The proof of this theorem is given in the Alvarado et al. (2020) paper, and argues that the mirror-reverse strategy can be proven as winning strategy for these boards.

Corollary 2.1. *Let G be a board with* 180° *rotational symmetry. No edge and its parter edge are part of the same cell.*

- *If there is no edge through the center of the board, then Player* 2 *has a winning strategy.*
- *If there is an edge through the center of the board, then Player* 1 *has a winning strategy.*

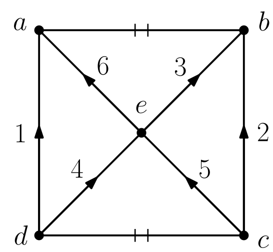

Figure 2.11 An example game on a boards with 180° rotational symmetry. Player 2 wins by using the mirror-reverse strategy and playing the last markable edge.

Let's check out an example of Corollary 2.1 and the mirror-reverse strategy in action! We see that the board in Figure 2.11 has 180° rotational symmetry and no edge through the center of the board. Therefore, by Corollary 2.1, Player 2 has a winning strategy. How does Player 2 use the mirror-reverse on the strategy? If Player 1 plays da, and Player 2 were to mirror that move under 180° rotational symmetry, they would play bc. To use the mirror-reverse strategy, Player 2 should reverse this, and play cb. If Player 1 then plays eb, then Player 2 would know that ed would mirror this move, and so reversing that to play de would correctly follow the mirror-reverse strategy. Next, Player 1 marks ce, and the correct mirror-reversed move for Player 2 to make is ea. The game is over since ab and cd are unmarkable edges at this point. Looks like Player 2 played the last markable edge, so they won on this board, as we hoped!

Corollary 2.2. *Let G be a board that is symmetric by reflection across some line. No edges are along this axis of symmetry, and no more than one edge crosses this axis of symmetry.*

- *If there is no edge that crosses the axis of symmetry, Player* 2 *has a winning strategy.*
- *If there is exactly one edge crossing the axis of symmetry, Player* 1 *has a winning strategy.*

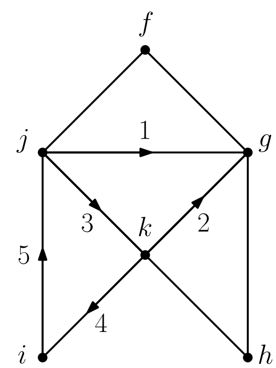

Figure 2.12 An example game on a boards with reflective symmetry (across a vertical axis of symmetry). Player 1 wins by using the mirror-reverse strategy and completing a cycle cell.

Here is an example of how we can use Corollary 2.2 and the mirror-reverse strategy to win. We see that the board in Figure 2.12 has reflective symmetry across a vertical line down the center, and that exactly one edge, jg, crossing the axis of symmetry. Therefore, by Corollary 2.2, Player 1 has a winning strategy. How does Player 1 use the mirror-reverse on the strategy? First, Player 1 should mark the self-involutive edge, jg, that crosses the axis of symmetry. Now if Player 2 plays kg, the Player 1 should play jk, since kj would mirror Player 2's move and so jk would be the mirror-reversed move. If Player 2 next plays ki, then Player 1 might be tempted to mirror-reverse this move and play hk. But be careful Player 1! Player 2 just made a death move! Remember that if you have the opportunity to complete a cycle cell and win, you always should. So, Player 1 can complete the cycle cell ijk by marking the edge ij. Player 1 wins (as we predicted). Yay!

So far we have seen a couple different strategies for winning the Game of Cycles. On a cycle, we know which player will win, and you can decide if you want to go first or second depending on the parity of the number of edges in the cycle. For boards with certain symmetries (and a few other requirements), we just saw that we can use this neat trick called the mirror-reverse strategy. The nice thing about all of these strategies is that they are easy to explain and remember. As we'll discuss later, every game has a player with a winning strategy. What isn't always apparent is who that player is, and furthermore, if there is a simple winning strategy for that player to follow. So, let us search for more game boards we can figure out how to win.

Chapter 3

Nimbers

> A pattern is an idea that has been useful in one practical context and will probably be useful in others.
>
> — Martin Fowler

While exploring solid examples of the Game of Cycles was good for developing intuition, it seemed like having some machinery to understand this game more generally would be useful. In this chapter, I will introduce some machinery from the field of combinatorial game theory, and explain how it can be used on the Game of Cycles.

3.1 Some Ideas in Combinatorial Game Theory

Let's start by defining what kinds of games we are looking at.

Definition 3.1. A game is a *perfect information game* if no aspect of the game is random or left to chance.

Games like chess and tic-tac-toe are perfect information games, whereas the card game War is not. Since there are no elements of unknown chance, the Game of Cycles is also a perfect information game. This is great news, because now any machinery that has been developed for impartial combinatorial games and perfect information games should work for the Game of Cycles!

Definition 3.2. A game is a *combinatorial game* if two players alternate turns until one player wins, and for each turn players have a well defined set of moves available to them as well as perfect information about the game (no hidden information or elements of chance).

Definition 3.3. A game is *impartial* if it is a combinatorial game where the same moves are available to each player when given the same configuration of that game.

Definition 3.4. A two player game is *finite* if the game is guaranteed to end after a finite number of moves.

Luckily, the Game of Cycles is a finite impartial combinatorial game. We play the Game of Cycles where two players alternate turns, and at any given turn, the players have a full understanding of the moves available to them (there are no random or secret surprises). Additionally, the two players play by the same rules, so given the same instance of any board, they would both have the same moves available to them. The game cannot go on forever, since there are a finite number of possible edges to be marked.

The Alvarado et al. (2020) paper used the fact that the Game of Cycles is a finite, two player, perfect information game, to apply Zermelo's Theorem, which tells us that either Player 1 or Player 2 has a *winning strategy* at the start of the game[Zermelo (1912)]. If you have a winning strategy, this means that you will always be able to move in response to any move by the other player in a way that guarantees you can still win.

3.1.1 N- and P-Positions

Every move a player makes puts them either in an N- or a P-position. A move to a P-position is a winning position (yay you have a winning strategy!), and a move to an N-position is a losing one (oof). A player who starts their turn in a P-position can only move to an N-position. However, a player can always move from an N-position to a P-position (and sometimes there is also an option to move from an N-position to another N-position). Players want to move to P-positions on their turns, because in the end, moving to a P-position means you win [Guy (1989)].

It is helpful to know what the N and P stand for in order to remember which is which. A P-position stands for a *previous*-player-winning position, and an N-position stands for a *next*-player-winning position [Guy (1989)]. Moving to a P-position is good, since once you make your move, *you* are the previous player, and that means *you* are the one in a winning position!

So if the game starts in an N-position, then Player 1 is guaranteed to have a winning strategy, and if the game starts in a P-position, then Player 2 has a winning strategy. Remember that having a winning strategy does not mean that you automatically win the game, but rather that you can move in response to any legal move made by your opponent and still guarantee a win.

In a perfectly played game, the player with a winning strategy will always move from an N-position to a P-position (even if they have the option to move to an N-position), because being in a P-position guarantees a winning strategy. By a similar logic, the player who has a winning strategy to start with could mess up by choosing to move from an N-position to another N-position on their turn, giving the other player an opportunity to move to a P-position on their next turn[Berlekamp et al. (2001)].

3.2 Nimbers and the Sprague-Grundy Theorem

Nim is a popular impartial combinatorial game played with piles of rocks. It seems as though not much is known for certain about its history, although it is hypothesized that Nim has roots in China but possibly also Europe. American mathematician Charles L. Bouton was the first to really explore the game mathematically, and published *Nim, A Game with a Complete Mathematical Theory*, which I used quite heavily to better understand Nim[Bouton (1901)].

To play Nim, players take turns removing a non-zero number of rocks from any single pile. The player who takes the last rock wins. The important thing to us is the fact that each game of Nim corresponds to a special number, called a Nimber. As explained in Bouton's paper, Nimbers correspond cleverly to the binary XOR of the size of the heaps in the game. The Nimbers help us easily determine what winning P-positions are, and how to get to them. Essentially Nimbers highlighted a nice, easy-to-understand, winning strategy for the game of Nim.

Recall that the big question is: "Can we find a nice, easy-to-understand, winning strategy for the Game of Cycles?" It sounds like the structure of Nimbers might be helpful in this endeavor.

Well... check out this theorem!

Theorem 3.1 (Sprague-Grundy Theorem)**.** *Every impartial game is equivalent to a game of single-heap Nim, and therefore corresponds to a single Nimber.*

This is really exciting! Maybe we can use the Sprague-Grundy Theorem on the Game of Cycles? (Spoiler alert, the answer is yes.)

