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The rationalist historians of mathematics consider the history of mathematics as the history of homogeneous growth of mathematical knowledge using essentially the unchanged axiomatic method. The fallibilists regard the growth of mathematics as a result of a dialectical process in which counter-examples to conjectures (theorems) lead to restructuring knowledge in mathematics or in its sub-fields.

INTRODUCTION

This paper addresses the question of the first paradigm in algebra, an achievement universally recognized to be that of the Arab mathematician Mohammed Ibn Musa Al-Khawarizmi (first half of the ninth century AD). First, the paper will discuss the usefulness of Kuhn's concept of paradigm in describing major developments in mathematics and education. Second, it will describe the pre-Al-Khawarizmi paradigm in algebra (henceforth, referred to as the prehistoric paradigm). Third, Al-Khawarizmi's algebra will be described. Then fourth, Al-khawarzmi's algebra will be analyzed as a paradigm, and finally some pedagogical implications will be discussed.

KUHN'S PARADIGM

Kuhn[1] contends that the development of science proceeds in paradigm shifts which involve revolutionary transitions. For Kuhn, "a paradigm is what the members of a scientific community share, and, conversely, a scientific community consists of men who share a paradigm" (p. 176). In the first sense (i.e. what a scientific community shares) a paradigm may be viewed as a "disciplinary matrix" whose components include:

> **1.** Symbolic generalizations (expressions universally accepted by members of a scientific community).
>
> **2.** Shared commitments to certain beliefs among members.
>
> **3.** Values which are widely shared among different communities belonging to the same discipline.
>
> **4.** Shared examples of concrete problem-solutions found in laboratories, textbooks, examinations, and journals (p. 182).

In the second sense, a paradigm is viewed as the shared examples themselves which provide the basis for acquired similarity relations that enable the members of a scientific community to regard similar situations as subjects for applying the same scientific law. According to Kuhn, a revolution is a special "sort of change involving a certain sort of reconstruction of group commitments" (p. 181). Kuhn argues that in normative science, this change is triggered by a crisis generated by incompatible ways of practicing the discipline by the particular scientific community.

To what extent do the constructs of *paradigm* and *paradigm shift* describe mathematics and its historical development? The paradigm construct seems to be applicable to mathematics in both meanings of *paradigm.* Mathematicians have always constituted a well-defined group that has shared a sophisticated system of symbolic generalizations, commitments to accepted beliefs and values, and a distinct body of examples with highly structured relations.

When it comes to the description of the historical development of mathematics, sharp disagreements arise. One can recognize two schools of thought in this regard. The rationalist historians of mathematics consider the history of mathematics as the history of homogeneous growth of mathematical knowledge using essentially the unchanged axiomatic method. The fallibilists regard the growth of mathematics as a result of a dialectical process in which counter-examples to conjectures (theorems) lead to restructuring knowledge in mathematics or in its sub-fields. To Lakatos[2],

who has best articulated the fallibilists' position, the inconsistencies and their refutations have led to changes in the dominant theory resulting in the reorganization of our knowledge. Thus, for instance, "the paradoxicality, and, indeed, seeming inconsistency of arithmetic induced the Greeks to abandon arithmetic

In a sense, these revolutions in education may be looked at as paradigm shifts not so much in the research concepts and methodology as in the conception and practices of education.

as the dominant theory and replace it by geometry" (p. 125). Lakatos's interpretation of the history of mathematics seems to be consistent with Kuhn's concept of paradigm shift in science.

Education, however, is a different matter. It is very difficult to argue for a paradigm in education in Kuhn's sense. Historically, educators have neither formed a distinct group with shared symbolic generalizations, beliefs, values and exemplars, nor have the latter constituted a well-defined basis for theory-building. Nevertheless, there have been throughout history basic changes in education resulting in revolutions in educational concepts and practices. Examples of such revolutions are: the introduction of the alphabet, the shift of responsibility for teaching from home to school, and the introduction of printing. A fourth revolution is predicted as a result of computer technology. In a sense, these revolutions in education may be looked at as paradigm shifts not so much in the research concepts and methodology as in the conception and practices of education.