3.3 Nimbers for the Game of Cycles

How do Nimbers work on the Game of Cycles? Given a specific instance of a game, how do we determine it's Nimber?

Let's start by learning a little bit more about Nimbers. The Nimbers are the set of numbers $\{*0, *1, *2, *3, \ldots, *n, \ldots\}$ for all $n \in \mathbb{N}$. The Nimber $*0$ always corresponds to a P-position, and $*n$ for $n \geq 1$ correspond to N-positions. The Nimbers are defined "recursive-ish-ly". Essentially, the Nimber for a position is determined by the Nimbers of all the positions that can be reached in the next move. The Minimum-Exclusion (Mex) Rule formalizes this saying:

Given a game board, and the current player has options to move to positions $*a, *b, *c, \ldots$*, then the Nimber of that board is the smallest natural number* $0, 1, 2, \ldots$ *that is not among the numbers* $a, b, c, \ldots$ [Berlekamp et al. (2001)].

The behavior of Nimbers is consistent with the behavior of P- and N-positions as we have previously described. We see that a P-position, with Nimber $*0$ has to move to an N-position. Why? Well suppose, for the sake of contradiction that we could move from a P-position to another P-position. This means we can move from a $*0$ position to a $*0$ position. However, by definition, a $*0$ position means that $*0$ is the smallest Nimber *not* reachable from that position (by the Mex Rule), and this is a contradiction. We also see that an N-position with Nimber $*n$ for some $n \geq 1$ can always move to a P-position, which has Nimber $*0$. This is because a Nimber of $*n$ means that we are guaranteed to be able to move to positions $\{*0, *1, *2, \ldots, *n-1\}$ (again by the Mex Rule). For $n \geq 2$, we see that a Player in a $*n$ position has the option to move to a winning $*0$ position, but also has the option to move to a position whose Nimber is $*m$ for some $1 < m < n$. This is where a player who is in a winning position could mess up.

Now how does this work for the Game of Cycles? We work backwards! We start by assigning all boards where the game has ended (either no remaining playable edges, or a cycle has been created) with $*0$. Now we proceed to label the other boards in a somewhat backwards recursive manner. For each board whose "next positions" have all been assigned a Nimber, we can assign this board its own corresponding Nimber using the Mex Rule. We continue until all possible positions have been labeled.

Figure 3.1 shows an example of assigning Nimbers to the Game of Cycles. The game tree illustrates game boards as moves are made, so the top of the tree shows the start of the game (no edges marked), and each path down represents a possible sequence of moves. To assign Numbers, we start by assigning $*0$ to every board that was the end of the game (either no markable edges left, or a cycle cell had been created). From there, we work our way up the tree to label assign each board its own Nimber using the Mex Rule. We see that the starting board has Nimber $*1$, which means that Player 1 has the opportunity to move to a $*0$ (or equivalently a P-position) which is a winning position. This tells us that Player 1 has a winning strategy at the start of the game.

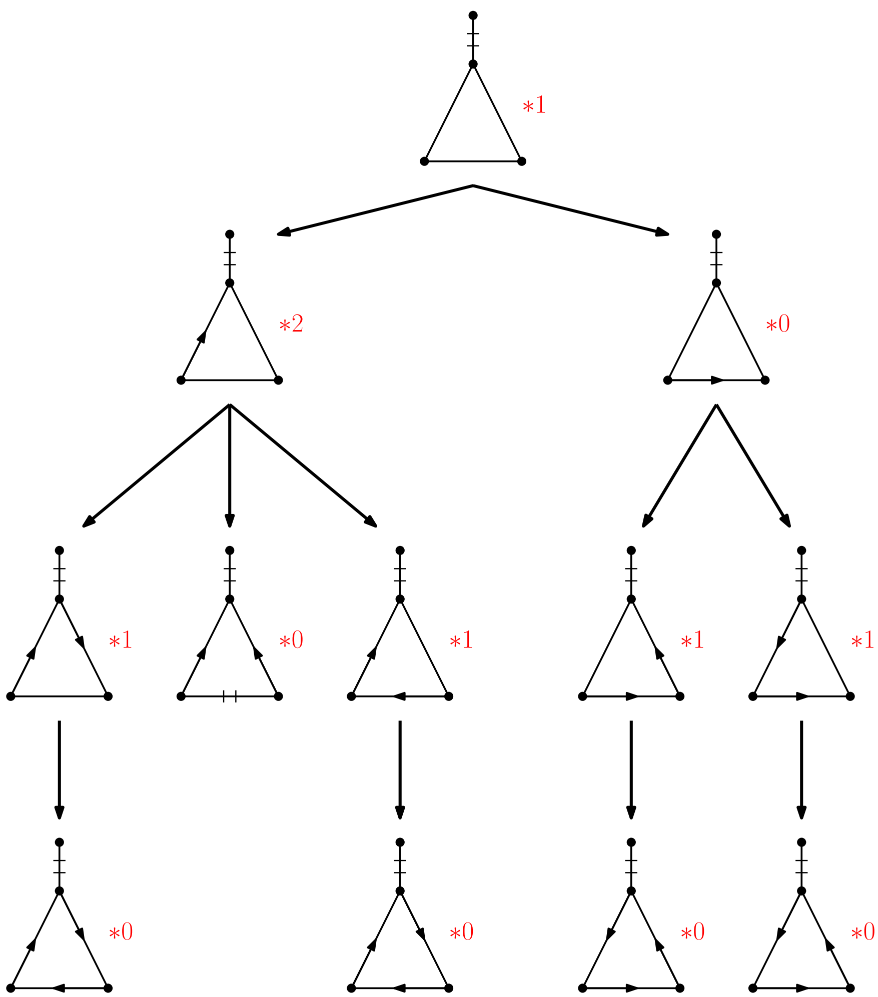

Figure 3.1 An example game tree showing how to compute the Nimbers (shown in red) for these game boards. Redundant boards as a result of symmetry have been left out.

In general, we see that if the original, unmarked board is a N-position, meaning its Nimber is $*n$ for some $n \geq 1$, then Player 1 has a winning strategy, since Player 1 must be able to move to a P-position from this original N-position. On the other hand, if the original, unmarked, board is a P-position, meaning its Nimber is $*0$, then Player 2 has a winning strategy, since Player 1 will be forced to move from this P-position to an N-position, and Player 2 will have the opportunity to move from this N-position to another P-position. So, assuming we can easily compute or determine the Nimber of any given board, we have a nice way to determine whether Player 1 or Player 2 has the winning strategy.

In the fall semester, I worked with Chris Thompson on a computer program that can compute the Nimbers for boards like n-cycles, cycles with rogue edges, and the square with a diagonal edge. However, the algorithm becomes very slow very fast, meaning as we increase the complexity of the boards we want to solve, the runtime of the algorithm shoots up hugely. We have tried running the program on the Two-Story House board (shown in Figure 2.4), and made only small amounts of progress after several hours. There are definitely improvements and optimizations that can be made to improve the runtime of the program to hopefully help spot patterns in Nimbers for more complicated boards.

At this time, I have not yet found a way to use Nimbers as a tool for finding winning strategies for general classes of boards for the Game of Cycles. Patterns were hard to spot in our results, so I did not pursue this approach during the spring semester. The rest of my thesis will focus on specific classes of boards, and explore which player wins, and how.

Chapter 4

Brainless Games

> I not only use all the brains that I have, but all I can borrow.
>
> Woodrow Wilson

4.1 Brainlessness

We know that every game will have a winner. It is simply a question of who the winner is. We learned in Chapter 3 using combinatorial game theory that Zermelo's Theorem tell us that for every instance of a Game of Cycles, either Player 1 or Player 2 will have a winning strategy[Zermelo (1912)]. If you have a winning strategy, this means that no matter how the other player moves, there exists a legal move for you to make such that you can guarantee a win. In most games, even if you have a winning strategy, you *can* mess up and lose the game.

Definition 4.1. A game is considered *brainless* if the player who has a winning strategy will win no matter what moves are played in the game (as long as they are legal). In a brainless game, the player who has a winning strategy *cannot* mess up, and is guaranteed to win.

Thinking in terms of N- and P-positions discussed in Chapter 3.1.1, the player with the winning strategy only ever has the option to move from an N-position to a P-position (and still the other player only has the option to move from P-position to N-positions).

For example, any game played on an n-cycle board is brainless. Why? Recall by Theorem 2.1 that the winner of a game played on any n-cycle is determined solely by the parity of n. Specifically if n is odd, Player 1 is guaranteed to win, and if n is even, Player 2 is guaranteed to win. This has no dependence whatsoever on how the players play, and so the game is brainless.

I was curious if I could find any other boards that were brainless. Let's table this thought for a moment, but don't worry we will be back to visit the world of brainlessness before the end of this chapter. First this chapter will discuss edges that are unmarkable from the start of the game, and then see how these lead to a neat example of a new class of brainless boards.

4.2 Rogue Edges

Definition 4.2. Let us call an edge that is connected to a vertex of degree 1 a *rogue edge*. Such edges are sometimes called *leaves*, but we'll stick with calling them *rogue edges*.

Imagine attaching some rogue edges to a board. Do these edges change the way this game is played? Are there moves we used to be able to make but can't make anymore? Or are there moves that we can make now but couldn't make before?

The edges are interesting because they allow an extra degree of freedom for the edges adjacent to it in the cycle, but cannot be played themselves (since they have an endpoint of degree 1 which would automatically become a sink/source as soon as the edge was labeled). So the answer to our first question is, slightly! We are not restricting the board in any way by adding rogue edges, so in answer to our second question, no! There are no moves we used to be able to make but can't make anymore. And finally, since we have added some extra flexibility in terms of available moves on the original board, the answer to our third question is yes!

For example, we see in Figure 4.1 that the edge af is a rogue edge, since vertex f only has degree 1. Without the added rogue edge af, vertex a would have been a source. However, with the addition of the rogue edge, these moves were completely legal!

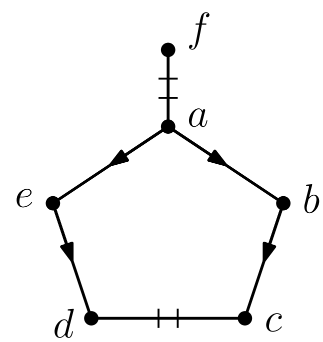

Figure 4.1 A 5-cycle board with 1 rogue edge, af. (Vertex a is not a source!)

4.3 Rogue Edges on n-Cycle Boards

I was curious to see how adding rogue edges affected game play. One question that came to mind was: Can adding rogue edges to a brainless game make it no longer brainless? And vice-versa, can adding rogue edges to a not brainless game make it brainless?

Studying an example within the family of n-cycles (which we know are brainless) seemed like a good idea, since we have a good amount of intuition about game play on n-cycle boards.

To gain some intuition, let us study adding rogue edges to a 5-cycle board (see Figure 4.2). We know, again by Theorem 2.1, that a 5-cycle board is a brainless game. Player 1 is guaranteed to win since 5 is odd.

If we add 1, 2, or 3 rogue edges in the positions shown in Figure 4.3, these boards are no longer brainless. Player 1 still has a winning strategy, but Player 1 is no longer guaranteed a win. They could mess up, and Player 2 has a chance of winning. It appears that adding rogue edges to a brainless board makes it not brainless anymore.

However, if we continue adding more rogue edges, and look at boards with 3, 4, or 5 rogue edges in the positions shown in Figure 4.4, these boards are all brainless again. Player 1 has regained a guaranteed win.

This struck me as odd... I was expecting to find that adding rogue edges either just made brainless games no longer brainless, or the other way around. Instead I found that when our "base" graph is a cycle, it goes both ways. Adding rogue edges initially makes the brainless game not brainless, but then adding enough rogue edges (and in some specific positions), can turn the game brainless again.

Let's see if we can understand a bit more intuitively why this happens

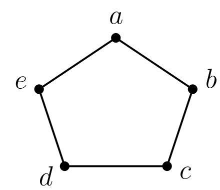

Figure 4.2 A 5-cycle board - Brainless

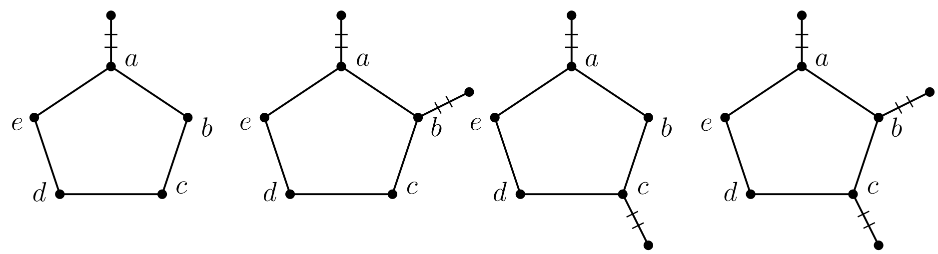

Figure 4.3 A 5-cycle board with 1, 2, and 3 rogue edges - Not Brainless

on a cycle. We know that the 5-cycle board is brainless because we are guaranteed to have an even number of unmarkable edges left at the end of the game. Adding any number of rogue edges breaks this rule. For example, Figure 4.1 shows a board with no markable edges remaining, but there is only one unmarkable edge, dc, left. So, it makes sense that Lemma 2.1, Lemma 2.2 and Theorem 2.1, which prove that the 5-cycle is brainless, break down once we have added rogue edges.

However, why is it that by adding some sufficient number of rogue edges (and in the correct positions), we can make the 5-cycle game brainless again? Well let us observe the 5-cycle boards with 3, 4, and 5 rogue edges. It appears that we not only want a sufficient number of rogue edges, but also that we want them spaced out in a certain way. For example, the 5-cycle with 3 rogue edges connected to vertices a, b, and c, is not a brainless game (Figure 4.3), but when we have a 5-cycle with 3 rogue edges somewhat "evenly" spaced and connected to vertices a, b and d, (Figure 4.4), out we do have a brainless game.

This observation led me to the following theorem. However, first let's talk about what a vertex cover of a graph is.

Definition 4.3. A subset of vertices from a graph is called a *vertex cover* if every edge in the graph has at least one endpoint included in that subset.

What does this have to do with rogue edges and brainlessness?

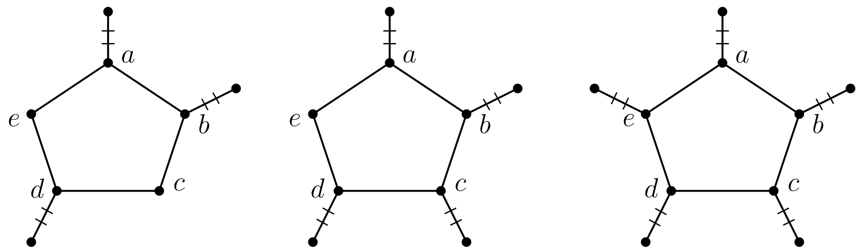

Figure 4.4 A 5-cycle board with 3, 4, and 5 rogue edges - Brainless

Theorem 4.1. *Suppose we are given a board that is an n-cycle with some number of rogue edges attached to the vertices of the cycle. Consider the subset of vertices of the n cycle that have rogue edges attached to them. If this subset forms a vertex cover of the original n-cycle, then the board is brainless.*

Proof. If the set of vertices with rogue edges attached to them is a vertex cover of the n-cycle, then we cannot have any unmarkable edges. Consider any vertex v that has a rogue edge. The extra "degree of freedom" afforded to v by the rogue edge, means that any edge in the cycle adjacent to v will never violate the Sink-Source Rule and can always be played legally. If the collection of vertices with rogue edges form a vertex cover, then this guarantees every edge can always be played legally.

Now, recall that the Game of Cycles is won in one of two ways:

1. The game can be won by the first player who forms a cycle cell. To do this on an n-cycle, all n edges must be played.

2. The game can be won by the player who plays on the last markable edge. We know that we cannot have any unmarkable edges on the n-cycle, so to do this all n edges must be played.

Therefore, regardless of which condition is used to win the game, the player who plays the n^{th} edge, or equivalently makes the n^{th} move, is guaranteed a win. This means our game is brainless again! Woah. □

4.4 A New Winning Strategy Conjecture

We have stumbled upon a counterexample to one of the open questions posed by Alvarado et al. (2020) which was:
"All of the theorems we have proved so far have shown, for various classes of boards,

that if the number of edges in the board is odd, Player 1 has a winning strategy, and otherwise Player 2 has a winning strategy. Is there a board that does not follow this pattern?"

We now know the answer is *yes*! There are boards that do not follow this pattern! Specifically boards with rogue edges do not follow this pattern! In the example of the 5-cycle, when we add 2, 4, or no rogue edges to the board, we have an odd total number of edges on the board. When we add 1, 3, or 5 rogue edges to the board, we have an even total number of edges on the board. However, in *all* of these boards, Player 1 has a winning strategy. These examples show the player who has the winning strategy is *not* in fact always determined by the parity of the number of edges on the board. This led me to amend the conjecture made by Alvarado et al. (2020). I used this conjecture to guide much of my work in the rest of my thesis exploration. This new working conjecture states that:

Conjecture 4.1. *At the start of any game, if the number of markable edges on the board is odd, Player* 1 *has a winning strategy, and if the number of markable edges on the board is even, Player* 2 *has a winning strategy.*

All the boards and winning strategies proven by Alvarado et al. (2020) as well as the ones that I will prove in my thesis follow this conjecture, and no counter example has not been found. However, this conjecture is still unproven. (This could be *you*!)