THE PRE-HISTORIC PARADIGM

Whatever algebra existed before Al-Khawarizmi had neither a specific form nor a specific name to distinguish it from other fields of knowledge. What actually existed was rudimentary knowledge of some concepts and techniques involved in quadratic equations not so much as independent and distinct techniques but rather as incidental solutions of specific and isolated problems. In the paragraph that follows, we present a brief historical account of algebraic analysis before Al-Khawarizmi. More details are given in Karpinski[3].

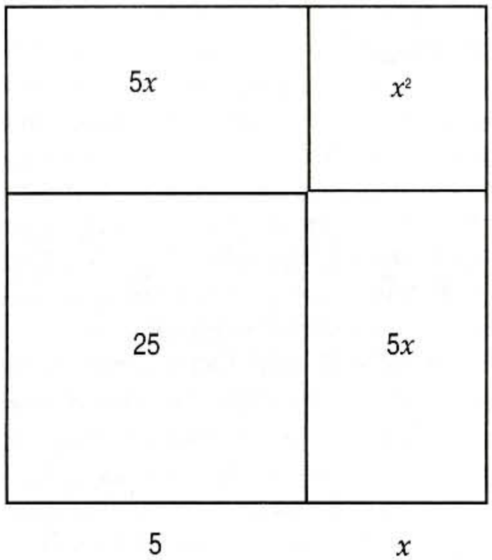
Simple equations of the first degree in one unknown are found in the oldest mathematical textbook, the Ahmes Papyrus of about 1700 B.C. Later quadratic equations appeared in Egypt in the context of area measurement. Square numbers such as $3^2 + 4^2 = 5^2$ were also used by Egyptians to construct right angles. Contemporary with the Egyptians, the ancient Babylonians also constructed tables of squares and cubes. Greek mathematicians were familiar with geometrical solutions of quadratic equations as early as the fifth century B.C. as it appears in the solutions of specific problems in the writings of Pythagoras, Hippocrates, and Euclid. Analytical solutions of quadratic equations appear around the beginning of the Christian Era in the works of Heron of Alexandria. Diophantus, the great Greek mathematician, solved analytically (about A.D. 250) the three types of quadratic equations ($ax^2 + bx = c$, $ax^2 + c = bx$, and $ax^2 = bx + c$, with positive coefficients and roots) in the context of solving other problems. In the fifth century B.C., Hindu mathematicians gave rules for the numerical solution of some quadratic equations using the method of completing the square. Though most of the algebraic ideas and techniques were known before Al-Khawarizmi, it is difficult to trace the algebra of Al-Khawarizmi to any of his predecessors. As Karpinski[3] comments:

> Yet we need to notice that we are dealing with the independent appearances of algebraic ideas and that the mathematics of Babylon, China, Greece, and India were developing from within (p. 11).

THE ALGEBRA OF AL-KHAWARIZMI

The algebra of Al-Khawarizmi will be briefly described using a photocopy (available at the Jafet Library of the American University of Beirut) of the English translation by Rosen[4] of the "Algebra of Mohammad Ben Musa" (Al-Khawarizmi). Rosen also included in his translation a printed version of the Arabic manuscript preserved in the Bodleian collection at Oxford.

Al-Khawarizmi starts his mathematical treatise by definitions of his basic mathematical terms: root or unknown, thing (variable), square (called 'money'), arithmetical operations, equality, equation. He then proceeds to define his mathematical concepts: first degree equation in one unknown, second degree equation in one unknown, binomial, trinomial, solution of an equation, proof. He ends this section by proving his first corollary that all six forms of quadratic equa-



$$(x+5)^2 = x^2 + 10x + 25$$
$$= 39 + 25$$
$$= 64$$

$$x+5 = 8$$
(negative roots were not recognized)

$$x = 3$$

Figure 1
Geometric proof for the case $x^2 + 10x = 39$

tions admissible under the conditions of positive solutions can be reduced to the three standard forms:

$$x^2 + px = q$$
$$x^2 + q = px$$
$$x^2 = px + q.$$

Al-Khawarizmi proceeds next to solve systematically each of the standard forms of quadratic equations using the method of completing the square, essentially in the form one would find in any high school algebra textbook three decades ago.