Chapter 5

Winning Strategies for More Kinds of Boards

> Winning isn't everything; it's the only thing.
>
> Vince Lombardi

Hey! You made it to Chapter 5! This is where it gets *good*, so you're in exactly the right place. In this chapter, I will show how the mirror-reverse strategy from Chapter 2.3 can be used and adapted to win on a bunch of new kinds of boards. Even better, these boards have fun names, like lollipops and steering-wheel-locks. A quick throwback moment, but remember going to malls and seeing those Build-a-Bear workshops where you can customize your own teddy bear? Well today I will gift you your very own Build-a-Board workshop, and give you the tools to build your *own* custom boards for the Game of Cycles that you can also win! What am I talking about? Stick with me just a bit longer and you will see, I promise.

5.1 Gameplay on a Tree

Here's a funny thought. This is a thesis about the Game of *Cycles*. What if we just... got rid of the cycles?

A graph with no cycles is called a *tree*. It turns out that we can play the Game of Cycles on a graph with no cycles, since the rules do not actually *require* a cycle to play the game (cycles just give players a fun opportunity to win the game by creating a cycle cell).

So what does playing the Game of *Cycles* on a *tree* look like? Recall that a player can win in either of the two following ways:

1. Win by completing a directed cycle cell on the board.

2. Win by playing the last markable edge on the board.

Since there are no cycle cells that can be completed when playing on a tree, players can only win by playing the last markable edge on the board. So the Game of Cycles really becomes some sort of "Game of Markable and Unmarkable Edges".

5.2 The Easiest Tree - Line

Let us consider any board that is a line with n edges. Call this board L_n. This is a game we can win using the mirror-reverse strategy, and I'll show you how right now!

Theorem 5.1. *For a board that is a line L_n with n markable edges, Player* 1 *has a winning strategy if n is odd and Player* 2 *has a winning strategy if n is even.*

Proof. Let us consider two cases, when n is even, and when n is odd. By Conjecture 4.1, if n is odd, Player 1 should have a winning strategy, and if n is even, Player 2 should have a winning strategy. We will see that the mirror reverse strategy can be used by the winning player in each of these cases.

We see that L_n has both 180° rotational symmetry as well as reflective symmetry. (Yes, there is an axis of reflective symmetry along the line, but we use the axis of symmetry that divides the line such that half the edges are on one side and half are on the other).

Let us consider the case when n is odd. The center edge of the line will be self-involutive. Conjecture 4.1 suggests that Player 1 should have a winning strategy in this case. We show that Player 1 does in fact have the following winning strategy: Player 1 should first mark the self-involutive edge in any direction. From then on, Player 1 should mirror-reverse any move Player 2 makes. We see this is a successful winning strategy since players can only win by marking the last markable edge (as there are no cycles to make a cycle cell in). The mirror reverse strategy guarantees there will be an even number of unmarkable edges at the end of the game, hence Player 1 must be the player to mark the last edge, and so Player 1 wins!

We similarly consider the case when n is even. There is no self-involutive edge. Conjecture 4.1 suggests that Player 2 should have a winning strategy in this case. We show that Player 2 does in fact have the following winning strategy: Player 2 should mirror-reverse any move Player 1 makes. We see this is a successful winning strategy since players can only win by marking the last markable edge (as there are no cycles to make a cycle cell in). The mirror reverse strategy guarantees there will be an even number of unmarkable edges at the end of the game, hence Player 2 must be the player to mark the last edge, and so Player 2 wins! □

5.3 3-Spoke Boards

So far we have studied cycles and lines, both of which only involve vertices of degree 1 or 2. We have seen that a vertex of degree 1 means that the incident edge cannot be marked, otherwise that vertex becomes either a source or a sink. We have seen that a vertex of degree 2 imposes no restrictions on the first incident edge that is marked, but demands that the second incident edge is either marked in the same direction, or perhaps not marked at all. What is the role of vertices with degree 3?

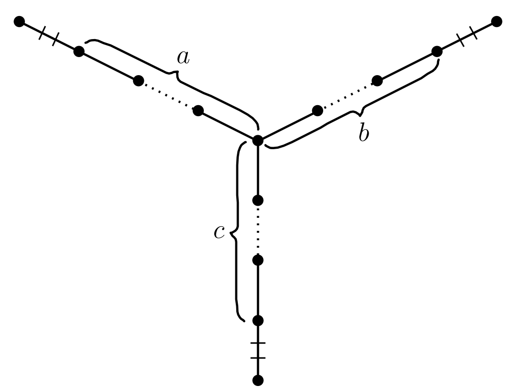

Figure 5.1 3-Spoke Tree - $T_{a,b,c}$. A central vertex with three spokes with a, b, and c markable edges.

A simple example to study might be a central vertex with 3 lines emanating from it. In general, we will call trees with this structure of three spokes connected by a central vertex a tree $T_{a,b,c}$ where a, b, and c denote the number of markable edges on each spoke as shown in Figure 5.1.

5.3.1 Some Small Cases

To explore this question, let us start with some simple examples.

Consider the board in Figure 5.2.

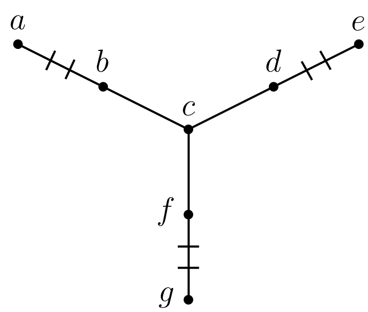

Figure 5.2 3-Spoke Tree - $T_{1,1,1}$. A central vertex c, with three spokes all with 1 markable edge.

For starters, there are three rogue edges, ab, de, and fg and three markable edges, bc, cd, and cg, so Conjecture 4.1 suggests that Player 1 should have a winning strategy. We see that every markable edge is adjacent to a rogue edge. This means players can never create another unmarkable edge during gameplay, since this would require a violation of the Sink-Source Rule at both ends of the edge. This means that all three edges will be markable. Since this is a tree, no player can win by making a cycle, and thus the only win condition is playing the last markable edge. Since there are guaranteed to be a total of three markable edges, Player 1 is actually sure to win this game - brainlessly!

Let's look at another example shown in Figure Figure 5.3.

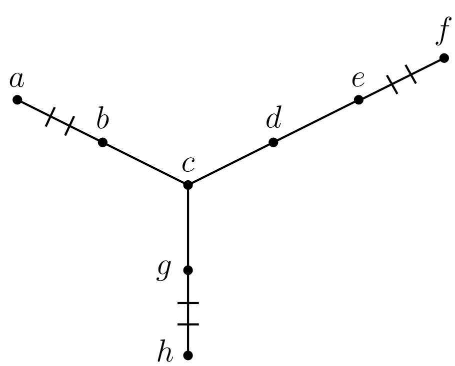

Figure 5.3 3-Spoke Tree - $T_{1,1,2}$. A central vertex with two spokes with 1 markable edge, and one spoke with 2 markable edges.

We still have three rogue edges, ab, ef, and fh. However, this time we have four markable edges, bc, cd, de, and cg, so we guess that Player 2 should have a winning strategy for this game. However this time, not every markable edge is adjacent to a rogue edge. Edge cd could become unmarkable if edges bc, de, and cg are marked all pointing towards or all pointing away from vertex c. This would be bad news for Player 2 because a single unmarkable edge at the end of the game means Player 1 gets to mark the last edge. But not to fear! This is easily preventable. No matter what Player 1 chooses to do on their first move, Player 2 should pick a different spoke, and play an edge in the oppose direction. For example, if Player 1 labels an edge towards the center vertex, Player 2 should label any edge on a different spoke pointing away from the center vertex. This will ensure that the center vertex cannot become a source or sink, and therefore no player can create any unmarkable edges. Guaranteeing four markable edges ensures that Player 2 will win with this strategy!

5.3.2 Winning Strategy for $T_{a,b,0}$

For any number of edges a and b, we can imagine $T_{a,b,0}$ to be a line with a markable edges, a second like with b markable edges, and a third line that is just one rogue edge. This is shown in Figure 5.4.

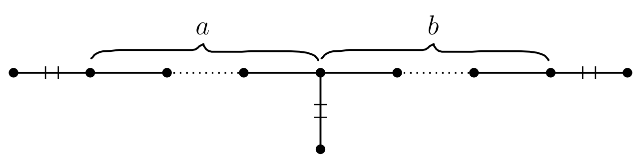

Figure 5.4 3-Spoke Tree - $T_{a,b,0}$. A central vertex with one spoke with a markable edges, one spoke with b markable edges, and one spoke that is a single rogue edge.

We can see that the vertex of degree 3 cannot ever be a source or a sink because of the rogue edge. This allows us to play the two sections of the line as somewhat independent games. Again, Conjecture 4.1 suggests that if the total number of markable edges $a + b$ is odd, we expect that Player 1 should have a winning strategy, and if $a + b$ is even, we expect that Player 2 should have a winning strategy. Let's prove this is true.

Theorem 5.2. *For the board $T_{a,b,0}$, if $a + b$ is odd, Player 1 has a winning strategy and if $a + b$ is even, Player 2 has a winning strategy.*

Proof. We show that the player with the winning strategy is guaranteed to win if they apply the mirror-reverse strategy separately on the a and b spokes of $T_{a,b,0}$. We can break this argument down into two cases.

- If $a+b$ is odd, then exactly one of a or b is odd. Player 1 should first play the self-involutive edge of the spoke with an odd number of edges. For their remaining turns, Player 1 will mirror-reverse on whichever spoke Player 2 has marked. This will guarantee an even number of unmarkable edges on each spoke, and therefore an even total number of unmarkable edges on the board by the end of the game. Thus when the game ends, an odd number of edges have been played, and this means Player 1 played the last edge and won!

- If $a+b$ is even, then either both a and b are even, or both a and b are odd. If both a and b are even, it will always be possible for Player 2 to mirror-reverse on whichever spoke Player 1 marks. Mirror reversing guarantees an even number of unmarkable edges, meaning an even number of edges have been marked whenever the game ends. This means Player 2 won since they must have marked the last edge! If both a and b are odd, Player 2 will follow the almost exactly the same mirror-reverse strategy as above. The only difference is if Player 1 plays the self-involutive edge of one of the spokes, then Player 2 should play the self-involutive edge of the other spoke. Because Player 2 will have been mirror-reversing any moves Player 1 has made up to this point in the game, it will always be possible to mark the self-involutive edge of both sections. Again, this guarantees an even number of unmarkable edges, hence an even number of played edges, at the end of the game, and again Player 2 is the winner!

□

So, we see that for $T_{a,b,0}$, we have found winning strategies with a slightly adjusted version of the mirror reverse strategy! Okay, but what about 3-spoke trees where all 3 edges incident to the central vertex are all actually markable?

5.3.3 Winning Strategy for $T_{a,a,b}$

Let us look at a 3-spoke tree where two spokes each have a markable edges, and the third spoke has b markable edges, for any a and b. The total number of markable edges is $2a + b$. Conjecture 4.1 suggests that Player 1 should have a winning strategy if b is odd, and Player 2 should have a winning strategy if b is even.

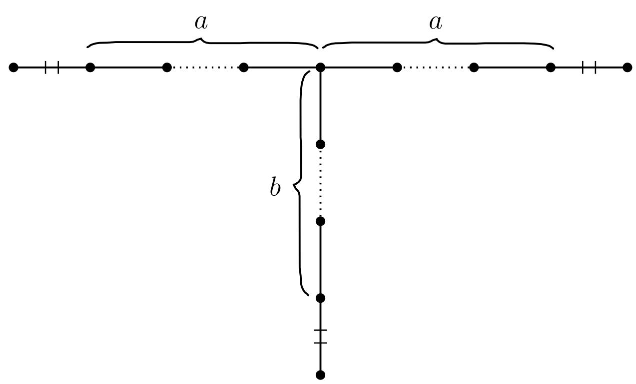

Figure 5.5 3-Spoke Tree - $T_{a,a,b}$. A central vertex with two spokes each with a markable edges, and one spoke with b markable edges.

Let's prove this.

Theorem 5.3. *For the board $T_{a,a,b}$, Player 1 has a winning strategy if b if odd, and Player 2 has a winning strategy if b is even.*

Proof. Let us imagine $T_{a,a,b}$ as two sections as seen in Figure 5.5. The first section is a line with $2a$ total markable edges, and the second section is the line with b markable edges appended adjacent to the middle of the line.

- If b is odd, then Player 1 should first play the self-involutive edge of the spoke with b edges. Now, they should mirror-reverse on the same section any move that Player 2 makes. As before, this guarantees an even number of unmarkable edges. Thus, Player 1 will play the last edge and win.
- If b is even, then Player 2 should mirror reverse on the same section any move that Player 1 makes. Again, this guarantees an even number of unmarkable edges, and so Player 2 will play the last edge and win.

Note that we will never violate the Sink-Source Rule at the degree 3 vertex with this strategy because mirror-reversing across the entire $2a$ line section ensures both adjacent edges will never be marked towards the same vertex at any point on the line (which includes the degree 3 vertex). □

These are a few types of boards where we can adapt the mirror reverse strategy to win these games. There are many further questions and generalizations left to explore. For example, can we find winning strategies for $T_{a,b,1}$, $T_{a,b,2}$, and even the most general $T_{a,b,c}$?

5.4 Lollipops

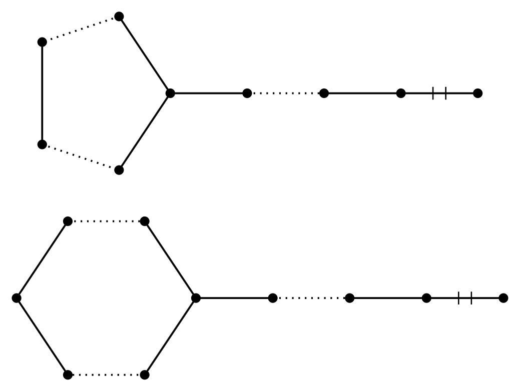

Figure 5.6 Top: A Lollipop with a cycle with an odd number of edges. Bottom: A Lollipop with a cycle with an even number of edge.

We call a cycle with a single line path appended to one vertex on that cycle a "Lollipop" as seen in Figure 5.6. We can imagine the cycle as the candy, and the line as the stick. We have one vertex of degree 3 where the cycle and line meet. Let us find winning strategies for either Player 1 or Player 2 as predicted by Conjecture 4.1. Let m denote the number of edges on the cycle and n denote the number of playable edges on the line (so n will always be one less than the number of edges on the line as the last edge will always be unmarkable).

Theorem 5.4. *For a Lollipop board, if $m + n$ is odd, Player* 1 *has a winning strategy, and if $m + n$ is even, Player* 2 *has a winning strategy.*

Proof. This proof will be broken into four cases, depending if m and n are each even or odd. In each case, we will show a version of the mirror-reverse strategy that the player with a winning strategy can use.

- *If both m and n are even:*
  The total number of playable edges is even, so we hypothesize by Conjecture 4.1 that Player 2 should have a winning strategy, and

indeed they do. Imagine an axis of symmetry that goes through the m-cycle as determined by the position of the line (or lollipop stick). If Player 1 plays a move on the cycle, Player 2 should mirror reverse on the cycle across this line of symmetry. If Player 1 plays a move on the edge, Player 2 should mirror reverse on the other half of that line. Since the cycle has an even number of edges, only Player 2 can win by completing the cycle, since they are always the second player to play and edge on the cycle. The mirror reverse strategy also guarantees that we have an even number of unmarkable edges on the cycle, and an even number of unmarkable edges on the line, for an even total number of unmarkable edges. This means there are always an even number of markable edges, hence Player 2 is also the only player who can win by playing the last markable edge. Thus, this is a good winning strategy for Player 2!

- *If m is even and n is odd:*
  The total number of playable edges is odd, so we hypothesize by Conjecture 4.1 that Player 1 should have a winning strategy. As you might guess, Player 1 should play the center edge on the line, since the line has an odd number of edges. After that, Player 1 should just mirror reverse whatever move Player 2 makes. If Player 2 marks an edge on the line, Player 1 should mirror reverse on the line using the center edge as the axis of symmetry. If Player 2 marks an edge on the cycle, Player 1 should mirror reverse on the cycle using the axis of symmetry determined by the position of the line. We see that using this strategy, only Player 1 can win by completing the cycle cell since they will always be the second player to mark an edge on the cycle. Also, this strategy guarantees an even number of unmarkable edges on each section, for an odd total number of markable edges by the end of the game. This, Player 1 is also the only player who can win by playing the last markable edge. So, Player 1 wins!