Geometric proofs are presented for each of the three cases. Figure 1 shows a geometric proof for the case of $x^2 + 10x = 39$.

Al-Khawarizmi, then, deals with binomials as numbers and introduces the four operations on them. Having developed the theory of quadratic equations, Al-Khawarizmi proceeds to apply his theory in four areas: numbers, mercantile transactions, measurement, and inheritance. The section on inheritance entitled "legacies" is by far the largest in the Al-Khawarizmi's book (more than half) and probably the most difficult to understand from a non-Islamic perspective. Inheritance laws are part of the Koran and hence are part of Islamic jurisdiction even in modern times. These laws—very sophisticated, detailed, and comprehensive—remain applicable even today. Because of the vastness of the Arab empire (from Morocco to China), many different algorithms existed in different parts of the empire. Developing standard techniques for dealing with inheritance problems was a priority for the judicial system of the state in order to work out inheritance deals and settle disputes. Al-Khawarizmi included in this section a comprehensive set of inheritance problems which model the various situations which may arise in Islamic inheritance laws. In many cases, the problems led to quadratic equations.

THE FIRST PARADIGM IN ALGEBRA

Historians of mathematics agree with the statement made by Karpinski[3] in the introduction to his translation to English of Chester's Latin manuscript:

> The activity of the great Arabic mathematicians Abu Abdallah Mohammed Ibn Musa Al-Khawarizmi marks the beginning of that period of mathematical history in which analysis assumed a place on a level with geometry; and his algebra gave a definite form to the ideas which we have been setting forth (p.13).

The work of Al-Khawarizmi in algebra constituted a core shared by a new scientific community of algebraists.

The prehistoric paradigm (prior to Al-Khawarizmi) was not a paradigm in Kuhn's sense. For instance, it is hard to establish that whatever algebraic knowledge had existed in the pre-historic period constituted a disciplinary matrix. None of the components of a disciplinary matrix as defined by Kuhn[1] is identifiable in the pre-historic algebraic paradigm. For one, no symbolic generalizations belonging to algebra had developed to be universally accepted by members of an identifiable scientific community. Not even a name

existed to describe the rudimentary and isolated algebraic knowledge that Greeks, Hindus, and Arabs had possessed in the pre-historic period. Since there was no distinct and independent body of knowledge in algebra in the pre-historic paradigm, one can safely say that there was no distinct scientific community (algebraists) identifiable within each of the cultures in which algebraic ideas were independently emerging. We present and provide support to the hypothesis that Al-Khawarizmi's algebra marked the first algebraic paradigm in Kuhn's sense. First, the technical language that Al-Khawarizmi used or developed constituted a well-defined set of symbolic generalizations which had been universally accepted by a scientific community of algebraists through the centuries. One example of the lasting impact of Al-Khawarizmi's symbolic generalizations is best represented by the name he gave to the new fledgling field of knowledge. Al-Jabr (Arabic root is "jabara" meaning either to "compel" or to "reduce a fracture") has been used to refer to the field all through the centuries and has been universally adopted by almost all languages. It is not conceivable to assume that the name algebra would have been universally adopted were it not for the significance of the referent field it denotes. The name algebra is unmistakably of Arabic origin and its etymology has been the subject of many investigations.[5] The controversy involves whether the meaning of word "jabr" in ordinary Arabic refers to a specific mathematical operation or the field of science itself. The work 'algorithm' itself derives from "Al-Khawarizmi".