- *If m is odd and n is even:*
  Very similar to the case above, the total number of playable edges is odd, so we hypothesize by Conjecture 4.1 that Player 1 should have a winning strategy. This time, Player 1 should play the edge exactly opposite the appended line on the cycle, since the cycle has an odd number of edges. After that, Player 1 should just mirror reverse whatever move Player 2 makes. If Player 2 marks an edge on the line, Player 1 should mirror reverse on the line using the halfway

vertex as the axis of symmetry. If Player 2 marks an edge on the cycle, Player 1 should mirror reverse on the cycle using the axis of symmetry determined by the position of the line. We see that using this strategy, after the first move there are an even number of edges left on the cycle, and only Player 1 can win by completing the cycle cell since they will always be the second player to mark a remaining edge on the cycle. Also, this strategy guarantees an even number of unmarkable edges on each section, for an odd total number of markable edges by the end of the game. This, Player 1 is also the only player who can win by playing the last markable edge. So, Player 1 wins!

- *If both m and n are odd:*
  This last case is a little bit tricker, but we will approach thinking about it in the exact same way as before. To start, we see that the total number of playable edges is even, so we hypothesize by Conjecture 4.1 that Player 2 should have a winning strategy. Generally, Player 2 should mirror reverse on the same section any move that Player 1 makes. If Player 1 plays either the center edge on the line or the edge opposite the appended line on the cycle, then Player 2 should play the other. This will always be possible if Player 2 has been playing by the mirror reverse strategy.

  However at this point we have to be careful! Following this strategy, Player 2 could make a death move on the cycle, allowing Player 1 to complete a cycle cell and win. So we slightly amend the strategy. Player 2 follows the mirror reverse strategy above, unless the mirror reverse move would be a death move! In this case, Player 2 should not make this move. Instead, Player 2 should mark the center edge of the line. How do we know that this is always going to be possible? If Player 2 has reached a situation where they are about to make a death move on the cycle, this means there would only be one unmarked edge left on the cycle if Player 2 made the death move. Since Player 2 has been following the mirror reverse strategy up to this point, it is only possible that the single unmarked edge on the cycle is the edge opposite the appended line (all other edges on the cycle have been played in pairs). Since this edge has not been played yet, we also know that the center edge has not been played yet, since the strategy says if one of these edges is played that Player 2 should play the other. So, if Player 2 is trying to avoid making this death move, we know that playing the center edge of the line is always an available option. (See

Figure 5.7 for an example of what this looks like.) From here on out, Player 2 continues to mirror reverse on the line if Player 1 marks an edge on the line, and should complete the cycle cell if Player 1 makes the death move on the cycle. So, with a little extra brainpower, Player 2 can win this kind of board using this strategy!

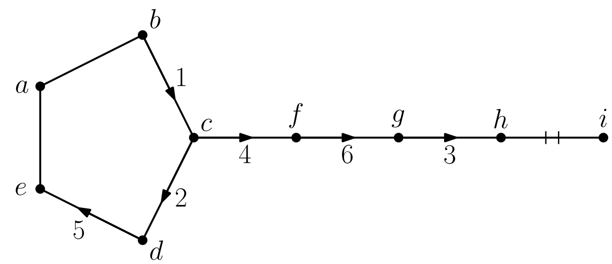

Figure 5.7 Death Move Example: If Player 2 simply followed the mirror-reverse strategy, they would make a death move by playing the 4^{th} out of 5 edges in a cycle cell. Instead, Player 2 marks the self-involutive edge on the line, forcing Player 1 to make the death move, so Player 2 wins!

□

5.5 Steering Wheel Lock

Now what if our lollipop had two sticks? Let us imagine a cycle with an even number of edges n, and two lines with a and b markable edges, appended to opposite vertices on the cycle like in Figure 5.8. This picture looks somewhat like a steering wheel lock, so that's what we will call this sort of board.

This is yet another example where we can easily apply the mirror reverse strategy to win the game!

Theorem 5.5. *For a Steering Wheel Lock board, Player* 1 *has a winning strategy if* $a + b$ *is odd, and Player* 2 *has a winning strategy if* $a + b$ *is even.*

Proof. We break this proof into three cases, depending on if a and b are both even, both odd, or if one is even and the other is odd.

- *If a and b are both even:*
  Player 2 should have a winning strategy since the total number of markable edges is even. Whichever section Player 1 marks, Player

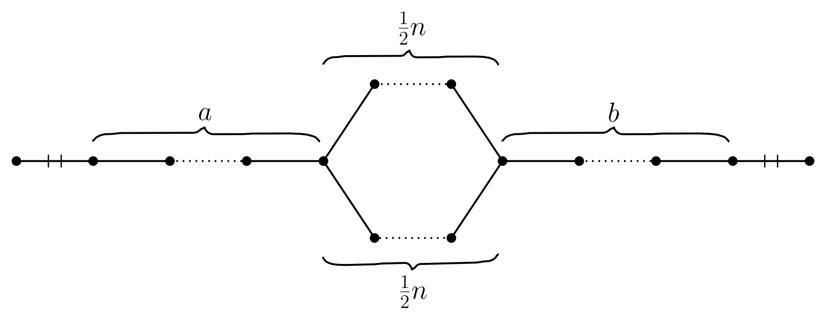

Figure 5.8 A Steering Wheel Lock board with a cycle with an even number n edges and two lines with a and b markable edges appended to opposite vertices on the cycle.

2 should respond by mirror reversing on the same section. This guarantees an even number of unmarkable edges on each of the lines, and on the cycle, for a total even number of markable edges, meaning Player 2 is the only player who can win by marking the last edge. Furthermore, Player 2 will always be the second player to mark an edge in the cell, so Player 2 is the only player who can possibly win by completing a cycle cell. Lastly, note that we will never worry about a source or sink vertex at the intersection of the cycle and a line because the mirror reverse strategy on the cycle guarantees that there will be at least one edge directed towards the vertex and another edge directed away. Thus, Player 2 wins!

- *If either a or b is even and the other is odd:*
  Player 1 should have a winning strategy since the total number of markable edges is odd. First, Player 1 should mark the center edge of the line with an odd number of unmarkable edges. Then, for any edge Player 2 marks, Player 1 should respond by mirror reversing on the same section. This guarantees an even number of unmarkable edges on each of the lines and on the cycle, for a total odd number of markable edges, meaning Player 1 is the only player who can win by marking the last edge. Furthermore, Player 1 will always be the second player to mark an edge in the cell, so Player 1 is the only player who can possibly win by completing a cycle cell. Again, note that we will never worry about a source or sink vertex at the intersection of the cycle and a line because the mirror reverse strategy on the cycle guarantees that there will be at least one edge directed towards the

vertex and another edge directed away. Thus, Player 1 wins!

- *If a and b are both odd:*
  Player 2 should have a winning strategy since the total number of markable edges is even. Whichever section Player 1 marks, Player 2 should respond by mirror reversing on the same section. If Player 1 marks the center edge of one of the lines, then Player 2 should mark the center edge on the other. There exactly 2 center edges (since both a and b are odd), so this will always be possible. This guarantees an even number of unmarkable edges on each of the lines, and on the cycle, for a total even number of markable edges, meaning Player 2 is the only player who can win by marking the last edge. Furthermore, Player 2 will always be the second player to mark an edge in the cell, so Player 2 is the only player who can possibly win by completing a cycle cell. Still, we will never worry about a source or sink vertex at the intersection of the cycle and a line because the mirror reverse strategy on the cycle guarantees that there will be at least one edge directed towards the vertex and another edge directed away. Thus, Player 2 wins!

□

5.6 Build-a-Board

The lollipop and steering wheel lock boards seem to suggest that we can append certain sections to make an overall board where the mirror reverse strategy still holds in a simple way. The lollipop appended a single line to a cycle, and the steering wheel lock appends two lines opposite one another on an even cycle. Mirror reversing on the cycle along the axis determined by these appended lines ensures that we do not have a problem (no chance or a source or a sink) at the vertices of degree 3 where these two sections meet. This nicely allows us to apply the mirror reverse strategy almost independently to each section of the board (with a small accommodation if we have multiple lines with odd numbers of edges).

In fact, we can use the following "building blocks" shown in Figure 5.9 and Figure 5.10 to construct boards where we can still use the mirror reverse strategy on each block like we did in the examples above.