The work of Al-Khawarizmi in algebra constituted a core shared by a new scientific community of algebraists. On the Arab side, some algebraists of whom the best known is Omar Khayyam (about 1045-1123 AD), who tried to extend Al-Khawarizmi's work to solutions of higher degree equations. Other algebraists, of whom Al-Karkhi (died about 1029) is best known, tried to surpass Al-Khawarizmi's work by attempting to "arithmetize" algebra; i.e., to apply arithmetical operations on algebraic expressions. For both trends, the work of Al-Khawarizmi was the starting point to the extent that many of the equations used by Al-Khawarizmi (for example, $x^2 + 10x = 39$) appeared in the algebras of Arab as well as European mathematicians[3]; "When towards the beginning of the twelfth century European scholars turned to Islam for light, the works of Mohammad Ibn Musa came to occupy a prominent place in their studies" (p. 23). Mohammad Ibn Musa, of course, refers to Al-Khawarizmi. The many translations of Al-Khawarizmi's treatise on algebra attest to its role as the core of shared algebraic knowledge of a growing scientific community of algebraists. The best known Latin translation of Al-Khawarizmi's treatise was done by Robert of Chester. Translations to other European languages were based on Chester's translation except for that of Rosen.[4] The translated manuscripts constituted the basic text for the study of algebra and a reference for scholars in the field.

The scientific community of algebraists came to share common beliefs and values regarding their field of study. One such basic belief was that of an existence of a mathematical system distinct from known mathematical systems at the time. The core of the value system is the appreciation of algebra as an applied field of mathematics besides being of value to mathematics itself. Such belief and value systems could not have developed in isolation from the work of Al-Khawarizmi.

Next we turn our attention to the body of knowledge (shared examples in Kuhn's sense) produced by Al-Khawarizmi to establish the extent to which this knowledge represented points of departure from the then existing mathematical knowledge.

A New Mathematical System

In retrospect, Al-Khawarizmi's algebra seems to be the first mathematical theory in algebra; i.e., the theory of the first and second degree equations in one unknown. An examination of the structure and development of Al-Khawarizmi's algebra (refer to section on "The Algebra of Al-Khawarizmi") will reveal a conscious effort to construct a coherent mathematical theory as we know it now:

1. Al-Khawarizmi's theory starts with clear definitions of technical terms, the most important of which for algebra are the concepts of "thing" (variable), "unknown", and "root." The basic concepts are then defined in terms of the technical terms and are independent of any application. The most significant concepts are those of first and second degree equations together with the related binomial and trinomial algebraic expressions.

2. Having laid the ground, Al-Khawarizmi charac-

terizes systematically the types of quadratic equations whose coefficients are positive rational numbers ($ax^2 = bx$, $ax^2 = c$, $bx = c$, $ax^2 + bx = c$, $ax^2 + c = bx$, $ax^2 = bx + c$). This development differs from pre-historic algebraic practice in that the purpose of quadratic equations was not to solve a series of problems but rather to characterize all types of quadratic equations as mathematical objects to be investigated independently of any application.

3. The development progresses to a higher level of abstraction by transforming the six types of quadratic equations to the three canonical forms ($x^2 + px = q$, $x^2 = px + q$, $x^2 + q = px$).

4. A general algorithm (completing the square) for solving the three canonical forms is then presented. The solution (in the set of positive rational numbers) is complete and covers all cases including the case of $x^2 + q = px$ where Al-Khawarizmi indicates that "the instance is impossible"[4] (p.12) when $q > (p/2)^2$.

5. Next, Al-Khawarizmi provides geometric proofs for each type of quadratic equation. Geometric proofs were very well known to the Greeks (see the section "Algebra before Al-Khawarizmi"). The value of Al-Khawarizmi's contribution in this regard is the systematic utilization of proof to validate statements (algorithms) that followed from a mathematical system.

6. The last step in the theory is the extension of the four operations, including the square root, to the binomial algebraic expressions of the form $ax + b$.