Theorem 5.6. *Consider any board created with any number of cycles with even edges or lines concatenated in a line and either cycles with an odd number of edges*

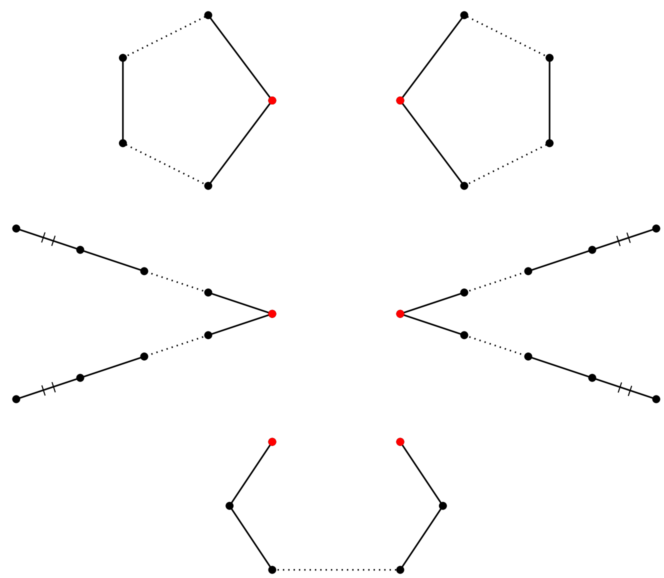

Figure 5.9 Three choices for end blocks. Connect to other blocks at the red vertices.
Top: Cycles with odd number of edges.
Middle: Two spokes with equal number of unplayable edges.
Bottom: A line of edges looping the two ends together.

or two spokes of equal length at each end, or one line looping between both ends. If there are an even number of markable edges on the board, Player 1 *has a winning strategy, and if there are an odd number of markable edges on the board, Player* 2 *has a winning strategy.*

Proof. We break this proof into two cases, if the number of markable edges is even or odd.

- *If the total number of markable edges is even:* We hypothesize that Player 2 should have a winning strategy for any of these boards. We know there will be an even number of "blocks" that have an odd number of edges. If Player 1 marks a self-involutive edge on a line or cycle with an odd number of edges, Player 2 should mark any other self-involutive edge. This will always be possible since there are an even number of such self-involutive edges, and our mirror reverse strategy will never make these edges unmarkable. Otherwise Player 2 should mirror reverse along the axis of symmetry on the same "block" that Player 1 played unless this would lead Player 2 to make a death move on a cycle.

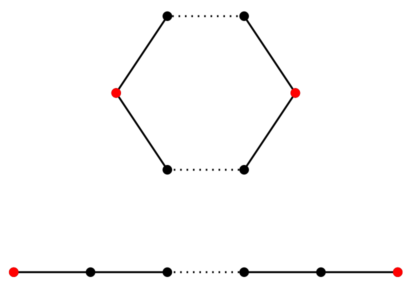

Figure 5.10 Two choices for middle blocks. Connect to other blocks at the red vertices.
Top: Cycles with even number of edges.
Bottom: A line of edges.

This can only happen on a cycle with an odd number of edges. Like the lollipop case where the cycle and stick both have an odd number of components, this will only happen if the self-involutive edge on the odd cycle has not been played, meaning that there is another self-involutive edge on the board that Player 2 can mark instead.

- *If the total number of markable edges is odd:* We hypothesize that Player 1 should have a winning strategy for any of these boards. We know that an odd number of "blocks" will have an odd number of sections. Player 1 should first mark a self-involutive edge on any of these "blocks", and then follow the strategy outlined above for Player 2. This works because we can effectively forget that first marked edge is on the board, and play as though there are an even number of "blocks" with an odd number of edges, and as though Player 1 is now Player 2 and vice versa.

□

Chapter 6

Conclusion

> Life should not be a journey to
> the grave with the intention of
> arriving safely in
> a pretty and well preserved
> body, but rather to skid
> in broadside in
> a cloud of smoke, thoroughly
> used up, totally worn out,
> and loudly proclaiming "Wow!
> What a Ride!"
>
> Hunter S. Thompson

Whew! That was a lot! So many new ideas, boards, and variations of the mirror-reverse strategy. So, what exactly happened in the past 40 pages? We learned how to play the Game of Cycles, and got familiar with the Sink-Source Rule, unmarkable edges, and practiced with a few example games. This groundwork is already enough background for you to start exploring the Game of Cycles yourself!

In my own thesis journey, this is exactly what I did. My own exploration was largely motivated by two questions left unanswered at the end of the Alvarado et al. (2020) paper.

- Is there a simple and nice winning strategy for the Two-Story House board? (Figure 2.4)
- Winning Strategy Conjecture: Is it true that if the number of edges on a board is odd, then Player 1 must have a winning strategy, and if the number of edges is even, then Player 2 does?

I was really intrigued not only by the Winning Strategy Conjecture suggested by Alvarado et al. (2020), but also the idea that for some boards, like cycles C_n, the winning player was guaranteed no matter what moves were made. I thought of what the next simplest board would be, and this led me to the idea of sticking rogue edges on cycles. This exploration led me to somewhat unintentially disprove the Winning Strategy Conjecture, and suggest my own new version, which counts the number of markable edges on the board (and excludes unmarkable ones).

At about the same time of this exploration, Professor Su also suggested I try to learn some basic combinatorial game theory, specifically the structure of Nimbers and using the Sprague-Grundy Theorem to apply Nimbers to the Game of Cycles. While I was able to lay the groundwork in terms of understanding how Nimbers work, and how to compute them for any Game of Cycles board, in practice this did not lead me far in terms of finding winning strategies. The hope was that some sort of pattern in the Nimbers might illuminate a new kind of winning strategy for certain kinds of boards. However, the computational method that I worked on was by no means optimal. It was not fast enough to run on larger graphs in any reasonable amount of time, and did not seem to highlight any special patterns pointing to winning strategies for the smaller graphs we were able to compute. This road block led me to table the idea after the fall semester, and move on in search of winning strategies in a new way. That being said, I definitely believe that if Nimbers or other ideas in combinatorial game theory are of interest to you, there is definitely more progress that can be made from this perspective.

After my midyear report and talk, many people had suggested I consider studying trees. This was an interesting though considering it is called the Game of Cycles after all... I started by studying a line, which was easy enough. The next simplest tree after that seemed like the 3-Spoke tree, essentially 3 separate lines (something I understood) joined at one central vertex. I was able to find a few interesting ways to apply the Alvarado et al. (2020) mirror-reverse strategy here, which was really exciting. Specifically, understanding the dynamics around a vertex of degree 3 might help build intuition about how to win on other boards with vertices of degree 3 or even higher. While I was studying these trees, my motivation was always to be able to take my understanding of 3-Spoke Trees, and apply them to other kinds of boards, even ones that might have cycles. Graphs with cycles are obviously different that graphs without any, but I was curious if there was a way to map between the two. I briefly explored an idea that I call "graph folding", which will be explained more in depth in Chapter 7 where I discuss further work.

After some time in tree land, I wanted to return to thinking about graphs that did have cycles. Coincidentally, at about this time, my graph theory class with Professor Omar had just worked through a proof of Dirac's Theorem on Hamiltonian cycles. While neither the theorem nor the proof necessarily applied to the Game of Cycles, one idea used in the proof did. This was my inspiration for studying Lollipop boards. From Lollipops came the idea of having two sticks, which led to studying Steering Wheel Locks. Once I realized how the mirror-reverse strategy could be applied to each section of these boards (with a small interplay between self-involutive edges), I wanted to see how else we could apply the mirror-reverse strategy to sections of boards. This led me to the Build-a-Board idea, where I chose building blocks that I had studied in previous sections of the chapter, and showed how they could fit together for a "choose your own adventure but still know the secret to winning" kind of game!

Anyway, that was a bit more of the narrative on what was going on with my thesis adventure this past year. I think the main lesson was to look anywhere and everywhere for inspiration. Even seemingly unrelated or random things could lead to fruitful exploration! Searching for winning strategies was always one of my main motivations, however my approach was not set in stone. The flexibility to pivot and change my own strategy to creatively approach these problems kept the ball rolling, and ultimately led to a really rewarding thesis experience.

Chapter 7

Further Work

> If there's one thing I've learned in my life, it's that curiosity might kill cats, but it doesn't kill people.
>
> Tracy Morgan

Even though I was able to learn a lot about the Game of Cycles this semester, there is still *so much more* to be understood about winning strategies for the Game of Cycles. Here, I offer some of the questions left open after my exploration this semester, as well as a few new avenues for exploration that I believe could be interesting.

7.1 Open Questions

There are a vast number of questions still to be answered about the Game of Cycles! Many questions, posed by Alvarado et al. (2020) are still unanswered, and my thesis work has raised new questions as well. The biggest question to rule all questions of course is: "*How do we win the Game of Cycles?*"

However, this is a *massive* question, so here are a few more manageable questions that could help us build towards eventually fully understanding the Game of Cycles.

- Is there a simple and nice winning strategy for the Two-Story House board? (Figure 2.4)

- New Winning Strategy Conjecture (Conjecture 4.1): Is it true that if the number of *markable* edges on a board is odd, then Player 1 must have a winning strategy, and if the number of *markable* edges is even, then Player 2 does?
- What other types of game boards are brainless?
- Is there a simple and nice winning strategy for a general 3-Spoke Tree board, $T_{a,b,c}$? (Figure 5.1)
- What are other classes of boards the mirror-reverse strategy can be applied to?
- What are other building blocks that can be added to the Build-a-Board collection?

7.2 Combinatorial Game Theory: Nimbers

In the fall semester I began studying the structure of Nimbers (from Chapter 3), but ultimately decided to use a different approach for my own research. Even though I did not end up finding winning strategies from Nimbers, I was able to lay the framework for assigning Nimbers to Game of Cycles boards. However, the process of actually computing the Nimbers was slow, and therefore unfruitful. Improvements to the existing code could include using memoization as well as recognizing and encoding symmetries of a board in order to reduce redundant computations. (A great project if you or your friends like coding, graph theory, math, or playing games!) If this tool could be improved, I believe it could help identify patterns in Nimbers for different types of game boards, and perhaps give us hints for new winning strategies (or adaptations of existing ones)!

It is totally possible that there are other techniques and theorems from combinatorial game theory that could also be applied to the Game of Cycles, and so considering new tools from combinatorial game theory could lead to new discoveries.

7.3 Graph Folding

During my exploration of trees, I was curious if there was a way to think about graphs with cycles and graphs without cycles as analogous to one another. Or perhaps if there was a way to turn one into the other. That would be kind of cool, right? While I did not get this thought fully worked out, here was a bit of what I was thinking.

What if we took the rogue edges on trees, and glued them together to create cycles? The two rogue edges and their respective pairs of endpoints get merged into one rogue edge. This idea is illustrated more clearly in Figure 7.1 and Figure 7.2. We see that in both examples, we start with the same graph, yet it can be folded in two different ways. Figure 7.1 shows a folding that yields a 4-cycle, and Figure 7.2 shows a folding that yields a 3-cycle.

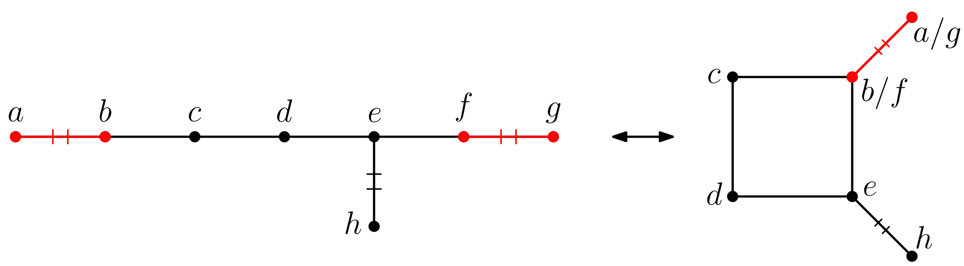

Figure 7.1 Graph Folding Example #1- Glue rogue edges ab and fg (shown in red) to form new graph with a 4-cycle.

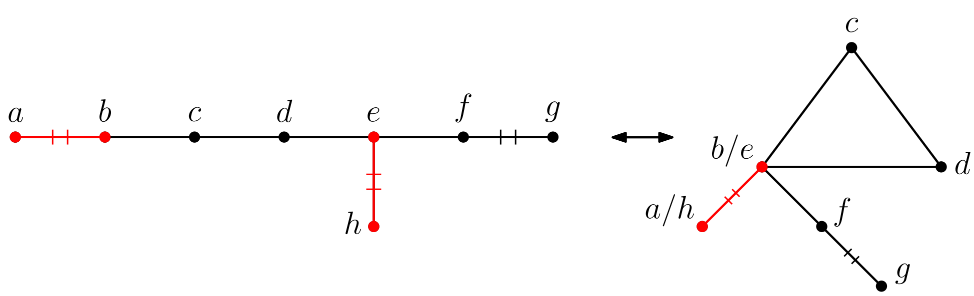

Figure 7.2 Graph Folding Example #2- Glue rogue edges ab and eh (shown in red) to form a new graph with a 3-cycle.

You might be wondering if we can play the game in exactly the same way?

Because we folded these graphs by pasting together rogue edges, any moves that are legal on board will be legal on the other. We do not have to worry about violating the Sink-Source Rule since the "degree of freedom" allowed by the rogue edge means the vertices that we are changing will never be a sink or source. The markable edges appear to be preserved under this folding action, so Conjecture 4.1 suggests that the same player will have a winning strategy at the start of both games.

Lots of this stuff makes it seem like the boards can be played the same way, and might even be equivalent. However, one component of the Game of Cycles is certainly different between the two boards. Recall that one of the winning conditions for the Game of Cycles is creating a cycle cell. This is not possible in the tree boards, yet it is possible in the folded versions of the boards. This may or may not lead to a different game outcome depending on how the board was folded.

For example, for the boards shown in Figure 7.1, the following set of moves would be a win in both the unfolded and folded versions of the board:

- bc
- cd
- de
- ef

However, for the boards shown in Figure 7.2, the following set of moves would be a win in the folded version of the board (since we have created a cycle cell) but not in the unfolded version (since we have a remaining markable edge, and no cycle cell):

- cd
- de
- ec

While this means that folded boards are not equivalent to their original tree boards, this does not necessarily mean that we cannot learn from one and apply it to the other. I have not fully explored this connected between trees and their foldings, but this could be an interesting connection to investigate in the future.

7.4 Linear Algebra

It really does seem like a big theme when exploring the Game of Cycles is learning about tools that can then help us learn about the Game of Cycles itself. Another tool that I think could potentially be useful is linear algebra!

I have been thinking that an oriented incidence matrix or a directed adjacency matrix could encode the information used in a Game of Cycles (vertices and directed edges), and then techniques from linear algebra could be used to uncover new insights.

For a directed graph (or in our case, board) on n vertices and m edges, an oriented incidence matrix δ is an $n \times m$ matrix whose rows are indexed by the vertices and columns are indexed by the edges of the graph. Let v be a vertex and e be an edge in the graph. The entries of δ are given by:

$$\delta_{v,e} = \begin{cases} 0 & \text{if vertex } v \text{ is not incident to edge } e \\ 1 & \text{if } v \text{ is the head of } e \\ -1 & \text{if } v \text{ is the tail of } e \end{cases}$$

For a directed graph (or board) on n vertices, a directed adjacency matrix M is an $n \times n$ matrix whose rows and columns are indexed by the vertices of the graph. Let i and j be vertices in our graph. The entries of M are given by:

$$M_{i,j} = \begin{cases} 1 & \text{if there is a directed edge from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases}$$

I wonder if examining eigenvalues, eigenvectors, inverses, sums across rows or columns, or any other properties of these matrices could lead to any new discoveries about the Game of Cycles. Linear algebra is certainly a powerful tool in many graph theoretic problems, and so I believe and hope that it could be an exciting way to explore the Game of Cycles too!

7.5 ... And So Much More!

So, it looks like there's great news and even better news. The great news we have now made it to the end of my thesis and you're still reading! The even better part is now *you* can go forth, play the Game of Cycles, beat your friends, and look for winning strategies like the mathematician you are. Run with any of the ideas for further exploration that I have mentioned in this thesis and in the above sections, or go rogue and follow your own curiosity!

Happy exploring!
- Kailee

Bibliography

Alvarado, Ryan, Maia Averett, Benjamin Gaines, Christopher Jackson, Mary Leah Karker, Malgorzata Aneta Marciniak, Francis Su, and Shanise Walker. 2020. The game of cycles. *American Mathematical Monthly* To appear, American Mathematical Monthly.

Berlekamp, Elwyn R., John Horton Conway, and Richard K. Guy. 2001. *Winning ways for your mathematical plays*, vol. 1. CRC Press LLC, 2nd ed. URL ebookcentral-proquest-com.ccl.idm.oclc.org/lib/claremont/detail.action?docID=3059511.

Bouton, Charles L. 1901. Nim, a game with a complete mathematical theory. *Annals of Mathematics* 3(1/4):35–39. URL http://www.jstor.org/stable/1967631.

Conway, John Horton. 1976. *On numbers and games*. Academic Press.

Guy, Richard. 1989. *Fair Game*. COMAP, Inc.

Su, Francis. 2021. *Mathematics for Human Flourishing*. Yale Univeristy Press.

Zermelo, Ernst. 1912. Über eine anwendung der mengenlehre auf die theorie des schachspiels. *Proceedings of the Fifth International Congress of Mathematicians* 2:501–504.