Algebraic expressions in their full generality are dealt with as numbers. In other words, we have a new mathematical system on a new set of mathematical objects.

A New Representation System

According to Kaput[6], a referential extension provides meaning to a notation system. For example, the meaning of second degree polynomial expressions in one unknown (a notation system) may be provided by their graphical referents ($x^2 + 2x + 3$ represents its graphical referent; i.e., the parabola). When there is a well-defined correspondence between the syntax of the notation system A and the syntax of referential extension B, "then we often refer to A and B and the correspondence between them as comprising a representation system" (p.169).

What then are the contributions of Al-Khawarizmi's algebra to representation systems? One can cite three specific contributions in this regard. First, Al-Khawarizmi extended the notation system of positive rational numbers to include the "unknown" or "root" and the "square" as new notations (though in words and not letters) in such a way that the syntax rules of positive rational numbers apply to the new notations. It is clear from the opening paragraphs of his book that such a coherent extension of the notation system was his first task. His extension of the notation system of positive rational numbers to include the "variable" is a milestone in the history of representation systems. Second, Al-Khawarizmi used natural language as a referential extension for his algebra. The correspondence between the syntax of the notational system and that of natural language was well-defined and consistent. The representation system thus developed was of such power that it dominated algebra for four centuries and was used not only in the original language of the text but also in the languages to which Al-Khawarizmi's book was translated (Latin and other European Languages). Third, Al-Khawarizmi used geometric figures to represent the extended positive rational numbers (a notation system). A number (positive rational or "unknown") was represented by a line segment and a product by a rectangle. The correspondence between the syntactic structures of the two systems is natural and well-defined. Al-Khawarizmi was not by any means the inventor of this kind of representation because it was very well known to the Greeks and others. The contribution of Al-Khawarizmi in this regard is that he pushed the idea of representing numbers by figures to the level of a well-developed representation system.

A Model of a Mathematical System

Up to Al-Khawarizmi's time, the thrust of the development of mathematics had been either to develop a purely mathematical system (Euclidean geometry, for example) or to find mathematical techniques to deal with specific types of situations (Egyptian and Babylonian mathematics). Al-Khawarizmi's work marks an early and rare example in which the full cycle of a model was achieved. The algebra of Al-

Khawarizmi moved from identifying situations to modeling such situations by a mathematical system and validating the mathematical system by applying the latter to a variety of situations much broader than the one with which he started. That Al-Khawarizmi had exactly that frame of mind is clear from his manuscript.[4] In the preface to his book, he mentions that his intention was to provide:

> What is easiest and most useful in arithmetic, such as men constantly require in cases of inheritance, legacies, partition, law-suits, and trade, and in all their dealings with one another, or where the measuring of lands, the digging of canals, geometrical computation, and other objects of various sorts and kinds are concerned-relying on the goodness of my intention therein... (p.3).

Next, Al-Khawarizmi proceeds to develop a mathematical system which, though motivated by the contexts mentioned earlier, transcends these contexts and is independent of any of them. When the mathematical system (quadratic equations) is fully developed it is again applied in a variety of contexts including those which had motivated the development of the mathematical system itself.

PEDAGOGICAL IMPLICATIONS

The pedagogical obstacles that students experience in learning algebra derive, to a large part, from two sources: prior arithmetical experience and translation from natural language. Research has indicated many pedagogical problems arising from prior arithmetical experience. One example of such difficulties is the process-product dilemma; i.e., the failure to accept that the process "$x + 3$" is itself the final answer (product).[7,8] Another example of such difficulties is concatenation, which denotes implicit addition in arithmetic (for example, 3 1/2) and multiplication in algebra (for example $6x$). Among the many difficulties that students encounter due to the translation of natural language is the misconception that a letter that represents a word ("b" for blue) represents a set in natural language but represents a number in algebra.[9]

Clement, Lochhead, and Soloway[10] have identified another difficulty which arises from syntactic translation from natural language to variable symbols (for example, "six times as many students as professors" is very often mistakenly translated into 6S = P, where S is the number of students and P the number of professors).

Algebra and Prior Arithmetical Experience

What lessons can we learn from the early stages of the evolution of algebra regarding the nature of such pedagogical obstacles? If one attempts to analyze the nature of the two pedagogical obstacles arising from prior arithmetical experience or translation from natural language, one is likely to observe that both obstacles relate to the representation of algebra. The pedagogical obstacle associated with prior arithmetical experience may probably be traced to the misconception that the symbol system of the alphabet with its morphology is a representation of algebra. Obviously, this is not the case because there is no well-defined correspondence between the morphology of the alphabet and that of algebra. The early algebra of Al-Khawarizmi provides a helpful insight into the relationship between arithmetic and algebra. Al-Khawarizmi views the relationship as that of extension rather than representation. In other words, the arithmetical system is extended by adding new symbols for new numbers; i.e., the root and the square. Consequently the syntax of arithmetic applies to the extended set in a natural way. In my judgment, this view of the relationship between arithmetic and algebra is critical in removing many misconceptions in the early stages of learning and teaching algebra.

Algebra and Natural Language

The relationship of algebra to natural language is another problematic area which may benefit from studying the historical development of algebra. The typical algebra curriculum has been characterized by an early introduction of variable and operation symbols. In the last three decades, a trend has been observed in which intermediate representations of place-holders are used as variable symbols in elementary classes in mathematics to prepare students for the introduction of the formal standard algebraic symbols in post-elementary grades. The early introduction of a symbol system does not seem to correspond to the historical development of algebra which was represented by natural language by Al-Khawarizmi and subsequently by other algebraists for four centuries. There is a natural correspondence between the syntax of the natural lan-

guage and that of the generalized arithmetic of algebra. According to Kaput[6], one major source of mathematical meaning is "via translations between mathematical representations and non-mathematical systems" (p.168). It would perhaps be worth considering the payoff of using natural language to represent introductory algebraic concepts. Natural language is normally fully developed at the time students start their study of algebra. It is plausible to assume, therefore, that representation via a familiar and well-developed system is more effective in providing meaning than via an unfamiliar system (like the alphabet). This possibility ought to be looked into seriously, particularly because it had served the science of algebra for more than four centuries.

Geometric Representation of Quadratic Equations

A promising system for representing algebra is the sub-system of geometry. Al-Khawarizmi adapted this representation from Greek mathematicians and used it extensively in providing "proofs" for his algorithms for solving quadratic equations. The power of this representation resides in its visual concrete form, a feature which renders it valuable for beginning algebra students. The correspondence between the two is simple and natural. Segments represent numbers and "unknowns" by matching the length of a segment to the number or "unknown"; addition corresponds to joining, and multiplication corresponds to the area of the rectangle whose sides are the factors. Dienes Blocks provide a concrete representation system of this kind. By using a flat square of side x (x square), a flat rectangle (one x) and a (1×1) small square, trinomials can be concretely represented.[11] This linkage between algebra and metric measures of geometric figures is a powerful pedagogical tool which merits more attention. It was an efficient tool in the hands of the shaper of algebra, and there should be no reason why it should not be as powerful in the hands of a beginning learner of algebra.

Algebra and Applications

One last lesson which may benefit the pedagogy of algebra is the fact that the genesis of algebra was that of a science and not of a symbol system. Al-Khawarizmi's algebra was grounded in the needs of the society at the time, such as mercantile transactions, measurement, and inheritance. Although Al-Khawarizmi elevated his algebra to a theory which transcended the applications that initially motivated it, he proceeded to show the power of his theory in the extensive real-life applications which constituted more than half his book. Perhaps there is merit in patterning the learning of algebra after that of its historical development. If this is the case, we might as well de-emphasize algebra as a symbol system whose syntax is to be mastered in favor of structuring algebra as a science which is grounded in real life applications.

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